Approximating a Sequence of Observations by a Simple Process

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Abstract

Given a sequence $(s_0, s_1, \ldots, s_N)$ of observations from a finite set S, we construct a process $(\mathbf{s}_n)_{n \leq N}$ that satisfies the following properties: (i) $(\mathbf{s}_n)_{n \leq N}$ is a piecewise Markov chain, (ii) the conditional distribution of $\mathbf{s}_n$ given $\mathbf{s}_0, \ldots, \mathbf{s}_{n-1}$ is close to the empirical transition given by the observed sequence, for most n's, (iii) under $(\mathbf{s}_n)_{n \leq N}$, with high probability the empirical frequency of the realized sequence is close to the one given by the observed sequence. We generalize this result to the case that the conditional distribution of $\mathbf{s}_n$ given $\mathbf{s}_0, \ldots, \mathbf{s}_{n-1}$ is required to be in some polyhedron $V_{\mathbf{s}_{n-1}}$.

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1 Introduction

We are interested in approximating a finite given sequence by a simple stochastic process. The basic problem can be summarized as follows.

Let $\sigma = (s_0, s_1, \ldots, s_N)$ be a sequence over a finite set S of *states*. For $s \in S$, the number of visits to s along σ is defined to be $N_s^\sigma = |\{n < N, s_n = s\}|$ (not counting the last state in the sequence). The *empirical frequency of* s in σ is $\nu_s^\sigma := N_s^\sigma / N$, and the *empirical transition* (along σ) out of s is

$$q^\sigma(t \mid s) = \frac{|\{n < N, (s_n, s_{n+1}) = (s, t)\}|}{N_s^\sigma}, \text{ for } t \in S.$$

Does there exist a "simple" process $(\mathbf{s}_n)_{n \leq N}$ such that

(i) the conditional law of $\mathbf{s}_{n+1}$ given $(\mathbf{s}_0, ..., \mathbf{s}_n)$ is close to $q^\sigma(\cdot \mid \mathbf{s}_n)$ a.s. for most n's;

(ii) with high probability under $(\mathbf{s}_n)_{n \leq N}$, the empirical frequency of $s \in S$ is close to the observed frequency ν_s^σ of s along σ ?

The naive solution is to define $(\mathbf{s}_n)_{n \leq N}$ as a Markov chain with the empirical transitions q^σ as transition function. As the next example illustrates, this solution fails.

Example 1.1 *Let $S = \{a, b\}$, and consider the sequence $\sigma = (a, a, \ldots, a, b, b, \ldots, b, a)$ of N a's followed by N b's, and one a at the end. The empirical transition q^σ is given by*

$$q^\sigma(a \mid b) = q^\sigma(b \mid a) = 1 - q^\sigma(b \mid b) = 1 - q^\sigma(a \mid a) = 1/N.$$

Let $(\mathbf{s}_n)_{n \leq 2N+1}$ be the Markov chain with transition q^σ, starting from a. With a probability bounded away from zero, $\mathbf{s}_n = a$ for every $n \leq 2N + 1$. In particular, condition (ii) is not satisfied. More generally, one can prove that for this example no Markov chain satisfies both conditions (i) and (ii).

In the sequel we show that, provided N is sufficiently large, there exists a *piecewise Markov chain* on S with at most $|S|$ pieces that approximates σ in the sense of (i) and (ii).

A remark is in order. We here insist on one-step transitions, by asking that the approximating process be a (piecewise) Markov chain on S. By this insistence, we potentially loose much information on the structure of the sequence. Indeed, the sequence 001100110011...will be approximated (for lack of a better term) by the Markov chain on $\{0, 1\}$ with transitions $(\frac{1}{2}, \frac{1}{2})$ in each state. Plainly, a Markov chain of order 2 would approximate perfectly the given sequence. More generally, there is a tradeoff between

the order of the chain and the quality of the approximation. Our results indicate that piecewise Markov chains (of order 1) are sufficient to get a good appromixation, when only one-step transitions matter.

Note that this restriction is natural when only the transition matrix of the sequence σ is known.

Our motivation stems from the analysis of zero-sum stochastic games, see e.g. Blackwell and Ferguson (1968), Mertens and Neyman (1981) or Rosenberg *et al.* (2002). Such games are Markov Decision Processes with two competing decision makers. When analyzing the optimal behavior of a player who does not perfectly observe the actions chosen by his opponent, one is led to the following problem. One player, called the adversary, controls a S-valued process $(\mathbf{s}_n)_{n \leq N}$, where S is a finite set of *states*. In each of finitely many stages, he chooses the law $\mathbf{y}_n$ according to which the next state $\mathbf{s}_{n+1}$ is selected. A second player, called the statistician, suffers a loss $r(\mathbf{s}_n, \mathbf{y}_n)$ where r is concave in y. The statistician gets only to observe the realized sequence of states, but he does not observe the value of $\mathbf{y}_n$ or his loss $r(\mathbf{s}_n, \mathbf{y}_n)$. One can further restrict the statistician to observe only the transition matrix $(N_{s \to t})_{s,t \in S}$, where $N_{s \to t} = \#\{n < N, (\mathbf{s}_n, \mathbf{s}_{n+1}) = (s, t)\}$. The statistician wishes to estimate *ex-post* his total loss $L := \sum_{n=0}^N r(\mathbf{s}_n, \mathbf{y}_n)$. On the basis of his information, the natural idea for the statistician is to compute, for each $s \in S$, the distribution $\widehat{y}(s) \in \Delta(S)$ that is closest to the empirical transitions out of state s, and to suggest the quantity $\widehat{L} = \sum_{s \in S} N_s r(s, \widehat{y}(s))$ as an estimate for the loss, where N_s is the number of visits to s.

For a given strategy τ of the adversary (i.e., a rule that dictates for every stage n which $\mathbf{y}_n$ to choose on the basis of the available information), the expectation $\mathbf{E}_\tau\left[\widehat{L}\right]$ of this estimator[1] is typically higher than the expected loss $\mathbf{E}_\tau[L]$, due to the concavity of r. In other words, $\widehat{L}$ will fail to be, even approximately, an unbiased estimator of the loss.

The basic question we are interested in is whether, for every observed transition matrix, there is a "simple" strategy $\widetilde{\tau}$ of the adversary such that $\mathbf{E}_{\widetilde{\tau}}[L]$ is close to $\widehat{L}$. This question reduces to the above problem.

In the actual game-theoretic motivating problem, the player's strategy choice is restricted: for each $n \in \mathbf{N}$, $\mathbf{y}_n$ has to belong to a given compact polyhedron $V(\mathbf{s}_n)$ of probability measures over S. This caveat makes the analysis in Section 3 of the corresponding problem substantially more difficult.

The question we study may be viewed as a variant of the following problem. Given a realization $(s_0, \ldots, s_N)$ of an unknown hidden Markov chain, find a hidden Markov chain that best approximates the given realization. This question was initially studied by Baum and Petrie (1966) and Baum et al. (1970). This problem has several application, including ecology (Baum

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[1]The expectation is taken w.r.t. the law of the process induced by the strategy.

and Eagon (1967), speech recognition (see, e.g., Rabiner (1989) and gene finding (see, e.g., Burge and Karlin (1997)). In construct to finding the optimal hidden Markov chain that approximates the given sequence, we find a piecewise hidden V-Markov chain that approximates the sequence.

The paper is organized as follows. In Section 2, we investigate the problem with no polyhedral restriction. Next, we turn in Section 3 to the general problem.

2 The Basic Problem

For every finite set K, let $|K|$ be the number of elements in K, and let $\Delta(K)$ be the space of probability distributions over K. Throughout the paper we fix a finite set S of *states*.

2.1 Presentation

Let $N \in \mathbf{N}$, and let $\sigma = (s_0, s_1, \ldots, s_N)$ be a finite sequence in S of length $N + 1$. For $s \in S$, let $N_s = |\{n < N | s_n = s\}|$ be the *number of visits* to s in σ (the last state of the sequence is not counted), and define the *empirical frequency* of s in σ as

$$\nu_s^\sigma = \frac{N_s}{N}.$$

The *(empirical) transitions* out of s along σ are defined by

$$q^\sigma(t \mid s) = \frac{|\{n < N, (s_n, s_{n+1}) = (s, t)\}|}{N_s}, t \in S. \tag{1}$$

$q^\sigma(t \mid s)$ is defined whenever the denominator in (1) does not vanish; that is, whenever the state s is visited by the sequence. If $N_s = 0$, we let $q^\sigma(\cdot \mid s)$ be arbitrary. Note that q^σ is a transition function over S.

A piecewise Markov chain is the concatenation of Markov chains. Formally,

Definition 2.1 *Let K be a positive integer. A process $(\mathbf{s}_n)_{n \leq N}$ is a piecewise Markov chain with K pieces if there exists a non-decreasing sequence $(n_k)_{0 \leq k \leq K}$ of integers with $n_0 = 0$ and $n_K = N$, such that for each $k = 1, \ldots, K$, the process $(\mathbf{s}_n)_{n_{k-1} \leq n \leq n_k}$ is a Markov chain.*

Given a S-valued process $(\mathbf{s}_n)_{n \leq N}$, $s \in S$, and $m \in \mathbf{N}$, we denote $\overline{F}_m^s = \frac{1}{m} |\{0 \leq n \leq m - 1 \mid \mathbf{s}_n = s\}|$ the empirical frequency of s from stage 0 up to stage $m - 1$ inclusive, and $F_m^s = \frac{1}{m} |\{1 \leq n \leq m \mid \mathbf{s}_n = s\}|$. We also denote by $\mathbf{P}$ the law of $(\mathbf{s}_n)_{n \leq N}$, and by $\mathbf{q}_n$ the conditional law of $\mathbf{s}_{n+1}$ given $(\mathbf{s}_1, ..., \mathbf{s}_n)$. Our basic theorem is the following.

Theorem 2.2 *For every $\varepsilon > 0$ sufficiently small, every $\rho \in (0, 1/2(4|S|+1)$, and every $\zeta \in (0, 2\rho)$, there exists $N_0 \in \mathbf{N}$ such that the following holds. For every sequence σ of length $N \geq N_0$, there is a piecewise Markov chain $(\mathbf{s}_n)_{n \leq N}$ with $|S|$ pieces over S such that, for each $s \in S$,*

B1 *If $\nu_s^\sigma \geq \frac{1}{N^\rho}$, then $\mathbf{P}(|\overline{F}_N^s - \nu_s^\sigma| \geq \varepsilon \nu_s^\sigma) \leq \frac{1}{N^\zeta}$.*

B2 *$\|\mathbf{q}_n - q^\sigma(\cdot \mid \mathbf{s}_n)\|_\infty < \varepsilon$, a.s. for at least $N - |S|$ values of $n < N$.*

2.2 On Markov chains

In the present section we present some general results on Markov chains, that have their own interest. We first provide a result on the speed of convergence of an irreducible Markov chain to its invariant measure. Next, we collect a few observations on the expected exit time from domains of S. Let $q : S \to \Delta(S)$ be a transition rule over S. Given $s \in S$ we denote by $\mathbf{P}_{s,q}$ the law of the Markov chain (S, q) starting from s. We denote by $\mathbf{E}_{s,q}$ the corresponding expectation operator. When there is no risk of confusion, we may abbreviate $\mathbf{P}_{s,q}$ and $\mathbf{E}_{s,q}$ to $\mathbf{P}_s$ and $\mathbf{E}_s$ respectively. The hitting time of $C \subseteq S$ is denoted $T_C := \min\{n \geq 0 : \mathbf{s}_n \in C\}$, the minimum of an empty set being $+\infty$. For $t \in S$, we abbreviate $T_{\{t\}}$ to T_t and we denote by $T_t^+ = \min\{n \geq 1, \mathbf{s}_n = t\}$ the first return to t. Finally, for $C \subset S$, $\overline{C} = S \backslash C$ denotes the complement of C in S.

Given a transition function q over S, $i = 1, 2$, we set for every non empty subset C of S

$$\nu_C^q(s) = \frac{\sum_{t \in \overline{C}} \mu_t q(s \mid t)}{\sum_{t \in \overline{C}} \mu_t q(C|t)} \text{ for every } s \in C, \text{ and} \tag{2}$$
$$K_C^q = \sum_{s \in C} \nu_C(s) \mathbf{E}_{s,q}[e_C].$$

The numerator (resp. the denominator) in the definition of ν_C^q is the long run frequency of transitions from $\overline{C}$ to s (resp. from $\overline{C}$ to C). Thus, $\nu_C^q(s)$ is the probability that the first stage in C the process visits is s, while K_C^q is the average length of a visit to C.

