When is there state independence?*

Brian Hill
HEC Paris†

November 22, 2007

Abstract

It has been noticed that whether a preference relation can be represented by state-independent utilities as opposed to state-dependent utilities may depend on which acts count as constant acts [Schervish et al., 1990]. Indeed, this remark underlies an extension of Savage's expected utility theory to the state-dependent case that was proposed by Edi Karni [Karni, 1993]. This paper contains a characterisation of the preference relations that permit a choice of acts which can play the role of constant acts, and relative to which there is a representation involving a state-independent utility function. This result applies both in the Savage and in the Anscombe & Aumann frameworks. It has as an immediate corollary an extension of Karni's representation theorem. Finally, it is of methodological interest, insofar that it harnesses techniques from mathematical logic to prove a theorem of interest to decision theorists and economists.

Keywords: Subjective expected utility; State-dependent utility; Monotonicity axiom

JEL classification: D81, C69.

*The author would like to thank Philippe Mongin for his useful comments.
†GREGHEC, HEC Paris. 1 rue de la Libération, 78351 Jouy-en-Josas, France. hill@hec.fr. www.hec.fr/hill.

1 Introduction

There has been much discussion of representations of preferences with state-dependent utilities [Karni et al., 1983, Karni and Schmeidler, 1993, Karni, 1993, Karni and Mongin, 2000, Drèze, 1987, Schervish et al., 1990]. In several of these discussions, it has been noted that, by redefining which acts count as "constant", one will transform a state-independent representation into a state-dependent one, and inversely, transform a state-dependent representation into a state-independent one. To take the example proposed in [Schervish et al., 1990], if the objects of choice are bets on the exchange rates between dollars and yen – in Savage's terminology, the states of the world are the exchange rates between dollars and yen, and the acts are functions taking exchange rates to monetary prizes – then a representation which is state-independent when the prizes are formulated in dollars will be state-dependent when the prizes are formulated in yen. Indeed, the idea that failures of Savage's state-independence axioms come about because the consequences do not yield the acts which are "really" constant is behind some theories of state-dependent utility. Most notably, the extension to Savage's expected utility theory proposed several years ago by Edi Karni [Karni, 1993] relies precisely on this idea. In particular, he introduces the notion of *constant valuation acts*, which, although they are not constant acts, play the role of constant acts in a Savage-like representation theorem; the theorem obtained does not assume state-independence with respect to the constant acts, but only state-independence with respect to the constant valuation acts.

The intriguing idea that state-dependence might merely be due to the choice of constant acts, or if you prefer, of consequences, poses the immediate question: under what conditions is there a choice of acts that can play the role of constant acts and that yield a state-independent utility representation? To put it another way, let us say that there is *essentially* a state-independent utility representation when there is a set of acts, which have all the necessary properties of constant acts, with respect to which the preference relation has a state-independent utility representation. The question is: when is there *essentially* a state-independent utility representation? Reformulated in the terms introduced by Karni [Karni, 1993], the question is: when do an appropriate set of constant valuation acts exist? To the extent that Karni assumes the existence of constant valuation acts in his result, this is a question about the cases where his result applies. In the current paper, this question shall be answered by characterising the set of preference orders which *essentially* admit state-independent utility representations. The result shall hold for any state space – from the finite to the atomless – and any set of consequences which is either finite or the set of lotteries over a finite number of outcomes. In other words, it applies in two of the main paradigms used in current literature: that of Savage [Savage, 1954] and that of Anscombe & Aumann [Anscombe and Aumann, 1963].

The situation with respect to the state-independence of utilities differs subtlely between the two paradigms. First of all, they share a common main axiom for state independence, which Savage calls P3 and Anscombe & Aumann call monotonicity. This axiom states that, for any pair of constant acts, the first is preferred to the second if and only if, for any non-null event, the first is preferred to the second given that event. It is this axiom that shall be the focus of attention in the present paper. The main result characterises the cases where an appropriate set of acts can be found

Table 1: No essential state independence

	s_1	s_2
c_3	first	first
c_2	second	second equal
c_1	third	

which satisfy this axiom; these are said to be the cases where monotonicity *essentially* holds. However, the axiom has different consequences in the two paradigms. In the Anscombe & Aumann paradigm, if one assumes reversal of order, the monotonicity axiom is sufficient to obtain a representation involving a state-independent utility function. By contrast, in the Savage framework, it is not sufficient: monotonicity (Savage's P3) only assures ordinal state-independence but not cardinal state-independence. For the latter, the extra axiom P4 is required. (For extended discussion of these points, see [Karni and Mongin, 2000, Karni, 1993].) The axiom P4 can only be stated when a set of constant acts is given: it enters the scene at a stage after it has been determined whether a set of constant acts exists which satisfies monotonicity. A traditional treatment of P4 shall be assumed in the paper where required, since it is not the main subject of the paper.

At first blush, it might seem surprising that the result obtained here is not trivial: one might expect that any preference relation which has a state-dependent utility representation would satisfy monotonicity, under appropriate modifications in what counts as constant acts. A simple example illustrates why this is not necessarily the case. Consider a decision problem with two states of the world, s_1 and s_2, and three consequences c_1, c_2 and c_3, and consider the acts f, which takes both states to c_1, g, which takes s_1 to c_1 and s_2 to c_2, h, which takes both states to c_2, i, which takes s_1 to c_2 and s_2 to c_3 and j, which takes s_1 to c_3 and s_2 to c_1. Consider a preference relation with $f \sim g \prec h \prec i \prec j$: there exist preference relations ordering the acts in this way and satisfying the basic postulates of Savagian decision theory (weak order and Savage's sure-thing principle, or Anscombe & Aumann's independence).[1] Such a preference relation establishes preferences orders on consequences, given states, which are shown in Table 1. For this preference relation, there is no set of acts that could count as constant acts (in the sense that other acts can be defined in terms of them) and satisfy monotonicity. A triple of acts which could be taken as constant acts must take different values on each state of the world; however, for each such triple, a pair of the acts must be indifferent given s_2 (those which take the values c_1 and c_2), although there is a definite preference between them given s_1. Since both states are evidently non null, it follows that the axiom cannot be satisfied.

The problem in the example seems to be that the preference orders given the different states do not have sufficient structure in common to permit a choice of constant acts which could satisfy monotonicity. The idea behind the result proposed in this paper is

[1]The continuity or Archimedianity axioms are left to one side in the example, since we are treating the finite case.

based on this observation. The monotonicity axiom implies that, to support a state-independent representation, a set of acts which may count as constant should be such that one of these acts is preferred to another given a particular state if and only if the former is preferred to the latter given any other state. This naturally yields the condition that, for any two states, the order on the consequences given one state is isomorphic to the order on the consequences given the other. If, for a particular state, there is a single maximal element in the preference order given that state, and for another state, there are two maximal elements in the preference order given the second state, there can be no set of essentially monotonic acts which can play the role of constant acts: for such acts need to take different values on each of the states, and so there will be ties given one state though there will be strict preference given the other state.

