

Market Informational Inefficiency, Risk Aversion and Quantity Grid*

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Abstract

In this paper we show that long run market informational inefficiency is perfectly compatible with standard rational sequential trade models. Our inefficiency result is obtained taking into account two features of actual financial markets: tradable quantities belong to a quantity grid and traders and market makers do not have the same degree of risk aversion. The implementation of our model for reasonable values of the parameters suggests that the long term deviations between asset prices and fundamental value are important. We explain the ambiguous role of the quantity grid in exacerbating or mitigating market inefficiency. We show that stock splits can improve the information content of the order flow and consequently increase price volatility. (JEL G1, G14, D82, D83)

Keywords: Informational efficiency, quantity grid, stock splits.

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1 Introduction

One of the central roles of financial markets is to provide information about asset's fundamentals through the price system. After the recent collapse of companies that were commonly considered, and priced, as worthy and safe, investors questioned the capacity of the market to perform this crucial task. For example, it is now clear that Enron's fall was long time coming and that pieces of information about the company's problems were spread among agents. Then, why have the price of Enron's shares remained for so long far above the company's fundamental value?

In financial economics, it is generally accepted that when it is possible to observe the actions of a sufficiently large number of rational investors, deviations between transaction prices and long-term market fundamental eventually vanish.¹ This happens because each investor's action discloses, at least partially, the investor's private information on fundamental. This information is incorporated into the trading prices. Thus, by observing these prices, it is ultimately possible to infer all the relevant private information that is dispersed among market participants. In other words, in the long run, the market is informational efficient. For this reason, some practitioners and financial economists attributed mispricing episodes to market exuberance or investors irrationality.

In this paper we show that long term mispricing is perfectly compatible with agent's rationality. To this purpose, in a standard microstructure model, we jointly consider two features of financial markets: tradable quantities belong to a quantity grid (in particular it is impossible to trade fractions of a share); traders and market makers do not have the same degree of risk aversion. We show that when these two factors are taken into account, then market is not informational efficient in the long run. In other words, surprisingly, in the long term the private information regarding the asset fundamental value cannot be completely incorporated into trading prices. As in the model we consider, trading prices are equal to the expected value of the risky asset given the history of trades, we can measure the long-term-mispricing with the distance between the trading prices in the long run and the expected value of the asset for someone who has the combined knowledge of all traders in the economy. We show that in general this distance cannot

¹This is a textbook result in financial economics. See for example O'Hara (1995) or Biais, Glosten and Spatt (2001) for a recent review of the financial microstructure literature.

vanishes, and that the resulting “long term pricing error” can be large. This could provide a partial explanation to the episodes mentioned above.

More precisely, the model we consider is a sequential trade model similar to Glosten and Milgrom (1985) and Glosten (1989): in each period risk neutral market makers quote a price schedule for a risky asset. Given the price schedule, informed risk averse traders choose the size of their trade. The difference with the existing literature is that we bring together on the one hand the existence of a grid for tradable quantities, on the other hand, discrepancy in risk aversion between dealers and traders.

In order to have an intuition of our result, notice that a risk averse privately informed trader’s order includes two components: an informational component and an inventory component. The first component comes from the trader’s informational advantage given by his private information on the asset’s fundamental. The latter component follows from the trader’s risk aversion and is not related to the asset’s fundamental. Note that when the past history of trades provides a sufficiently precise information about the asset’s fundamental, then an additional partially informative private signal will affect slightly a trader’s belief. Thus, as a trader can demand only discrete quantities of the asset, a small change in his belief will in general not be sufficient to affect his demand and so, eventually all traders’ demands will only reflect their inventory components. From this point on, the flow of trades will no longer be informative, the social learning process stops and trading prices will be bounded away from market fundamental. Long-run-mispricing will increase with traders’ risk aversion and with the fundamental’s volatility that cannot be explained through private information. It will decrease with the precision of traders’ private signals.

In short, when the market is quite sure about the asset fundamental, the equilibrium is unique and such that the flow of trade does not provide information because orders only reflects traders’ inventory concerns. Moreover, if the learning process stops when the market is quite sure about the asset’s fundamental but in a completely wrong direction, then prices will be trapped far away from the asset fundamental value, and consequently the long term pricing error will be large.

Other papers in the financial microstructure literature have considered separately the discrepancy in risk aversion and discrete trading without obtaining informational inefficiency. For example, in Glosten and Milgrom (1985) or Easley and O’Hara (1992) traders can only trade discrete quantities (buy 1 asset, sell 1 asset, no trade) but in these models both market

makers and informed traders are risk neutral, so trades are always informative because of the absence of the inventory component. In Glosten (1989) and Biais, Martimort and Rochet (2000) risk neutral market makers face risk averse informed traders, but these models assume that it is possible to trade a continuum of quantities of the asset so that even a tiny information component can affect the trader's order, and for this reason the order flow is always informative. Thus, our contribution is to show that the combination of risk aversion and discrete trading generates informational inefficiency as learning process stops at wrong price. Moreover, we show that mispricing can be large even for realistic calibrations of the model.

Our main result is in line with the theoretical literature on “herd behavior” that proves that sequential interaction of rational investors can generate rational imitative behavior and this prevents agents from learning the market fundamental.² However, most of the results in this literature are based on the assumption that transaction prices are exogenously fixed and are not affected by the information provided by past trades. Therefore, the herding literature cannot be directly applied to stock markets, and it is clearly unfit to study the issue of the informational content of prices. An informational inefficiency result in presence of endogenous price is obtained by Lee (1998). He shows that information aggregation failure is due to the existence of exogenous transaction costs. When the profit from trade is smaller than the transaction cost, investors stop trading and this prevents the complete learning of market fundamental. Décamps and Lovo (2002) and Cipriani and Guarino (2002) show that in a model where traders strategies are restricted (buy one lot, sell one lot, no trade) herd behavior and long run inefficiency can occur because of differences in agents' valuation for the asset.

In this paper we show that informational inefficiency is not necessarily linked to the presence of exogenous frictions in transaction prices due to inelastic prices (as in the rational herding literature), transaction cost (as in Lee (1998)) or exogenous difference in agents valuation for the object (as Cipriani and Guarino (2002)).

One might wonder whether our theoretical result can account for relevant long run mispricing even when the lot size is one share. Indeed, with the advent of on-line trading, traders can trade any integer size without much of a problem. Can a one-share-discreteness actually induce significant long run mispricing? In order to answer this question, we implement our model for

²See Chamley (2001) for an extensive study on the causes of rational herding.

reasonable values of parameters. We measure the market inefficiency of an odd lots trading mechanism for a share with expected value 35\$ and standard deviation 7\$. We find that, for reasonable level of traders' risk aversion and private information, the minimum long-run pricing error is about 1.16 \$ (i.e. 3.31% of the expected value of the asset), the maximum pricing error can be about 5,84 \$ (i.e. 16.93% of the expected value of the asset) and the expected pricing error is about 1.93\$. (i.e. 5.52% of the expected value of the asset).