We shall use below the identity (easily derived from the ergodic Theorem), that holds whenever the invariant measure μ of q exists.

$$\sum_{s \in D} \nu_D^q(s) \mathbf{E}_{s,q^\sigma}[T_{\overline{D}}] = \frac{\sum_{s \in D} \mu_s}{\sum_{s \in D} \mu_s q^\sigma(\overline{D} \mid s)}. \tag{3}$$

2.2.1 Convergence to the invariant measure

Definition 2.3 *Given $k > 0$, $q : S \to \Delta(S)$ over S is k-mixing if $\mathbf{E}_{s,q}[T_t^+] \leq k$, for every $s, t \in S$.*

Note that every mixing transition rule is irreducible.

Theorem 2.4 *Assume that $q : S \to \Delta(S)$ is n-mixing, with invariant measure μ. Let $m \in \mathbf{N}$ and $\varepsilon \in (0, \frac{1}{4})$ be such that $\varepsilon m > 4n$. Then, for every $s, t \in S$,*

$$\mathbf{P}_t(|F_m^s - \mu_s| > \varepsilon \mu_s) < \frac{9(2n+1)}{m\varepsilon^2}. \tag{4}$$

Remark 2.5 *Inspection of the proof shows that inequality (4) holds more generally for each state $s \in S$ such that $\max_{t \in S} \mathbf{E}_{t,q}[T_s^+] \leq n$.*

Remark 2.6 *Since $\left|\overline{F}_m^s - F_m^s\right| \leq \frac{1}{m}$, one has, under the assumptions of Theorem 2.4,*

$$\mathbf{P}_t(|\overline{F}_m^s - \mu_s| > \varepsilon \mu_s + \frac{1}{m}) < \frac{9(2n+1)}{m\varepsilon^2}. \tag{5}$$

Remark 2.7 *Theorem 2.4 is related to a recent generalization of Hoeffding's inequality to uniformly ergodic chains by Glynn and Ormoneit (2002).*

Proof. Denote by $T_s^{+,1} + ... + T_s^{+,p}$ the pth return time to s. For each m, the event $|F_m^s - \mu_s| \geq \varepsilon \mu_s$ is included in the union of the two events $\left\{T_s^{+,1} + ... + T_s^{+,\lceil m\mu_s(1-\varepsilon)\rceil} \geq m\right\}$ and $\left\{T_s^{+,1} + ... + T_s^{+,\lfloor m\mu_s(1+\varepsilon)\rfloor} \leq m\right\}$. For notational convenience, set $m_\varepsilon := \lceil m\mu_s(1-\varepsilon)\rceil$ and $m^\varepsilon := \lfloor m\mu_s(1+\varepsilon)\rfloor$.

We first deal with the case $s = t$. In this case the variables $T_s^{+,k}$ are *iid*, and share the law of T_s^+ under $\mathbf{P}_s$.

By Chebycheff inequality, since $\mathbf{E}_s[T_s^+] = \frac{1}{\mu_s}$ and the variables are independent,

$$\mathbf{P}_s(T_s^{+,1} + ... + T_s^{+,m_\varepsilon} \geq m) = \mathbf{P}_s\left(T_s^{+,1} + ... + T_s^{+,m_\varepsilon} - \frac{m_\varepsilon}{\mu_s} \geq m - \frac{m_\varepsilon}{\mu_s}\right)$$

$$\leq \frac{m_\varepsilon var_s T_s^+}{\left(m - \frac{m_\varepsilon}{\mu_s}\right)^2} \leq \frac{m_\varepsilon var_s T_s^+}{\left(m\varepsilon - \frac{1}{\mu_s}\right)^2},$$

where the second inequality holds since $m - \frac{m_\varepsilon}{\mu_s} \geq m\varepsilon - \frac{1}{\mu_s}$, and

$$\mathbf{P}_{s,q}(T_s^{+,1} + ... + T_s^{+,m^\varepsilon} \leq m) \leq \mathbf{P}_s\left(\frac{m^\varepsilon}{\mu_s} - (T_s^{+,1} + ... + T_s^{+,m^\varepsilon}) \geq \frac{m^\varepsilon}{\mu_s} - m\right)$$

$$\leq \frac{m^\varepsilon var_s T_s^+}{\left(\frac{m^\varepsilon}{\mu_s} - m\right)^2} \leq \frac{m^\varepsilon var_s T_s^+}{\left(m\varepsilon - \frac{1}{\mu_s}\right)^2}.$$

Hence,

$$\mathbf{P}_{s,q}(|F_m^s - \mu_s| \geq \varepsilon \mu_s) \leq \frac{(m_\varepsilon + m^\varepsilon) var_s T_s^+}{\left(m\varepsilon - \frac{1}{\mu_s}\right)^2}. \tag{6}$$

Since q is n-mixing, $\frac{1}{\mu_s} = \mathbf{E}_s\left[T_s^+\right] \le n < \varepsilon m/4$. Therefore, the denominator in (6) is at least $\frac{9}{16}m^2\varepsilon^2$. On the other hand, by Aldous and Fill (2002, Chapter 2, page 21, identity (22))

$$var_s T_s^+ \times \mu_s = 2\mathbf{E}_\mu T_s + 1 - \frac{1}{\mu_s}. \tag{7}$$

Since q is n-mixing, $\mathbf{E}_\mu T_s \le \mathbf{E}_\mu T_s^+ \le n$, hence $var_s T_s^+ \times \mu_s \le 2n + 1$. Since $m\mu_s \ge m/n > 4/\varepsilon > 1$, and $m_\varepsilon + m^\varepsilon \le 2m\mu_s + 1 \le 3m\mu_s$, we obtain

$$\mathbf{P}_s(|F_m^s - \mu_s| \ge \varepsilon\mu_s) \le \frac{16 \times 3(2n+1)}{9m\varepsilon^2}.$$

This concludes the proof in the case $s = t$.

Assume now $s \ne t$. We estimate $\mathbf{P}_t(T_s^{+,1} + ... + T_s^{+,m_\varepsilon} \ge m)$ and $\mathbf{P}_t(T_s^{+,1} + ... + T_s^{+,m^\varepsilon} \le m)$ in turn. Since q is n-mixing, we obtain by Markov inequality

$$\mathbf{P}_t(T_s^+ \ge \varepsilon^2 m) \le \frac{n}{m\varepsilon^2}. \tag{8}$$

On the other hand, by following the steps of the previous computation,

$$\mathbf{P}_t(T_s^{+,1} \le \varepsilon^2 m, T_s^{+,1} + ... + T_s^{+,m_\varepsilon} \ge m) \le \mathbf{P}_s(T_s^{+,2} + ... + T_s^{+,m_\varepsilon} \ge m(1 - \varepsilon^2))$$
$$\le \frac{(m_\varepsilon - 1)var_s T_s^+}{(m(1-\varepsilon^2) - \frac{m_\varepsilon - 1}{\mu_s})^2} \le \frac{(m_\varepsilon - 1)var_s T_s^+}{(m(\varepsilon - \varepsilon^2))^2}, \tag{9}$$

and

$$\mathbf{P}_t(T_s^{+,1} + ... + T_s^{+,m^\varepsilon} \le m) \le \mathbf{P}_s(T_s^{+,2} + ... + T_s^{+,m^\varepsilon} \le m - 1)$$
$$\le \frac{(m^\varepsilon - 1)var_s T_s^+}{(\frac{m^\varepsilon - 1}{\mu_s} + 1 - m)^2} \le \frac{(m^\varepsilon - 1)var_s T_s^+}{(m\varepsilon + 1 - \frac{2}{\mu_s})^2}. \tag{10}$$

In both (9) and (10), the denominator is at least $\left(\frac{1}{2}m\varepsilon\right)^2$. Therefore, summation of (8), (9) and (10) yields

$$\mathbf{P}_t(|F_m^s - \mu_s| \ge \varepsilon\mu_s) \le \frac{4var_s T_s^+(m_\varepsilon + m^\varepsilon - 2) + nm}{m^2\varepsilon^2}.$$

Since $m_\varepsilon + m^\varepsilon - 2 \le 2m\mu_s$, one gets

$$\mathbf{P}_t(|F_m^s - \mu_s| \ge \varepsilon\mu_s) \le \frac{4 \times 2(2n+1) + n}{m\varepsilon^2},$$

hence the result. ∎

2.2.2 Expected exit times

We assume throughout this section that q is irreducible. We use repeatedly the following inequality

$$\mathbf{E}_s\left[T_{\overline{L}}\right] \le \mathbf{E}_s\left[T_{\overline{L}\cup t}\right] + \mathbf{E}_t\left[T_{\overline{L}}\right],\tag{11}$$

that holds for every $L \subset S$ and every $s, t \in L$.

Proposition 2.8 *Let $C \subset S$, with $|C| > 1$. Define $\rho_1(C) = \max_{D \subset C} \min_{s \in D} \mathbf{E}_s\left[T_{\overline{D}}\right]$ and $\rho_2(C) = \max_{s \in C} \mathbf{E}_s\left[T_{\overline{C}}\right]$. One has*

$$\mathbf{E}_s\left[T_{\overline{D}}\right] \le |D|\,\rho_1(C) \text{ for every } D \subset C \text{ and every } s \in D, \text{ and}\tag{12}$$
$$\mathbf{E}_s\left[T_{\overline{C}}\right] \ge \rho_2(C) - (|C|-1)\rho_1(C) \text{ for every } s \in C.\tag{13}$$

Proof. We prove (12) by induction over $|D|$. Plainly, the inequality holds for singletons. Assume that the result holds for every subset of size k. Let $D \subset C$ be of size $k+1$, and let $s \in D$. By the definition of $\rho_1(C)$, there is $t \in D$, such that $\mathbf{E}_t\left[T_{\overline{D}}\right] \le \rho_1(C)$. By (11) and the induction hypothesis for $D \setminus t$,

$$\mathbf{E}_s\left[T_{\overline{D}}\right] \le \mathbf{E}_s[T_{\overline{D}\cup t}] + \mathbf{E}_t[T_{\overline{D}}] \le (|D|-1)\rho_1(C) + \rho_1(C).$$

We now prove (13). Let $s \in C$ be given. For $t \ne s \in C$, one has, by (11) and (12)

$$\mathbf{E}_s\left[T_{\overline{C}}\right] \ge \mathbf{E}_t\left[T_{\overline{C}}\right] - \mathbf{E}_t\left[T_{\overline{C}\cup s}\right] \ge \mathbf{E}_t\left[T_{\overline{C}}\right] - (|C|-1)\rho_1(C).\tag{14}$$

The result now follows by taking the maximum over t in (14). $\blacksquare$

Corollary 2.9 *Under the notations of Proposition 2.8, one has*

$$\mathbf{P}_s\left(T_{\overline{C}} < T_t\right) \le 2\,|C|\,\frac{\rho_1(C)}{\rho_2(C) - (|C|-1)\rho_1(C)} \text{ for each } C \subset S,\, s, t \in C.\tag{15}$$

Proof. Let $C \subset S$, and $s, t \in C$ be given. We modify the Markov chain by collapsing $\overline{C}$ to a single state, still denoted $\overline{C}$, and we set $q(t \mid \overline{C}) = 1$, so that $\mathbf{E}_{\overline{C}}[T_t] = 1$. This modification does not affect $\mathbf{P}_s\left(T_{\overline{C}} < T_t\right)$. By Aldous and Fill (2002, Chapter 2, Corollary 10),

$$\mathbf{P}_s(T_{\overline{C}} < T_t) = \frac{\mathbf{E}_s\left[T_t\right] + \mathbf{E}_t\left[T_{\overline{C}}\right] - \mathbf{E}_s\left[T_{\overline{C}}\right]}{\mathbf{E}_{\overline{C}}\left[T_t\right] + \mathbf{E}_t\left[T_{\overline{C}}\right]}.\tag{16}$$

Since $\mathbf{E}_{\overline{C}}[T_t] = 1$, one has $\mathbf{E}_s\left[T_t\right] \le \mathbf{E}_s\left[T_{t \cup \overline{C}}\right] + 1$. By (11), $\mathbf{E}_t\left[T_{\overline{C}}\right] - \mathbf{E}_s\left[T_{\overline{C}}\right] \le \mathbf{E}_t\left[T_{s \cup \overline{C}}\right]$. By (12), the numerator in (16) is at most $1 + \mathbf{E}_t[T_{\overline{C}\cup s}] + \mathbf{E}_s[T_{\overline{C}\cup t}] \le 2(|C|-1)\rho_1(C) + 1$.

