Portfolio diversification and internalization of production externalities through majority voting

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May 2005

Abstract

A general equilibrium model with uncertainty and production externalities is studied. In absence of markets for externalities, we look for governances and conditions under which majority voting among shareholders is likely to give rise to efficient internalization. We argue that the financial market clearing conditions endogenously set up, within the firms, social choice configurations where the (perfectly diversified) market portfolio, which gives the right incentives for economic efficiency, happens to be a good (and sometimes the best) candidate in the political process, i.e., a candidate with good stability properties with respect to the majority rule. The central and natural role played by a governance of stakeholders is underlined and benchmarked.

Keywords: Production externalities, majority voting, portfolio diversification, general equilibrium, stakeholder governance, mean voter.

JEL-classification: D21, D52, D71, G39.

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1 Introduction

Since markets fail in efficiently allocating public goods, mechanisms founded on majority voting appear as natural and legitimate alternatives to the market mechanism. Whether such collective-decision mechanisms can implement efficient allocations is a question that has been studied since Bowen (1943). In the latter, Bowen shows that if the voters' marginal rates of substitution are symmetrically distributed, then Pareto optimal allocations happen to be the optimal choice of the median voter, hence stable under the 50%-majority voting rule; this argument has been extended in Bergstrom (1979). Other papers studying this problem are, e.g., Barlow (1970).

Bowen's argument is fundamental and will reveal very usefull to explain the contributions of the present paper. Consider an economy with one private good, taken as the numéraire, and one public good, produced using the private good as the input through a technology with decreasing returns to scale. At a given allocation, let p_i be the marginal rate of substitution (gradient) between the public and the private goods for consumer/voter i, and let p be the marginal rate of technical substitution for the firm. The so-called Bowen-Lindahl-Samuelson (necessary) condition for the allocation to be optimal is $\sum_i p_i = p$. In words, p should be collinear to the *average* (or mean) of the p_i's, where all voters have equal weights (and equal contribution to the input) in the aggregation process. Of course, gradient vectors $(p_i)_i$ here are unidimensional, therefore a median voter can be defined, and if the voters' gradients are symmetrically distributed around p, then the *mean is the median*. Hence the unique political equilibrium (the median) happens to be the efficient allocation (the mean).

Bowen's approach has two caveats: (1) in general, the median is not the mean; (2) as soon as there are more than one public good, the concept of median voter is tricky to generalize[1], and one knows since Plott (1967) that a equilibrium usually does not exist[2]. But there is a comforting observation: caveat (2) makes caveat (1) not so embarassing after all. There is more: Among the successful attempts made in social choice theory to go beyond the one-dimensional setup, Caplin and Nalebuff (1988, 1991) give a strong argument in favor of the mean: under some conditions on individual preferences and on the distribution of these individual preferences, the mean voter is a stable outcome, hence a political equilibrium, for super majority rules with a not too conservative rate of super majority (inferior to 64%); and under some additional conditions, the mean happens to be the min-max, i.e., the political outcome which remains stable under the lowest possible

[1] Greenberg (1979) may be seen as a way to generalize this median voter argument to multidimensional settings.

[2] Grandmont (1978) gives an extension of the conditions on preferences, and on the distribution of preferences under which a 50%-majority political equilibrium exists in a multi-dimensional setup.

rate of super majority. In the light of Bowen's argument, we believe that the latter strand of research reinforces the hope that *super majority voting can lead to an efficient allocation of public goods*.

In the present paper, we address the problem of managing production externalities. Externalities are ordinary public goods. If there is no market mechanism to internalize them, then shareholders do not agree on how the firm should be managed: there is no unanimity for profit maximization. Actually only an undiversified shareholder, with shares only in the considered firm, would favor profit maximization. Another way to state it is that, like in the incomplete financial market case (and to some extent, the absence of markets to price external effects is a form of market incompleteness) shareholders disagree on the price with respect to which profit should maximized.

Therefore, unlike in the traditional neo-classical approach, in this paper firms do not have a specified objective function. Their behavior is modeled as representing the shareholders' interests in the following sense: the firm provides a production plan that suits its shareholders inasmuch as no alternative production plan makes a (super) majority of them better off. A simple observation drives the analysis: when comparing, within a firm, the incumbent production plan with a proposed challenger, a *diversified* shareholder will take into account the impact of the proposed change on his own welfare not only through his share in the concerned firm, but also through his shares *in the other firms*. Hence, when expressing his opinion on whether or not the challenger should be implemented instead of the status quo, a diversified shareholder *internalizes* production externalities. He might do it in a biased way, since the internalization goes though his portfolio which is not necessarily diversified enough, but he internalizes. And the question is: under which governance and conditions is the aggregated shareholders' choice (through majority voting) efficient?

Hansen and Lott (1996) studies this problem mostly from an applied point of view. In a simple and crisp model with two firms and two non-consuming shareholders, it shows that if shareholders are *perfectly* diversified, in the sense that both have fifty percent of both firms in their portfolios, then they both have the right incentives for perfect internalization of externalities. In other terms, a perfectly diversified shareholder has the same incentives as a public planer and his own maximization program will necessarily entail the Bowen-Lindahl-Samuelson condition. Hansen and Lott (1996) documents the extent of diversification and cross-ownership of stocks among companies where production externalities are likely to be large. They argue that besides the traditional benefit of risk reduction, portfolio diversification offers additional benefits to shareholders through helping internalize externalities.

The present paper builds on this argument, and extends the approach to a general equilibrium framework. Of course we generalize Hansen and Lott (1996)'s observation and characterize governances and conditions under which efficient internalization comes with perfect diversification (Proposition 1 and Corollary 2); not surprisingly, these conditions are closely linked to the conditions under which muti-fund separation theorems hold true: roughly, all agents are perfectly diversified because they hold the market portfolio, and the perfectly diversified market portfolio, like a global merger, gives the right incentives for efficient internalization.

Outside these conditions, we study conditions under which the (imperfect) portfolio diversification endogeneously sets up, within the firms, social choice configurations in which the efficient choice of external effects has good stability properties in the political process, and is likely to be the outcome of that process. Another simple observation gives some intuition about the avenue we follow: Even though individual portfolios are not perfectly diversified, the financial market clearing condition guarantees that they add up to the market portfolio; hence the portfolio that gives the right incentives for efficient internalization happens to be *the mean portfolio.* (This parallels the introductory example, where the price giving the right incentives for efficient public good provision is the mean individual MRS.) The link between efficient internalization and the social choice literature on the mean voter lies in this observation.

Since the market clearing condition gives equal weight to all consumers in the economic exchange mechanism, we are naturally led to the notion of **stakeholder democracy** (denoted μ^d in the sequel) where all consumers have equal voting weight in the political process of firms. It appears to be the governance that is the most likely to promote the 'political first welfare theorem' we aim at, namely that voting gives rise to efficient internalization of production external effects. Beside the traditional arguments of social choice theory, we provide two original statistical approaches which allow us to underline the fundamental role played by the stakeholder democracy in promoting economic efficiency (Theorem 3 and Theorem 4). Both approaches yield the same (asymptotic) claim: efficient internalization almost surely results from the 50%-majority voting rule in a stakeholder democracy.

There are strands of the literature obviously linked with the present one. Between-firms production external effects are just one instance in which markets fail in leading the shareholders to unanimously support some specified objective function. Another instance is when financial markets are incomplete. Of course, collective decision mechanisms have been studied for that problem too. A mechanism based on Lindahl pricing (side payments) was proposed by Drèze (1974), and then Grossman and Hart (1979), in order to recover (constrained) efficiency. Along a different avenue, following Gevers (1974), mechanisms based on majority voting have naturally been proposed and studied (see, e.g., Benninga and Muller (1979), Drèze (1985), Sadanand and Williamson (1991), De Marzo (1993), Kelsey and Milne (1996)). Crès and Tvede (2004) reconciles these two approaches along the line proposed in the present paper: the Drèze (1974) criterion to recover (the first order conditions of constrained) efficiency indicates that production should be optimized with respect to the gradients of the *mean* shareholder; and Caplin and Nalebuff (1991) gives conditions under which this efficient choice is likely to be the outcome of the voting process.

The paper is constructed as follows: Section 2 introduces the model, defines the market and political equilibrium concepts, sets the assumptions and provides the conditions for efficient internalization. In Section 3 the fundamental structure of the political problem is exposed; two polar cases are provided: one with maximal disagreement between shareholders, another one with no disagreement: conditions are given under which shareholders are unanimous, a unanimity which is shown to always results in efficient internalization; based on the distribution of individual portfolios and their relative diversification, we provide an index of political stability for the social choice problem within the firms. Then Section 4 provides our main results, obtained in stochastic environments, which underline the efficiency property of the stakeholder democracy. Finally, the latter is benchmarked in Section 5 through the design of a better performing governance.