We also prove that a change in the minimum trading unit (or lot size)³ has an ambiguous effect on informational inefficiency. On the one hand, an appropriate increase in the minimum trading unit can eliminate the long run mispricing. However, the choice of such an "informational-efficient lot size" is not robust to perturbations of the fundamentals of the economy. This suggests that it can be actually difficult to restore efficiency through the choice of an appropriate grid of tradable quantities. On the other hand, decreasing the lot size reduces, but does not eliminate, the long term inefficiency. The latter observation allows to relate our analysis to the literature on stock splits. Indeed, a stock split corresponds to a reduction of the minimum trading unit and therefore stock splits reduce market inefficiency. More precisely, a stock split can temporarily restore the informativeness of trades and consequently increase price volatility. This could give reasons for the empirical findings that a stock split generates higher volatility in the stock's return (Ohlson and Penman's (1985), Koski (1998)). Moreover, the same mechanism could motivate manager with favorable information about their company to split their share in order to allow market prices to further incorporate this positive information. This could be an explanation of the empirical observation that stock splits are associated with significant increases in the stock prices (Lamoureux and Poon (1987), Amihud et al. (1999)).

In Section 2 the notations, the assumptions and the basic structure of the model are presented. Section 3 shows the main result. In Section 4 we implement the model and we discuss stock splits. In Section 5 we generalize the inefficiency result to a broader class of economies. Section 4 concludes. The proofs are in the Appendix.

³The minimum trading unit corresponds to the tick of the quantity grid.

2 The model

We consider a sequential trade model in the style of Glosten and Milgrom (1985) and Glosten (1989): a risky asset is exchanged for money among market makers and traders. We denote with $\mathbf{v} = \mathbf{V} + \varepsilon$ the fundamental value of one share of the asset. \mathbf{v} is the sum of two components: a realized shock \mathbf{V} on which agents are asymmetrically informed, and a noise ε that represents the shocks on fundamental whose realization is unknown to everybody such as for example future shocks⁴. For expositional clarity we introduce some simplifying assumptions on the distribution of \mathbf{V} and ε . The general case is discussed in Section 5. We assume that \mathbf{V} and ε are independently distributed and that \mathbf{V} is equal to \bar{V} with probability π_0 and to $\underline{V} < \bar{V}$ with probability $1 - \pi_0$, moreover ε has zero mean and strictly positive standard deviation σ_ε . Remark that \mathbf{V} is an unbiased estimator of \mathbf{v} , but knowing \mathbf{V} is not sufficient to know the exact value of \mathbf{v} . Each trader receives a private partially informative signal $\mathbf{s} \in \{l, h\}$. Signals are conditionally i.i.d. across traders and independent from ε and from the compositions of traders' portfolios. We assume $\Pr(\mathbf{s} = l | \mathbf{V} = \underline{V}) = \Pr(\mathbf{s} = h | \mathbf{V} = \bar{V}) = p$ with $1/2 < p < 1$. The parameter p represents the precision of the signal. Signal l is more likely when $\mathbf{V} = \underline{V}$ and it can be interpreted as a “Bearish” signal. Similarly, $s = h$ can be interpreted as a “Bullish” signal. In other words, $E[\mathbf{v} | s = l] < E[\mathbf{v}] < E[\mathbf{v} | s = h]$ ⁵.

Trading mechanism. Trading occurs sequentially and time is discrete. Each time interval is long enough to accommodate the trade of at most one trader. At the beginning of each trading period a trader receives a private signal \mathbf{s} and comes to the market with an endowment of shares known only to him. The trader submits a market order and market makers compete to fill the trader's order without knowing the trader's signal and portfolio composition. We assume that traders leave the market after they have had the opportunity to trade. We restrict the tradable quantities to belong to a quantity grid. We denote by δ the *minimum trading unit*. In other words, a

⁴This way of modelling the information structure is borrowed from Biais, Martimort and Rochet (2000). The noise ε takes into account that, as in reality, uncertainty is never completely resolved.

⁵The results of the paper do not rely on the independence between \mathbf{V} and ε , their binomial distribution nor on the fact that the precision of the signal is the same for all the agents. See Section 5 for the treatment of the general case.

trader's market order can be any integer multiple Q (positive or negative) of a lot of δ shares of the asset. Note that our restriction to discrete quantities reflects the intrinsic nature of financial markets. If the exchange's rules allow to trade any integer number of shares, then $\delta = 1$. This is the case for odd lots trading mechanisms. By contrast, if only round lots can be traded, then δ is greater than one and it represents the amount of shares in a round lot.⁶

Market participants. Market makers are risk neutral and traders are risk averse.⁷ A trader's expected utility obtained from a portfolio that contains an amount X of the risky assets and M of cash is $E[u(M + X\mathbf{v})]$, where $u' > 0$ and $u'' < 0$. For simplicity, we assume that all traders have the same utility function⁸ but they can differ for the initial compositions of their portfolios that are assumed to be independently and identically distributed. We denote x and m the initial amounts of risky asset and money respectively for a given trader. Note that x is an integer number (positive or negative) as traders cannot hold fractions of shares of the asset, hence $x \in \mathbb{Z}$ and $m \in \mathbb{R}$.⁹ We will refer to x as the trader's *inventory*. For a set $\Theta \subset \mathbb{Z} \times \mathbb{R}$, we use $F(\Theta) = \Pr((x, m) \in \Theta)$ to denote the probability that in a generic period t , the portfolio composition of a trader is in Θ . We assume that there exist a bounded set $\hat{\Theta} \subset \mathbb{Z} \times \mathbb{R}$ such that $F(\hat{\Theta}) = 1$.

Public and private belief. We denote H_t the history of trades up to time $t - 1$. All the agents observe H_t but they do not know the identity of past traders. As private signals provide information on the realization of \mathbf{V} but not on the realization of ε , the learning process on the asset's fundamental only regards \mathbf{V} . The presence of ε guarantees that the uncertainty on \mathbf{v} remains even when the realization of \mathbf{V} is commonly known. We denote $\pi_t = \Pr[\mathbf{V} = \bar{\mathbf{V}} | H_t]$ the public belief at time t . If in period t a trader submits an order of size Q , then public belief will evolve according to Bayes' rule: $\pi_{t+1} = \Pr[\mathbf{V} = \bar{\mathbf{V}} | H_t, Q]$. We denote $v(\pi_t) = E[\mathbf{v} | H_t] = E[\mathbf{V} | H_t]$ the expectation of \mathbf{v} when the belief is π_t . A trader refines public information with the one provided by his private signal. We denote $\pi_t^s = \Pr[\mathbf{V} = \bar{\mathbf{V}} | H_t, s]$, $s \in \{h, l\}$,

⁶Usually, a round lot consists of a lot of 100 shares or a multiple thereof.

⁷Though the crucial assumption is that market makers and traders have different degrees of risk aversion, the assumption that market makers are risk neutral simplifies the analysis.

⁸See Section 5 for the case of heterogenous traders.

⁹ \mathbb{Z} denotes the set of integer numbers, positive and negative.

an informed traders' belief at time t .