Naturally, this basic idea needs to qualified when applied to the general case. For one, the reasoning only applies to non-null states. Furthermore, the case of atomless state spaces, where it is illegitimate to reason in terms of states (since they are not events) will be more complicated. Finally, the isomorphism condition as expressed above may seem an unnatural "axiom" for essential state-independence; in any case, it becomes unwieldy, especially in the atomless case. For this reason, we shall use some basic tools of modern mathematical logic to formulate and prove the result. These shall allow us to represent rigorously the set of sentences which correctly describe the properties of the relevant orders. Such sets of sentences shall be called theories. So, for a given event, there will be a theory of the preference order over constant acts given that event, and this theory captures many of the important properties of the order. Under a technical condition, such theories may be used to define a theory for each state of the state space. The condition for essential monotonicity proposed in the main result of the paper (Theorem 2 in Section 3) is the following: the theories for different states are identical. When and only when this condition is satisfied, a set of acts exists that one may take as constant acts, and that satisfy monotonicity.

In Section 2, a definition of what is required for a set of acts to be able to be taken as constant acts is given, and conditions for monotonicity and (for the case of the Savage framework) cardinal state independence on such sets of acts are stated. These conditions is satisfied if and only if there is a representation with a state-independent utility function on these acts. In Section 3, some basic logical notions are introduced, and a characterisation theorem is proven, giving necessary and sufficient conditions for the existence of a set of acts on which the monotonicity condition defined in the Section 2 holds. Furthermore, the set of constant acts satisfying monotonicity has rather strong uniqueness properties (Corollary 1). The final section, Section 4, contains a representation theorem which follows immediately from this characterisation (Theorem 3), and some concluding remarks regarding the interest of the result. Proofs of the main results will be found in the Appendix.

2 Essential monotonicity

Let S be a set of states, with a σ-algebra of events, and C a finite set of consequences or a finite set of outcomes; in the latter case the consequences are considered to be the lotteries over these outcomes. Throughout the paper, n will denote the number of elements of C. Note that there is a naturally defined σ-algebra in the (Boolean) algebra generated by $S \times C$: namely the product of the σ-algebra of events with the "discrete" σ-algebra, containing all singletons of C. Let $\mathcal{A}$ be the set of measurable functions from S into C. The acts are the measurable functions from the set of states S into the set of consequences. If C is the set of consequences, then $\mathcal{A}$ is the set of acts; if the set of consequences are the set of lotteries over C, the set of acts is basically the set of mixtures of the set $\mathcal{A}$.[2] Note that, each element of $\mathcal{A}$ is canonically associated with a subset of $S \times C$; furthermore, since the elements of $\mathcal{A}$ are measurable functions, these subsets are measurable with respect to the measure structure described above. In this paper, the same symbol will be used to denote the element of $\mathcal{A}$ and the corresponding subset of $S \times C$. Finally, $\preceq$ is a preference relation on the set of acts (and thus on $\mathcal{A}$), with $\prec$ and $\sim$ being the related strict preference relation and indifference relation. Throughout the paper, it will be assumed that this relation is non-trivial (there are $f, g \in \mathcal{A}$ such that $f \prec g$).

For the rest of the paper, the functions mentioned (f, g, and so on) shall be assumed to be measurable, as will the sets of states (A, B, E and so on): in other words, they will be assumed to be events.

Finally, for an event $A \subseteq S$, $f_A g$ will be the function which takes the values of f on A and the values of g on A^c.

The set of constant acts has two properties which render is important with respect to the set of all acts: firstly, any act can be expressed as an appropriate "mix" of constant acts; secondly, for each act, there is a unique way to do so. However, the set of constant acts is not the only such set with these properties; any such set will be called a basis.

Definition 1 (Basis)**.** A *basis* $\mathcal{B}$ is a set $\{b^i \in \mathcal{A} \mid i = 1, \ldots, n\}$, such that, for each $s \in S$ and for each $c \in C$, there is a unique b^i with $b^i(s) = c$.

The set C defines a canonical basis: namely, the set of acts taking a given element of C for any state of the world. Let this basis be called $\mathcal{B}_C$.

The following result assures that the notion of basis is well-defined even in the case where the measurability of acts is non-trivial (in particular in the Savage paradigm).

Proposition 1. *Consider a basis $\mathcal{B}$. For each $f \in \mathcal{A}$, there is a unique measurable function $f^b : S \to \mathcal{B}$ such that $f(s) = f^b(s)(s)$.*

For a basis $\mathcal{B}$, $\preceq^{\mathcal{B}}$ will be used to denote the restriction of the preference relation $\preceq$ to the elements of $\mathcal{B}$. Similarly, for any event A, $\preceq_A^{\mathcal{B}}$ will be used to denote the restriction of the preference relation $\preceq_A$ – the preference relation on acts given A – to the elements of $\mathcal{B}$.

The traditional notion of null event shall be employed: an event A is *null* iff, for any pair $f, g \in \mathcal{A}$, $f \sim_A g$.

[2] As shall be discussed below, in the case where the set of consequences is the set of lotteries over C, the Reversal of Order axiom shall be assumed, so every act is equivalent to a mixture of acts yielding only certain lotteries (lotteries where one outcome has probability 1), that is, to a mixture of elements of $\mathcal{A}$ [Fishburn, 1970, Ch 13]. It is thus legitimate to focus attention on $\mathcal{A}$; this will permit formulation of results which apply to both the Savage and Anscombe & Aumann paradigms.

As discussed in the Introduction, there is an axiom for state-independence which is common to the the Savage and the Anscombe & Aumann paradigms: P3 or monotonicity. This axiom shall be the centre of attention in this paper. In particular, we will say that monotonicity essentially holds if there is a basis with respect to which the preference order is monotonic.

Definition 2 (Essential monotonicity)**.** *Monotonicity essentially* holds iff there is a basis $\mathcal{B}$ such that, for every non-null event A, and for all i, j, $b^i \preceq_A b^j$ iff $b^i \preceq b^j$.

Call this basis an *essentially monotonic basis*.

The definitions have been formulated so as to cover both "frameworks" or "paradigms" of Savage-style decision theory: that proposed by Savage himself [Savage, 1954] and those proposed by Anscombe & Aumann [Anscombe and Aumann, 1963]. Let us introduce a little notation to allow us to refer to these paradigms.