2 The model

Consider an economy with 2 dates, $t \in \{0, 1\}$, 1 state at the first date $s = 0$, and S states at the second date $s \in \{1, \dots, S\}$. There are: 1 commodity at every state, a finite number of consumers with $i \in \mathcal{I}$ where $\mathcal{I} = \{1, \dots, I\}$ and J firms where $J = S$ with $j \in \mathcal{J}$ where $\mathcal{J} = \{1, \dots, J\}$. Consumers are characterized by their identical consumption sets $\mathcal{X} = \mathbb{R}^{S+1}$, initial endowments $\omega_i \in \mathbb{R}^{S+1}$, utility functions $u_i : \mathcal{X} \to \mathbb{R}$, and initial portfolio of shares in firms $\delta_i = (\delta_{i1}, \dots, \delta_{iJ})$, where $\delta_{ij} \in \mathbb{R}$ and $\sum_{i \in \mathcal{I}} \delta_{ij} = 1$ for all j. Firms are characterized by their sets of action $\mathcal{A}_j$ and production functions $F_j : \mathcal{A} \to \mathbb{R}^{S+1}$ where $\mathcal{A} = \prod_{j \in \mathcal{J}} \mathcal{A}_j$, so if the action of firm k is a_k then $y_j = F_j(a_1, \dots, a_J)$ is the production plan of firm j. The sets of possible action are described by maps $G_j : \mathbb{R}^{n_j} \to \mathbb{R}$ such that $\mathcal{A}_j = \{a_j \in \mathbb{R}^{n_j} \,|\, G_j(a_j) \leq 0\}$.

Individual programs and equilibrium concepts

Let $q = (q_1, \dots, q_J)$ where $q_j \in \mathbb{R}$ is the price of shares in firm j, be the price system. Consumers choose consumption plans $x_i \in \mathcal{X}$ and portfolios $\theta_i \in \mathbb{R}^J$. Firms choose action $a_j \in \mathcal{A}_j$.

The program of consumer i given a price system for shares and a collection of individual actions (q, a) where $a = (a_1, \ldots, a_J)$ is

$$\max_{x_i, \theta_i} \quad u_i(x_i)$$
$$\text{s.t.} \quad \begin{cases} x_i^0 - \omega_i^0 = \sum_j q_j \delta_{ij} - \sum_j (q_j - y_j^0)\theta_{ij} \\ x_i^s - \omega_i^s = \sum_j y_j^s \theta_{ij} \text{ for all } s \geq 1. \end{cases} \tag{1}$$

There are no strategic considerations involved in the choice of consumption plans and portfolios.

Definition 1 *For a collection of individual actions a, a* **stock market equilibrium with fixed actions** *denoted SME(a), is a price system for shares and a collection of individual consumption plans and portfolios $(\bar{q}, \bar{x}, \bar{\theta})$ where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_I)$ and $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_I)$, such that:*

- *consumers maximize their utilities: $(\bar{x}_i, \bar{\theta}_i)$ is a solution to the program of consumer i given $(\bar{q}, a)$;*
- *markets clear: $\sum_i \bar{x}_i = \sum_i \omega_i + \sum_j F_j(a)$ and $\sum_i \bar{\theta}_i = \sum_i \delta_i$.*

The study will be restricted to the case of complete financial markets. Therefore at a SME(a) the gradients of the consumers/shareholders are collinear. Let $\bar{p} \in \mathbb{R}^{S+1}$ denote the common normalized identical gradient vector (by normalizing its date zero component to one).

Although financial markets are complete, there is market incompleteness in the present model because there are no market for externalities. Hence shareholders disagree on how firms should be managed. In particular, there is no unanimity of shareholders for profit maximization with respect to $\bar{p}$. Indeed the only case where a shareholder wants a firm to maximize profits with respect to $\bar{p}$ is the case where he is not affected by the externalities of the firm, so his portfolio is totally undiversified in the sense that he only has shares in the considered firm. In order to formalize this intuition consider the following formalization.

Assuming differentiability, for all j and k let $\bar{p}_{jk} \in \mathbb{R}^{n_j}$ denote the 'price' vector of marginal externalities of firm j on firm k, so

$$\bar{p}_{jk} = \bar{p} D_{a_j} F_k(a). \tag{2}$$

This price vector prices the action of firm j according to their marginal impact on firm k. Profit maximization in production plans with respect to $\bar{p}$ is equivalent to profit maximization in actions with respect to $\bar{p}_{jj}$. Next, let $\bar{p}_{ij}$ denote the 'price' vector of marginal changes in action on consumer i, so $\bar{p}_{ij}$ is the price vector that firm j should use to price actions according to consumer i. Therefore

$$\bar{p}_{ij} \;=\; \sum_k \bar{\theta}_{ik}\bar{p}_{jk}. \tag{3}$$

From this construction it becomes clear that externalities make shareholders disagree on the optimal production plans for the firms: Each consumer wants the action of firm j to be chosen optimally with respect to his idiosyncratic pricing vector $\bar{p}_{ij}$[3].

The fact that markets fail to push shareholders to agree on the way to price actions in firms naturally leads to the study of whether collective decision mechanisms can help reduce the market failure. In the present paper we focus on majority voting mechanisms. But the first obstacle that arises is the generic non-existence of 50%-majority voting equilibria in multidimensional setups. A way out is to consider super majority voting.

Firms are not modelled as optimizing some specific objective function. Indeed their behavior is modelled as representing the shareholders' interests through a centralized political process: the firm provides a production plan that is suitable to its shareholders in the sense that no alternative production plan makes a (super) majority of them better off. The political process allows us to define preferences for the firms as follows.

Let $\mathcal{A}_{ij}(x_i, \theta_i, a) \subset \mathcal{A}_j$ denote the set of actions for firm j that at the consumption bundle, portfolio and collection of individual actions (x_i, θ_i, a) make consumer i better off, so

$$\mathcal{A}_{ij}(x_i, \theta_i, a) \;=\; \{a'_j \in \mathcal{A}_j | u_i(x_i + \textstyle\sum_k \theta_{ik}[F_k(a'_j, a_{-j}) - F_k(a)]) > u_i(x_i)\}.$$

Next, at (x, θ, a) let $\mathcal{I}_j(x, \theta, a, a'_j - a_j)$ denote the set of consumers who are better off with action a'_j than with action a_j for firm j, so

$$\mathcal{I}_j(x, \theta, a, a'_j - a_j) \;=\; \{i \in \mathcal{I} | a'_j \in \mathcal{A}_{ij}(x_i, \theta_i, a)\}.$$

Finally let $\mu_j = (\mu_{1j}, \cdots, \mu_{Ij})$ where $\mu_{ij} \geq 0$, be a collection of individual voting weights for decision making in firm j. Then for a rate of majority $\rho \in [0, 1]$ preferences of firms are described by correspondences $P^\rho_j : \mathcal{X}^I \times \mathbb{R}^{IJ} \times \mathcal{A} \times \mathbb{R}^I_+ \to \mathcal{A}_j$ defined by

[3]This is the case also with incomplete financial markets: A shareholder wants profit to be maximized with respect to his idiosyncratic equilibrium gradient; and at equilibrium shareholders' gradients are typically not collinear because utility maximization only make gradients orthogonal to the subspace of possible income transfers, which has codimesion two at least in case of incomplete financial markets.

$$P_j^\rho(x,\theta,a,\mu_j) = \begin{cases} \emptyset & \text{for } \sum_i \mu_{ij} = 0 \\ \{a_j' \in \mathcal{A}_j | \frac{\sum_{i \in \mathcal{I}_j(x,\theta,a,a_j'-a_j)} \mu_{ij}}{\sum_i \mu_{ij}} > \rho\} & \text{for } \sum_i \mu_{ij} > 0. \end{cases}$$

Thus a_j is a solution to the program of firm j if $P_j^\rho(x,\theta,a,\mu_j) = \emptyset$.

Definition 2 $(\bar{q},\bar{x},\bar{\theta},\bar{a})$ *is a* ρ**-majority stable equilibrium** *for governance* μ*, denoted* ρ*-MSE(*μ*), if*

- $(\bar{q},\bar{x},\bar{\theta})$ *is a SME(*$\bar{a}$*), and;*
- $\bar{a}_j$ *is a solution to the program of firm* j*, so* $P_j^\rho(\bar{x},\bar{\theta},\bar{a},\mu) = \emptyset$.

A collection of individual consumption bundles and actions (x,a) is denoted a *state* and a collection of individual consumption bundles, portfolios and actions (x,θ,a) is denoted an *extended state.*

Assumptions

Consumer i is supposed to satisfy the following assumptions:

(A.1) $u_i \in C^1(\mathcal{X},\mathbb{R})$ with $Du_i(x) \in \mathbb{R}_{++}^{S+1}$ for all $x \in \mathcal{X}$.

(A.2) u_i is quasi-concave and $u_i^{-1}(r)$ is bounded from below for all $r \in \mathbb{R}$.

Both assumptions are standard (see Balasko (1988)).

For firm j the set of actions is supposed to satisfy the following assumptions:

(A.3) $\mathcal{A}_j$ is compact and convex.

(A.4) $G_j \in C^1(\mathbb{R}^{n_j},\mathbb{R})$ with $DG_j(b) \neq 0$ for all $b \in \mathbb{R}^{n_j}$.

Assumption (A.4) implies that the set of actions is a n_j-dimensional manifold. For firm j the production function is supposed to satisfy the following assumptions

(A.5) $F_j : \mathcal{A} \to \mathbb{R}^{S+1}$ is concave in each variable.

(A.6) $F_j \in C^1(\mathcal{A},\mathbb{R}^{S+1})$.