Agents' behavior and equilibrium concept. When a trader comes to the market, he expects a pricing schedule $P_\delta(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}$, with the interpretation that if he submits a market order $Q \in \mathbb{Z}$ positive (negative), then he will buy δQ shares (resp. sell δQ shares) and pay (resp. receive) $P_\delta(Q)$ per share. Thus, if at the time t trader has portfolio (x, m) , received the signal s and expects a price schedule $P_\delta(Q)$, then he will demand the quantity

$$Q^*(x, m, P_\delta, \pi_t, s) = \arg \max_{Q \in \mathbb{Z}} E[u(m + (x + \delta Q)\mathbf{v} - P_\delta(Q)\delta Q) | H_t, s].$$

Apart from the discreteness in the tradable quantities, competition among market makers is modeled as in Glosten (1989) or in Kyle (1985). Following these papers, as market makers are risk neutral and competitive, any trade of δQ shares must lead to a zero conditional expected profit. Considering that market makers are ignorant of the portfolio composition and information of the trader who is trading, a price schedule must satisfy

$$P_\delta(Q) = E[\mathbf{v} | H_t, Q^* = Q]. \quad (1)$$

That is the market clearing price is equal to the market makers' expectation of \mathbf{v} conditional on what they learn about \mathbf{v} from the past and current trades.

3 Informational inefficiency

This section contains our main result showing that if: (i) traders are risk averse and market makers are risk neutral, (ii) agents can trade only discrete quantities; (iii) all the private information is not sufficient to completely resolve uncertainty, i.e. $\sigma_\varepsilon > 0$;¹⁰ then in general the market is not informational efficient.

In the long run, the market is strong-form informational efficient if all the information dispersed among the traders in the economy is eventually incorporated into market prices. Considering that in our model, traders' private information only regards \mathbf{V} , $E[\varepsilon] = 0$ and market makers are risk neutral,

¹⁰Consequently, uncertainty cannot be completely resolved even in the long term. We show however in Section 4 that this assumption is not necessary to generate inefficiency.

we have informational efficiency if the trading prices eventually converge to the realization of \mathbf{V} .

Definition 1: The market is strong-form informational efficient in the long run, if

$$\lim_{t \rightarrow \infty} E[|P_\delta(Q) - \mathbf{V}|] = 0.$$

Note that trading prices reflect the information content of past and current trades, and that the information content of a trader's order is bounded by the information content of the trader's private signal. As signals are not perfectly correlated with \mathbf{V} , in order to achieve full efficiency, trades must never cease to be informative. We provide a formal definition of not informative trade:

Definition 2: A trader with portfolio (x, m) who expects a price schedule P_δ is said to place a *not informative order* if the order is not affected by the trader's private signal, i.e.

$$Q^*(x, m, P_\delta, \pi_t, h) = Q^*(x, m, P_\delta, \pi_t, l).$$

According to this definition, a trader's order is not informative when even knowing the trader's portfolio composition (x, m) , the observation of his order does not allow to infer whether he received a bullish or a bearish signal. In other words, if a trade of size Q is not informative, then $\Pr(Q^* = Q | \mathbf{V} = \bar{V}) = \Pr(Q^* = Q | \mathbf{V} = \underline{V})$.

We will show that under some conditions on the distribution F , if in period t the public belief π_t is sufficiently close to 1 or to 0, the orders of all traders in the economy are not informative. In this instance, the learning process stops and public belief and prices will not change anymore. Namely, trading prices will remain at level $P_\delta(Q) = E[\mathbf{V} | H_t]$ for all Q and all following periods. This is usually referred as an informational cascade in the herding literature (see for instance Bikchandani, Hirshleifer and Welch (1992)). Considering that no single order can fully reveal \mathbf{V} , eventually belief π_t will be close either to 1 or to 0, and so an informational cascade will occur before market makers have completely learned \mathbf{V} . Thus, contrary to the common wisdom, the trading prices cannot aggregate completely private information and the market is not informational efficient in the sense of

Definition 1. This phenomenon can lead to important long-run mispricing episodes. When, for instance, π_t is sufficiently close to 1 but the actual fundamental \mathbf{V} is equal to \underline{V} , the long term pricing error will be close to $v(\pi) - \underline{V} \simeq \overline{V} - \underline{V}$.

In order to understand why traders' orders eventually cease to be informative, it is useful to distinguish two components in the trading motivations of a risk averse agent: the *inventory component* and the *information component*. The inventory component reflects the agent's preference for low-risk-portfolios. It increases with the agent's degree of risk aversion, and the unresolved uncertainty about the asset's fundamental. The information component reflects the changes in traders' belief that follows a bearish or a bullish signal and can be measured by $\pi_t^h - \pi_t^l$.¹¹ As signals are not perfectly informative about \mathbf{V} , the information component will decrease as the public belief π_t approaches 0 or 1.¹² In other words, if the trader is quite sure about the realization of \mathbf{V} , a partially informative private signal will affect his belief just slightly. Now, as a trader can demand only discrete quantities of the asset, a small change in his belief will in general not be sufficient to affect his demand¹³ and so we will have $Q^*(x, m, P_\delta, \pi_t, h) = Q^*(x, m, P_\delta, \pi_t, l)$. That means that, when the public belief is sufficiently close to 0 or to 1, in general a trader's demand only reflects his inventory component.

The formal proof is slightly more complex. Indeed, it is always possible to imagine risk averse traders whose demand is informative no matter how close to 1, or to 0, is the public belief π_t . Thus, in order to characterize inefficient markets, we proceed as follows: firstly we identify the traders that submit informative orders even when π_t is arbitrarily close to 1 or to 0. Secondly, we show that the market is informational inefficient if the probability of observing such "informative traders" is zero.

Suppose that the belief π_t is almost equal to 1, or to 0, and take a trader that before receiving the private signal, was indifferent between demanding an amount of Q^* lots or $Q^* + 1$. The demand of this trader will be informative. Indeed, after receiving a tiny informative signal this trader will demand Q^* if the signal is bearish, whereas he will demand $Q^* + 1$ if the signal is bullish. The following lemma characterizes the set of such traders:

¹¹Indeed, if signals are informative we have $\pi_t^l < \pi_t < \pi_t^h$.

¹²That is to say, $\lim_{\pi_t \rightarrow 0} (\pi_t^h - \pi_t^l) = 0$ and $\lim_{\pi_t \rightarrow 1} (\pi_t^h - \pi_t^l) = 0$.

¹³Note that this would not be that case if traders could demand a continuum of the asset.

Lemma 1 *Take $\pi_t = 1$ or $\pi_t = 0$. For any $n \in \mathbb{Z}$, there exist $x^*(n)$, with $\delta n < x^*(n) < \delta(n + 1)$, such that if a trader's inventory is $x^*(n)$, then trading $-n$ lots or $-n - 1$ lots is optimal. If ε is symmetrically distributed then $x^*(n) = \delta(n + 1/2)$.*

In other words, when π_t is almost equal to 0 or to 1, the only traders whose orders are informative, are those whose inventories are sufficiently close to $x^*(n)$ for some $n \in \mathbb{Z}$.¹⁴

Consequently, if the quantity grid δ and -or- the traders' portfolio distribution F are such that inventories of all traders are bounded away from $x^*(n)$ for all $n \in \mathbb{Z}$, then, by a continuity argument, when the public belief will be close enough to 0 or to 1, the demand of all traders in the economy will reflect only the inventory component and will provide no information on \mathbf{V} . In this instance the flow of trade will be no more informative and long run inefficiency will occur. Now we turn to the formal statement of our result.