On the other hand, the denominator is equal to $1 + \mathbf{E}_t\left[T_{\overline{C}}\right]$, hence, by (13), at least $\rho_2(C) - (|C| - 1)\rho_1(C)$. ∎

The next result deals with the transition function q^C of the Markov chain q watched on C (see Aldous and Fill (2001, Chapter 2, Section 7.1)):

$$q^C(t \mid s) = q(t \mid s) + \sum_{u \notin C} q(u \mid s)\mathbf{P}_u(T_C = T_t), \text{ for every } s, t \in C. \quad (17)$$

By Aldous and Fill, q^C is irreducible, and its invariant measure μ^C coincides with the invariant measure of q, conditioned on C.

Corollary 2.10 *For $s, t \in C$, one has $\mathbf{E}_{s,q^C}[T_t] \leq \frac{(|C|-1)\rho_1(C)}{\min_{u \in C} \mathbf{P}_{u,q}(T_t < T_{\overline{C}})}$.*

Proof. Let $t \in C$ be given. For convenience, set $\alpha := \max_{s \in C} \mathbf{E}_{s,q^C}[T_t]$. Let $s \in S$ achieve the maximum in the definition of α. By (12)

$$\alpha = \mathbf{E}_{s,q^C}[T_t] \leq \mathbf{E}_{s,q}\left[T_{\overline{C} \cup t}\right] + \mathbf{P}_{s,q}(T_{\overline{C}} < T_t)\alpha.$$
$$\leq (|C| - 1)\rho_1(C) + \alpha \mathbf{P}_{s,q}(T_{\overline{C}} < T_t).$$

Then, for every $s' \in C$,

$$\mathbf{E}_{s',q^C}[T_t] \leq \alpha \leq \frac{(|C| - 1)\rho_1(C)}{1 - \mathbf{P}_{s,q}(T_{\overline{C}} < T_t)} \leq \frac{(|C| - 1)\rho_1(C)}{\min_{u \in C} \mathbf{P}_{u,q}(T_t < T_{\overline{C}})},$$

as desired. ∎

2.2.3 A structure theorem

Here we prove a structure result which states that for every finite sequence of states in S there is a partition of S such that the number of times the sequence exits a given atom of the partition is much smaller than the number of visits to any strict subset of this atom. The sequence moves around inside the atom much more quickly than from one atom to another.

For every positive integer $N \in \mathbf{N}$, every sequence $(s_0, s_1, \ldots, s_N)$ of states, and every subset $C \subset S$, define

$$R_C = |\{n < N \mid s_n \notin C, s_{n+1} \in C\}| + 1_{s_0 \in C}.$$

R_C is the number of C-runs along the sequence (see Feller (1968, II.5)). For convenience of notations, we omit the dependency of R_C on the sequence. Note that $R_{C \setminus D} \leq R_C + R_D$ for every proper subset D of C, and that $|R_C - R_{S \setminus C}| \leq 1$. Note also that $R_C \geq |\{n < N | s_n \in C, s_{n+1} \notin C\}|$

Theorem 2.11 *For every positive integer N, every sequence $(s_0, s_1, \ldots, s_N)$, of states in S, and every $a > 0$, there is a partition $\mathcal{C}$ of S such that the following holds for every $C \in \mathcal{C}$.*

    P1 *$R_C \le (a+1)^{|\mathcal{C}|}$.*
    P2 *For each proper subset D of C, $R_D > aR_C$.*

    Proof. Observe that the trivial partition $\mathcal{C} = \{S\}$ satisfies **P1**, since $R_S = 1$.

Among all the partitions that satisfy **P1**, let $\mathcal{C}$ be one with maximal number of atoms. Denote $k = |\mathcal{C}|$. We prove that $\mathcal{C}$ satisfies **P2**. Otherwise, there is $C \in \mathcal{C}$, and there is a proper subset D of C, such that $R_D \le aR_C$.

Consider the partition $\mathcal{C} \setminus \{C\} \cup \{D, C \setminus D\}$; that is, we further partition the set C into two sets D and $C \setminus D$. We show that this new partition, that has $k+1$ elements, satisfies **P1** as well, contradicting the maximality of $\mathcal{C}$. Indeed, $R_D \le aR_C \le (a+1)^{k+1}$, and $R_{C\setminus D} \le R_C + R_D \le R_C(a+1) \le (a+1)^{k+1}$. ∎

Remark 2.12 *The partition need not be unique. Indeed, if $|S| = 2$ and $a < \frac{n}{2} \le (a+1)^2$, the two partitions of S satisfy **P1** and **P2**.*

Remark 2.13 *When $a > 2$, the collection of all partitions that satisfy **P2** is a lattice in the following sense: if $\mathcal{C}$ and $\mathcal{D}$ are two partitions that satisfy **P2**, then for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, if the intersection $C \cap D$ is not empty then it is equal to C or D. In particular, the partition considered in the proof of Theorem 2.11 is unique.*

*Indeed, let $\mathcal{C}$ and $\mathcal{D}$ be two partitions that satisfy **P2** such that, for some $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the intersection $P = C \cap D$ is not empty, and a strict subset of both C and D.*

For every set A which is disjoint of P set

$$k_A = \#\{n < N \mid s_n \in P, s_{n+1} \in A\}.$$

Then $k_{\overline{D}} + k_{D\setminus P} = k_{\overline{C}} + k_{C\setminus P} = R_P - 1_{s_N \in P}$, $k_{C\setminus P} + k_{D\setminus P} \le R_P - 1_{s_N \in P}$, $k_{\overline{C}} \le R_C - 1_{s_N \in P}$, and $k_{\overline{D}} \le R_D - 1_{s_N \in P}$. It follows that

$$R_P - 1_{s_N \in P} \ge k_{C\setminus P} + k_{D\setminus P} \ge 2R_P - 2 \times 1_{s_N \in P} - k_{\overline{C}} - k_{\overline{D}} \ge 2R_P - R_C - R_D.$$

*In particular, by **P2***

$$R_C + R_D - 1_{s_N \in P} \ge R_P \ge a \times \max\{R_C, R_D\},$$

a contradiction when $a > 2$.

2.3 Proof of Theorem 2.2

To prove Theorem 2.2 it is sufficient to consider only exhaustive sequences; namely, sequences that visit all states in S (by dropping from S states that are never visited). However, as the proof of the more general Theorem 3.6 below refers to the proof of Theorem 2.2, it is more convenient not to make this assumption.

We prove Theorem 2.2 first by considering periodic and exhaustive sequences, and then by looking at a general sequence.

Let $\varepsilon > 0$ be small enough, let $\rho \in (0, 1/2(4|S| + 1))$, and let $\zeta \in (0, 2\rho)$ be fixed.

2.3.1 The case of periodic exhaustive sequences

We choose $N_0 \in \mathbf{N}$ such that (N.i) $N_0^{(4|S|+1)\rho-1} \leq \varepsilon/(2^{|S|} + 1)$, (N.ii) $N_0^{4\rho} \geq \max\{11|S|, 2/\varepsilon\}$, (N.iii) $N_0^{2\rho-\zeta} \geq 4 \times 19\,|S|\,/\varepsilon^2$, and (N.iv) $N_0^\rho \geq 4\,|S| + 1$. Let $N \geq N_0$, and we set $a = N^{4\rho}$.

We assume here that the sequence $\sigma = (s_0, s_1, \ldots, s_N)$ is periodic and exhaustive: $s_N = s_0$ and $N_s \geq 1$ for every $s \in S$. The proof of the following lemma is left to the reader.

Lemma 2.14 *The empirical transition function q^σ is irreducible. Its invariant measure is $\mu_s = \frac{N_s}{N}$.*

Let $\mathcal{C} = (S_1, \ldots, S_K)$ be a partition of S obtained when applying Theorem 2.11 to σ and a. For $C \subset S$, we let $n_C := \sum_{s \in C} N_s$ denote the number of stages spent in C along σ. We abbreviate n_{S_k} to n_k.

Proposition 2.15 *With the notations of Proposition 2.8, one has*

$$\rho_1(S_k) \leq \max_{D \subset S_k} \frac{n_D}{R_D - 1} \leq \frac{2}{a}\rho_2(S_k), \text{ for every } k \text{ such that } |S_k| > 1.$$

Proof. By Lemma 2.14 $\sum_{s \in D} \mu_s = \frac{n_D}{N}$ and $\sum_{s \in D} \mu_s q^\sigma(\overline{D} \mid s) = \frac{|\{n < N | s_n \in D, s_{n+1} \notin D\}|}{N}$. Therefore,

$$\frac{n_D}{R_D} \leq \sum_{s \in D} \nu_D(s)\mathbf{E}_{s,q^\sigma}[T_{\overline{D}}] \leq \frac{n_D}{R_D - 1} \tag{18}$$

Fix $k \in \{1, \ldots, K\}$. By (N.ii) $a \geq 2$, hence $\frac{1}{aR_{S_k}-1} \leq \frac{2}{a} \times \frac{1}{R_{S_k}}$. Hence, for every $D \subset S_k$, by (18), the definition of $\mathcal{C}$, and (18) again,

$$\min_{s \in D} \mathbf{E}_{s,q^\sigma}[T_{\overline{D}}] \leq \frac{n_D}{R_D - 1} \leq \frac{n_{S_k}}{aR_{S_k} - 1} \leq \frac{2}{a}\frac{n_k}{R_{S_k}} \leq \frac{2}{a}\max_{s \in S_k}\mathbf{E}_{s,q^\sigma}\left[T_{\overline{S_k}}\right] = \frac{2}{a}\rho_2(S_k).$$

The result follows, by taking the supremum over $D \subset S_k$. ∎

We now construct a piecewise Markov chain with K pieces. The kth piece is used for n_k stages, and its goal is to approximate the empirical transitions on S_k. In those stages, the process will remain in S_k.

Let $m_0 = 0$, and for every positive integer $k \leq K$ define $m_k = n_1 + n_2 + \cdots + n_k$.

For every $k = 1, \ldots, K$ define a transition rule $q'_k : S \to \Delta(S)$ as follows. If $n_k < N^{1-\rho}$ we define $q'_k = q^\sigma$. Otherwise we define

$$q'_k(t \mid s) = \begin{cases} q_k(t \mid s) & s \in S_k, t \in S_k \\ \mu_k(t) & s \notin S_k, t \in S_k \end{cases}$$

where q_k is the transition function of the Markov chain q^σ watched on S_k (see Eq. (17)) and μ_k is the invariant measure of q_k. Let $(\mathbf{s}_n)_{0 \leq n \leq N}$ be the piecewise Markov chain that starts in S_1 and follows the transition rule q'_k from stage m_k up to m_{k+1}, for each k. The exact way the initial state is chosen is irrelevant. We will show that it satisfies the requirements of Theorem 2.2.

We first show that condition **B2** is satisfied. Fix $k \in \{1, \ldots, K\}$, and let $s \in S_k$. If $n_k < N^{1-\rho}$ then $q'_k(\cdot \mid s) = q^\sigma(\cdot \mid s)$. Otherwise,

$$\|q'_k(\cdot \mid s) - q^\sigma(\cdot \mid s)\| \leq \sum_{u \notin S_k} q^\sigma(u \mid s) \leq \frac{R_{S_k}}{N_s}.$$

If $S_k = \{s\}$ is a singleton, then by Theorem 2.11(**P1**) and (N.i) the right hand side is bounded by $\frac{(a+1)^{|S|}}{N^{1-\rho}} < \varepsilon$, while if $|S_k| \geq 2$, by (N.ii) the right hand side is bounded by $\frac{R_{S_k}}{a R_{S_k}} < \varepsilon$. Therefore, $\|\mathbf{q}_n - q^\sigma(\cdot \mid \mathbf{s}_n)\| < \varepsilon$ holds a.s. whenever $n \neq m_k$, for $k = 0, .., K - 1$.

We now prove that condition **B1** is satisfied. Let $k \in \{1, \ldots, K\}$ be given. If $n_k < N^{1-\rho}$, then $\nu_s^\sigma < \frac{1}{N^\rho}$ for every $s \in S_k$, hence **B1** holds for such states. If $n_k \geq N^{1-\rho}$ and $S_k = \{s\}$ is a singleton then $F_N^s = \nu_s^\sigma$, and **B1** holds as well.

We may thus assume that $n_k \geq N^{1-\rho}$ and $|S_k| \geq 2$. We establish the claim by proving first that q_k is mixing, and by using Theorem 2.4.