Assumptions (A.3) and (A.5) ensure that, for fixed actions a_{-k} of all firms but firm k, the production set of firm j

$$\mathcal{Y}_j(a_{-k}) \;=\; \{Y_j \in \mathbb{R}^{S+1} | Y_j = F_j(a_k, a_{-k}) \text{ for some } a_k \in \mathcal{A}_k\}$$

is concave.

The production sector is supposed to satisfy the following assumptions:

(A.7) For all $a = (a_1, \dots, a_J)$ the matrix

$$Y(a) = \begin{pmatrix} F_1^1(a) & \cdots & F_J^1(a) \\ \vdots & & \vdots \\ F_1^S(a) & \cdots & F_J^S(a) \end{pmatrix}$$

has full rank.

(A.8) For all actions $a = (a_j, a_{-j})$ and firms j if $a_j \in \operatorname{int} \mathcal{A}_j$ then there exists $a'_j \in \mathcal{A}_j$ such that $\sum_k F_k(a'_j, a_{-j}) \geq \sum_k F_k(a_j, a_{-j})$ and $\sum_k F_k(a'_j, a_{-j}) \neq \sum_k F_k(a_j, a_{-j})$.

(A.9) For all a, for all j, the matrix $D_{a_j} F(a)$ has rank n_j.

Assumption (A.7) excludes that firms are able to replicate production plans of each other. (A.8) ensures that only collections of individual actions in the boundaries of action sets produce efficient production plans. Finally (A.9) excludes superfluous actions: a change in the action of a firm must produce a change in the production plan of some firm.

All assumptions are supposed to be satisfied in the sequel.

Let us close this subsection by showing that our assumptions ensure the **principle of minimal differentiation**: To maximize the support for a challenger against the status quo, infinitesimal changes of action perform better than large changes. This principle is secured by two facts: (1) for any given technologically feasible change of action Δa_j for firm j, all infinitesimal changes $da_j = \epsilon \Delta a_j$, with $\epsilon \to 0$, are also feasible (Lemma 1); and (2) the quasi-concavity of the utility functions guarantees that $\mathcal{I}_j(x, \theta, a, \Delta a_j) \subset \mathcal{I}_j(x, \theta, a, da_j)$: a challenger never looses support by shortening the length of the change of action it proposes.

Let $T_j(a_j) \subset \mathbb{R}^{n_j}$ be defined by

$$T_j(a_j) \;=\; \begin{cases} \mathbb{R}^{n_j} & \text{for } a_j \in \operatorname{int} \mathcal{A}_j \\ \{b_j \in \mathbb{R}^{n_j} \,|\, DG_j(a_j) \cdot b_j \leq 0\} & \text{for } a_j \in \operatorname{bd} \mathcal{A}_j. \end{cases}$$

Lemma 1 *For all $a_j, a'_j \in \mathcal{A}_j$ and $a_{-j} \in \prod_{k \neq j} \mathcal{A}_k$,*

$$F(a'_j, a_{-j}) - F(a_j, a_{-j}) \;\in\; D_{a_j} F(a_j, a_{-j}) T_j(a_j).$$

Proof: Clearly $\mathcal{A}_j \subset \{a_j\} + T_j(a_j)$ according to (A.3). So

$$\mathcal{Y}(a_{-j}) \ \subset \ \{F(a_j, a_{-j})\} + D_{a_j}F(a_j, a_{-j})T_j(a_j)$$

according to (A.5). Therefore $F(a'_j, a_{-j}) - F(a_j, a_{-j}) \in D_{a_j}F(a_j, a_{-j})T_j(a_j)$.

Q.E.D

Efficiency conditions

Let us first define Pareto optimal states.

Definition 3 *A state $(x, a) \in \mathcal{X}^I \times \mathcal{A}$ is Pareto optimal if there does not exist another state $(x', a') \in \mathcal{X}^I \times \mathcal{A}$ such that:*

- $\sum_i x'_i \leq \sum_i \omega_i + \sum_j F_j(a')$.
- $u_i(x'_i) \geq u_i(x_i)$ *for all i with "$>$" for at least one consumer.*

The following lemma provides the usual necessary conditions for Pareto optimality.

Lemma 2 *A state $(x, a) \in \mathcal{X}^I \times \mathcal{A}$ is Pareto optimal only if:*

- *There exist a normalized vector of 'state prices' $p \in \{1\} \times \mathbb{R}^S_{++}$, and a collection of individual multipliers $(\nu_i)_i$ where $\nu_i > 0$ such that $Du_i(x_i) = \nu_i p$.*
- *Let $p_j \in \mathbb{R}^{n_j}$ be defined by $p_j = DG_j(a_j)$ and let $(p_{jk})_k$ be defined by Equation (2), then*

$$\sum_{k \in \mathcal{J}} p_{jk} \in \langle p_j \rangle \tag{4}$$

  where $\langle p_j \rangle$ is the span of p_j.

Moreover if the aggregate production function $a \to \sum_j F_j(a)$ is concave, then the conditions are sufficient.

Proof: If the state $(\bar{x}, \bar{a})$ is Pareto optimal, then it is a solution of the following optimization program:

$$\max_{x,a} \quad u_1(x_1)$$

$$\text{s.t.} \quad \begin{cases} u_i(x_i) \leq u_i(\bar{x}_i) & \text{for } i \geq 2 \\ G_j(a_j) \leq 0 & \text{for all } j \\ \sum_i (x_i - \omega_i) \leq \sum_j F_j(a) & \end{cases}$$

Thanks to assumptions (A.1) and (A.8), one has at the solution: $G_j(a_j) = 0$ for all j, and $\sum_i (x_i - \omega_i) = \sum_j F_j(a)$. First order derivatives of the Lagrangean with respect to x and a give the conditions of the lemma.

Q.E.D

Corollary 1 *For a SME(a), $(\bar{q}, \bar{x}, \bar{\theta})$, the state $(\bar{x}, a)$ is Pareto optimal only if Equation (4) holds.*

3 Fundamentals of firms' politics

In this section we focus on the structural aspects of the political game inside the firms since, at stock market equilibria, there typically is some disagreements between shareholders on the aim of a firm. Firstly we introduce two extreme cases where the degree of conflict is maximal, resp. minimal. Secondly we characterize coalitions which are never unanimous when proposed some alternative to the status quo; based on this characterization, we provide a measure of the degree of disagreement between shareholders. Thirdly this construction is applied to provide a measure of portfolio diversification.

Illustrations

Consider a SME(a) at which the $(\bar{p}_{ij})_{i\in\mathcal{I}}$ are linearly independent. Then shareholders completely disagree on the price that the firm should maximize its action with respect to. Indeed there exist alternatives to a_j such that all shareholders but one are better off; moreover no other action is stable for a lower rate of majority. Hence, in a stakeholder democracy only super majority rules with $\rho > (I-1)/I$ support perfect internalization. This result is in line with the result in Greenberg (1979) which states that $(I-1)/I$ is the lowest super majority rule that ensures existence of equilibrium in voting models where the dimension of conflict is $I-1$ as in the present case.

Alternatively, at the other extreme, consider a SME(a) where shareholders have collinear pricing vectors $\bar{p}_{ij}$ pointing in the same direction. This happens, e.g., when they are perfectly diversified: for each shareholder there exists a $\tau_i > 0$ such that $\theta_i = \tau_i \mathbf{1}_J$ where $\mathbf{1}_J \in \mathbb{R}^J$ is the market portfolio[4]. Then all shareholders agree that every firm should

[4] The condition that all shareholders are perfectly diversified can be weakened. Suppose that firms are partitioned into L *clusters*, $\mathcal{J}_1$, $\mathcal{J}_2$, ..., $\mathcal{J}_L$, of respective size $J_1, \ldots, J_L$ such that $\mathcal{J}_\ell \cap \mathcal{J}_{\ell'} = \emptyset$ if $\ell \neq \ell'$ and $\cup_\ell \mathcal{J}_\ell = \mathcal{J}$, so that if two firms pertain to two different clusters, they do not inflict external effects upon each other: let $\bar{p}_{jk} = 0$ whenever $j \in \mathcal{J}_\ell$ and $k \in \mathcal{J}_{\ell'}$ with $\ell \neq \ell'$. It is easy to define the finest clustering of the economy. Then a weaker sufficient condition for shareholders' unanimity for efficient internalization is that shareholder are all perfectly diversified *within clusters*: there exists a L-vector $\tau_i = (\tau_{i\ell})_\ell$ such that $\bar{\theta}_i = (\tau_{i1}\mathbf{1}_{J_1}, \ldots, \tau_{iL}\mathbf{1}_{J_L})$. Such a generalization of the paper is trivial and will in

maximize its action with respect to the efficient pricing vector $\bar{p}_j = \sum_k \bar{p}_{jk}$. Indeed, market clearing yields: $\sum_i \bar{p}_{ij} = \sum_k \bar{p}_{jk} = \bar{p}_j$, and the $\bar{p}_{ij}$'s being collinear, for all i $\bar{p}_{ij}$ is collinear to $\bar{p}_j$ pointing in the same direction. Therefore we make the following remarkable observation that underlines the special role played by the market portfolio in the political game:

Observation 1 *If all shareholders agree, then they unanimously support perfect internalization.*

Conditions on consumers under which they unanimously support efficient internalization are studied now. We consider here economies where one firm, say firm 1, does not inflict not receive any production external effects, and whose only possible action leads to the provision of the riskless security $(0, \mathbf{1}_J) \in \mathbb{R}^{S+1}$. So firm 1 forms a cluster on its own and all other firms form a second cluster. Such an economy is known in the finance literature as a *bond-equity* economy. For shareholders' unanimity for efficient internalization to obtain, it is sufficient that shareholders be perfectly diversified within the second cluster. This is exactly what happens when consumers have von Neumann-Morgenstern additively separable utility functions with linear risk tolerance and the same marginal risk tolerance, as proved in Cass and Stiglitz (1970) (see also Magill and Quinzii (1997), section 16, for a modern, integrated treatment of that case).