Proposition 1 *If for all $n \in \mathbb{Z}$ the distribution of traders' portfolio composition F is such that there is a zero probability that a trader's inventory is close to $x^*(n)$, then there exists $\underline{\pi} > 0$ and $\bar{\pi} < 1$ such that for $\pi_t < \underline{\pi}$ or $\pi_t > \bar{\pi}$,*

(i) *all traders's orders at date t are not informative about \mathbf{V} . As a consequence $\pi_\tau = \pi_t$ for all $\tau > t$.*

(ii) *the equilibrium is unique and the price schedule satisfies $P_\delta(Q) = v(\pi_t)$ for all $Q \in \mathbb{Z}$.*

Proposition 1 shows that, when the public belief π_t is sufficiently large (or small) then the equilibrium exists, it is unique and not informative. Precisely the equilibrium price schedule must be $P_\delta(Q) = v(\pi_t)$ for all tradable quantities $Q \in \mathbb{Z}$. The result is fairly robust as it is obtained without specifying the traders' utility function nor the precise distributions of ε .¹⁵

When the hypothesis of Proposition 1 are satisfied, the financial market cannot be informational efficient as the learning process stops as soon as the public belief π_t crosses one of the threshold $\underline{\pi}$ or $\bar{\pi}$. We call the regions $(0, \underline{\pi})$

¹⁴Note that even when $\pi_t = 1$ or $\pi_t = 0$ the asset is still risky because of the ε component. Thus traders trade in order to hedge the risk of their portfolio.

¹⁵In order to obtain our inefficiency result we do not study the equilibrium for all levels of the public belief π . Therefore we do not need to restrict our analysis to the case of CARA utility function and normal distribution of ε , that are the usual assumptions in financial microstructure literature.

and $(\bar{\pi}, 1)$ *information traps*. Indeed if after a trading history the public belief π_t belongs to one of these two regions, it will not move anymore. In this case, all quantities of the asset will be traded at $v(\pi_t)$ per share, and the trading price will not change for all the subsequent periods $\tau > t$. This can potentially lead to highly inefficient markets. For example, suppose that $\mathbf{V} = \underline{V}$ and that $\pi_t \in (\bar{\pi}, 1)$, then no matter the trading history observed after t , prices will remain at level $P(Q) = v(\pi_t)$ much larger than \underline{V} .

The following two corollaries enlighten the role of the lot size in exacerbating and mitigating informational inefficiency. For example it turns out that it could be optimal to increase quantity grid in order to restore the market informational efficiency. Precisely, Corollary 1 states that if the quantity grid is the finest one, that is $\delta = 1$, then long run informational inefficiency occurs almost surely for all discrete distribution F of the traders' portfolios. Corollary 2 shows that when the noise ε is symmetrically distributed it is possible to find a quantity grid δ that guarantees long run market efficiency.

Corollary 1 *An odd lot trading mechanism is informational inefficient.*

Corollary 2 *If ε is symmetrically distributed, then in a round-lot mechanism, long run informational efficiency can be obtained only by choosing a minimum trading unit δ such that*

$$\sum_{n \in \mathbb{Z}} F \left(\left\{ x = \delta \left(n + \frac{1}{2} \right) \right\} \right) > 0. \quad (2)$$

In order to have informational efficiency in a round lot mechanism, the size of the round lot δ must be chosen so that the probability of observing informative order is positive also when the public belief π_t reaches extreme levels. Thus, a regulator that is mainly concerned with the problem of informational efficiency could choose the minimum trading unit that maximizes the probability of observing orders from traders whose inventory is $x^*(n)$. For Lemma 1, when ε is symmetrically distributed, we have $x^*(n) = \delta(n + 1/2)$ that implies that the optimal δ only depends on the distribution function F and not on traders' utility functions. However, it is worth stressing that a mean-preserving asymmetric perturbation of the distribution of ε would change the value $x^*(n)$ and this would restore informational inefficiency in the economy. Roughly speaking, informational efficiency appears to be very fragile.

4 Implementation

In order to understand whether the inefficiency result of the previous section can actually account for relevant pricing errors, in this section we consider a specification of our model and we measure the predicted long term pricing error for reasonable value of the parameters.

Definition 1 and equation (1) suggest that informational efficiency properties of the market can be measured by the distance between the realization of \mathbf{V} and the trading price in the long run. Hence, the long-term pricing error (*LTPE* thereafter) can be defined as the random variable $LTPE = \lim_{t \rightarrow \infty} |\mathbf{V} - v(\pi_t)|$. For Proposition 1, as soon as the public belief π reaches one of the two information traps, we have $P_\delta(Q) = v(\pi)$ for all Q . Therefore, in the long run trading price will be either close to $v(\underline{\pi})$ or to $v(\bar{\pi})$. Thus, a threshold $\underline{\pi}$ close to 0 and a threshold $\bar{\pi}$ close to 1 correspond to a relatively efficient market. Indeed, on the one hand the prices can reach a region that is relatively close to the true value of \mathbf{V} and, on the other hand, the probability of observing a trading history that lead the public belief into the “wrong” information trap is low.

In this section we study how $\underline{\pi}$, $\bar{\pi}$ and *LTPE* are affected by traders’ degree of risk aversion, the volatility of the fundamental and stock splits. To this purpose, we consider an odd lot trading mechanism ($\delta = 1$), and we adopt the standard assumptions of the microstructure literature for what regards traders utility function and the distribution of ε . That is to say that traders have negative exponential utility function (with risk aversion coefficient γ) and that ε is normally distributed. From Corollary 1, we already know that such a market cannot be informational efficient. The following lemma allows us to characterize the belief thresholds $\underline{\pi}$ and $\bar{\pi}$ and to measure the degree of inefficiency for reasonable parameters value.

Lemma 2 *Let $u(W) = -e^{-\gamma W}$, let $\varepsilon \hookrightarrow N(0, \sigma_\varepsilon)$ and let $\delta = 1$, then $\bar{\pi}$ (resp. $\underline{\pi}$) is the minimum $\pi > 1/2$ (resp. maximum $\pi < 1/2$) such that the following two expressions are satisfied*

$$e^{-\gamma(v(\pi) + \gamma\sigma_\varepsilon^2/2)} \leq \pi^h e^{-\gamma\bar{V}} + (1 - \pi^h) e^{-\gamma\underline{V}}, \quad (3)$$

$$e^{\gamma(v(\pi) - \gamma\sigma_\varepsilon^2/2)} \leq \pi^l e^{\gamma\bar{V}} + (1 - \pi^l) e^{\gamma\underline{V}}. \quad (4)$$

If for a given level π , the inequalities (3) and (4) are satisfied, then an informed traders chooses to trade exactly his inventory ($Q^* = -x$) no matter

the signal he received,¹⁶ and so his order will only reflect his inventory concerns.

Note first that if γ is sufficiently large, then inequalities (3) and (4) will be met¹⁷. This happens because when traders are sufficiently risk averse the informational content of their order vanishes as they only trade to reduce the risk of their portfolio.

Similarly, when the information content of signals is low, i.e. π^h is close to π^l , inequalities (3) and (4) will be satisfied even if σ_ε is arbitrarily small.¹⁸ This implies that the presence of the additional noise ε is not a necessary condition to obtain informational inefficiency. Thus, even if the aggregation of all private information could resolve uncertainty almost completely, when traders' information is not precise¹⁹, the existence of a minimum trading size will induce traders to neglect their information and this will impede the convergence of prices to fundamental.