Lemma 2.16 *The transition function q_k on S_k is $N^{1-3\rho}$-mixing.*

Proof. By Corollary 2.10, for every $s, t \in S_k$

$$\mathbf{E}_{s,q_k}[T_t] \leq \frac{(|S_k| - 1)\rho_1(S_k)}{1 - \max_{u \in S_k} \mathbf{P}_{u,q^\sigma}(T_{\overline{S}_k} < T_t)}.$$

Abbreviate $\rho_1(S_k)$ and $\rho_2(S_k)$ to ρ_1 and ρ_2 respectively. By Corollary 2.9, the denominator is at least $1 - 2|S_k|\frac{\rho_1}{\rho_2 - (|S_k| - 1)\rho_1}$. Therefore,

$$\mathbf{E}_{s,q_k}[T_t] \leq (|S_k| - 1)\rho_1 \times \frac{\rho_2 - (|S_k| - 1)\rho_1}{\rho_2 - (3|S_k| - 1)\rho_1} \leq 2|S_k|\rho_1, \qquad (19)$$

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where the second inequality follows by Proposition 2.15 and (N.ii).

By Proposition 2.15 and Theorem 2.11(**P2**)

$$\rho_1 \leq \max_{D \subset S_k} \frac{n_D}{R_D - 1} < \frac{N}{a-1} \leq \frac{N^{1-3\rho}}{2\,|S_k|} - 1, \tag{20}$$

since $N^\rho \geq 4\,|S| + 1$. The result follows by (19) and (20). ∎

By Section 2.2.2, the invariant measure of q_k is $\nu^\sigma(\cdot \mid S_k)$, where $\nu^\sigma(t \mid S_k)) = N_t/n_k$ for $t \in S_k$. By (N.i), Remark 2.6, and (N.iii),

$$\mathbf{P}_{s,q_k}\left(|\overline{F}^t_{n_k} - \nu^\sigma(t \mid S_k)| > \varepsilon\nu^\sigma(t \mid S_k)\right)$$

$$\leq \mathbf{P}_{s,q_k}\left(|\overline{F}^t_{n_k} - \nu^\sigma(t \mid S_k)| > \frac{\varepsilon}{2}\nu^\sigma(t \mid S_k) + \frac{1}{n_k}\right)$$

$$\leq 4 \times \frac{19N^{1-3\rho}}{n_k\varepsilon^2} \leq 4 \times \frac{19N^{1-3\rho}}{\varepsilon^2 N^{1-\rho}} \leq \frac{1}{|S|} \times \frac{1}{N^\zeta}. \tag{21}$$

Since the process $(\mathbf{s}_n)$ does not visit $t \in S_k$ except in the kth phase, **B1** follows from (21) by summation over t.

2.3.2 The sequence $\sigma = (s_0, s_1, \ldots, s_N)$ is arbitrary

Choose $N_0 \in \mathbf{N}$ such that (N'.i) $N_0^{2\rho(4|S|+1)-1} \leq \frac{\varepsilon-2\varepsilon^2}{2^{|S|}+1}$, (N'.ii) $N_0^{8\rho} \geq \max\{11|S|, 2/\left(\varepsilon - 2\varepsilon^2\right))\}$, (N'.iii) $N_0^{4\rho-\zeta} \geq 4 \times 19\,|S| / \left(\varepsilon - 2\varepsilon^2\right)^2$, (N'.iv) $N_0^{1-2\rho} \geq \frac{|S|}{\varepsilon^2}$, (N'.v) $N_0^\rho \geq \frac{1}{1-\varepsilon^2}$, and (N'.vi) $N_0^{2\rho} \geq 4\,|S| + 1$.

Let $N \geq N_0$, and let $\sigma = (s_0, ..., s_N)$ be an arbitrary sequence in S. We will add few states to σ, so as to get a periodic and exhaustive sequence. We then apply the results of Section 2.3.1 to the new sequence, and then prove that similar estimates hold for the original sequence.

Let $S^* = \cup_{n=0}^{N}\{s_n\} \subseteq S$ be the set of states visited by σ. Consider the sequence $\sigma^* = (s_0, s_1, \ldots, s_N, s_1^*, \ldots, s_r^*, s_0)$, where $r = |S| - |S^*|$ is the number of states not visited by σ, and $S \backslash S^* = \{s_1^*, \ldots, s_r^*\}$. By construction, this new sequence is periodic and exhaustive. The length $N_* + 1$ of this sequence is $N + r + 2 < N + |S| + 2$.

One can verify that N_* satisfies (N.i-iv) with $\rho' := 2\rho$ and $\varepsilon' := \varepsilon - 2\varepsilon^2$. Therefore there is a piecewise Markov chain[2] $(\mathbf{s}_n)_{n \leq N_*}$ such that **B1** and **B2** hold w.r.t. ν^{σ^*}. Observe that each state $s_j^* \in S \setminus S^*$ constitutes a singleton in the partition $\mathcal{C}$ associated with σ^*. We assume that the last r stages are devoted to these elements of the partition, and we now check that the restriction $(\mathbf{s}_n)_{n \leq N}$ of the process to the first N stages satisfies **B1** and **B2**.

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[2]In Section 2.3.1, we set $q_k' = q^\sigma$ whenever $n_k < N^{1-\rho}$. We still set here q_k' to be q^σ and not q^{σ^*}. This does not affect conclusions **B1** and **B2** for σ^*.

We start with **B1**. Let $s \in S$ with $\nu_s^\sigma \geq \frac{1}{N^\rho}$. By (N'.iv),

$$\left| \nu_s^\sigma - \nu_s^{\sigma^*} \right| \leq \frac{r+1}{N} \leq \nu_s^\sigma \frac{r+1}{N^{1-2\rho}} \leq \varepsilon^2 \nu_s^\sigma. \tag{22}$$

By (N'.v),

$$\nu_s^\sigma \geq \frac{1}{N^\rho} \Longrightarrow \nu_s^{\sigma^*} \geq \frac{1}{N^{2\rho}} \Rightarrow \mathbf{P}\left(\left| \overline{F}_{N_*}^s - \nu_s^{\sigma^*} \right| \geq \varepsilon' \nu_s^{\sigma^*} \right) \leq \frac{1}{N^\zeta}. \tag{23}$$

In such a case, by (N'.iv)

$$\left| \overline{F}_{N_*}^s - \overline{F}_N^s \right| \leq \frac{r+1}{N} \leq \varepsilon^2 \nu_s^\sigma.$$

Hence, by (22), $\left| \overline{F}_N^s - \nu_s^\sigma \right| \leq \left| \overline{F}_{N_*}^s - \nu_s^{\sigma^*} \right| + 2\varepsilon^2 \nu_s^\sigma$. Condition **B1** follows using (23).

We now prove **B2**. By construction, except for at most $|S|$ stages, $\mathbf{q}_n = q^\sigma(\cdot \mid \mathbf{s}_n)$, or both $\left| \mathbf{q}_n - q^{\sigma^*}(\cdot \mid \mathbf{s}_n) \right| \leq \varepsilon'$ and $N_{\mathbf{s}_n}^* \geq \min\{a, N^{1-\rho}\} \geq N^{4\rho}$. In the latter case, by (N'.iii), $\left| q^{\sigma^*}(\cdot|\mathbf{s}_n) - q^\sigma(\cdot \mid \mathbf{s}_n) \right| \leq 1/N^{4\rho} \leq \varepsilon'$, which concludes the proof.

3 The General Problem

3.1 Presentation

For every state $s \in S$ let $V_s \subseteq \Delta(S)$ be a non-empty polyhedron,[3] and set $V = (V_s)_{s \in S}$. The set V_s should be thought of as the set of permissible transitions from s.

Throughout this section, V is fixed.

Definition 3.1 *A* V-*process is a* S-*valued process* (X_n) *such that for every* $n \geq 1$, *the conditional distribution of* $\mathbf{s}_n$ *given* $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{n-1}$ *is in* $V_{\mathbf{s}_{n-1}}$.

We here generalize the question addressed in the previous section. Given a sequence σ, does there exist a simple V-process that approximates σ, in the sense of Theorem 6. In general, such an approximation needs not exist. Indeed, as the following two examples show, if all V-processes are reducible, or if the sequence is not "typical", meaning that the empirical transitions are "far" from any V-process, such a construction is not possible. In the following two examples, V_s is a singleton for each $s \in S$, so there is a unique V-process, which is a Markov chain.

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[3]A *polyhedron* is a convex hull of finitely many points.

Example 3.2 (A reducible Markov chain) *Consider a problem with three states $\{a, b, c\}$. Assume V is such that for any V-process, states b and c are absorbing, whereas if the process is in state a, with equal probabilities it moves to states b and c. When the initial state is a, there are two possible sequences under the unique V-process, each is realized with probability $1/2$: $(a, b, b, b, \ldots, b)$ and $(a, c, c, c, \ldots, c)$. But if the given sequence is $(a, b, b, b, \ldots, b)$ there is no V-process that satisfies both (i) and (ii).*

Example 3.3 (A non-typical sequence) *Assume there are two states $\{a, b\}$, and $V_a = V_b = \{\frac{1}{2}a + \frac{1}{2}b\}$.*

Assume the given sequence is $(a, a, \cdots, a)$. There is no V-process that satisfies (i) and (ii), provided N is sufficiently large.

Thus:

- The sequence σ may be completely atypical of any V-process.

- Transitions out of states that are transient under any V-process may not be approximated.

We now define the notion of typical sequences w.r.t. $V = (V_s)_{s \in S}$, prove that for every V-process, the probability that the realized sequence is typical is close to 1, and prove that for every typical sequence there is a hidden piecewise V-Markov chain with at most $|S|$ pieces that approximates the typical sequence in the sense of (i) and (ii) above.

Therefore, we will assume that V contains an irreducible transition function $b = (b_s)_{s \in S}$, and limit our analysis to sequences that are typical, in the following sense.

Definition 3.4 *Let $N \in \mathbf{N}$, and $\delta, \varepsilon > 0$. A sequence $(s_0, s_1, \ldots, s_N)$ is (N, δ, ε)-typical if there exists $v \in V$ such that $\left| 1 - \frac{v(t|s)}{q(t|s)} \right| < \varepsilon$ for every $s, t \in S$ that satisfy $N_s q(t \mid s) \geq N^\delta$ or $N_s v(t \mid s) \geq N^\delta$. The set of (N, δ, ε)-typical sequences is denoted by $T_{\delta, \varepsilon}^N$.*

As we prove in the sequel, under some constraints on the parameters the probability of the typical sequences is close to 1, under any V-process.

Definition 3.5 *A process $(\mathbf{s}_n)_{n \leq N}$ is a (piecewise) hidden Markov chain over S if there exists an auxiliary finite set T and a (piecewise) Markov chain $(\mathbf{z}_n)_{n \leq N}$ over $S \times T$ such that $(\mathbf{s}_n)_{n \leq N}$ is the marginal of $(\mathbf{z}_n)_{n \leq N}$ over S.*

Following the notations of Definition 3.5, let $p(\cdot \mid (s, t))$ be the transition function of $(\mathbf{z}_n)_{n \leq N}$. If the marginal over S of $p(\cdot \mid (s, t))$ belongs to V_s for each $(s, t) \in S \times T$, the process $(\mathbf{s}_n)_{n \leq N}$ is a V-process, due to the convexity of V_s. It is typically not a Markov chain. In such a case, we say that $(\mathbf{s}_n)_{n \leq N}$ is a *hidden (piecewise) V-Markov chain*.

Theorem 3.6 *Assume that there is an irreducible transition function $b \in V$, and set $B := \max_{s,t \in S} \mathbf{E}_{s,b}[T_t]$. Let $\psi, \eta \in (0,1)$ be given. There exist $\delta, \varepsilon > 0$ and $N_1 \in \mathbf{N}$ such that the following holds. For every $N \geq N_1$ and every (N, δ, ε)-typical sequence $(s_0, s_1, \ldots, s_N)$, there exists a hidden piecewise V-Markov chain with at most $|S|$ pieces such that*

G1 *If $\nu_s^\sigma \geq \frac{1}{N^\delta}$, then $\mathbf{P}(|\overline{F}_N^s - \nu_s^\sigma| \geq \eta \nu_s^\sigma) \leq \frac{1}{N^\delta}$.*

G2 *Let $\mathbf{N}_0 = |\{n < N : \|\mathbf{q}_n - q^\sigma(\cdot \mid \mathbf{s}_n)\| > \eta \}|$. Then $\mathbf{E}[\mathbf{N}_0] \leq N^\psi B$.*

3.2 Typical sequences

Theorem 3.7 below states that most sequences are typical, provided the parameters are chosen properly. Its proof uses the following large deviation estimate for Bernouilli variables. Let $(X_n)_{n \in \mathbf{N}}$ be an infinite sequence of i.i.d. Bernouilli r.v.s with parameter p, and denote for every positive integer n, $\overline{X}_n = \sum_{i=1}^n X_i/n$. By Alon et al (2000, Corollary A.14),

$$\mathbf{P}(|\overline{X}_n - p| > \varepsilon p) \leq 2 \exp(-c_\varepsilon p n),$$

where $c_\varepsilon = \min\{\varepsilon^2, -\varepsilon + (1 + \varepsilon) \ln(1 + \varepsilon)\}$ is independent of n and p. In particular, for every positive integer k,

$$\mathbf{P}\left(\sup_{pn \geq k} |\overline{X}_n - p| > \varepsilon p \right) \leq 2 \sum_{n=\lceil k/p \rceil}^\infty \exp(-c_\varepsilon p n) \leq \frac{2 \exp(-c_\varepsilon k)}{1 - \exp(-c_\varepsilon p)}. \quad (24)$$

Observe that for every ε sufficiently small, $\varepsilon^2/3 < c_\varepsilon \leq \varepsilon^2/2$.