(A.10) Utility functions are

$$u_i(x_i) = u_{i0}(x_i^0) + \sum_{s=1}^{S} \pi^s u_{i\mathbf{1}}(x_i^s)$$

where u_{i0} and $u_{i\mathbf{1}}$ are strictly increasing and strictly concave. The risk tolerance

$$T_i(x) = -\frac{u'_{i\mathbf{1}}(x)}{u''_{i\mathbf{1}}(x)}, \quad x \in \mathbb{R}$$

is linear: there exists $(\alpha_i, \beta_i) \in \mathbb{R}_+ \times \mathbb{R}$ such that $T_i(x) = \alpha_i + \beta_i x$ on the relevant domain, i.e., whenever $\alpha_i + \beta_i x > 0$. And agents all have the same marginal risk tolerance: $\beta_i = \beta$ for all i.

This assumption yields the hyperbolic constant absolute risk aversion (HARA) class of utility functions:

$$u_{i\mathbf{1}}(x) = \begin{cases} \dfrac{(\alpha_i + \beta x)^{1-1/\beta}}{1/\beta(1-1/\beta)} & \text{if} \quad \beta \neq 0, \beta \neq 1 \\ -\alpha_i e^{-x/\alpha_i} & \text{if} \quad \beta = 0 \\ \log(\alpha_i + x) & \text{if} \quad \beta = 1 \end{cases}$$

general be omitted, for the sake of the lightness of the notation, by sticking to only one cluster.

The case $\beta \geq 0$ includes power functions with power less than 1, the log and the negative exponential. When $\beta > 0$ and $\alpha_i = 0$ the utility function exhibits constant relative risk aversion. The quadratic case corresponds to $\beta = -1$.

Proposition 1 *Under the additional assumption (A.10), at a SME(a), the equilibrium allocations satisfy a linear sharing rule and the two-fund separation property holds: for all i, the portfolio $\bar{\theta}_i$ is of the form $(\bar{\theta}_{i1}, t_i \mathbf{1}_{J-1})$, with $\sum_i \bar{\theta}_{i1} = \sum_i t_i = 1$; i.e., agent i invests $(\bar{\theta}_{i1} - \delta_{i1})$ in the riskless bond and t_i in the market portfolio.*

Proof: Classical result (see, e.g., proposition 16.15 in Magill and Quinzii (1997)).

Q.E.D

Corollary 2 *Assume (A.10), a ρ-MSE(μ), $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$, satisfies the first-order conditions for Pareto optimality if:*

- $\rho < 1$ *for governances where only sharehoders with positive amounts of shares can participate in the voting process*[5]*;*
- $\rho < 0.5$ *in a stakeholder democracy where all shareholders with $t_i \neq 0$ have some voting right.*

Proof: Consider a ρ-MSE(μ): $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$. Then $(\bar{q}, \bar{x}, \bar{\theta})$ is a SME($\bar{a}$) and therefore, thanks to Corollary 1, one only has to check that equations (4) hold; thanks to Proposition 1, for all $j \in \mathcal{J} \backslash \{1\}$, for all $i \in \mathcal{I}$, $\bar{p}_{ij} = t_i \sum_{k \geq 2} \bar{p}_{jk}$: all shareholders such that $t_i > 0$ are unanimous on the way action of firm j should be priced. So are shareholders such that $t_i < 0$.

Suppose that in firm j, equations (4) do not hold. Then there exists an infinitesimal change of action $da_j \in T_{\bar{a}_j} \mathcal{A}_j$, the tangent space at $\bar{a}_j$ to $\mathcal{A}_j$ (which is orthogonal to $\bar{p}_j$), such that for all i with $t_i > 0$, $\bar{p}_{ij} \cdot da_j > 0$. Hence $\mathcal{I}_j(\bar{x}, \bar{\theta}, \bar{a}, da_j) = \{i \in \mathcal{I} \mid t_i > 0\}$ and $\mathcal{I}_j(\bar{x}, \bar{\theta}, \bar{a}, -da_j) = \{i \in \mathcal{I} \mid t_i < 0\}$.

If only shareholders with positive amounts of shares can participate in the voting process, $\sum_{i \in \mathcal{I}} \mu_{ij} = \sum_{i \in \mathcal{I}_j(\bar{x}, \bar{\theta}, \bar{a}, da_j)} \mu_{ij}$, therefore $\bar{a}_j + da_j \in P_j^{\rho}(\bar{x}, \bar{\theta}, \bar{a}, \mu_j)$ as soon as $\rho < 1$, a contradiction to the assumption that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a ρ-MSE(μ).

Consider a stakeholder democracy. Then, since $\rho < 0.5$, either $\bar{a}_j + da_j \in P_j^{\rho}(\bar{x}, \bar{\theta}, \bar{a}, \mu_j)$ or $\bar{a}_j - da_j \in P_j^{\rho}(\bar{x}, \bar{\theta}, \bar{a}, \mu_j)$ (or both) depending on which of the two groups $\mathcal{I}_j(\bar{x}, \bar{\theta}, \bar{a}, da_j)$ and $\mathcal{I}_j(\bar{x}, \bar{\theta}, \bar{a}, -da_j)$ has the highest agregate voting weight. (If the two groups have the same aggregate voting weight, then we have both.) Hence a contradiction.

Q.E.D

[5] Note that $\rho < 1$ includes *infra* majority voting rules, and that such governances include the traditional 'one share-one vote' and 'one shareholder-one vote'.

Political stability

Consider a SME(a), $\mathcal{E} = (\bar{q}, \bar{x}, \bar{\theta})$ and let p_j be the supporting price of action a_j. The following construction shows that if the current action a_j is optimized with respect to a supporting price p_j which somewhat *averages* the idiosyncratic prices $\bar{p}_{ij}$ of the members of the coalition (in the sense that p_j is in the positive convex cone of the $\bar{p}_{ij}$'s), then there does not exist a change Δa_j of action that is *unanimously* supported by the coalition members.

Let us begin with some drawings. Figures 1.a and 1.b show two possible political configurations within firm j (where $n_j = 3$): p_j is the supporting price of the action a_j and there are five shareholders having individual pricing vectors $(\bar{p}_{ij})_{1 \leq i \leq 5}$. (To avoid three-dimensional pictures, without loss of generality all pricing vectors are supposed to be 'normalized' so that they lie in a two-dimensional hyperplane.)

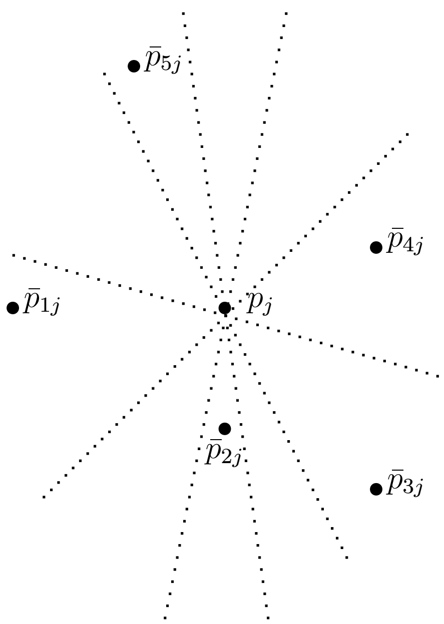

Figure 1.a

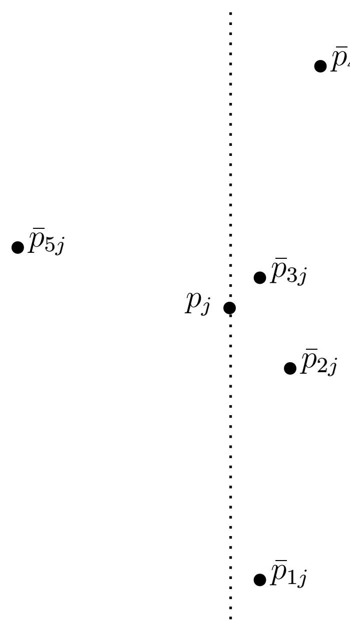

Figure 1.b

Figure 1.a represents an optimistic scenario. Indeed, the maximal coalitions $\mathcal{C}$ such that p_j does not lie inside the convex hull of the $(p_{ij})_{i \in \mathcal{C}}$ are of size 3: $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{4, 5, 1\}$, $\{5, 1, 2\}$. Hence, in the political game, any change of action will divide the electoral population in three voters against two, as shown by the five dotted lines, one for each possible division. In such a configuration, in a stakeholder democracy we will say that the 'score' of p_j with respect to the family $(\bar{p}_{ij})_{1 \leq i \leq 5}$ is 3/5, i.e. 60% (see Definition 4 below). Theorem 1 below formalizes this approach.