We shall be said to be working with the Anscombe & Aumann framework when the set of states S is assumed to be finite and the set of consequences is assumed to be a set of lotteries over a finite set of outcomes C. Furthermore, the basic axioms in [Anscombe and Aumann, 1963] except the principal axiom for state-independence – monotonicity – shall be assumed: namely weak order, independence, continuity and reversal of order.[3]

We shall be said to be working with the Savage framework when the set of states S is a suitably rich (infinite) set, and the set of consequences is the finite set C, which does not possess any particular structure. Furthermore, the basic axioms in [Savage, 1954] except those which assure state-independence – P3 and P4 – shall be assumed: namely weak order, the sure-thing principle, and axioms to insure continuity and Archimedianity, such as Savage's P6 and P7 (see also [Gilboa, 1987, Hill, 2007]).

Both these frameworks require, for state-independence of utility, a monotonicity axiom (Savage calls it P3, Anscome & Aumann call it Monotonicity). The main difference between their axioms and the notion of essential monotonicity in Definition 2 is that, while the former demand that a property holds with respect to a given basis, namely $\mathcal{B}_C$, the latter only requires that there is a basis with respect to which the property holds.

However, whereas monotonicity is enough to ensure a state-independent utility representation in the Anscombe & Aumann framework, it only ensures *ordinal* state independence in the Savage framework, and not *cardinal* state independence; in particular, it is not sufficient to yield a state-independent utility function [Karni, 1993]. The further axiom required is Savage's P4, reproduced below in the notation of the current paper.

Definition 3 (Cardinal state independence)**.** Suppose that monotonicity essentially holds with essentially monotonic basis $\mathcal{B}$. There is, in addition, *cardinal state independence* iff for every pair of events A and B and every $b^i, b^j, b^k, b^l \in \mathcal{B}$ such that $b^i \prec b^j$ and $b^k \prec b^l$, $b^i_A b^j \preceq b^i_B b^j$ iff $b^k_A b^l \preceq b^k_B b^l$.

The following theorem is little more than a rewording of the theorems in [Savage, 1954, Anscombe and Aumann, 1963].

[3] Reversal of Order shall be assumed throughout this paper. Although removing this assumption is one way to gain state-dependence [Drèze, 1987], this paper only concentrates on the Monotonicity axiom.

Theorem 1. *Suppose that there is a representation of $\preceq$ by a measurable function*[4] $U : S \times C \to \Re$ *which is unique up to a positive affine transformation.*

If the setup is the Anscombe & Aumann framework, suppose that monotonicity essentially holds, with an essentially monotonic basis $\mathcal{B}$.

If the setup is the Savage framework, suppose that monotonicity essentially holds and cardinal state-independence holds, with an essentially monotonic basis $\mathcal{B}$.

Then there is a probability measure p on S and a function u on $\mathcal{B}$ such that, for any $f, g \in \mathcal{A}$, $f \preceq g$ if and only if

$$\int_S p(s)u(f^b(s))ds \leqslant \int_S p(s)u(g^b(s))ds$$

Furthermore, p is unique and u is unique up to a positive affine transformation.[5]

Note that the converse direction is trivially true: if there is such a representation, then monotonicity essentially holds (and cardinal state-independence holds).

3 Characterising essential monotonicity

The goal of this section, the main section of the paper, is to characterise essential monotonicity. This characterisation (Theorem 2), combined with Theorem 1 above, will yield a generalised representation theorem (Theorem 3 in Section 4).

As above, it shall be assumed in this section that a representation of $\preceq$ has been obtained by a measurable function $U : S \times C \to \Re$; it is thus assumed that the axioms required for such a representation (weak order, the sure-thing principle or independence, and so on) hold.

To state and prove the characterisation result (Theorem 2), it will be necessary to introduce some basic logical machinery. In what follows, the necessary elements are briefly presented. The reader is referred to any textbook of mathematical logic, such as [Chang and Keisler, 1990], for further details.

Definition 4 (Language)**.** For $x, y, \ldots$ a set of variables, $\leqslant$ and $=$ two binary relations on variables (in the case below, on variables and constants), $\neg, \wedge, \vee, \rightarrow, \forall, \exists$ the logical connectives of first-order logic (respectively read as not, 'and,' 'or', 'if ... then', 'for all', 'there exists'), the *language* $\mathcal{L} = (x, y, \ldots, \leqslant, =, \neg, \wedge, \vee, \rightarrow, \forall, \exists)$ is defined to be the set of sentences constructed from sentences of the form $x = y$ and $x \leqslant y$ (for x, y variables) using the connectives.

Furthermore, $\mathcal{L}_c$ is defined to be the extension of $\mathcal{L}$ obtained by adding n constants $\mathbf{c_1}, \ldots, \mathbf{c_n}$; that is, it is the set of sentences constructed from sentences of the form $x = y$, $x \leqslant y$, $x = \mathbf{c_j}$, $x \leqslant \mathbf{c_j}$, $\mathbf{c_i} = \mathbf{c_k}$ and $\mathbf{c_i} \leqslant \mathbf{c_k}$ using the connectives.

[4] Measurable, that is, with respect to the product σ-algebra mentioned above.

[5] Naturally, in the Anscombe & Aumann framework, this representation can be reformulated in a simpler but extended form. First of all, since the state space is finite, the integral reduces to a sum. Furthermore, since acts can be expressed as mixtures of acts yielding certain lotteries (footnote 2), it is common to consider acts as functions from $S \times C$ to the reals satisfying certain conditions [Karni et al., 1983, Karni and Mongin, 2000, Fishburn, 1970]. It is thus possible to recover a representation of the following, standard, form from the representation given in the text: for any acts f, g, $f \preceq g$ iff $\sum_S \sum_{\mathcal{B}} p(s)u(b^i)f^b(s, b^i) \leqslant \sum_S \sum_X p(s)u(b^i)g^b(s, b^i)$.

A sentence ϕ of $\mathcal{L}$ (resp. $\mathcal{L}_c$) is said to be *closed* iff all the variables which appear in ϕ appear within the scope of a quantifier. That is, for each variable x appearing in ϕ, it appears in one of the following formats: $\ldots \forall x(\ldots x \ldots) \ldots$ or $\ldots \exists x(\ldots x \ldots) \ldots$

As a point of notation, we shall treat $\geqslant$ as the reverse relation ($x \geqslant y$ defined as $y \leqslant x$) and $<$ as the strict relation ($(x \leqslant y) \wedge \neg(y \leqslant x)$), and $\sim$ is indifference with respect to the order ($(x \leqslant y) \wedge (y \leqslant x)$).