Theorem 3.7 *Let $\delta, \varepsilon > 0$ be given. For each $\xi \in (0, \delta/4)$, there exists $N_0 \in \mathbf{N}$ such that, for every $N \geq N_0$ and every V-process π,*

$$\mathbf{P}(T_{\delta, \varepsilon}^N) \geq 1 - \frac{1}{N^\xi}.$$

Proof. Let $\delta, \varepsilon \in (0,1)$ and $\xi \in (0, \delta/4)$ be given. For each $s \in S$, let V_s^* be the (finite) set of the extreme points of V_s. Choose $\xi' \in (\xi, \delta/4)$. Set $\varepsilon' = \varepsilon/(1 + \varepsilon)$. Let $N_0 \in \mathbf{N}$ be large enough so that the following conditions are satisfied for each $N \geq N_0$: (i) $\frac{2 \exp(-c_{\varepsilon'} N^{\delta/4})}{1 - \exp(-c_{\varepsilon'} N^{\delta/4-1})} \leq 1/N^{\xi'}$, (ii) $N^{\xi'-\xi} \geq 3 |S| \sum_{s \in S} |V_s^*|$ and (iii) $N^{\delta/2} \geq 1/\varepsilon$. Let $N \geq N_0$, and let $(\mathbf{s}_n)_{n \leq N}$ be any V-process.

We first present the V-process $(\mathbf{s}_n)_{n \leq N}$ in an alternative way, by writing the conditional distribution of $\mathbf{s}_{n+1}$ given $\mathbf{s}_0, \ldots, \mathbf{s}_n$ as a convex combination $\sum_{v \in V_{\mathbf{s}_n}^*} \mathbf{b}_n(v)v$ of the extreme points of $V_{\mathbf{s}_n}$ (the weights $\mathbf{b}_n(v)$ being random themselves).

Consider the process $\pi' = (\mathbf{s}_n, \mathbf{v}_n)_{n \leq N}$: given the past, $\mathbf{v}_n \in V_{\mathbf{s}_n}^*$ is selected according to $\mathbf{b}_n$, then $\mathbf{s}_{n+1}$ is selected according to $\mathbf{v}_n$. Plainly, the

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law of the sequence of states is the same under both processes. We shall deal with the process π'.

Define $\mathbf{n}_{s,v} = |\{n < N, (\mathbf{s}_n, \mathbf{v}_n) = (s, v)\}|$ to be the number of times the extreme point v was chosen at s, $\mathbf{n}_s = \sum_{v \in V^*} \mathbf{n}_{s,v}$ the number of visits to s, and $\mathbf{q}(t \mid s, v) = |\{n < N, (\mathbf{s}_n, \mathbf{v}_n, \mathbf{s}_{n+1}) = (s, v, t)\}| / \mathbf{n}_{s,v}$. Note that the empirical transitions out of s are given by $\mathbf{q}(t \mid s) = \frac{\sum_{v \in V^*} \mathbf{n}_{s,v} \mathbf{q}(t|s,v)}{\mathbf{n}_s}$, and define $\mathbf{v}_s^* = \frac{\sum_{v \in V^*} \mathbf{n}_{s,v} v}{\mathbf{n}_s}$. Both are defined whenever $\mathbf{n}_s > 0$. As V_s is convex, $\mathbf{v}_s^* \in V_s$. We will show that with high probability, $\mathbf{v}^* = (\mathbf{v}_s^*)$ is close to $\mathbf{q}$ in the sense of Definition 3.4.

Fix for a moment $s, t \in S$ and $v \in V_s^*$. Plainly, $\mathbf{n}_{s,v} v(t) < N^{\delta/4}$ if $v(t) < N^{\delta/4-1}$. We now assume that $v(t) \geq N^{\delta/4-1}$. Let $(X_n)_{n \leq N}$ be a sequence of i.i.d. Bernouilli r.v.s with parameter $v(t)$. By (24) and (i)

$$\mathbf{P}\left(\mathbf{n}_{s,v} v(t) \geq N^{\delta/4} \text{ and } |\mathbf{q}(t \mid s, v) - v(t)| > \varepsilon' v(t)\right) \leq \frac{2 \exp(-c_{\varepsilon'} N^{\delta/4})}{1 - \exp(-c_{\varepsilon'} N^{\delta/4-1})} \leq \frac{1}{N^{\xi'}}.$$
(25)

We now claim that

$$\mathbf{P}\left(\mathbf{n}_{s,v} v(t) < N^{\delta/4} \text{ and } \mathbf{n}_{s,v} q(t \mid s, v) \geq N^{\delta/2}\right) \leq 2/N^{\delta/4}. \qquad (26)$$

Indeed, setting $n = \lfloor N^{\delta/4}/v(t) \rfloor$, the left hand side in (26) is at most

$$\mathbf{P}\left(\sup \left\{X_1 + \cdots + X_k, kv(t) < N^{\delta/4}\right\} \geq N^{\delta/2}\right) \leq \mathbf{P}\left(X_1 + \cdots + X_n \geq N^{\delta/2}\right).$$

By Markov inequality, the right-hand side is at most $nv(t)/N^{\delta/2} \leq 2/N^{\delta/4}$.

Eqs. (25) and (26) yield together

$$\mathbf{P}\left(\mathbf{n}_{s,v} \max\{v(t), \mathbf{q}(t \mid s, v)\} \geq N^{\delta/4} \Rightarrow |\mathbf{q}(t \mid s, v) - v(t)| \leq \varepsilon' v(t)\right) \geq 1 - \frac{3}{N^{\xi'}}.$$
(27)

Let T be the set of all sequences $(s_0, v_0, s_1, v_1, \ldots, s_N)$ that satisfy the implication in (27), for every $s, t \in S$ and every $v \in V_s^*$. By (ii), $\mathbf{P}(T) \geq 1 - \frac{1}{N^\xi}$. We will show that every sequence in T is (N, δ, ε)-typical.

Let us be given a sequence in T. Let $n_{s,v}, n_s, q(t \mid s, v), q(t \mid s)$ and v^* be the values of the r.v.s. $\mathbf{n}_{s,v}, \mathbf{n}_s, \mathbf{q}(t \mid s, v), \mathbf{q}(t \mid s)$ and $\mathbf{v}^*$ respectively for this sequence. Assume that $s, t \in S$ satisfy $n_s q(t \mid s) \geq N^\delta$ (the same argument is valid also in the case $n_s v(t) \geq N^\delta$). We prove that $|q(t \mid s) - v(t)| \leq \varepsilon q(t \mid s)$. We first claim that for every $v \in V_s^*$,

$$n_{s,v}|q(t \mid s, v) - v(t)| \leq \frac{\varepsilon'}{1 - \varepsilon'} n_s q(t \mid s). \qquad (28)$$

Indeed, if $n_{s,v} \max\{q(t \mid s, v), v(t)\} \geq N^{\delta/2}$ then by (27) $v(t) \leq \frac{1}{1-\varepsilon'} q(t \mid s, v)$, and therefore

$$n_{s,v}|q(t \mid s, v) - v(t)| \leq \varepsilon' n_{s,v} v(t) \leq \frac{\varepsilon'}{1 - \varepsilon'} n_{s,v} q(t \mid s, v) \leq \frac{\varepsilon'}{1 - \varepsilon'} n_s q(t \mid s),$$

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where the last inequality holds since $n_s q(t \mid s) = \sum_{v \in V_s^*} n_{s,v} q(t \mid s, v)$. If, on the other hand $n_{s,v} \max\{q(t \mid s, v), v(t)\} < N^{\delta/2}$ then

$$n_{s,v}|q(t \mid s, v) - v(t)| \le N^{\delta/2} \le n_s q(t \mid s)/N^{\delta/2},$$

and (28) holds by (iii).

By summing (28) over all $v \in V_s^*$ we get,

$$|q(t \mid s) - v_s^*(t)| \le \sum_{v \in V_s^*} \frac{n_{s,v}}{n_s} |q(t \mid s, v) - v(t)| \le \frac{\varepsilon'}{1 - \varepsilon'} q(t \mid s) = \varepsilon q(t \mid s),$$

as desired. ∎

The requirement $\xi < \delta/4$ arises from the use of Markov inequality. A slight modification of the argument would improve the bound to $\delta/2$. It is not clear whether this latter bound is optimal.

3.3 Proof of Theorem 3.6

We here prove Theorem 3.6. The proof mostly follows the proof of Theorem 2.2. The main complication is the following. Each piece q_k of the Markovian approximation that was constructed in Section 2 was obtained by watching the empirical transition function q on a specific subset S_k. Characteristics of the corresponding chain q_k (invariant measure, mixing time) were then easily derived from the properties of q and of the partition of S. By constrast, each piece of the approximation is here required to be a V-process. Thus, the former choice for q_k may no longer be admissible, and one is led to choose the V-process that is closest (in some sense) to q_k on S_k. Properties of this process are obtained from results on perturbations of Markov chains to be found in Solan and Vieille (2002).

3.3.1 Fixing parameters

Let $\psi, \eta \in (0,1)$ be given. Choose $\varepsilon > 0$ small enough so as to satisfy the following conditions, with $L = \sum_{n=1}^{|S|} \binom{|S|}{n} n^{|S|}$. (**E1**) $\varepsilon < \eta/56L < \eta$, (**E2**) $\varepsilon < \frac{1}{20L^2|S|^2}$, (**E3**) $\varepsilon < \frac{1}{3 \times 2^{|S|}}$ and (**E4**) $\frac{4}{1-\varepsilon}(1 + 54\varepsilon L) \le L$.

Fix $\beta \in (0, \frac{1}{2}\left(\frac{A}{L}\right)^{|S|} \times \frac{\varepsilon(1-\varepsilon)}{L \times |S|^4})$, where $A = 1/2$. Set $\alpha = \frac{1}{2\beta|S|L^2}$, and $\alpha' = \frac{\alpha/2 - |S|}{2|S|}$. Then $\beta < 1/20|S|^2 L^2$, so that $\alpha' \ge 2$.

Choose $\psi' \in (0, \psi)$, $\xi \in (0, \psi'/(|S|+1))$, $\delta' \in (0, \xi/2)$, and $\delta \in (0, \min\{\delta', (1-\psi)/2\})$. Set $a = N^\xi$. This choice implies that for every $N \in \mathbf{N}$ sufficiently large the following inequalities hold. (**C1**) $N^{\delta'} \ge N^\delta + 1$, (**C2**) $N^\delta \ge \frac{3}{\varepsilon(1-3\varepsilon)}$, (**C3**) $2 + 16L|S|(N+S+2)^{\psi'} \times \frac{1}{1-|S|N^{-\delta}} < N^\psi/|S|$, (**C4**) $\eta N^{1-2\delta} \ge 1$, (**C5**) $4BL|S|N^{\delta+\psi'-1} \le (N + |S| + 2)^{-\delta} \le \varepsilon/2$, (**C6**) $N^{1-2\delta-\psi} \ge 42B|S|/\varepsilon^2$,

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(A1) $\beta(a-1) \geq (N+S+2)^{\delta'}$, **(A2)** $a - 1 \geq \frac{1}{2\beta|S|}$, **(A3)** $2a \leq N$, **(A4)** $(a+1)^{|S|} \leq N^{\psi'} \leq N$, **(A5)** $\frac{4\times72}{\varepsilon^2 N^{1-\delta}}\left(\frac{N}{a-1}+B\right) \leq \frac{N^{-\delta}}{L|S|^2}$, **(A6)** $B(1+3\varepsilon)\frac{(a+1)^{|S|}}{N^{1-\delta}} \leq \frac{1}{2N^\delta}$, and **(A7)** $N^\psi/|S| \geq 1 + 2(1+3\varepsilon)N^\delta(a+1)^{|S|}$.