Figure 1.b on the other hand is a more pessimistic scenario: p_j does not lie inside

the convex hull of the $(p_{ij})_{i\in\mathcal{C}}$ where $\mathcal{C} = \{1,2,3,4\}$. Hence there exists a change of action (indicated by the vertical dotted line) which rallies 4 votes against the status quo a supported by p_j. Here the 'score' of p_j with respect to the family $(\bar{p}_{ij})_{1\le i\le 5}$ is 4/5, i.e. 80%. This configuration is more pessimistic to the extent that if one wants, for the sake of productive efficiency, to guarantee the political stability of p_j (which in these figures satifies Equations 4), then a super majority rate of more than 80% has to be adopted (see Corollary 3 below); hence a very conservative voting rule.

Some pieces of notation are needed: For a given finite collection of H vectors $V = (v_h)_{h\in\mathcal{H}}$ where $v_h \in \mathbb{R}^n$, let $K(V)$ denote the convex cone generated by V

$$K(V) \;=\; \{v \in \mathbb{R}^n \mid \exists\, \lambda \in \mathbb{R}^H : v = \textstyle\sum_h \lambda_h v_h \text{ and } \lambda \ge 0\},$$

and let $K_+(V)$ denote the strictly positive convex cone

$$K_+(V) \;=\; \{v \in \mathbb{R}^n \mid \exists\, \lambda \in \mathbb{R}^H : v = \textstyle\sum_h \lambda_h v_h,\ \lambda \ge 0 \text{ and } \lambda \ne 0\}.$$

Theorem 1 *At a SME(a), for a coalition of consumers $\mathcal{C} \subset \mathcal{I}$, there does not exist a possible change of action Δa_j for firm j (so $\bar{p}_j \cdot \Delta a_j \le 0$) that is unanimously supported by all members of $\mathcal{C}$ if and only if $\bar{p}_j \in K((\bar{p}_{ij})_{i\in\mathcal{C}})$ or $0 \in K_+((\bar{p}_{ij})_{i\in\mathcal{C}})$.*

Proof: Let the $n_j \times J$ matrix $\bar{P}_j$ be defined by $\bar{P}_j = (\bar{p}_{jk})_k$. For firm j, a change of action Δa_j where $\Delta a_j \in \mathbb{R}^{n_j}$ is feasible if and only if

$$\bar{p}_j^t \Delta a_j \;\le\; 0$$

where $\bar{p}_j^t \Delta a_j = 0$ corresponds to an efficient change and $\bar{p}_j^t \Delta a_j < 0$ corresponds to an inefficient change.

For a coalition $\mathcal{C} \subset \mathcal{I}$ of consumers let the $J \times |\mathcal{C}|$ matrix $\bar{\theta}_{\mathcal{C}}$ be defined by $\bar{\theta}_{\mathcal{C}} = (\bar{\theta}_i)_{i\in\mathcal{C}}$. Then coalition $\mathcal{C}$ supports a change Δa_j if and only if

$$\bar{\theta}_{\mathcal{C}}^t \bar{P}_j^t \Delta a_j > 0.$$

From Theorem 22.2 in Rockafellar (1970) it follows that either there exists a solution to (which is $\Delta a_j \in \mathbb{R}^{n_j}$ such that)

$$(a)\quad \begin{cases} p_j^t \Delta a_j & \le\; 0 \\ -\bar{\theta}_{\mathcal{C}}^t \bar{P}_j^t \Delta a_j & <\; 0 \end{cases}$$

or there exists a solution to (which is $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{|\mathcal{C}|}$ such that)

$$
(b)\quad \begin{cases} \bar{p}_j\mu - \bar{P}_j\bar{\theta}_{\mathcal{C}}\lambda & = & 0 \\ I_{|\mathcal{C}|}\lambda & \geq & 0 \\ \mu & \geq & 0 \\ (1,\ldots,1)\lambda & > & 0 \end{cases}
$$

where $I_{|\mathcal{C}|}$ is the $|\mathcal{C}| \times |\mathcal{C}|$ identity matrix.

Clearly there exists a solution to (b) if and only if there exists a solution to

$$
(b.1)\quad \begin{cases} \bar{P}_j\bar{\theta}_{\mathcal{C}}\lambda & = & 0 \\ I_{|\mathcal{C}|}\lambda & \geq & 0 \\ (1,\ldots,1)\lambda & > & 0 \end{cases}
$$

or

$$
(b.2)\quad \begin{cases} \bar{P}_j\bar{\theta}_{\mathcal{C}}\lambda & = & \bar{p}_j \\ I_{|\mathcal{C}|}\lambda & \geq & 0 \end{cases}
$$

However $(b.1)$ is equivalent to the zero price vector being in the strictly positive convex cone generated by the individual pricing vector of the members of $\mathcal{C}$. And $(b.2)$ is equivalent to $\bar{p}_j$ being the convex cone generated by individual pricing vector of the members of $\mathcal{C}$.

Q.E.D

Obviously as a consequence of Theorem 1 we have $p_j \notin K((\bar{p}_{ij})_{i\in\mathcal{C}})$ and $0 \notin K_+((\bar{p}_{ij})_{i\in\mathcal{C}})$ if and only if there exists a change that coalition $\mathcal{C}$ unanimously supports. Therefore Theorem 1 enables us to define the rate of super majority that is necessary and sufficient within firm j for the current action to be majority stable.

Definition 4 *At a SME(a) the* **score** *of the supporting price p_j with respect to the collection $(\bar{p}_{ij})_{i\in\mathcal{I}}$ (also called the score of action a_j) is defined as the maximum size of a coalition unanimously supporting some change within firm j:*

$$
\rho_\mu(p_j;(\bar{p}_{ij})_{i\in\mathcal{I}}) \;=\; \max\left\{\frac{\sum_{i\in\mathcal{C}}\mu_{ij}}{\sum_i \mu_{ij}} \mid \mathcal{C}\subset\mathcal{I} \text{ and } p_j \notin K((\bar{p}_{ij})_{i\in\mathcal{C}}) \text{ and } 0 \notin K_+((\bar{p}_{ij})_{i\in\mathcal{C}})\right\}.
$$

Corollary 3 *For a SME(a) action a_j is ρ-majority stable within firm j if and only if $\rho \geq \rho_\mu(p_j;(\bar{p}_{ij})_{i\in\mathcal{I}})$. A SME(a) is a ρ-MSE(μ) if and only if $\rho \geq \rho_\mu(a) = \max_j \rho_\mu(p_j;(\bar{p}_{ij})_{i\in\mathcal{I}})$.*

Proof: Immediate.

Q.E.D

In case all consumers are unanimous in a firm (because, e.g., they all have the market portfolio), the score is 0 if p_j is collinear to $\bar{p}_j$, 1 otherwise. In that case, efficient internalization is the only ρ-MSE, and it is so even for infra majority rules, whatever the governance. The score of a supporting price with respect to individual pricing vectors measures the degree of disagreement between shareholders, at a SME(a). The score as defined here is in the line of the traditional Simpson-Kramer approach: it is the maximum size of a coalition that unanimously supports some alternative action to the status quo. And the 'best' initial position for the status quo is the one with lowest score: the so-called 'min-max'.

Definition 5 *The* **min-max score** *over all actions a is $\rho^*_\mu = \min_{a\in\mathcal{A}} \rho_\mu(a)$. The min-max set is $\mathcal{A}^* = \{a \in \mathcal{A} | \rho_\mu(a) = \rho^*_\mu\}$.*

A first classical class of social choice results (see, e.g., Grandmont (1978)) allows us to underline once again the special role played by the market portfolio and perfect internalization in the political process, even in case consumers are not unanimous. This deals with the case where the collection $(\bar{p}_{ij})_{i\in\mathcal{I}}$ is axially balanced: within each firm j there exists a vector p^c_j such that every agent $i \in \mathcal{I}$ can be pairwise matched with another one[6], $\bar{i} \in \mathcal{I}$, such that $\bar{p}_{\bar{i}j} + \bar{p}_{ij} = \lambda_i p^c_j$ and $\bar{i}$ is matched with i.

Proposition 2 *Suppose that at a SME(a), $\mathcal{E}$, for all j the collection of pricing vectors $(\bar{p}_{ij})_{i\in\mathcal{I}}$ is axially balanced; then $\mathcal{E}$ is a 0.5-MSE(μ^d) if and only if it satisfies the first-order condition (4) of Pareto optimality.*

Proof: Obviously, if the collection $(\bar{p}_{ij})_{i\in\mathcal{I}}$ is axially balanced about p^c_j, it has to be that the p^c_j is collinear to the efficient pricing vector $\bar{p}_j = \sum_k \bar{p}_{jk}$ (when different from zero). Indeed, market clearing yields $\sum_i \bar{p}_{ij} = \bar{p}_j$ and axial balancedness gives $\sum_i \bar{p}_{ij}$ is collinear to p^c_j.

Suppose that $\mathcal{E}$ satisfies equations (4). Then any infinitesimal change of action within firm j will be supported by one, and only one agent of each pair. Both have the same voting weight under μ^d. Hence stability with respect to the 50% rule.

Suppose in turn that $\mathcal{E}$ is is a 0.5-MSE(μ^d). It has to be that, for all j, no halfspace in $\mathbb{R}^{n_j}$ defined by a hyperplane containing p_j contains a pair of matched agents. Hence p_j must be on the axis of symmetry.