These are particular cases of the standard definition of first-order languages – they are languages with only two relations, no functions and, in the case of $\mathcal{L}$, no constants. See [Chang and Keisler, 1990] for more details. The definition of a language makes explicit its expressive power – the things it can say – in terms of the sentences which belong to it. The language $\mathcal{L}_c$ is more expressive than $\mathcal{L}$ since the former can refer directly to particular elements, using the constants $\mathbf{c_i}$, whereas the latter cannot. On the other hand, the language $\mathcal{L}$ is rich enough to say something useful about the structure of the relation $\leqslant$. This is illustrated in the following example.

Example 1. $\psi = \exists x \forall y(y \leqslant x)$ says that there is a maximal element of $\leqslant$.
$\psi' = \forall x \forall y(x \sim y)$ says that the relation $\leqslant$ ranks all elements equivalently.
$\psi'' = \exists x(\mathbf{c_1} < x)$ says that there is an element above $\mathbf{c_1}$.
The first two sentences belong to $\mathcal{L}$ and $\mathcal{L}_c$, whereas the last one only belongs to $\mathcal{L}_c$. These three sentences are closed, whereas $\psi''' = \mathbf{c_1} < x$ is not closed: x does not appear in the scope of any quantifier.

The languages defined above will be useful insofar as they can be used to talk about various orders (such as the order $\preceq^{\mathcal{B}c}$ on the constant acts). The abstract definition given above does not say anything about how they can be used to talk about orders on sets. For this, the notion of interpretation of a language and truth in a structure is required.

Definition 5 (Interpretation)**.** Let an *order structure* $\mathcal{M} = (X, \preceq)$ be a set X with an order $\preceq$. $\mathcal{L}$ (resp. $\mathcal{L}_c$) can be interpreted on $\mathcal{M}$ by interpreting $\leqslant$ as $\preceq$. Formally: (for $\mathcal{L}_c$) let h be a function from the set of constants into X – it is called an *interpretation* of the constants – and (for $\mathcal{L}$ and $\mathcal{L}_c$) let g be a function from the set of variables of the language into X – it is called an *assignment* of the variables. $\mathcal{M}, h, g \vDash \phi$ is the notation used to express the fact that the sentence $\phi \in \mathcal{L}$ (resp. $\phi \in \mathcal{L}_c$) is *true in* $\mathcal{M}$ under the interpretation h and the assignment g. $\vDash$ is defined inductively as follows:

- $\mathcal{M}, h, g \vDash x \leqslant y$ iff $g(x) \preceq g(y)$
- $\mathcal{M}, h, g \vDash c_i \leqslant y$ iff $h(c_i) \preceq g(y)$
- $\mathcal{M}, h, g \vDash x = y$ iff $g(x) = g(y)$
- $\mathcal{M}, h, g \vDash c_i = y$ iff $h(c_i) = g(y)$[6]
- $\mathcal{M}, h, g \vDash \neg\phi$ iff it is not the case that $\mathcal{M}, h, g \vDash \phi$
- $\mathcal{M}, h, g \vDash \phi \wedge \psi$ iff it is not the case that $\mathcal{M}, h, g \vDash \phi$

[6] And similarly for other combinations of order, equality, constant and variable.

- $\mathcal{M}, h, g \vDash \forall x\psi$ iff, for any assignment g', $\mathcal{M}, h, g' \vDash \psi$[7]

The interpretation of the closed sentences of the language in an order structure $\mathcal{M}$ is completely specified by the interpretation h of the constants.

Throughout the rest of the paper, an interpretation of the relevant language in the appropriate order structure is assumed. For the case of $\mathcal{L}_c$, the only interpretations of interest will be in order structures where the set is the set of acts taking the same value in C for all $s \in S$. Henceforth, it will be assumed that the interpretation h takes a constant of the language $\mathbf{c_i}$ to the act which takes the value $c_i \in C$ for each $s \in S$; this interpretation will not be explicitly mentioned below.[8]

Having defined what it is to interpret a language in an order structure, one may begin to talk about the sentences of a language which are true in a given order structure, and about which order structures are such that a given set of sentences are true in them. For this, the following definitions shall prove useful.

Definition 6 (Theory and model)**.** A set of closed sentences of $\mathcal{L}$ (resp. $\mathcal{L}_c$) is called a *theory* of the language $\mathcal{L}$ (resp. $\mathcal{L}_c$).

An order structure $\mathcal{M}$ is a *model* of a theory T if, for all $\phi \in T$, $\mathcal{M} \vDash \phi$.

A theory T is said to be *consistent* if there exists a model of it.

A consistent theory T is said to be *complete* if, for each closed sentence ϕ of the language, either $\phi \in T$ or $\neg\phi \in T$.

Finally, the theory $T(\mathcal{M})$ of an order structure $\mathcal{M}$ is defined as follows: $T(\mathcal{M}) = \{\phi \in \mathcal{L} \mid \mathcal{M} \vDash \phi\}$ (resp. $T_c(\mathcal{M}) = \{\phi \in \mathcal{L}_c \mid \mathcal{M} \vDash \phi\}$).

The following facts follow almost immediately from the definitions.

Fact 1. *Let T_c be a theory of the language $\mathcal{L}_c$; there is a unique restriction of T_c to the language $\mathcal{L}$, the theory $T = \{\phi \in \mathcal{L} | \phi \in T_c\}$. Furthermore, if T_c is consistent (resp. complete), then so is T.*

Fact 2. *If a consistent theory T_c in the language $\mathcal{L}_c$ contains, for each $\mathbf{c}, \mathbf{d} \in \mathcal{L}_c$, one of $\mathbf{c} < \mathbf{d}$, $\mathbf{c} \sim \mathbf{d}$ or $\mathbf{c} > \mathbf{d}$, then it is complete.*

The following example illustrates the notion of a theory of an order structure.

Example 2. If $(X, \preceq)$ has a maximal element, then the sentence ψ in Example 1 is true in it; it belongs to $T((X, \preceq))$.

These definitions are standard in the logical literature; see [Chang and Keisler, 1990] for example. That work gives proofs of the following well-known facts.

[7] As is standard practice, the negation, the conjunction and the universal quantification are treated as primitive, and the other connectors as definable in terms of them ($\phi \vee \psi$ defined as $\neg(\neg\phi \wedge \neg\psi)$, $\phi \to \psi$ defined as $\neg(\phi \wedge \neg\psi)$, $\exists x Fx$ defined as $\neg\forall x \neg Fx$). This implies that, in definitions such as this, only the clauses for the negation, the conjunction and the universal quantification need be stated.