3.3.2 The periodized sequence

Let σ be an (N, δ, ε)-typical sequence. For every $s \in S$ choose $v(\cdot \mid s) \in V_s$ such that, for every t,

$$N_s^\sigma \max\{q^\sigma(t \mid s), v(t \mid s)\} \geq N^\delta \Rightarrow \left|1 - \frac{v(t \mid s)}{q^\sigma(t \mid s)}\right| \leq \varepsilon. \qquad (29)$$

Let $\sigma^* = (s_0^*, \ldots, s_{N_*}^*)$ be the periodic and exhaustive sequence that is generated from σ as in the proof of Theorem 2.2. Following the notations used in Section 2, we let $N_s := N_s^\sigma$, $N_s^* := N_s^{\sigma^*}$, and $q^* := q^{\sigma^*}$ denote the empirical transitions along σ^*, and $n_C^* := \sum_{s \in C} N_s^*$ denote the number of stages spent in $C \subset S$ along σ^*. By **(C1)**,

$$N_s^* \max\{q^*(t \mid s), v(t \mid s)\} \geq N_*^{\delta'} \Rightarrow N_s \max\{q^\sigma(t \mid s), v(t \mid s)\} \geq N_*^\delta.$$

In that case, by **(C2)**, $\left|1 - \frac{N_{s\to t}}{N_{s\to t}^*}\right| \leq \frac{\varepsilon}{3}$, where $N_{s\to t}$ and $N_{s\to t}^*$ are the number of transitions from s to t along σ and σ^* respectively, and $\left|1 - \frac{N_s}{N_s^*}\right| \leq \frac{\varepsilon}{3}$. Hence $\left|1 - \frac{q^*(t|s)}{q^\sigma(t|s)}\right| \leq \varepsilon$ (see Lemma 15 in Solan and Vieille (2002)). Therefore,

$$N_s^* \max\{q^*(t|s), v(t|s)\} \geq N_*^{\delta'} \Rightarrow \left|1 - \frac{v(t|s)}{q^*(t \mid s)}\right| \leq 3\varepsilon. \qquad (30)$$

In other words, σ^* is $(N_*, \delta', 3\varepsilon)$-typical.

Lemma 3.8 *Let $s \in S$ such that $N_s \geq N^{1-\delta}$. One has $\|v(\cdot|s) - q^\sigma(\cdot|s)\| \leq \eta$.*

    Proof. Let $t \in S$ be given. If $\max\{v(t|s), q^\sigma(t|s)\} \leq \eta$, one has $|v(t|s) - q^\sigma(t|s)| \leq \eta$. Otherwise, by **(C4)**, $N_s \max\{v(t|s), q^\sigma(t|s)\} \geq \eta N^{1-\delta} \geq N^\delta$. Therefore, by (29) and (E1), $|v(t|s) - q^\sigma(t|s)| \leq \varepsilon q^\sigma(t|s) < \eta$. ∎

3.3.3 The approximating process

Let $(S_1, S_2, \ldots, S_K)$ be a partition of S that is given by Theorem 2.11 w.r.t. σ^* and a. We abbreviate $n_{S_k}^*$ to n_k^*. Note that every state that is not visited by σ constitutes a singleton in this partition. Let $K_0 = \{k, n_k^* \geq N_*^{1-\delta}\}$ be the atoms that are visited many times and, for $k \in K_0$, set $\bar{n}_k =$

$\left\lceil \frac{n_k^*}{\sum_{k \in K_0} n_k^*} N \right\rceil$. The approximating process π has $|K_0|$ pieces. For convenience, assume that $K_0 = \{1, ..., |K_0|\}$.

Let $m_0^* = 0$, and for every $k = 1, \ldots, K$ define $m_k^* = \overline{n}_1 + \overline{n}_2 + \cdots + \overline{n}_k$. It follows a hidden V-Markov chain p_k from stage m_k^* up to m_{k+1}^*. All auxiliary Markov chains are defined on the same set $S \times T$, with $T = S \cup \{\Box\}$, where $\Box$ is an additional symbol. The initial state of the process is irrelevant. Unless otherwise stated, $\mathbf{E}$ stands for the expectation w.r.t. the law of π.

For $k \in K_0$, it is convenient to introduce the auxiliary transition function q_k defined by

$$q_k(\cdot \mid s) = \begin{cases} v(\cdot|s) & s \in S_k \\ q^*(\cdot \mid s) & s \notin S_k \end{cases}. \tag{31}$$

Thus, q_k coincides with v on S_k and with q^* on $\overline{S}_k$.

Next, we define a transition function p_k over $S \times T$ as follows:

- From state $(s, \Box)$, where $s \in S_k$: first $s' \in S$ is drawn according to $v(\cdot|s)$; if $s' \in S_k$, p_k moves to $(s', \Box)$; if $s' \notin S_k, t \in S_k$ is drawn with probability $\mathbf{P}_{s', q_k}(T_{S_k} = T_t)$, and p_k moves to (s', t).

- From state (s, t), where $s \neq t$ and $t \in S_k$: first $s' \in S$ is drawn according to $b(\cdot|s)$; if $s' = t$, p_k moves to $(s', \Box)$; if $s' \neq t$, p_k moves to (s', t).

- From state (s, t), where $s \notin S_k$ and $t \in \overline{S}_k \cup \{\Box\}$: (s', t') is drawn with probability $b(s'|s) \times \mathbf{P}_{s, q_k}(T_{S_k} = T_{t'})$. If $s' = t'$, p_k moves to $(s', \Box)$. Otherwise, p_k moves to (s', t').

All other transitions from these states receive probability zero. Transitions in other states are irrelevant. Note that the marginal over S of $p_k(\cdot|(s, t))$ belongs to V_s.

We now loosely describe the behavior of the S-coordinate. Starting from S_k, this coordinate evolves according to v until exit from S_k occurs. Then, the entry state in S_k is chosen at random, and the S-coordinate evolves according to b until that particular state is reached. The behavior resumes from the beginning. The T-coordinate of the auxiliary chain serves as an indicator of whether b or v is currently used and, if relevant, specifies which entry state in S_k has been selected.

The third item in the definition of p_k is introduced to take care of the initial stage in phase k, where the current state is inherited from the previous phase. It is used only at stage m_k^*.

Note that $\mathbf{q}_n = v(\cdot|\mathbf{s}_n)$ holds whenever $\mathbf{t}_n = \Box$ and $n \neq m_k^*$, for $k = 1, \ldots, K$. Observe that there is an ergodic set E_k for p_k that contains $S_k \times \{\Box\}$. Let ν_k be the invariant measure of p_k on E_k.

We proceed by proving several properties of the hidden Markov chain p_k.

The case where S_k is a singleton is albeit simpler, but also has some specific features. Therefore we shall postpone it and assume first in sections 3.3.4-3.3.7 that $|S_k| \geq 2$.

3.3.4 Perturbation of Markov chains: reminder

We here introduce a result due to Solan and Vieille (2002). Given $C \subseteq S$ with $|C| \geq 2$, and an irreducible transition rule q^1 over S with invariant measure μ^1, set

$$\zeta_{q^1}^C = \min_{\emptyset \subset D \subset C} \sum_{s \in D} \mu_s^1 q^1(\overline{C} \mid s).$$

This is a variation of the conductance of a Markov chain, that was originally defined by Jerrum and Sinclair (1989), and was used in the study of the rate of convergence to the invariant measure (see also Lovasz and Kannan (1999), Lovasz and Simonovits (1990)).

Definition 3.9 *Let q^1 be an irreducible transition function on S with invariant measure μ^1, let $C \subseteq S$ with $|C| \geq 2$, and let $\beta, \varepsilon > 0$. A transition rule q^2 is (β, ε)-close to q on C if (i) $q^2(\cdot \mid s) = q^1(\cdot \mid s)$ for every $s \notin C$; (ii) $\left|1 - \frac{q^2(t|s)}{q^1(t|s)}\right| < \varepsilon$ for every $s, t \in C$ such that $\mu_s^1 \max\{q^1(t \mid s), q^2(t|s)\} \geq \beta \zeta_{q^1}^C$.*

The next result summarizes Theorems 4 and 6 in Solan and Vieille (2002). Recall that $L = \sum_{n=1}^{|S|-1} \binom{|S|}{n} n^{|S|}$.

Proposition 3.10 *Let $\varepsilon \in (0, 1/2^{|S|})$, $A > 0$ and $\beta \in (0, \frac{1}{2}\left(\frac{A}{L}\right)^{|S|} \times \frac{\varepsilon(1-\varepsilon)}{L \times |S|^4})$. Let q^1 be an irreducible transition function on S. Assume that $|C| \geq 1$ and that $\mathbf{P}_{s,q^1}(T_t^+ < T_{\overline{C}}^+) \geq A$, for every $s, t \in C$. Let q^2 be (β, ε)-close to q^1 on C. Then all states of C belong to the same ergodic set E for q^2. Moreover, for every $s \in C$ and every $D \subset C$,*

$$|\mu^2(s \mid C) - \mu^1(s \mid C)| < 18\varepsilon L \mu^1(s \mid C), \tag{32}$$

$$L^{-1} \leq \frac{\mathbf{E}_{s,q^2}\left[T_{\overline{D}}\right]}{\mathbf{E}_{s,q^1}\left[T_{\overline{D}}\right]} \leq L, \text{ and } L^{-1} \leq \frac{K_D^{q_2}}{K_D^{q_1}} \leq L. \tag{33}$$

In addition, let $\chi \in (0, \beta \zeta_{q^1}^C]$ be any number such that, for every $s, t \in C$,

$$\mu_s^1 \max\{q^1(t \mid s), q^2(t|s)\} \geq \chi \Rightarrow \left|1 - \frac{q^2(t \mid s)}{q^1(t \mid s)}\right| < \varepsilon. \tag{34}$$

Then at least one of the following holds.

$(i) L^{-1} K_C^{q_1} \leq K_C^{q_2} \leq L K_C^{q_1}, \text{ or } (ii) K_C^{q_1} \geq \frac{1}{2|S|} \times \frac{\mu_C^1}{\chi} \text{ and } K_C^{q_2} \geq \frac{1}{L} \times \frac{1}{2|S|} \times \frac{\mu_C^1}{\chi}.$
$$\tag{35}$$

3.3.5 Perturbation of Markov chains: application

We here apply Proposition 3.10 to the transition functions q^* and q_k (defined in (31)), and to $C = S_k$.

Lemma 3.11 *If $|S_k| \geq 2$ then the transition function q_k is $(\beta, 3\varepsilon)$-close to q^* on S_k.*

Proof. Using (30) it now suffices to prove that $\frac{N_*^{\delta'}}{N_*} \leq \beta \zeta_{q^*}^{S_k}$.
Let C be an arbitrary non-empty subset of S_k. One has

$$\sum_{s \in C} \mu_s q^*(\overline{C} \mid s) = \frac{R_{\overline{C}} - 1_{s_0 \in \overline{C}}}{N_*} \geq \frac{R_C - 1}{N_*} \geq \frac{a - 1}{N_*}. \tag{36}$$

By taking the minimum over C, this yields $\zeta_{q^*}^{S_k} \geq \frac{a-1}{N_*}$. The result follows by (A1). ∎

We denote below by $\widehat{\rho}_i$, $i = 1, 2$, the value of the mixing constant ρ_i (defined in Proposition 2.8) for the transition function q_k. We abbreviate K for K^{q^*}, and $\widehat{K}$ for K^{q_k}.

Lemma 3.12 *If $|S_k| \geq 2$ then $\widehat{\rho}_1(S_k) \leq \frac{2}{\alpha} \widehat{\rho}_2(S_k)$, where $\alpha = 1/(2\beta |S| L^2)$.*

Proof. By (33), $\widehat{\rho}_1(S_k) \leq L \rho_1(S_k)$. We argue now that

$$K_{S_k} \geq \frac{1}{2\beta |S|} \times \frac{\mu_{S_k}}{\zeta_{q^*}^{S_k}}. \tag{37}$$

For $C \subset S_k$ one has by (3)

$$\frac{K_{S_k}}{\mu_{S_k}} = \frac{1}{\sum_{s \in S_k} \mu_s q^*(\overline{S}_k|s)} \geq \frac{N_*}{R_{S_k}} \geq a \frac{N_*}{R_C} \geq (a-1) \frac{1}{\sum_{s \in C} \mu_s q^*(\overline{C}|s)},$$

where the last inequality follows by (36) and since $R_C \geq a R_{S_k} \geq a$. Eq. (37) follows by optimizing over C, using (A2). By (35) and (37),

$$\widehat{\rho}_2(S_k) \geq \widehat{K}_{S_k} \geq \frac{1}{L} \times \frac{1}{2\beta |S|} \times \frac{\mu_{S_k}}{\zeta_{q^*}^{S_k}}. \tag{38}$$

Fix $C \subset S_k$. By (38),

$$\widehat{\rho}_2(S_k) \geq \frac{1}{L} \times \frac{1}{2\beta |S|} \times \frac{\mu_C}{\sum_{s \in C} \mu_s q^*(\overline{C}|s)} \geq \frac{1}{L} \times \frac{1}{2\beta |S|} \times K_C$$

$$\geq \frac{1}{L^2} \times \frac{1}{2\beta |S|} \times \widehat{K}_C \geq \frac{1}{L^2} \times \frac{1}{2\beta |S|} \times \min_{s \in C} \mathbf{E}_{s, q_k} \left[T_{\overline{C}} \right].$$

The result follows by taking the maximum over C. ∎

3.3.6 Estimates on $\mathbf{E}\left[\mathbf{N}_0^k\right]$

Let $\mathbf{N}_0^k = \left|\left\{m_k^* \leq n < m_{k+1}^* : \|\mathbf{q}_n - q^\sigma(\cdot|\mathbf{s}_n)\| > \eta\right\}\right|$. The goal of this section is to prove Lemma 3.16 below.