Q.E.D

[6] An agent such that $\bar{p}_{ij}$ is collinear to p^c_j can be matched with himself.

From pricing vectors to portfolios: the role of diversification

The special structure of the present set-up allows us to study the political stability of SME(a)'s *in all firms at once in the same space* by working with portfolios rather than with pricing vectors.

Theorem 2 *At a SME(a), for a coalition of consumers $\mathcal{C} \subset \mathcal{I}$, there does not exist a possible change of action Δa in any firm that is unanimously supported by all members of $\mathcal{C}$ if $\mathbf{1}_J \in K((\bar{\theta}_i)_{i\in\mathcal{C}})$ or $0 \in K_+((\bar{\theta}_i)_{i\in\mathcal{C}})$.*

Proof: If there exists a solution to

$$(b.3) \quad \begin{cases} \bar{\theta}_{\mathcal{C}}\lambda &= 0 \\ I_{|\mathcal{C}|}\lambda &\geq 0 \\ (1,\ldots,1)\lambda &> 0 \end{cases}$$

resp.

$$(b.4) \quad \begin{cases} \bar{\theta}_{\mathcal{C}}\lambda &= \mathbf{1}_J \\ I_{|\mathcal{C}|}\lambda &\geq 0 \end{cases}$$

then there exists a solution to (b.1) resp. (b.2) in the proof of Theorem 1. The fact that (b.4) implies (b.2) follows from $\sum_k \bar{p}_{jk} = \bar{p}_j$. Hence if there exists a solution to (b.3) or (b.4), then there exists a solution to (b) and there exists no solution to (a) both in the proof of Theorem 1.

Q.E.D

Note that Theorem 2 only depends on portfolios rather than consumers' pricing vectors (which in turn depend on firms' pricing vectors and portfolios) as in Theorem 1. However Theorem 2 provides a sufficient condition for a coalition not to support any change in *any* firm, while Theorem 1 provides a necessary and sufficient condition for a coalition not to support any change in *some* firm.

For all j, let $V_j(\mathcal{E})$ be the (possibly empty) subspace of portfolios that generate the supporting price, p_j, of action a_j from the pricing vectors $(\bar{p}_{jk})_k$ so for all j

$$V_j(\mathcal{E}) \;=\; \{\theta_j \in \mathbb{R}^J \mid p_j = \textstyle\sum_k \theta_{jk}\bar{p}_{jk}\}.$$

As in Definition 4 we define the score of the portfolio $\theta_j \in V_j(\mathcal{E})$ with respect to the collection $(\bar{\theta}_i)_{i\in\mathcal{I}}$, as:

$$\rho_\mu(\theta_j; (\bar{\theta}_i)_{i\in\mathcal{I}}) \;=\; \max\left\{ \frac{\sum_{i\in\mathcal{C}}\mu_{ij}}{\sum_i \mu_{ij}} \mid \mathcal{C} \subset \mathcal{I} \text{ and } \theta_j \notin K((\bar{\theta}_i)_{i\in\mathcal{C}}) \text{ and } 0 \notin K_+((\bar{\theta}_i)_{i\in\mathcal{C}}) \right\}.$$

Corollary 4 *At a SME(a) for all j:*

$$\rho_\mu(p_j; (\bar{p}_{ij})_{i\in\mathcal{I}}) \ \leq \ \rho_\mu(\theta_j; (\bar{\theta}_i)_{i\in\mathcal{I}}).$$

Proof: Immediate consequence of the proof of Theorem 2.

Q.E.D

The score of portfolio θ within a collection of portfolios measures how well it is diversified *relative* to the collection: The smaller its score, the better it is diversified relative to the collection. When all shareholders disagree, the best rate of super majority guaranteeing existence of equilibrium we can hope for is 0.5. According to the preceding corollary, this happens when the score of θ is 0.5.

This scoring approach with portfolios rather than pricing vectors is particularly appealing in the case of an efficient SME(a) $\mathcal{E}$: The first-order conditions (4) then entails that for all j, $V_j(\mathcal{E})$ contains a portfolio collinear to the market portfolio $\mathbf{1}_J$. Therefore we have the following result.

Corollary 5 *Suppose that $\mathcal{E}$ is efficient. Then $\mathcal{E}$ is a ρ-MSE(μ) if $\rho \geq \rho_\mu(\mathbf{1}_J; (\bar{\theta}_i)_{i\in\mathcal{I}})$.*

Proof: Immediate consequence of Corollary 3 and Corollary 4.

Q.E.D

A portfolio is well diversified relative to the collection when it is placed in such a way that it lies in the convex cone of all 'big' coalitions. Intuitively, such a portfolio should be a good average of the collection. This intuition indicates that the center of gravity (or mean) of the collection of individual portfolios is well diversified and therefore has good stability properties with respect to the voting mechanism. Caplin and Nalebuff (1988, 1991) provide strong arguments in favor of this intuition: in their search for a dimension-free upper bound to the min-max in a classical multidimensional spatial voting model, they were led by the geometric structure of the problem to use the center of gravity of the individual preferred alternative as a proxy of the min-max.

At a SME(a), the mean of individual portfolios is collinear to the market portfolio; the one resulting in perfect internalization of production externalities. In the following section, we give conditions under which perfect internalization is likely to come out of the voting mechanism.

4 The political stability of efficient internalization

Ideally, from an economic viewpoint, we would like efficient internalization to come out of the voting mechanism. To study the relative political stability of the efficient internalization amounts to study the stability property, in the space $\mathbb{R}^{n_j}$ of the efficient pricing vector $\sum_k \bar{p}_{jk}$, or, alternatively, to study the stability property in the space $\mathbb{R}^J$ of the market portfolio $\mathbf{1}_J$. Once again the market portfolio has remarkable properties: we have already argued that if the agents are unanimous about the production policy of the firms, then this policy must satisfy the FOC of efficient internalisation. The driving force in this result is the market clearing condition. We argue in present section that, although market clearing does not ensure the first welfare theorem (because of the absence of markets for externalities), *market clearing gives remarkable (relative) stability properties to the efficient internalization in the voting mechanism for the stakeholder democracy.* The object of the sequel is to qualify this 'political first welfare theorem'.

We propose two statistical approaches. The first one randomizes over the direction of marginal external effects and builds on Theorem 1. The second approach randomizes over endowments and builds on Theorem 2. Through both approaches, we show that the 'score' of perfect internalization concerges to 0.5 as the number of consumers increases.

A first statistical approach: random political configurations

Let us consider a *worst-case scenario* as far as individual portfolio diversification is concerned: suppose that at a SME (a) agents have shares in at most one firm. Without loss of generality, one can reduce the study to an economy with $I = J$ agents (and identify $\mathcal{I}$ with $\mathcal{J}$), each agent owning one firm and only one: this economy is called a **sole proprietorship**.

The collection of portfolios $(\theta_i)_{i\in\mathcal{J}}$ is thus composed of the J vertices of a spherico-regular $(J-1)$-dimensional simplex, Δ^{J-1}. (It is regular since $\|\theta_i - \theta_j\| > 0$ has the same value for all $i \neq j$; moreover it is spherico-regular since the vertices lie of a sphere centered at the origin.) That case is also a worst-case scenario from a social choice perspective: all portfolios θ in the positive orthant (the positive convex cone of the θ_i's), *and thus also the market portfolio*, has the same score under the stakeholder democracy μ^d: $\rho_{\mu^d}(\theta; (\theta_i)_{i\in\mathcal{I}}) = 1 - 1/J$ (the others have a score equal to one). We know from Greenberg (1979) that $1 - 1/J$ is an upper bound to the min-max score in a very general spatial voting model. If we were to stick to the space of portfolios to study the political process within firms, we would get to the conclusion that no ρ-MSE(μ^d) exists for $\rho < 1 - 1/J$, and thus no criterion does better than the mere Pareto criterion.

But it is still the case that this extremal collection of portfolios gives rise, within each firm $j \in \mathcal{J}$, to a n_j-dimensional social choice problem characterized by the collection of individual pricing vectors $(\bar{p}_{ij})_{i\in\mathcal{J}}$, where $\bar{p}_{ij}$ is the linear transformation of θ_i through the matrix $\bar{P}_j$ (as defined in the proof of Theorem 1). The operator $\bar{P}_j$ projects the simplex of portfolios in a subspace of strictly smaller dimension if $n_j < J$, and in that subspace it is not at all the case that the collection $(\bar{p}_{ij})_{i\in\mathcal{J}}$ is distributed according to a worst-case scenario similar to that of portfolio (e.g., as vertices of a $(n_j - 1)$-dimensional simplex). On the contrary, we argue that if the image subspace of the operator $\bar{P}_j$ is randomly chosen (in a sense made precise below), then *up to an affine transformation, the resulting point set, $(\bar{p}_{ij})_{i\in\mathcal{J}}$, coincides in distribution with a standard centered Gaussian sample in that subspace*. Hence, if J is high enough compared to n_j, the worst-case scenario in the space of portfolio gives rise, within each firm, to a *best-case scenario*: the Gaussian distribution being symmetric one can hope for existence of ρ-MSE(μ^d) for ρ's close to 0.5; and the sample being centered, *statistically the min-max point is a vector collinear to the market portfolio*, hence efficient internalization occurs at the min-max.