[8] Note that this interpretation implies that, firstly, each pair of constants has a different interpretation, and secondly, that for each element of the ordered structures considered here, there is a constant which is interpreted by this element. Thus, each of the theories in the language $\mathcal{L}_c$ considered below will contain the sentences $\neg(\mathbf{c_i} = \mathbf{c_j})$, for each $i, j \in \{1, \ldots, n\}$, and $\forall x(x = \mathbf{c_1} \vee x = \mathbf{c_2} \vee \cdots \vee x = \mathbf{c_n})$.

Fact 3. *For any order structure $\mathcal{M}$, the theory of $\mathcal{M}$ is consistent and complete. Furthermore, for any complete and consistent theory T, there is an order structure $\mathcal{M}$ such that T is the theory of $\mathcal{M}$. (This is true for both $\mathcal{L}$ and $\mathcal{L}_c$.)*

Fact 4. *Any two finite order structures have the same theory in $\mathcal{L}$ iff they are isomorphic.*

Furthermore, given any two finite order structures $(X, \preceq_1)$ and $(X, \preceq_2)$ with X having n elements, and given a fixed interpretation of the constants of $\mathcal{L}_c$ in X,[9] the two order structures have the same theory in $\mathcal{L}_c$ iff they are identical.

This fact indicates the expressivity of the languages $\mathcal{L}$ and $\mathcal{L}_c$. On the one hand, using sentences of $\mathcal{L}$, it is possible to distinguish any pair of (finite) sets equipped with an order. Or, to put it another way, $\mathcal{L}$ can describe completely the relative positions of the elements in the order on a given (finite) set. On the other hand, given an appropriate interpretation, $\mathcal{L}_c$ can distinguish between isomorphic orders on the same set which order the elements of the set differently. The power of these languages will be harnessed in the characterisation of essential monotonicity.

These preparatory remarks being made, the application of these notions to the case in hand may now begin. The following definitions will prove crucial.

Definition 7. For an event $A \subseteq S$, let $T(A) = \bigcap_{A' \subseteq A} T(C, \preceq_{A'}^{\mathcal{B}_C})$; that is, the intersection of the theories of $(C, \preceq_{A'}^{\mathcal{B}_C})$, for A' subevents of A. Similarly for $T_c(A)$.

Note that, by construction, $T(A)$ is consistent for all A.

Definition 8. For any $s \in S$, define $T(s)$ as follows: for any sentence $\phi \in \mathcal{L}$, $\phi \in T(s)$ iff there exists an event A such that $s \in A$ and $\phi \in T(A)$. Similarly for $T_c(s)$.

Proposition 2. *For all $s \in S$, $T(s)$ and $T_c(s)$ are consistent.*

The following technical notion will be required in the theorem.

Definition 9. The pair $(\mathcal{A}, \preceq)$ is said to be *monotonic complete* if and only if, for any $f, g \in \mathcal{A}$, there is a partition $\{E_i\}$ of S into events E_i which are disjoint and whose union is S, such that, for each i, $f \preceq_{E_i} g$ if and only if, for any event $A \subseteq E_i$, $f \preceq_A g$.

Proposition 3. *Suppose that $(\mathcal{A}, \preceq)$ is monotonic complete. Then $T(s)$ is complete for any $s \in S$. Similarly for $T_c(s)$.*

Monotonic completeness is only required in the case of infinite state spaces (the Savage paradigm): if S is finite, then it applies automatically, and the condition in the proposition above (and in Theorem 2 below) is empty. Intuitively, monotonic completeness demands that, for any pair of acts, there be a set of events covering the state space such that the acts are in a stable preference order given these events: for any subevent, the order given this subevent is the same as the order given the original event. It implies that there is no infinite strictly decreasing sequence of events such that the preference order on the acts given the events does not settle as one goes down the sequence.

[9] Recall that this interpretation satisfies the assumption stated after Definition 5; namely, that the theories involved contain the sentences in footnote 8.

For the purposes of the characterisation below, monotonic completeness essentially guarantees that the basis which one constructs consists only of measurable functions. It is noteworthy that the set of consequences (and thus constant acts) determines which functions from states to consequences are measurable (ie. those functions where the inverse image of each consequence is measurable). Thus, if one replaces the set of constant acts by a set of *unmeasurable functions* satisfying the condition in Definition 1, and the set of consequences by the elements of this set (as in Proposition 1), the functions which were measurable according to the notion of measurability implied by the set of consequences are no longer measurable with respect to this new set of functions, and those which are measurable with respect to the new set are not measurable with respect to the old. Since the preference relation is defined on the set of functions measurable with respect to the original set of consequences, the new set of unmeasurable functions cannot be used in the representation of this relation; for this reason, Definition 1 demands that elements of the basis be acts (ie. functions measurable with respect to the set of consequences). Monotonic completeness implies that a basis of acts can be constructed, and thus used in the representation of the preference relation.

Finally, the classic definition of null events can be extended to states, as follows.

Definition 10. $s \in S$ is *null* iff $T(s)$ contains $\forall x \forall y (x \sim y)$.[10]

We may now state the main theorem of the paper.

Theorem 2. *Monotonicity essentially holds iff* $(\mathcal{A}, \preceq)$ *is monotonic complete and, for any non-null* $s_1, s_2 \in S$*,* $T(s_1) = T(s_2)$*.*

The proof of the theorem operates by constructing a basis $\mathcal{B}$ that is essentially monotonic. It has, as a corollary, that if there is essentially monotonicity, then the essentially monotonic basis has the following uniqueness property.

Corollary 1. *Suppose that monotonicity essentially holds, and suppose that* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are two essentially monotonic bases. Then, for each* $b_1^i \in \mathcal{B}_1$*, there is a* $b_2^i \in \mathcal{B}_2$ *such that, for any non-null event* A*,* $b_1^i \sim_A b_2^i$*. Furthermore, for each* $b_1^i \in \mathcal{B}_1$*,* $|\{b_2^i \in \mathcal{B}_2 |$ *for any non-null event* A*,* $b_1^i \sim_A b_2^i\}| = |\{b_1^j \in \mathcal{B}_1 | \, b_1^i \sim b_1^j\}|$*.*

This corollary may be understood as indicating the difference in degree of freedom between essential monotonicity and cardinal state-independence: it implies that if there is essential monotonicity, there is cardinal state independence with respect to one of the essentially monotonic bases if and only if there is cardinal state independence with respect to them all.

4 A representation theorem and concluding remarks

The results in this paper may have more widespread implications than at first seems. Firstly, they have a representation theorem as an immediate corollary: if one has determined necessary and sufficient conditions on the preference relation for it to be represented by a function from state–consequence pairs into the real numbers (an "expected

[10] For the informal reading of this sentence, see Example 1.

utility" function), the results presented here reveal a supplementary condition, which, when added, is necessary and sufficient for the existence of a choice of a set of constant acts relative to which there is a representation of the preference relation by a probability and a state-independent utility function. The representation theorem, stated below, is thus an immediate corollary of Theorems 1 and 2.