Recall that E_k is the ergodic set for p_k that contains $S_k \times \{\square\}$, and that ν_k is the invariant measure of p_k on E_k.

Lemma 3.13 *If $|S_k| \geq 2$ then $\nu_k(S_k \times \{\square\}) \geq 1 - \frac{2B}{\widehat{\rho}_2(S_k)}$.*

Proof. We shall use the following fact. Let $\bar{q}$ be an irreducible transition function over a finite set Ω, with invariant measure $\bar{\mu}$. Let $C \subset \Omega$, and let $\overline{C} = \Omega \backslash C$. Then

$$\frac{\bar{\mu}(C)}{\bar{\mu}(\overline{C})} \geq \frac{\min_{s \in C} \mathbf{E}_{s,\bar{q}}\left[T_{\overline{C}}\right]}{\max_{s \in \overline{C}} \mathbf{E}_{s,\bar{q}}\left[T_C\right]}. \tag{39}$$

We apply this observation to p_k and E_k, with $C = S_k \times \{\square\}$. Plainly, $\mathbf{E}_{(s,t),p_k}\left[T_{S_k \times \{\square\}}\right] = \mathbf{E}_{s,b}\left[T_t\right] \leq B$ for each (s,t) with $t \neq \square, s$, while by (13), Lemma 3.12, and (E2),

$$\min_{s \in S_k} \mathbf{E}_{(s,\square),p_k}\left[T_{\overline{S_k \times \{\square\}}}\right] = \min_{s \in S_k} \mathbf{E}_{s,q_k}\left[T_{\overline{S}_k}\right]$$

$$\geq \widehat{\rho}_2(S_k) - (|S_k| - 1)\widehat{\rho}_1(S_k) \geq \widehat{\rho}_2(S_k)\left(1 - 4\beta L^2(|S| - 1)|S|\right) \geq \frac{1}{2}\widehat{\rho}_2(S_k).$$

By (39), one gets

$$\frac{\nu_k(\overline{S_k \times \{\square\}})}{\nu_k(S_k \times \{\square\})} \leq \frac{2B}{\widehat{\rho}_2(S_k)},$$

hence $\nu_k(\overline{S_k \times \{\square\}}) \leq \frac{2B}{\widehat{\rho}_2(S_k)}$. $\blacksquare$

Lemma 3.14 *If $|S_k| \geq 2$ then*

$$\widehat{\rho}_2(S_k) \geq \frac{1}{2L\,|S|}\frac{n_k^*}{N_*^{\psi'}}.$$

Proof. We will use the fact that $K_{S_k} \geq n_k^*/R_{S_k}$ (see (3)).

By (30), and since $\delta' < \psi'$, (34) holds with $\chi = N_*^{\psi'-1}$. We distinguish two cases. If $K_{S_k} \geq \frac{1}{2|S|}\frac{n_k^*}{N_* \times \chi}$, then by (35)

$$\widehat{\rho}_2(S_k) \geq \widehat{K}_{S_k} \geq \frac{1}{L} \times \frac{1}{2\,|S|} \times \frac{n_k^*}{N_* \times \chi} = \frac{1}{L} \times \frac{1}{2\,|S|} \times \frac{n_k^*}{N_*^{\psi'}}.$$

If on the other hand $K_{S_k} < \frac{1}{2|S|}\frac{n_k^*}{N_* \times \chi}$, then

$$\widehat{\rho}_2(S_k) \geq \widehat{K}_{S_k} \geq \frac{1}{L}K_{S_k} \geq \frac{1}{L} \times \frac{n_k^*}{R_{S_k}} \geq \frac{1}{L} \times \frac{n_k^*}{(a+1)^{|S|}},$$

which gives also the result by (A4). $\blacksquare$

In particular, by (**C5**), $\nu_k(S_k \times \{\square\}) \geq 1/2$.

Lemma 3.15 *For every $\omega \in E_k$ and every $s \in S_k$ one has*

$$\mathbf{E}_{\omega,p_k}\left[T^+_{(s,\square)}\right] \leq \frac{(|S_k|-1)\widehat{\rho}_1(S_k)+2B}{\min_{u\in S_k}\mathbf{P}_{u,v}(T_s < T_{\overline{S}_k})} + 1.$$

Proof. It is a simple adaptation of the proof of Corollary 2.10. We repeat it, with few modifications. Let $\omega \in E_k$ and $s \in S_k$ be given. Note that

$$\mathbf{E}_{\omega,p_k}\left[T^+_{(s,\square)}\right] \leq 1 + \max_{\omega'\in E_k}\mathbf{E}_{\omega',p_k}\left[T_{(s,\square)}\right]. \tag{40}$$

For convenience, set $\alpha := \max_{t\in S_k}\mathbf{E}_{(t,\square),p_k}\left[T_{(s,\square)}\right]$. Let $t \in S_k$ achieve the maximum in the definition of α. By (11) and (12)

$$\alpha = \mathbf{E}_{(t,\square),p_k}\left[T_{(s,\square)}\right] \leq \mathbf{E}_{t,v}\left[T_{\overline{S}_k\cup s}\right] + \mathbf{P}_{t,v}(T_{\overline{S}_k} < T_s)(\alpha + B)$$

$$\leq (|S_k|-1)\widehat{\rho}_1(S_k) + B + \alpha \times \max_{u\in S_k}\mathbf{P}_{u,v}(T_{\overline{S}_k} < T_s)$$

Therefore,

$$\alpha \leq \frac{(|S_k|-1)\widehat{\rho}_1(S_k)+B}{\min_{u\in S_k}\mathbf{P}_{u,v}(T_s^+ < T_{\overline{S}_k})}. \tag{41}$$

For $\omega \in E_k\backslash(S_k \times \square)$,

$$\mathbf{E}_{\omega,p_k}\left[T_{(s,\square)}\right] \leq B + \alpha. \tag{42}$$

The result follows from (40), (41) and (42). ■

$$\overline{n}_k$$

Lemma 3.16 *If $|S_k| \geq 2$ then*

$$\mathbf{E}\left[\mathbf{N}_0^k\right] \leq \frac{1}{|S|}BN^{\psi}.$$

Proof. We introduce $\widetilde{\mathbf{N}}_0^k = |\{n \leq \overline{n}_k : \mathbf{t}_n \neq \square\}|$. Thus, by Lemmas 3.8, $\mathbf{E}\left[\mathbf{N}_0^k\right] \leq \sup_\omega \mathbf{E}_{\omega,p_k}\left[\widetilde{\mathbf{N}}_0^k\right]$. By Lemma 3.13, $\mathbf{E}_{\nu_k,p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \leq \overline{n}_k\nu_k(E_k\backslash(S_k \times \{\square\})) \leq 2B \times \frac{\overline{n}_k}{\widehat{\rho}_2(S_k)}$. Therefore,

$$\min_{\omega\in S_k\times\{\square\}}\mathbf{E}_{\omega,p_k}\left[\widetilde{\mathbf{N}}_0^k\right]\nu_k(S_k\times\{\square\}) \leq \sum_{\omega\in S_k\times\{\square\}}\mathbf{E}_{\omega,p_k}\left[\widetilde{\mathbf{N}}_0^k\right]\nu_k(\omega)$$

$$\leq \mathbf{E}_{\nu_k,p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \leq 2B \times \frac{\overline{n}_k}{\widehat{\rho}_2(S_k)}.$$

Since $\nu_k(S_k \times \{\square\}) \geq 1/2$, this yields

$$\min_{\omega\in S_k\times\{\square\}}\mathbf{E}_{\omega,p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \leq 4B \times \frac{\overline{n}_k}{\widehat{\rho}_2(S_k)}. \tag{43}$$

Let $\gamma = \max_{\omega \in S_k \times \{\square\}} \mathbf{E}_{\omega, p_k}\left[\widetilde{\mathbf{N}}_0^k\right]$, and let $\omega_1 \in S_k \times \{\square\}$ be a state that achieves the maximum. Since p_k follows b once the process leaves $S_k \times \{\square\}$, one has, for each $\omega_2 \in S_k \times \{\square\}$,

$$\gamma = \mathbf{E}_{\omega_1, p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \leq \mathbf{E}_{\omega_2, p_k}\left[\widetilde{\mathbf{N}}_0^k\right] + \mathbf{P}_{\omega_1, p_k}(T_{E_k \setminus (S_k \times \{\square\})} < T_{\omega_2})(B + \gamma).$$

By Corollary 2.9 and Lemma 3.12, $\mathbf{P}_{\omega_1, p_k}(T_{E_k \setminus S_k \times \{\square\}} < T_{\omega_2}) \leq \frac{2|S|}{\alpha/2 - |S|} = 1/\alpha'$. Since $\alpha' \geq 2$, one gets by letting ω_2 vary

$$\gamma \leq \frac{\alpha'}{\alpha' - 1} \min_{\omega \in S_k \times \{\square\}} \mathbf{E}_{\omega, p_k}\left[\widetilde{\mathbf{N}}_0^k\right] + \frac{B}{\alpha' - 1} \leq 2 \min_{\omega \in S_k \times \{\square\}} \mathbf{E}_{\omega, p_k}\left[\widetilde{\mathbf{N}}_0^k\right] + B. \tag{44}$$

Finally, for each $\omega' \in S \times T$, by (43), (44), Lemma 3.14 and (**C3**),

$$\begin{aligned}
\mathbf{E}_{\omega', p_k}\left[\widetilde{\mathbf{N}}_0^k\right] &\leq \mathbf{E}_{\omega', p_k}\left[T_{S_k \times \{\square\}}\right] + \max_{\omega \in S_k \times \{\square\}} \mathbf{E}_{\omega, p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \\
&\leq B + B + 2 \times \min_{\omega \in S_k \times \{\square\}} \mathbf{E}_{\omega, p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \\
&\leq 2B + 8B \times \frac{\overline{n}_k}{\widehat{\rho}_2(S_k)} \\
&\leq 2B + 16L\,|S|\,BN_*^{\psi'} \times \frac{N_*}{N_* - |S|\,N_*^{1-\delta}} \leq BN^\psi/\,|S|\,.
\end{aligned}$$

∎

3.3.7 Estimates on $\widetilde{F}_{\overline{n}_k}^s$

Let $\widetilde{F}_{\overline{n}_k}^s$ denote the frequency of visits to $(s, \square)$ during phase k. The lemma below is a mixing-type result. It is very close to Lemma 2.16.

Lemma 3.17 *If $|S_k| \geq 2$ then for every $\omega \in E_k$ and every $s \in S_k$,*

$$\mathbf{E}_{\omega, p_k}\left[T_{(s, \square)}^+\right] \leq 2|S|L\frac{N}{a - 1} + 4B + 1$$

Proof. We repeat the proof of Lemma 2.16 with minor adjustments. Abbreviate $\widehat{\rho}_1(S_k)$ and $\widehat{\rho}_2(S_k)$ to $\widehat{\rho}_1$ and $\widehat{\rho}_2$ respectively. By Lemma 3.15,

$$\mathbf{E}_{\omega, p_k}[T_{(s, \square)}^+] \leq \frac{(|S_k| - 1)\widehat{\rho}_1 + 2B}{\min_{u \in S_k} \mathbf{P}_{u, v}(T_s^+ < T_{\overline{S}_k})} + 1.$$

By Corollary 2.9, the denominator is at least $1 - 2\,|S_k|\,\frac{\widehat{\rho}_1}{\widehat{\rho}_2 - (|S_k| - 1)\widehat{\rho}_1}$. Therefore,

$$\mathbf{E}_{\omega, p_k}[T_{(s, \square)}^+] \leq ((|S_k| - 1)\widehat{\rho}_1 + 2B) \times \frac{\widehat{\rho}_2 - (|S_k| - 1)\widehat{\rho}_1}{\widehat{\rho}_2 - (3\,|S_k| - 1)\widehat{\rho}_1} + 1 \leq 2\,|S_k|\,\widehat{\rho}_1 + 4B + 1,$$

where the second inequality follows by Lemma 3.12 and (E3).