Finally, remark that there exists σ_ε sufficiently large such that no matter the level of public belief or the information content of the private signal, the two inequalities are satisfied. This means that if the uncertainty coming from the noise ε is sufficiently large with respect to the information provided by the component \mathbf{V} , then even signals that are perfectly informative about \mathbf{V} will not be reflected in traders' orders. Indeed, the asset will be too risky to be held even by traders that are perfectly informed about one component of the asset fundamental value.

To sum up, when i) the traders' risk aversion is high; or ii) the precision of private signals is low; or iii) the volatility in market fundamental is mostly due to shocks on which there is no information, then even an infinite sequence of trades will not allow the market to aggregate the relevant private information dispersed among traders.

Starting from Lemma 2, it is possible to compute numerically $\underline{\pi}$ and $\bar{\pi}$ for different levels of the contribution of the private information component to the fundamental's volatility, and for different levels of traders' risk aversion. For Proposition 1 in the long run trading price will be either close to $v(\underline{\pi})$ or to $v(\bar{\pi})$, furthermore, given the symmetry of the parameters, we have

¹⁶Remark that as $x \in \mathbb{Z}$ and $\delta = 1$, it is possible to trade $\delta Q = x$.

¹⁷Indeed, an increase in γ increases the convexity of the exponential. Moreover, a sufficiently large increase in γ reduces the left hand sides of expressions (3) and (4).

¹⁸This follows from the convexity of the exponential.

¹⁹From Bayes rule, the difference between π^h and π^l increases with the precision of the signal p .

$\underline{\pi} = 1 - \bar{\pi}$. Consequently, the minimum $LTPE$ is close to $|\bar{V} - v(\bar{\pi})| = |\underline{V} - v(\underline{\pi})| = (1 - \bar{\pi})(\bar{V} - \underline{V})$ while the maximum $LTPE$ is close to $|\bar{V} - v(\underline{\pi})| = |\underline{V} - v(\bar{\pi})| = \bar{\pi}(\bar{V} - \underline{V})$. Finally, it is possible to approximate the expected long term pricing error²⁰ with $E[LTPE] \simeq 2\bar{\pi}(1 - \bar{\pi})(\bar{V} - \underline{V})$.

We consider thereafter an asset whose ex-ante expected fundamental value is $E[\mathbf{v}] = 35\$$, and whose ex-ante standard deviation is $\sigma = \sqrt{\sigma_{\mathbf{V}}^2 + \sigma_{\varepsilon}^2} = 7\$$ that is 20% of its ex-ante value, where $\sigma_{\mathbf{V}}^2 = \frac{1}{4}(\bar{V} - \underline{V})^2$. This corresponds to the magnitude of the average share price and annual volatility in the New York Stock Exchange.

Figure 1 depicts $\underline{\pi}$ and $\bar{\pi}$ when keeping constant $E[\mathbf{v}] = 35\$$ and $\sigma = 7\$$ and varying $\sigma_{\mathbf{V}}/\sigma$.²¹ The ratio $\sigma_{\mathbf{V}}/\sigma$ represents the proportion of the total volatility of asset's fundamental that could be explained by aggregating all the private information. The parameters set is ($\gamma = 0.08$, $p = 0.8$). When $\sigma_{\mathbf{V}}/\sigma$ increases, the information traps shrink and the market improves its informational efficiency. For example when $\sigma_{\mathbf{V}}/\sigma = 0.5$, then $\underline{\pi} \simeq 0,165$ and $\bar{\pi} \simeq 0,835$, whereas for $\sigma_{\mathbf{V}}/\sigma = 0.9$ we have $\underline{\pi} \simeq 0,021$ and $\bar{\pi} \simeq 0,979$. Interestingly we also deduce from Figure 1 that, when the information in the economy can explain less than 42% of the fundamental's volatility, the market mechanism fails completely to aggregate any private information.²²

Table 1 reports approximations of the minimum $LTPE$, of the maximum $LTPE$ and of the expected long run pricing error when varying $\sigma_{\mathbf{V}}/\sigma$. When for example 50% of the standard deviation of \mathbf{v} could be explained with the private information, then the minimum $LTPE$ is 1.16 \$ and the maximum $LTPE$ is 5.84 \$. Saying it differently, in the long term the trading price is either "wrong" by 1.16\$ in one direction or by 5.84\$ in the opposite direction, and the expected pricing error is about 2.25\$. These pricing errors correspond to 3.31%, 16.93% and 5.52% of the expected fundamental value of the asset respectively. The larger the ratio $\sigma_{\mathbf{V}}/\sigma$, the smaller the expected long term error.

Figure 2 depicts $\underline{\pi}$ and $\bar{\pi}$ when changing the traders' degree of risk aversion γ . The parameters set is ($\sigma_{\mathbf{V}}/\sigma = 0.6$, $p = 0.8$). As the traders' risk aversion increases, traders will exchange the asset mainly for inventory reason, the inventory component overwhelms the information component, and the market informational inefficiency increases. For example when γ moves

²⁰See the Appendix.

²¹Note that in order to vary $\sigma_{\mathbf{V}}/\sigma$ keeping constant $\sigma = 7\$$ and $E[\mathbf{v}] = 35\$$ it is necessary to vary σ_{ε} , \underline{V} and \bar{V} .

²²That is for all $\pi \in (0, 1)$ the inequalities of Lemma 2 is satisfied.

from 0.02, to 0.08, $\underline{\pi}$ and $\bar{\pi}$ move from 0.015 and 0.985 respectively to 0,106 and 0.894.

Table 2 reports the approximations for the minimum, maximum *LTPE* and the expected long run pricing error for different levels of traders' risk aversion. For $\gamma = 0,06$ the minimum and maximum *LTPE* are of 1.53% and 22.45% of the ex-ante value of the asset respectively. The expected long run error is about 1 \$ that represents 2.87% of $E[\mathbf{v}]$. An increase in traders risk aversion reduces the informational efficiency of the market.

4.1 Stock Splits

The market value of a firm's equity is independent from the number of shares outstanding. Thus, a stock splits should not affect the distribution of stocks returns. Nevertheless, Ohlson and Penman (1985) and Koski (1998) find that stock return volatility increases after stock splits. Ohlson and Penman interpret this phenomenon as an increase in the presence of noise traders. Gottlieb and Kalay (1985) attributed this increase in volatility to the presence of a price grid.²³ However, in Koski (1998) the volatility increase does not appear to be due to rounding to discrete price level as suggested by Gottlieb and Kalay. Our model provides an alternative explanation of the effects of a stock split on price volatility and consequently return volatility.

In order to see how a stock split increases price volatility, note first that a stock split corresponds to a reduction of the minimum unit of trade. Indeed, before the stock split the fundamental value of one unit of trade is $\delta\mathbf{v}$. After splitting each share into n new shares, the fundamental value of each new share will be $\mathbf{v}' = \mathbf{v}/n$, and so the value of one lot of δ new shares is $\delta\mathbf{v}' = \frac{\delta}{n}\mathbf{v}$. But this is perfectly equivalent to reducing the minimum unit of trade from δ shares to δ/n shares without splitting the stock, in this case, the value of the new unit of trade would be $\frac{\delta}{n}\mathbf{v}$.