Theorem 3. *Suppose that the necessary and sufficient axioms hold for a representation of $\preceq$ by a measurable function $U : S \times C \to \Re$ which is unique up to a positive affine transformation. Suppose furthermore that monotonic completeness holds and, for any non-null $s_1, s_2 \in S$, $T(s_1) \subseteq T(s_2)$. Finally, if the setup is the Savage framework, suppose that cardinal state independence holds with respect to any essentially monotonic basis.*[11]

Then there is a basis $\mathcal{B}$, a probability measure p on S and a function u on $\mathcal{B}$ such that, for any $f, g \in \mathcal{A}$, $f \preceq g$ if and only if

$$\int_S p(s)u(f^b(s))ds \leqslant \int_S p(s)u(g^b(s))ds$$

Furthermore, p is unique and u is unique up to a positive affine transformation.[12]

As anticipated in the Introduction, this theorem can be thought of as a generalisation of the theorem proposed by Karni [Karni, 1993, Theorem 3.1]. In a word, Karni's theorem assumes an essentially monotonic basis as given, whereas the Theorem 3 does not suppose such a basis, but instead contains a condition under which such a basis exists.[13] Another difference between the two theorems is perhaps worthy of note. Whereas the theorem in [Karni, 1993] deals exclusively with the Savage framework, replacing some of Savage's other axioms (notably P5, P6 and P7) by versions expressed in terms of the essentially monotonic basis, the results obtained in this paper, and notably Theorem 3, apply to both the Savage and the Anscombe & Aumann frameworks. To do this, it supposes that necessary and sufficient conditions have been obtained for the representation of the preference relation by an "expected utility" function U. While such conditions are well-known for the Anscombe & Aumann framework [Fishburn, 1970, p146], they have proved more difficult to obtain for the Savage framework. Some advances have however been made in this direction, most notably in [Wakker and Zank, 1999, Hill, 2007]. They seem to suggest that the reformulation of the Savage axioms P5, P6 and P7 in terms of an essentially monotonic basis which Karni proposes in [Karni, 1993] is unnecessary: these axioms are basically required to get the "expected utility" representation, and this can be done at a stage before the question of state independence enters onto the scene.

As noted, in the Savage framework, monotonicity does not guarantee cardinal state-independence. Several state-dependent utility theories have been pro-

[11] Note that, by Corollary 1, demanding cardinal state independence with respect to any essentially monotonic basis is equivalent to demanding it with respect to a single essentially monotonic basis.

[12] The remarks in footnote 5 about the simplifications of the formulation in the Anscombe & Aumann paradigm apply here.

[13] The terminology "essentially monotonic basis" has been introduced here and does not appear in [Karni, 1993]; however the assumption of such a basis is what is effectively underlying the result in that paper.

posed in that framework relying on this fact: they only drop cardinal state independence whilst keeping ordinal state independence (monotonicity); examples are [Karni and Schmeidler, 1993, Wakker and Zank, 1999]. The characterisation theorem proved here provides an immediate extension of these results, requiring not monotonicity but the conditions which guarantee essential monotonicity. (This is analogous to the extension of Savage's and Anscombe & Aumann's representation theorem obtained in Theorem 3.) Moreover, as noted in the remarks following Corollary 1, the result indicates the limited "degree of freedom" between monotonicity and cardinal state independence: if monotonicity essentially holds, the uniqueness properties of the set of acts which may count as an essentially monotonic basis are so strong that if one such set satisfies cardinal state independence, they all do. This paper is concerned with the possibility of producing monotonicity (ordinal state independence) by redefining what counts as constant acts; the uniqueness result implies that there is no analogous way of obtaining cardinal state independence just by redefining the constant acts.

It is noteworthy that the results proven here imply that not all preference relations which admit state-dependent representations are essentially monotonic. It follows that any theorem which elicits a state-dependent representation employing a technique which relies on the fact that the preference is monotonic but with respect to another set of constant acts, such as that in [Karni, 1993], will not apply to all preference relations admitting state-dependent representations. This opens the question of the degree to which given representations for state-dependent utilities treat "essentially" state-independent utilities or not: it is perhaps important to distinguish the state-dependent utility theorems which can deal with preference relations which are not essentially monotonic from those which cannot. Most of the results mentioned here, with the possible exception of [Karni et al., 1983, Drèze, 1987], are in the latter category.

Let us make a final remark on the logical techniques employed above. It would not be surprising if such techniques could be applied beyond the problem dealt with here: after all, an important branch of logic, model theory, is concerned precisely with the possibilities of characterising mathematical structures and classes of mathematical structures in terms of sets of sentences (or "axioms") which they satisfy. It may turn out that some of the notions and results from that domain will prove useful to decision theorists, and perhaps economists more generally. Given this possibility, the contribution of this article is not limited to the characterisation of situations where sets of acts could be found which support state-independent utility representations, or to the extension of the theorem in [Karni, 1993]. It makes an important technical contribution, by introducing logical techniques in the field of decision theory.

Appendix

Throughout this appendix, f^c, f^{c_i}, f^d will be used to denote the constant acts yielding elements $c, c_i, d \in C$.

Proof of Proposition 1. Set $f^b(s) = b^i$ where b^i is the unique element of the $\mathcal{B}$ where $b^i(s) = f(s)$. It remains to show that f^b is an act (ie. a measurable function). For each i and for each $c \in C$, $\{s \mid f(s) = c = b^i(s)\} = \{s \mid f(s) = c\} \cap \{s \mid b^i(s) = c\}$: that is, it is the intersection of two measurable sets (since both f and b^i are measurable), and thus itself measurable. For each i, $\{s \mid f(s) = b^i(s)\} = \bigcup_{c \in C}\{s \mid f(s) = c = b^i(s)\}$: this is a finite union of measurable sets and thus measurable. Hence f^b is measurable. □

Sketch of proof of Theorem 1. The proof is essentially a combination of parts of those in [Anscombe and Aumann, 1963, Savage, 1954]. For this reason, only a brief sketch shall be presented here; the reader is referred to those works for details.

Given the uniqueness properties of U, it can be supposed that it takes value zero on null events. Define $u(b^i) = \int_{b^i} dU$ and $p(A) = \int_{b^i \cap (A \times C)} dU/u(b^i)$ for all events A and $b^i \in \mathcal{B}$. It is necessary to show that p is a probability measure on S: that is, firstly, that it is a function independent of $\mathcal{B}$, and secondly, that it satisfies the probability axioms.