<u>A natural distribution of random points:</u>

More generally, when dealing with a d-dimensional spatial voting problem with, say, n voters, one is left with a combinatorial problem about n-tuples of (random) points in $\mathbb{R}^d$. Many 'natural' distributions of these random points have been proposed in the mathematical literature (see Schneider (2004)). Among them, the one described above takes a central place: Every configuration of $n > d$ numbered points in general position in $\mathbb{R}^d$ is affinely equivalent to the orthogonal projection of the set of numbered vertices of a fixed spherico-regular $(n-1)$-dimensional simplex onto a unique d-dimensional linear subspace in $\mathbb{R}^{n-1}$. This construction builds a one-to-one correspondence between the (orientation-preserving) affine equivalence classes of such point set configurations and an open dense subset of the Grassmanian $G(n-1, d)$ of oriented d-spaces in $\mathbb{R}^{n-1}$. The so-called *Grassmann approach* (sometimes refered as the Goodman-Pollack model) considers the probability distribution on the set of affine equivalence classes of n-tuples in general position in $\mathbb{R}^d$ that stems from the unique rotation-invariant probability measure on $G(n-1, d)$. Baryshnikov and Vitale (1994) (following an observation of Affentranger and Schneider (1992)) proved that under the Grassmann approach, the resulting point set coincides in distribution with a standard Gaussian sample in that subspace. As a consequence, an affine-invariant functional of n-tuples with this distribution is stochastically equivalent to the same functional taken at an i.i.d. n-tuple of standard normal points in $\mathbb{R}^d$.

<u>Random production external effects:</u>

Suppose $n_j < J$ and consider the image subspace of the operator $\bar{P}_j$, Im$\bar{P}_j$: the subspace of $\mathbb{R}^{J-1}$ generated by the pricing vectors $(\bar{p}_{jk})_{k\in\mathcal{J}}$ (which has dimension smaller or equal to n_j; without loss of generality, we assume it has dimension n_j and take the first n_j vectors as a basis). Recall that the vector $\bar{p}_{jk}$ measuring the marginal impact of firm j's action on firm k's output: it is the image of the unique state prices vector by the Jacobian matrix of the production function of firm k, F_k, with respect to action of firm j (see Equations 2). These marginal impacts are fixed by the exogenously fixed production function. Randomizing over the production function amounts to randomizing over Im$\bar{P}_j$. Let us use the Grassmann approach.

The Grassmann approach amounts to randomly rotate the simplex Δ^{J-1} and project it orthogonally on a fixed n_j-dimensional subspace. Witout loss of generality, the spherico-regular simplex Δ^{J-1} can be suitably translated so that it becomes centered: the sum of its vertices (the scaled-down market portfolio $(1/J)\mathbf{1}_J$, center of gravity under μ^d) is translated at zero. Once more without loss of generality, the fixed n_j-dimensional subspace onto which the rotated simplex is projected can be taken as the first n_j coordinates of $\mathbb{R}^J$. But our operator $\bar{P}_j$ is not this orthogonal projection, but a linear (thus affine) transformation of the latter (one can pass from one to the other using the Gram-Schmidt procedure between the standard orthonormal basis and the basis defined by $(\bar{p}_{jk})_{1\leq k\leq n_j}$). The composition of affine transformation being affine, we can directly apply Baryshnikov and Vitale (1994) to our problem.

Theorem 3 *Fix n_j. When J tends toward infinity, then almost surely the min-max score converges to 0.5 for the stakeholder governance, and the min-max set of directions of the $(\bar{p}_{ij})_{i\in\mathcal{J}}$ shrinks to the direction of the efficient price vector $\bar{p}_j$.*

Proof: Under the Grassman approach on production externalities, within each firm j, the point set $(\bar{p}_{ij})_{i\in\mathcal{J}}$ coincides in distribution with a standard centered Gaussian sample. Therefore the convergence of the min-max of the sample to the min-max of the Gaussian distribution is a consequence of Theorem 3 in Caplin and Nalebuff (1988).

Q.E.D

Another statistical approach: random initial characteristics

In this approach, actions of firms $\bar{a} = (\bar{a}_j)_j$ and prices of shares $\bar{q} = (\bar{q}_j)_j$ are supposed to be fixed while the distribution of initial endowments and initial portfolios $(\omega_i, \delta_i)_i$ is supposed to be variable. However we only consider distributions of initial endowments and initial portfolios such that there exists a stock market equilibrium with fixed actions $(\bar{a}_j)_j$ where the prices of shares are $(\bar{q}_j)_j$.

Let $\bar{y} = (\bar{y}_j)_j$ be defined by $\bar{y}_j = F_j(\bar{a})$ for all j. Then there exists a unique vector of state prices $\bar{p} = (\bar{p}_s)_s$ where $\bar{p}_0 = 1$ and $\bar{p}_s > 0$ for all s such that $\bar{q}_j = \sum_s \bar{p}_s \bar{y}_j^s$ (because markets are complete and prices of shares are part of a stock market equilibrium with fixed actions, so arbitrage is not possible). Indeed the vector of state prices is defined as follows

$$\begin{pmatrix} \bar{p}_1 \\ \vdots \\ \bar{p}_S \end{pmatrix} = \begin{pmatrix} \bar{y}_1^1 & \cdots & \bar{y}_J^1 \\ \vdots & & \vdots \\ \bar{y}_1^S & \cdots & \bar{y}_J^S \end{pmatrix}^{-1} \begin{pmatrix} \bar{q}_1 - \bar{y}_1^0 \\ \vdots \\ \bar{q}_J - \bar{y}_J^0 \end{pmatrix}.$$

Let $f_i : \mathbb{R}^{S+1}_{++} \times \mathbb{R} \to \mathbb{X}$ be the demand function of consumer i, so $f_i(p, m_i)$ is the solution to the following problem

$$\begin{aligned} \max_x \quad & u_i(x_i) \\ \text{s.t.} \quad & p \cdot x_i \leq m_i. \end{aligned}$$

Fix the income distribution $\bar{m} = (\bar{m}_i)_i$; and for $\bar{x} = (\bar{x}_i)_i$, where $\bar{x}_i = f_i(\bar{p}, \bar{m}_i)$, only consider distributions of initial endowments and initial portfolios $(\omega_i, \delta_i)_i$ such that the budget constraints are satisfied:

$$\forall i \quad \bar{p} \cdot \omega_i \;=\; \bar{m}_i - \bar{q} \cdot \delta_i,$$

and markets clear:

$$\sum_i \omega_i \;=\; \sum_i \bar{x}_i - \sum_j \bar{y}_j.$$

The distribution of portfolios $\bar{\theta} = (\bar{\theta}_i)_i$ depends on the net-trades of the consumers $\bar{x} - \omega = (\bar{x}_i - \omega_i)_i$, so for the considered distributions of initial endowments and portfolios, only the distribution of initial endowments is relevant. Indeed if the initial distribution of initial endowments and portfolios is $(\omega_i, \delta_i)_i$ then the distribution of portfolios is $(\bar{\theta}_i)_i$ where

$$\begin{pmatrix} \bar{\theta}_{i1} \\ \vdots \\ \bar{\theta}_{iJ} \end{pmatrix} = \begin{pmatrix} \bar{y}_1^1 & \cdots & \bar{y}_J^1 \\ \vdots & & \vdots \\ \bar{y}_1^S & \cdots & \bar{y}_J^S \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}_i^1 - \omega_i^1 \\ \vdots \\ \bar{x}_i^S - \omega_i^S \end{pmatrix}.$$

Therefore if $\Omega \subset \mathbb{R}^{(S+1)I}$ is defined by

$$\Omega \;=\; \{\, (\omega_i)_i \in \mathbb{R}^{(S+1)I} \,|\, \textstyle\sum_i \omega_i = \sum_i \bar{x}_i - \sum_j \bar{y}_j \,\}$$

then Ω is the relevant set of distributions of initial endowments in the sense that $(\omega_i)_i \in \Omega$ if and only if there exists a distribution of initial portfolios $(\delta_i)_i$ such that $(\bar{q}, (\bar{x}_i)_i, (\bar{\theta}_i)_i)$ is a stock market equilibrium with fixed actions $\bar{a} = (\bar{a}_j)_j$ and income distribution $(\bar{m}_i)_i$.

Let $\pi_i : \mathbb{R}^{S+1} \to \mathbb{R}^J$ be defined by

$$\pi_i(\omega_i) \;=\; \begin{pmatrix} \bar{y}_1^1 & \cdots & \bar{y}_J^1 \\ \vdots & & \vdots \\ \bar{y}_1^S & \cdots & \bar{y}_J^S \end{pmatrix}^{-1} \begin{pmatrix} \bar{x}_i^1 - \omega_i^1 \\ \vdots \\ \bar{x}_i^S - \omega_i^S \end{pmatrix}.$$

then $\pi_i(\omega_i)$ is the portfolio of consumer i. Therefore, if $\Theta \subset \mathbb{R}^{IJ}$ is defined by

$$\Theta \;=\; \{\, (\theta_i)_i \in \mathbb{R}^{IJ} \,|\, \textstyle\sum_i \theta_{ij} \;=\; 1 \text{ for all } j \,\}$$

then every probability measure Γ on Ω induces a probability measure Φ on Θ.