Second observe that a reduction of the unit of trade decreases $\underline{\pi}$ and increases $\bar{\pi}$. Figure 3 depicts $\underline{\pi}$ and $\bar{\pi}$ following a split of the stock. On the horizontal axis there is the number of new share obtained from one old share after the stock split. The parameters set is ($\sigma_{\mathbf{v}}/\sigma = 0.6$, $p = 0.8$, $\gamma = 0.1$). For the given levels of parameters, in the absence of a stock split we have $\underline{\pi} \simeq 0,178$ and $\bar{\pi} \simeq 0,822$ that correspond to an expected *LTPE* of 2.46\$

²³Gottlieb and Kalay (1985) show that when continuous prices are rounded to discrete price levels, the variance of return computed using the round prices exceeds the variance of unrounded returns.

(see also Table 3). However, it is sufficient to split each share into two new shares in order to obtain a dramatic reduction of the inefficiency as the expected $LTP E$ drops to 0.77\$. Splitting each share into 10 new shares, we have $\underline{\pi} \simeq 0,005$ and $\bar{\pi} \simeq 0,995$ that is a huge informational efficiency improvement.

How is this result related to price volatility? Note that as long as π_t does not lie into an information trap, trading prices can vary within a range of about $v(\bar{\pi}) - v(\underline{\pi})$. Thus, after a stock split this range increases allowing a higher volatility for prices. Note also that a stock split can increase the price volatility by restoring the informativeness of trade in case of informational cascade. For example, suppose that before the split, the public belief π_t was in an information trap. Then the asset is traded only for inventory reasons, trades do not transmit information on the asset fundamental, and trading prices will be steady. In case of stock split, the information traps shrink, and for the same level of public belief π_t , informativeness of trade can be temporarily restored. Thus the volatility of trading prices increases.²⁴

5 Extensions

In order to simplify the analysis, in the previous sections we introduced some strong assumptions on agent characteristics and on the distribution of the risky asset's fundamentals. Namely we assumed homogeneity of traders' utility functions, binomial distribution for \mathbf{V} and \mathbf{s} , and independence between \mathbf{V} and ε . In this section we discuss the robustness of our result when these three assumptions are relaxed. We denote by $\mathbf{v}(\mathbf{Z}, \mathbf{N})$ the fundamental value of the asset that depends on two components: a realized shock \mathbf{Z} on which agents are asymmetrically informed, and a noise \mathbf{N} that represents the shocks on fundamentals whose realization is unknown to everybody. Random variables \mathbf{Z} and \mathbf{N} may lay in any measurable space, whereas \mathbf{v} takes value in \mathbb{R}^+ . We assume that the aggregation of all the private information that is dispersed among investors allows to know the realization of \mathbf{Z} . Still, knowing \mathbf{Z} will not be sufficient to completely resolve the uncertainty on the funda-

²⁴Besides, the same mechanism can induce managers with favorable information about their companies to split their share in order to allow a positive reaction of prices to the order flow. This would provide a further explanation to the empirical observation that stock splits lead to higher stock prices as shown by Lamoureux and Poon (1987) and Amihud et al. (1999).

mental value of the asset because of \mathbf{N} . We denote $\mathbf{V} = E[\mathbf{v}|\mathbf{Z}]$ the expected fundamental value of the asset after aggregating all the private information. And we denote $\varepsilon = \mathbf{v} - \mathbf{V}$ the remaining error, where ε has zero mean and positive standard deviation $\sigma_\varepsilon > 0$. Thus, we can assume without loss of generality that $\mathbf{v} = \mathbf{V} + \varepsilon$ and that agents private information regards \mathbf{V} but not ε .

We assume that the random variable \mathbf{V} takes value in a bounded set $\Omega \in \mathbb{R}^+$. The random variables \mathbf{V} and ε are not independently distributed. Still $E[\mathbf{V}]$ is an unbiased estimator of \mathbf{v} : $E[\varepsilon] = 0$ and $\sigma_\varepsilon > 0$. Each trader receives a partially informative private signal \mathbf{s} that takes value in a bounded set Σ . Without loss of generality, we assume that conditional on the realization of \mathbf{V} , private signals are independent. We assume that for all $V \in \Omega$ and $s \in \Sigma$, we have $g(s|V) = \Pr(\mathbf{s} = s|\mathbf{V} = V) > 0$.²⁵ That means that private signals are not perfectly informative as each realization of the signal is compatible with all realizations of \mathbf{V} .²⁶ Finally, we assume that knowing a trader's inventory does not change the expectation of \mathbf{V} , that is to say that for all levels of inventory x and x' we have

$$E[\mathbf{V}|\text{a trader's inventory is } x] = E[\mathbf{V}|\text{a trader's inventory is } x'].$$

This last assumption guarantees that whenever a trader exchanges only for inventory reason his order will provide no additional information about \mathbf{V} .²⁷ Finally we assume that traders are risk averse but they can differ in their utility functions and not only in the composition of their portfolio. The following proposition extends Corollary 1 to the general set up considered in this section.

Proposition 2 *An odd lot trading mechanism is informational inefficient.*

Proposition 2 shows that market inefficiency does not rely on the simplifying assumptions we have introduced in the previous sections. Indeed, even the less risk averse trader who received the most precise signal, will eventually trade only for inventory reason once the public beliefs are sufficiently

²⁵If \mathbf{s} is continuously distributed $g(\cdot)$ shall be interpreted as the conditional density of \mathbf{s} .

²⁶This condition is equivalent the condition $p_{ql} > 0$ at page 1000 in Bikhchandani et al. (1992).

²⁷Note that this is weaker than assuming independence between the distribution of a trader's private signal and his inventory. For instance, it is possible that traders with large inventory in absolute value have more precise signals.

precise about market fundamentals. Thus, we can say that the informational inefficiency arises when there is a general agreement on the asset's fundamental. In these cases, informed traders are prone to ignore their signals and trade only for inventory reasons. Note also that our result is obtained assuming that there is a zero measure of risk neutral traders. Décamps and Lovo (2002) in a simplified model show that inefficiency can also occur when traders are risk neutral provided that dealers are risk averse. This suggests that what lead to inefficiency is not the absence of risk neutral traders but the absence of traders whose utility functions are identical to those of market makers.

6 Conclusion

We studied the informational efficiency properties of a financial market when one takes into account two factors: first, agents can trade only integer quantities of assets; second, traders and market makers do not have the same degree of risk aversion. We show that an odd lot trading mechanism leads to market inefficiency in the sense that in the long run, prices are bounded away from the value of the asset given all the information dispersed in the market. Indeed, when public belief are sufficiently precise, traders' order only reflect their inventory concerns and thus provide no information about the asset fundamental value. Implementing our model for reasonable values of the parameters leads to large long run pricing errors. Long run market inefficiency increases with traders' risk aversion, with the proportion of fundamental's volatility that cannot be explained with private information and it decreases with the precision of informed traders' signals.

We show that decreasing the unit of trade can reduce but does not eliminate market inefficiency. This provides an alternative explanation of the empirical observation that stock splits increase stock return volatility. We show that an appropriate increase of the minimum trading unit can restore completely long run informational efficiency. Still, the choice of an "efficient quantity grid" is not robust to small perturbation of the fundamentals' distribution.

The fact that our results are obtained within a fairly general framework and by introducing reasonable assumptions into standard microstructure models, suggests that the informational efficiency hypothesis is not com-

patible with the way economists are used to model the trading process in financial markets.