Consider first independence of p with respect to $\mathcal{B}$ in the Anscombe & Aumann framework. Essential monotonicity implies that $\preceq_A^{\mathcal{B}}$ and $\preceq^{\mathcal{B}}$ coincide for any non-null event A, so that u represents $\preceq_A^{\mathcal{B}}$, as does the restriction of U to $A \times C$. By the uniqueness properties of u in the Anscombe & Aumann framework, it follows that there is a constant α_A such that $\int_{b^i \cap (A \times C)} dU = \alpha_A u(b^i)$ for any b^i: $p(A) = \alpha_A$ and is thus independent from $\mathcal{B}$.

Now consider independence of p with respect to $\mathcal{B}$ in the Savage framework. If an event A is such that $\int_{b^i \cap (A \times C)} dU/u(b^i) = \frac{1}{2}$ for some i, then $\int_{b^j \cap (A \times C)} dU/u(b^j) = \frac{1}{2}$ for all $j \in \{1, \ldots, n\}$: if not, then there will be j, k and l such that $b^k \prec b^i$ and $b^l \prec b^j$ (or both $\succ$) and $b^i_A b^k \sim b^i_{A^c} b^k$ but $b^j_A b^l \succ b^j_{A^c} b^l$ contradicting cardinal state independence. Reasoning by induction, it can be shown that similar properties hold for events A' with $\int_{b^i \cap (A' \times C)} dU/u(b^i) = \frac{m}{2^p}$ for any positive integers m and p. For any event B, using increasingly accurate bounds by events A' such that $\int_{b^i \cap (A' \times C)} dU/u(b^i) = \frac{m}{2^p}$, the value of $\int_{b^i \cap (B \times C)} dU/u(b^i)$ can be fixed as accurately as one would like, and is the same for all $i \in \{1, \ldots, n\}$. This value is $p(B)$ and is thus well-defined.

Finally, it is easy to see that p takes values between 0 and 1, that $p(S) = 1$, and that it is additive, since U is additive. The uniqueness properties of p are immediate; those of u follow easily from those of U and the specific properties of the frameworks (reversal of order in the case of the Anscombe & Aumann framework; cardinal state independence in the case of the Savage framework).

□

Proof of Proposition 2. Suppose that $T(s)$ is not consistent. Then there exist A_1, A_2, $s \in A_1$, $s \in A_2$, with $\phi \in T(A_1)$ and $\neg\phi \in T(A_2)$. But $A_1 \cap A_2$ is an event: $A_1 \cap A_2 \subseteq A_1$ so $\phi \in T(A_1 \cap A_2)$ and $A_1 \cap A_2 \subseteq A_1$ so $\neg\phi \in T(A_1 \cap A_2)$ contradicting the fact that $T(A)$ is consistent for every event A. So $T(s)$ is consistent. Similarly for $T_c(s)$. □

Proof of Proposition 3. Reason in $\mathcal{L}_c$. Consider a pair of constant acts f^c and f^d. By monotonic completeness, there is a partition $\{E_i\}$, such that, for each i, $f^c \preceq_{E_i} f^d$ if and only if, for any $A \subseteq E_i$, $f^c \preceq_A f^d$. So, for each E_i, one and only one of $\mathbf{c} < \mathbf{d}$, $\mathbf{c} \sim \mathbf{d}$ or $\mathbf{c} > \mathbf{d}$ is in $T_c(E_i)$. So exactly one of these sentences is in $T_c(s)$, for $s \in E_i$. By this reasoning, for each pair of constants $\mathbf{c_i}$, $\mathbf{c_j}$ in $\mathcal{L}_c$, exactly one of $\mathbf{c_i} < \mathbf{c_j}$, $\mathbf{c_i} \sim \mathbf{c_j}$ or $\mathbf{c_i} > \mathbf{c_j}$ is in $T_c(s)$. So, by Fact 2, $T_c(s)$ is complete. It follows from Fact 1 that $T(s)$ is complete. $\square$

In the proof of Theorem 2, the following lemma will be useful.

Lemma 1. *Suppose that $(\mathcal{A}, \preceq)$ is monotonic complete. Then, for each $s \in S$, there is an event E, $s \in E$ such that $T(s) = T(E)$.*

Proof. Reason in $T_c(s)$. In the proof of Proposition 3, it was shown that, for each pair of constants $\mathbf{c}$, $\mathbf{d}$, there is an event E_{cd} with $s \in E_{cd}$ such that the sentence $\mathbf{c} < \mathbf{d}$ (resp. $\mathbf{c} \sim \mathbf{d}$, $\mathbf{c} > \mathbf{d}$) is in $T(E_{cd})$ iff it is in $T(s)$. Let E be the intersection of E_{cd} over all pairs $c, d \in C$. E is an event since it is an intersection of events; it is non-empty because it contains s. Furthermore, by construction $T_c(E) = T_c(s)$, and so (by Fact 1) $T(E) = T(s)$. $\square$

Proof of Theorem 2. Necessity. Suppose there is essentially state independence with basis $\mathcal{B}$ and let T be the theory of $(\mathcal{B}, \preceq^{\mathcal{B}})$. It shall be shown (1) that monotonic completeness holds, and (2) that $T(s) = T$ for all non-null $s \in S$.

(1) Consider any pair of acts $f, g \in \mathcal{A}$. By Proposition 1, there are corresponding measurable functions $f^b, g^b : S \to \mathcal{B}$. Since these acts are measurable, there are two partitions $\{E^f_j\}$, $\{E^g_k\}$ of S such that f^b (respectively, g^b) takes constant values on each of the E^f_j (resp. E^g_k). Let $\{E_i\}$ be the coarsest refinement of $\{E^f_j\}$ and $\{E^g_k\}$.[14] On each of the elements E_i, both f^b and g^b take constant values in $\mathcal{B}$. Moreover, since $\mathcal{B}$ is an essentially monotonic basis, for any element E_i, $f \preceq_{E_i} g$ iff, for any event $A \subseteq E_i$, $f \preceq_A g$. So $(\mathcal{A}, \preceq)$ is monotonic complete.