Indeed let Φ be defined by

$$\Phi(A) \;=\; \Gamma(\{\, (\omega_i)_i \in \Omega \,|\, (\pi_i(\omega_i))_i \in A \,\}).$$

Hence probability measures on Θ rather than probability measures on Ω are considered.

Suppose that Ψ is a probability measure with density ψ on $\mathbb{R}$ such that the mean E is positive, so

$$E \;=\; \int t\psi(t)\, dt \;>\; 0,$$

and the variance V is finite, so

$$V \;=\; \int (t - E)^2 \psi(t)\, dt \;<\; \infty.$$

If $(\xi_{ij})_{ij}$ is the result of IJ trials then let the associated distribution of portfolios $(\bar{\theta}_i)_i$ be defined by

$$\bar{\theta}_{ij} \;=\; \frac{\xi_{ij}}{\sum_k \xi_{kj}}.$$

Clearly $\sum_k \xi_{kj} \neq 0$ with probability 1 for all j, so $\bar{\theta}_{ij}$ is well-defined for all i and j and $\sum_i \bar{\theta}_{ij} = 1$ for all j.

Theorem 4 *Suppose that $(\bar{q}, \bar{x}, \bar{\theta})$ is a Pareto optimal stock market equilibrium with actions $\bar{a}$. For all $\rho > 1/2$ if I tends to ∞ then the probability that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a ρ-MSE converges to 1 for the stakeholder governance μ^d.*

Proof: If the sequence $(\xi_{ij})_{ij}$ is the result of an infinite number of trials then let $((\theta_i^I)_i)_I$ be sequence of associated distributions of portfilios. Let the unit interval $[0, 1[$ with the Lebesgue measure be the set of consumers, so for each I the interval $[(i-1)/I, i/I[$ is consumer i, then $((\theta_i^I)_i)_I$ induces a signed vector-valued probability measure Λ_I on $[0, 1[$. Indeed the vector-valued density $\lambda_I : [0, 1[\to \mathbb{R}^J$ of Λ_I on the interval $[(i-1)/I, i/I[$ is $I\theta_i$, because then

$$I \int_{\frac{i-1}{I}}^{\frac{i}{I}} \theta_i \, dt \;=\; \theta_i.$$

Moreover Λ_I induces a signed probability measure Υ_I on $\mathbb{R}^J$. Indeed for $A \subset R^J$ let

$$\Upsilon_I(A) \;=\; \frac{|\{i \in \{1,\ldots,I\}\,|\,I\theta_i \in A\}|}{I}.$$

According to Theorem 4.5.3 (Kolmogorov's strong law of large numbers) in Ito (1984) the sequence $((\sum_i \xi_{ij})/I)_I$ converges to E almost surely. Therefore the sequence of signed probability measures $(\Upsilon_I)_I$ converges almost surely to a signed probability measure Υ on $\mathbb{R}^J$ with density $\upsilon : \mathbb{R}^J \to \mathbb{R}$ defined by

$$\upsilon(t) \;=\; \prod_j \frac{\psi(t_j)}{E}.$$

Clearly the distribution is symmetric with respect to the diagonal. Therefore for all p_j if $p_j \neq \bar{p}_j$ then less than 50 pct. of the consumers support p_j against $\bar{p}_j$. Hence for all $\rho > 1/2$ if I tends to ∞ then the probability that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a ρ-MSE converges to 1 for the stakeholder democracy μ^d.

Q.E.D

Corollary 6 *Suppose that $(\bar{q}, \bar{x}, \bar{\theta})$ is a Pareto optimal stock market equilibrium with actions $\bar{a}$ where $(\bar{p}_{jk})_k$ are linearly independent for at least one firm. Furthermore suppose that the measure of $]-\infty, 0[$ is positive, so*

$$P(t < 0) \;=\; \int_{-\infty}^{0} \psi(t)\,dt \;>\; 0$$

Then there exists $\bar{\rho} > 1/2$ such that for all $\rho < \bar{\rho}$ if I tends to ∞ then the probability that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a ρ-MSE converges to 0 for the shareholder governance θ^+.

Proof: According to the proof of Theorem 4 if I tends to infinity, then the sequence of distributions of portfolios converges to a distribution which is symmetrical with respect to the diagonal. Therefore if

$$P(t < 0) \;=\; \int_{-\infty}^{0} \psi(t)\,dt \;>\; 0$$

then distribution for the shareholder governance θ^+ is not symmetric with respect to the diagonal. Hence if $(\bar{p}_{jk})_k$ are linearly independent then there exists p_j such that more than 50 pct. of the shareholders supports p_j against $\bar{p}_j$. Thus if I tends to ∞ then the probability that $(\bar{q}, \bar{x}, \bar{\theta}, \bar{a})$ is a ρ-MSE converges to 0 for the shareholder governance θ^+.

Q.E.D

5 Internalizing through 50%-majority voting

In this section, we benchmark the stakeholder democracy with respect to its ability to promote efficient internalization through 50%-majority voting. We design a governance which always results in perfect internalization. In this governance, voting weights are endogeneously fixed at equilibrium and depend on the proposed challenger. This governance gives indications about the type and quantity of information that is needed to always get efficient internalization as the unique political outcome.

Efficient internalization as a permanent median voter

At a SME(a), $\mathcal{E}$, for a change of action Δa_j within firm j, define endogeneously the shareholders' voting weights by the following rule:

$$\mu^s_{ij}(\mathcal{E}, \Delta a_j) = \left| \bar{p}_{ij} \cdot \frac{\Delta a_j}{\|\Delta a_j\|} \right| .$$

The major weakness of this governance is that shareholders' voting weights depend on the alternative proposed to the status quo. Another (milder) one is that individual pricing devices, $\bar{p}_{ij}$, should be disclosed. But this problem seems not as serious as for ordinary public goods, since Equations (3) show that these price vectors are disclosed with the portfolio as soon as the price vectors for firms, $\bar{p}_{jk}$, are known. We argue here that portfolios and price vectors for firms are easier to disclose than individual willingness to pay for a public good. As the following proposition shows, this governance has the virtue of making the efficient internalization the only 0.5-MSE.

Proposition 3 *Under the governance defined by the mapping μ^s, if a state $(\bar{x}, a)$ associated with a SME(a), $\mathcal{E}$, is Pareto optimal, then $\mathcal{E}$ is 50%-majority stable. Inversely, any 0.5-MSE must satisfy the first order conditions of Pareto optimality as given by equations (4).*

Proof: Consider a Pareto optimal SME(a), $\mathcal{E}$. Suppose it is not 0.5-majority stable with respect to μ^s. By the minimum differentiation Lemma 1, it entails that there exists a firm, j, and an infinitesimal change of action, $da_j \in p_j^{\perp}$ such that

$$\textstyle\sum_{\mathcal{I}_j(da_j)} \mu^s_{ij} > \sum_{\mathcal{I}_j(-da_j) \cup \mathcal{I}_j(da_j)} \mu^s_{ij}$$

where, for shorter notation, $\mathcal{I}_j(da_j)$, resp. $\mathcal{I}_j(-da_j)$, stands for $\mathcal{I}_j(\bar{x}, \bar{\theta}, a, da_j) = \{i \in \mathcal{I} \mid \bar{p}_{ij} \cdot da_j > 0\}$, resp. $\mathcal{I}_j(\bar{x}, \bar{\theta}, a, -da_j) = \{i \in \mathcal{I} \mid \bar{p}_{ij} \cdot da_j < 0\}$, and $\mathcal{I}_j(da_j) = \{i \in \mathcal{I} \mid \bar{p}_{ij} \cdot da_j = 0\}$. The latter inequality is equivalent to

$$\textstyle\sum_{\mathcal{I}_j(da_j)} \bar{p}_{ij} \cdot da_j > \sum_{\mathcal{I}_j(-da_j) \cup \mathcal{I}_j(da_j)} - \bar{p}_{ij} \cdot da_j \quad \Longleftrightarrow \quad [\sum_{\mathcal{I}} \bar{p}_{ij}] \cdot da_j > 0 .$$

But $(\bar{x}, a)$ being Pareto optimal, one has that the vector $\sum_{\mathcal{I}} \bar{p}_{ij}$, which is equal to $\sum_{k\in\mathcal{J}} \bar{p}_{jk}$ thanks to financial market clearing, is collinear to p_j, a contradiction to the later inequality.

Reciprocally, consider a 0.5-MSE(μ^s). Suppose it is does not satisfy equations (4): there exists a firm, j, such that p_j is not collinear to $\sum_{k\in\mathcal{J}} \bar{p}_{jk}$. Then there exists $da_j \in p_j^{\perp}$ such that $da_j \cdot \sum_{k\in\mathcal{J}} \bar{p}_{jk} > 0$. On the other hand, being at a 0.5-MSE(μ^s), one has

$$\sum_{\mathcal{I}_j(da_j)} \mu_{ij}^s \leq \sum_{\mathcal{I}_j(-da_j)\cup\mathcal{I}_j(da_j)} \mu_{ij}^s \quad \Longleftrightarrow \quad [\sum_{\mathcal{I}} \bar{p}_{ij}] \cdot da_j \leq 0 \ .$$

But financial market clearing give $\sum_{\mathcal{I}} \bar{p}_{ij} = \sum_{k\in\mathcal{J}} \bar{p}_{jk}$, hence a contradiction with the former strict inequality.

Q.E.D

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