7 Appendix

Proof of Lemma 1: Take $\pi_t = 1$, in this case $P_\delta(Q) = \bar{V}$. Let $U(x, Q)$ be a traders' expected utility from trading Q at price \bar{V} when his initial inventory is x , i.e. $U(x, Q) = E[u(m + x\bar{V} + (x + \delta Q)\varepsilon)]$. Then from risk aversion and from the fact that traders wealth is bounded, we have that $U(\delta n, -n) > U(\delta n, -(n+1))$ and $U(\delta(n+1), -n) < U(\delta(n+1), -(n+1))$. Thus, from the continuity of U in x there exists $x^*(n) \in (\delta n, \delta(n+1))$ such that if $x = x^*(n)$, then the trader is indifferent between trading $-n$ lots or $-(n+1)$ lots.

In order to see that when $x = x^*(n)$ both these quantities are optimal, note that if the trader could trade a continuum of quantities, then he would trade exactly $-\frac{x}{\delta}$. The trader is however constrained to trade integer multiples of δ . Taking advantage from the concavity in Q of $U(x, Q)$, the constrained optimal tradable quantities are the closest to $-\frac{x}{\delta}$ that is $-\delta n$ and $-\delta(n+1)$. Finally if ε is symmetrically distributed, then ε and $-\varepsilon$ are identically distributed and so $U(\delta(n+1/2), -n) = U(\delta(n+1/2), -(n+1))$ that means $x^*(n) = \delta(n+1/2)$. The proof for the case $\pi_t = 0$ is symmetric.

Proof of Proposition 1: Note first that as quotes must satisfy equation (1) and the informativeness of an order is bounded by the precision of a traders' order, we have that in equilibrium, at any date t , $v(\pi_t^l) \leq P_\delta(Q) \leq v(\pi_t^h)$ for all $Q \in \mathbb{Z}$. Thus, for any price schedule $P_\delta(Q)$ satisfying this property and for a trader with portfolio (x, m) we have:

$$\begin{aligned} E [u(m + (x + \delta Q)\mathbf{v} - \delta Q v(\pi_t^l)) | s] &\leq E [u(m + (x + \delta Q)\mathbf{v} - \delta Q P_\delta(Q)) | s] \leq \\ &\leq E [u(m + (x + \delta Q)\mathbf{v} - \delta Q v(\pi_t^h)) | s] \end{aligned} \quad (5)$$

for Q positive, and

$$\begin{aligned} E [u(m + (x + \delta Q)\mathbf{v} - \delta Q v(\pi_t^l)) | s] &\leq E [u(m + (x + \delta Q)\mathbf{v} - \delta Q P_\delta(Q)) | s] \leq \\ &\leq E [u(m + (x + \delta Q)\mathbf{v} - \delta Q v(\pi_t^h)) | s] \end{aligned} \quad (6)$$

for Q negative.

Note that π_t^s is continuous in π_t and that u is a continuous function. Moreover, when π_t is close to 1 or to 0, an informative signal affects slightly the informed trader belief, indeed $\pi_t^l < \pi < \pi_t^h$ and $\lim_{\pi_t \rightarrow 1} (\pi_t^h - \pi_t^l) = \lim_{\pi_t \rightarrow 0} (\pi_t^h - \pi_t^l) = 0$. Thus, we have that:

$$\begin{aligned} \lim_{\pi_t \rightarrow 1} E \left[u \left(m + (x + \delta Q) \mathbf{v} - \delta Q v(\pi_t^h) \right) | s \right] &= \lim_{\pi_t \rightarrow 1} E \left[u \left(m + (x + \delta Q) \mathbf{v} - \delta Q v(\pi_t^l) \right) \right] \\ &= E \left[u \left(m + x \bar{V} + (x + \delta Q) \varepsilon \right) \right]. \end{aligned} \quad (7)$$

From Lemma 1 we know that if $x \neq x^*(n)$ for all $n \in \mathbb{Z}$, then there exist a unique \widehat{Q} , such that for all $Q \neq \widehat{Q}$, we have

$$E \left[u \left(m + x \bar{V} + (x + \delta \widehat{Q}) \varepsilon \right) \right] > E \left[u \left(m + x \bar{V} + (x + \delta Q) \varepsilon \right) \right].$$

Thus, from expression (7), it must be that for π_t sufficiently close to 1 and for all $Q \neq \widehat{Q}$

$$E \left[u \left(m + (x + \delta \widehat{Q}) \mathbf{v} - \delta \widehat{Q} v(\pi_t^h) \right) | s \right] > E \left[u \left(m + (x + \delta Q) \mathbf{v} - \delta Q v(\pi_t^l) \right) | s \right], \quad (8)$$

$$E \left[u \left(m + (x + \delta \widehat{Q}) \mathbf{v} - \delta \widehat{Q} v(\pi_t^l) \right) | s \right] > E \left[u \left(m + (x + \delta Q) \mathbf{v} - \delta Q v(\pi_t^h) \right) | s \right]. \quad (9)$$

Now take an informed trader whose inventory x is bounded away from $x^*(n)$ for all $n \in \mathbb{Z}$, and suppose he expects a price schedule $P_\delta(Q)$. His maximization problem will be:

$$\arg \max_{Q \in \mathbb{Z}} E \left[u \left(m + (x + \delta Q) \mathbf{v} - \delta Q P_\delta(Q) \right) | s \right].$$

Then expressions (5), (6), (8) and (9) imply

$$E \left[u \left(m + (x + \delta \widehat{Q}) \mathbf{v} - \delta \widehat{Q} P_\delta(\widehat{Q}) \right) | s \right] > E \left[u \left(m + (x + \delta Q) \mathbf{v} - \delta Q P_\delta(Q) \right) | s \right]$$

for all $Q \neq \widehat{Q}$. That is if π_t is sufficiently close to 1, this trader will trade a quantity \widehat{Q} no matter he received a bearish or a bullish signal. Therefore his action will provide no information on \mathbf{V} . To conclude the proof it is

sufficient to observe that because of the hypothesis on F , for all $n \in \mathbb{Z}$ there is no trader whose inventory is not bounded away from $x^*(n)$. Thus, traders' demand is not informative, and so, from equation (1), the price schedule must be $P_\delta(Q) = E[\mathbf{v}|H_t]$ for all $Q \in \mathbb{Z}$. In other words, when π_t is sufficiently close to 1, there exist no equilibrium where the traders orders are informative. In order to prove that a not informative equilibrium exist, it is sufficient to observe that for π_t close to 1, $Q^*(x, m, v(\pi_t), \pi_t, s) = \widehat{Q}$ for all s . An identical argument applies for π_t sufficiently close to 0.

Proof of Corollary 1: Simply remark that from Lemma 1, when $\delta = 1$ we have $x^*(n) \in (n, n + 1)$. That means that when π_t reaches extreme levels, the only traders whose orders are informative are traders that hold fractions of the asset.²⁸ However, all traders in the economy hold only integer amounts of the asset, $x \in \mathbb{Z}$, and thus, from Proposition 1, eventually trade will stop providing information on \mathbf{V} .