(2) Suppose that $T(s)$ is not equal to T for every non-null $s \in S$, and pick non-null $s \in S$ such that $T(s) \neq T$. Note that, since $T(s)$ and T are complete (Proposition 3 and Fact 3), it follows that they are inconsistent. By Lemma 1, there is an event E, $s \in E$ such that $T(E) = T(s)$. By the expressivity of the language $\mathcal{L}$ (Fact 4), it follows that there are disjoint pairs of subsets $K, K' \subseteq \{1, \ldots, n\}$ and $L, L' \subseteq \{1, \ldots, n\}$ such that

1. for every $k' \in K'$, $k \in K$, $k'' \in \{1, \ldots, n\} \setminus (K \cup K')$, and for all $A \subseteq E$, $f^{c_{k'}} \succ^{\mathcal{B}_C}_A f^{c_k} \succ^{\mathcal{B}_C}_A f^{c_{k''}}$; for every $l' \in L'$, $l \in L$, $l'' \in \{1, \ldots, n\} \setminus (L \cup L')$, $b^{l'} \succ^{\mathcal{B}} b^l \succ^{\mathcal{B}} b^{l''}$; and $|K'| = |L'|$.
2. for all $k_1, k_2 \in K$ and for all $A \subseteq E$, $f^{c_{k_1}} \sim^{\mathcal{B}_C}_A f^{c_{k_2}}$; for every $l_1, l_2 \in L$, $b^{l_1} \sim^{\mathcal{B}} b^{l_2}$; and $|K| \neq |L|$.

Suppose that $|L| < |K|$ (the case $|L| > |K|$ is treated similarly). Let $b^j \in \mathcal{B}$ be a largest element below $\{b^l |\, l \in L\}$ (for all $l \in L$, $b^j \prec^{\mathcal{B}} b^l$, for every $b^{j'}$ such that $b^{j'} \prec^{\mathcal{B}} b^l$ for all $l \in L$, $b^{j'} \preceq^{\mathcal{B}} b^j$). Since $T(E)$ describes the properties common to orders $\preceq^{\mathcal{B}_C}_A$, for all non-null $A \subseteq E$, and since T describes $\preceq^{\mathcal{B}}_A$ for these A, by state-independence, the elements of $\mathcal{B}$ above the elements of $\{b^l |\, l \in L\}$ must take values

[14] That is, the coarsest partition each of whose elements are contained in a single element of $\{E^f_j\}$ and in a single element of $\{E^g_k\}$.

in the set of consequences $\{c_{k'} |\, k' \in K'\}$. Similarly, the elements of $\{b^l |\, l \in L\}$ must take values in $\{c_k |\, k \in K\}$. Since the element b^j is the largest element below $\{b^l |\, l \in L\}$, the values on subsets of E must be the most preferred among the consequences which are not in $\{c_{k'} |\, k' \in K'\} \cup \{c_k |\, k \in K\}$. However, since $|K| > |L|$, these will all be c_{k_i} such that $k_i \in K$. So, for any non-null $A \subseteq E$, $b_j \sim_A f^{c_k} \sim_A b^l$ for any $k \in K$ and $l \in L$, contradicting the definition of b^j. This contradicts the assumption that $\mathcal{B}$ is an essentially state-independent basis. So $T(s) = T$.

Sufficiency. Since monotonic completeness holds, $T_c(s)$ is complete for any s (Proposition 3); by Facts 3 and 4, there is an unique order $\preceq_s$ on C such that $(C, \preceq_s)$ is a model of $T_c(s)$.

Since, for any non-null $s_1, s_2 \in S$, $T(s_1) = T(s_2)$, it follows from fact 4, that for any non-null $s_1, s_2 \in S$, the orders $(C, \preceq_{s_1})$ and $(C, \preceq_{s_2})$ are isomorphic. So they have the same structure for all non-null $s \in S$: the same number of maximal elements, the same number of second best elements and so on. Hence one can define a set of acts b^i by induction as follows:

- $b^1(s)$ is the maximal element in $(C, \preceq_s)$, where, if there are several, then any one is chosen.
- for $i > 1$, $b^i(s)$ is the maximal element in $(C \setminus \{b^k(s) \mid k < i\}, \preceq_s)$, where, if there are several, then any one is chosen.

It is easy to see that the b^i are well-defined, and form a basis $\mathcal{B}$.

To prove essential monotonicity of this basis, suppose not; that is, suppose that there is b^i, b^j and a non-null event A such that $b^i \preceq b^j$ but $b^i \succ_A b^j$. Either there is a non-null event $A' \subseteq A$ such that, for any non-null $A'' \subseteq A'$, $b^i \succ_{A''} b^j$, or there is none.

Consider the former case: using the hypothesis and the the fact that $(\mathcal{A}, \preceq)$ is monotonic complete, pick a sufficiently small non-null event A', such that there exist constant functions f^{c_1} and f^{c_2} with $b^i \succeq_{A'} f^{c_1} \succ_{A'} f^{c_2} \succeq_{A'} b^j$ and such that the same inequalities hold for any $A'' \subseteq A'$. But this contradicts the construction of b^i and b^j on $s \in A'$: since the construction was supposed to yield b^j more preferred than b^i, b^i should have the value c_2 or lower and b^j should have the value c_1 or higher. So this case cannot hold.

Suppose on the other hand that no such A' exists. Construct the following sequence of non-null events: pick any $A_1 \subset A$ such that $b^i \preceq_{A_1} b^j$ – this exists because there is no $A' \subseteq A$ such that, for any non-null $A'' \subseteq A'$, $b^i \succ_{A''} b^j$. Since $b^i \succ_A b^j$, it follows from the sure-thing principle that $b^i \succ_{A \setminus A_1} b^j$. However, there is no non-null $A' \subseteq A \setminus A_1$ such that for any non-null $A'' \subseteq A'$, $b^i \succ_{A''} b^j$; because if there were, this would contradict the assumption that there is no $A' \subseteq A$ with this property. So there exists non-null $A_2 \subseteq A \setminus A_1$ with $b^i \preceq_{A_2} b^j$. Furthermore, $A \setminus (A_1 \cup A_2)$ inherits from A the property that there is no non-null $A' \subseteq A \setminus (A_1 \cup A_2)$ such that for any non-null $A'' \subseteq A'$, $b^i \succ_{A''} b^j$, so a non-null subset A_3 may be chosen, such that $b^i \preceq_{A_3} b^j$. One thus constructs a sequence of disjoint non-null events A_k which are subsets of A and such that $b^i \preceq_{A_k} b^j$ for all k. Furthermore, the limit of the unions of these events is A, for if not, there would be a non-null $A' \subseteq A$ such that, for any non-null $A'' \subseteq A'$, $b^i \succ_{A''} b^j$, and this is not the case by assumption. By the sure thing principle, it follows that $b^i \preceq_A b^j$, contrary to the hypothesis.[15] This completes the proof of state-independence, and indeed, of the theorem.

□

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[15] Recall that the basic axioms of the Savage and Anscombe & Aumann frameworks are being assumed. Furthermore, remark that the case considered here can only occur if the state space is infinite – that is, in the Savage framework – so it is the sure thing principle which is relevant.