Eq. (33) implies that $\widehat{\rho}_1 < L\rho_1(S_k)$, and by Proposition 2.15, $\rho_1(S_k) \leq \max_{D \subset S_k} \frac{n_D}{R_D - 1} \leq \frac{N}{a-1}$. The result follows. ∎

Define $\nu_k^\square(s) = \nu_k((s, \square))/\nu_k(S_k \times \{\square\})$. This is the invariant measure of p_k conditioned on $S_k \times \{\square\}$.

Proposition 3.18 *If* $|S_k| \geq 2$ *then* $\mathbf{P}\left(\left|\widetilde{F}_{\overline{n}_k}^s - \nu_k^\square(s))\right| > \frac{2\varepsilon}{1-\varepsilon}\nu_k^\square(s)) + \frac{1}{\overline{n}_k}\right) \leq \frac{1}{2|S|} \times \frac{1}{N^\delta}$.

Proof. By Remarks 2.5, 2.6 Lemma 3.17, (A3) and (A5), for each $\omega \in E_k$,

$$\mathbf{P}_{\omega, p_k}\left(|\overline{F}_{\overline{n}_k}^{(s,\square)} - \nu_k((s, \square))| > \varepsilon\nu_k((s, \square)) + \frac{1}{\overline{n}_k}\right)$$

$$\leq \frac{9}{\varepsilon^2 \overline{n}_k}\left(2(2|S|L\frac{N}{a-1} + 4B + 1) + 1\right)$$

$$\leq 2 \times 72\frac{1}{\varepsilon^2 \overline{n}_k}(|S|L\frac{N}{a-1} + B) \leq \frac{1}{2|S|} \times \frac{1}{N^\delta}. \qquad (45)$$

Since $\nu_k(S_k \times \{\square\}) \geq 1/2$, by Lemmas 3.13 and 3.14, and by (**C5**),

$$\left|\nu_k((s, \square)) - \nu_k^\square(s)\right| \leq 2\nu_k(E_k \backslash (S_k \times \{\square\})) \times \nu_k((s, \square)) \leq \varepsilon\nu_k((s, \square)).$$

Therefore if $\left|\widetilde{F}_{\overline{n}_k}^s - \nu_k((s, \square))\right| \leq \varepsilon\nu_k((s, \square)) + \frac{1}{\overline{n}_k}$ then $\left|\widetilde{F}_{\overline{n}_k}^s - \nu_k^\square(s))\right| \leq \frac{2\varepsilon}{1-\varepsilon}\nu_k^\square(s) + \frac{1}{\overline{n}_k}$. The result follows by (45). ∎

Corollary 3.19 *If* $|S_k| \geq 2$ *then*

$$\mathbf{P}\left(\left|\widetilde{F}_{\overline{n}_k}^s - \nu^{\sigma^*}(s \mid S_k))\right| > 55\varepsilon L\nu^{\sigma^*}(s \mid S_k)\right) \leq \frac{1}{2|S|} \times \frac{1}{N^\delta}.$$

Proof. Recall that $\nu_k^\square$ is the invariant measure of q_k conditioned on S_k. On the other hand, the invariant measure of q^* conditioned on S_k is simply $\nu^{\sigma^*}(\cdot|S_k)$. By Lemma 3.11 and Proposition 3.10,

$$\left|\nu_k^\square(s) - \nu^{\sigma^*}(s \mid S_k)\right| \leq 18 \times 3\varepsilon L\nu^{\sigma^*}(s \mid S_k),$$

The claim follows by Proposition 3.18, (**E4**) and since $\nu^{\sigma^*}(s \mid S_k) \geq \frac{a}{N_*} \geq \frac{1}{N^{1-\delta}}$. ∎

3.3.8 The singleton case: $S_k = \{\overline{s}\}$

By Lemma 3.8, $\mathbf{N}_0^k$ is at most $|\{n : (\mathbf{s}_n, \mathbf{t}_n) \neq (\overline{s}, \square)\}|$. The next lemma is an analog of Lemma 3.13. Its proof is however significantly different.

Lemma 3.20 *One has* $\nu_k((\overline{s}, \square)) \geq 1 - B(1 + 3\varepsilon)\frac{(a+1)^{|S|}}{N^{1-\delta}}$.

Proof. We first provide a lower bound for $v(\overline{s} \mid \overline{s})$. By Theorem 2.11

$$q^*(S \backslash \{\overline{s}\} \mid \overline{s}) \leq \frac{R_{\overline{s}}}{N_{\overline{s}}^*} \leq \frac{(a+1)^{|S|}}{N^{1-\delta}}.$$

Using (30), this yields

$$v(S \backslash \{\overline{s}\} \mid \overline{s}) \leq (1 + 3\varepsilon)\frac{(a+1)^{|S|}}{N^{1-\delta}}.$$

We apply (39) to $\overline{q} = p_k$, $\Omega = E_k$ and $C = \{(\overline{s}, \square)\}$ to get

$$\frac{\nu_k(E_k \backslash (\overline{s}, \square))}{\nu_k((\overline{s}, \square))} \leq Bv(S \backslash \overline{s} | \overline{s}) \leq B(1 + 3\varepsilon)\frac{(a+1)^{|S|}}{N^{1-\delta}}.$$

∎

The rest of the proof for the singleton case follows closely the proof for $|S_k| \geq 2$.

Corollary 3.21 *One has* $\mathbf{P}\left(\left|\widetilde{F}_{\overline{n}_k}^{\overline{s}} - 1\right| > 2\varepsilon\right) \leq \frac{1}{2K} \times \frac{1}{N^{\delta}}$.

Proof. By definition of p_k, $\max_{t \in E_k} \mathbf{E}_{t,p_k}\left[T_{(\overline{s}, \square)}^+\right] \leq B + 1$. Therefore, using Remark 2.5 to p_k, ε and $s = (\overline{s}, \square)$,

$$\mathbf{P}\left(\left|\widetilde{F}_{\overline{n}_k}^{\overline{s}} - \nu_k((\overline{s}, \square))\right| > \varepsilon\nu_k((\overline{s}, \square))\right) \leq \frac{9(2B+3)}{\overline{n}_k \varepsilon^2}.$$

By Lemma 3.20 and (**A6**), $|\nu_k((\overline{s}, \square)) - 1| \leq \varepsilon$. The result follows using (**C6**). ∎

Lemma 3.22 *One has* $\mathbf{E}\left[\mathbf{N}_0^k\right] \leq \frac{BN^{\psi}}{K}$.

Proof. We follow the proof of Lemma 3.16, and define $\widetilde{\mathbf{N}}_0^k$ accordingly. For each $\omega \in E_k$ one has, by Lemma 3.20 and (**A7**)

$$\mathbf{E}_{\omega,p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \leq \mathbf{E}_{\omega,p_k}\left[T_{(\overline{s}, \square)}\right] + \mathbf{E}_{(\overline{s}, \square),p_k}\left[\widetilde{\mathbf{N}}_0^k\right]$$

$$\leq B + \mathbf{E}_{(\overline{s}, \square),p_k}\left[\widetilde{\mathbf{N}}_0^k\right] \leq B + 2\mathbf{E}_{\nu_k,p_k}\left[\widetilde{\mathbf{N}}_0^k\right]$$

$$\leq B + 2\overline{n}_k \nu_k(E_k \backslash (\overline{s}, \square))$$

$$\leq B + 2NB(1 + 3\varepsilon)\frac{(a+1)^{|S|}}{N^{1-\delta}}$$

$$\leq BN^{\psi}/K.$$

∎

3.3.9 Conclusion

We here conclude the proof of Theorem 3.6. Note first that $\mathbf{N}_0 = \sum_k \mathbf{N}_0^k$. Therefore, **G2** follows from Lemmas 3.16 and 3.22. Let now $s \in S_k$ with $\nu_s^\sigma \geq \frac{1}{N^\delta}$. Plainly, $N\overline{F}_N^s - \overline{n}_k \widetilde{F}_{\overline{n}_k}^s$ is the total number of visits to s that are not counted in $\widetilde{F}_{\overline{n}_k}^s$: there are at most $\sum_k \mathbf{N}_0^k$ many of them.

Since $\mathbf{E}\left[\sum_k \mathbf{N}_0^k\right] \leq BN^\psi$ one has by Markov inequality and (**C6**)

$$\mathbf{P}\left(\sum_k \mathbf{N}_0^k > \varepsilon \frac{N}{N^\delta}\right) \leq \frac{BN^{\delta+\psi-1}}{\varepsilon} \leq \frac{1}{2N^\delta}.$$

Therefore, using Corollaries 3.19 and 3.21, the probability that both inequalities $\sum_k \mathbf{N}_0^k \leq \varepsilon N^{1-\delta}$ and $\left|\widetilde{F}_{\overline{n}_k}^s - \nu^{\sigma^*}(s \mid S_k))\right| \leq 55\varepsilon L \nu^{\sigma^*}(s \mid S_k)$ hold for every k and every $s \in S_k$, is at least $1 - \frac{1}{N^\delta}$. On this event, by (E1),

$$\left|\overline{F}_N^s - \nu_s^{\sigma^*}\right| \leq \left|\overline{F}_N^s - \frac{\overline{n}_k}{N}\widetilde{F}_{\overline{n}_k}^s\right| + \frac{1}{N}\widetilde{F}_{\overline{n}_k}^s |\overline{n}_k - n_k^*| + \frac{n_k^*}{N}\left|\widetilde{F}_{\overline{n}_k}^s - \nu^{\sigma^*}(s \mid S_k)\right|$$

$$\leq 56\varepsilon L\nu_s^{\sigma^*}.$$

This proves **G1**.

References

[1] Aldous D.J. and Fill J.A. (2002) Reversible Markov Chains and Random Walks on Graphs, Book in preparation; draft available via homepage http://www.stat.berkeley.edu/users/aldous

[2] Alon N., Spencer J.H. and Erdos P. (2000) The Probabilistic Method, John Wiley and Sons

[3] Baum L.E. and Eagon J.A. (1967) An Inequality with Applications to Statistical Estimation for Probabilistic Functions of a Markov Process and to a Model for Ecology, *Bull. Amer. Math. Soc.*, 73:360–363

[4] Baum L. E. and Petrie T. (1966) Statistical Inference for Probabilistic Functions of Finite State Markov Chains, *Ann. Math. Stat.*, 37:1554 – 1563

[5] Blackwell D. and Ferguson T. (1968) The "Big Match", *Ann. Math. Stat.*, 39:159 – 163

[6] Baum L.E., Petrie T., Soules G. and Weiss N. (1970) A Maximization Technique Occurring in the Statistical Analysis of Probabilisitc Functions of Markov Chains, *Ann. Math. Stat.*, 41:164–171

[7] Burge C. and Karlin S. (1997) Prediction of Complete Gene Structures in Human Genomic DNA, *J. Mol. Biol.*, 268:78–94

[8] Feller W. (1968) An Introduction to Probability Theory and its Applications, John Wiley and sons

[9] Glynn P.W. and Ormoneit D. (2002) Hoeffding's Inequality for Uniformly Ergodic Markov Chains, *Statist. Probab. Lett.* **56**, 143–146

[10] Jerrum M. and Sinclair A. (1989) Approximating the permanent. *SIAM Journal on Computing* , 18:1149–1178

[11] Lovasz L. and Kannan R. (1999) Faster mixing via average conductance. *Annual ACM Symposium on Theory of Computing* , 282–287

[12] Lovasz L. and Simonovits M. (1990) The mixing rate of Markov chains, an isoperimetric inequality, and computing the volume. *31st Annual Symposium on Foundations of Computer Science, Vol. I, II* , 346–354

[13] Mertens J.-F. and Neyman A. (1981) Stochastic Games, *Int. J. Of Game Th.*, 10:53 – 66

[14] Rabiner L.R. (1989) A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition, *Proc. IEEE*, 77:257–285

[15] Rosenberg D., Solan E. and Vieille N. (2002), Stochastic Games with Imperfect Monitoring, Discussion Paper #1341, The Center for Mathematical Studies in Economics and Management Science, Northwestern University

[16] Solan E. and Vieille N. (2002), Perturbed Markov Chains, Discussion Paper #1342, The Center for Mathematical Studies in Economics and Management Science, Northwestern University