Proof of Corollary 2: From Lemma 1 and Proposition 1 we know that the only traders whose orders are informative even when π_t is arbitrarily close to 0 or to 1 are those whose inventory is equal to $x^*(n)$ for some $n \in \mathbb{Z}$. Moreover, if ε is symmetrically distributed we know that $x^*(n) = \delta(n + \frac{1}{2})$. Therefore δ should be chosen such that there exist a positive probability of observing these traders, thus the inequality (2).

Proof of Lemma 2: From Proposition 1 we know that when all traders' orders are not informative, in equilibrium $P(Q) = v(\pi_t)$ for all Q . From Corollary 1 we deduce that when π_t is sufficiently close to 1 or to 0 and $P(Q) = v(\pi_t)$, then a trader with inventory $x \in \mathbb{Z}$ will trade exactly $-x$ no matter he received a bullish or a bearish signal. As the expression $E[u(m + (x + q)\mathbf{v} - v(\pi_t)q)|\mathbf{s}]$ is a strictly concave function in the traded quantity $q \in \mathbb{R}$, then it will have a unique maximum. Thus in order to find $\bar{\pi}$ (resp. $\underline{\pi}$), it is sufficient to find the minimum $\pi > 1/2$ (resp. maximum $\pi < 1/2$) such that the trader prefers to trade $-x$ rather than $-x - 1$ or $-x + 1$ for both $s = h$ and $s = l$. That is to say

$$u(m+v(\pi)x) > \max\{E[u(m + \mathbf{v} + (x - 1)v(\pi)) |s], E[u(m - \mathbf{v} + (x + 1)v(\pi)) |s]\} \quad (10)$$

²⁸Moreover, as the traders wealth is bounded, we know that $x^*(n)$ is bounded away from n or $n + 1$.

for $s = h$ and $s = l$. Considering that $u(W) = -e^{-\gamma w}$ and that $\varepsilon \hookrightarrow N(0, \sigma_\varepsilon)$, we have that expression (10) is satisfied only if both inequalities in Lemma 2 are met.

Approximation of the long term pricing error: Proposition 1 shows that, at the equilibrium, the long term price $\tilde{P} \equiv \lim_{t \rightarrow \infty} E[\mathbf{v}|H_t]$ will be either close to $v(\underline{\pi})$ or to $v(\bar{\pi})$. Assuming the initial public belief is $\frac{1}{2}$, the ex ante expectation of the long term pricing error can be approximated by

$$\begin{aligned} E[LTPE] &\simeq \frac{1}{2}(\bar{V} - v(\bar{\pi}))Pr(\tilde{P} \geq v(\bar{\pi})|V = \bar{V}) + \\ &\quad + \frac{1}{2}(\bar{V} - v(\underline{\pi}))Pr(\tilde{P} \leq v(\underline{\pi})|V = \bar{V}) + \\ &\quad + \frac{1}{2} \left((v(\bar{\pi}) - \underline{V})Pr(\tilde{P} \geq v(\bar{\pi})|V = \underline{V}) + (v(\underline{\pi}) - \underline{V})Pr(\tilde{P} \leq v(\underline{\pi})|V = \underline{V}) \right). \end{aligned}$$

Indeed the probability that the long term price being close to $v(\bar{\pi})$ is equal to the probability of the public belief reaches the level $\bar{\pi}$. Moreover, from the symmetry of the model we deduce that given that $(\tilde{P} \geq v(\bar{\pi})|V = \underline{V}) = Pr(\tilde{P} \leq v(\underline{\pi})|V = \bar{V})$. Based on the first passage time approach developed in Diamond and Verrechia (1987) we obtain a proxy for the long term pricing error:

Claim: *Assuming the initial public belief is $\frac{1}{2}$, a proxy for the expectation of the long term pricing error is given by*

$$2\underline{\pi} \bar{\pi} (\bar{V} - \underline{V}).$$

We prove this Claim as follows: Let consider the likelihood ratio $L_t = \frac{\pi_t}{1-\pi_t}$. We have $L_t = \prod_{i=1}^t r(Q_i, \pi_i)$ where $r(Q_i, \pi_i) = \frac{Pr(Q_i|V=\bar{V}, H_i)}{Pr(Q_i|V=\underline{V}, H_i)}$. The ratio $r(Q_i, \pi_i)$ is interpreted as the informative content of trader i 's order of size Q_i . As we do not know the equilibrium strategies when $\pi_t \in (\underline{\pi}, \bar{\pi})$, we approximate the information content of a trade assuming that as long as $\pi_t \in (\underline{\pi}, \bar{\pi})$ market makers can perfectly infer the traders' signals from their orders. Thus, as long as $\pi_t \in (\underline{\pi}, \bar{\pi})$, the ratio $r(Q_t, \pi_t)$ takes the value $\frac{p}{1-p}$ when the trader's order Q_t reveals a bullish signal and the value $\frac{1-p}{p}$ when the order Q_t reveals a bearish signal. Consequently, under this approximation, $r(Q_t, \pi_t)$ does not depend on π_t . Now, consider $N = \inf\{t \text{ s.t. } L_t \notin (\frac{\underline{\pi}}{1-\underline{\pi}}, \frac{\bar{\pi}}{1-\bar{\pi}})\}$,

the random variable representing the number of periods before bound $\frac{\pi}{1-\pi}$ or bound $\frac{\bar{\pi}}{1-\bar{\pi}}$ is passed. The probability that the public belief π_t reaches the threshold $\bar{\pi}$ given that the fundamental value of the asset is \bar{V} is equal to $Pr(L_N \geq \frac{\bar{\pi}}{1-\bar{\pi}} | V = \bar{V})$. Using the same techniques²⁹ than Diamond and Verrechia (1987), we obtain $Pr(L_N \geq \frac{\bar{\pi}}{1-\bar{\pi}} | V = \underline{V}) = \frac{(1-2\bar{\pi})(1-\bar{\pi})}{\bar{\pi}-\underline{\pi}}$, and $Pr(L_N \leq \frac{\pi}{1-\pi} | V = \bar{V}) = \frac{(2\bar{\pi}-1)\underline{\pi}}{\bar{\pi}-\underline{\pi}}$ from which we deduce our result.

Proof of Proposition 2: The proof is similar to the proof of Proposition 1. First we show that the price schedule has an upper and a lower bound that converge to $E[\mathbf{V}]$ as public information becomes sufficiently precise. Then we show that, whenever trading prices are sufficiently close to $E[\mathbf{V}]$ and public information is sufficiently precise, all risk averse traders will optimally choose to trade $Q^* = -x$ no matter the signal they received. Thus, the informational content of the order flow vanishes and prices will not converge to fundamental.

We say that the public ex-ante belief is sufficiently precise at time t if $Var(\mathbf{V}|H_t)$ is positive but sufficiently close to 0. Remark that preciseness of ex-ante public belief has nothing to do with the fact that agents' belief are actually correct. Indeed, very precise public belief can turn out to be completely wrong.

Now, for any finite history H_t it is always possible to find two signals in Σ that ,with an abuse of notation, we will denote l and h , such that

$$E[\mathbf{V}|H_t, \mathbf{s} = l] \leq E[\mathbf{V}|H_t, \mathbf{s} = s] \leq E[\mathbf{V}|H_t, \mathbf{s} = h]$$

for all $s \in \Sigma$. Note that, from standard property of the conditional variance we have $Var(\mathbf{V}|H_t) = E[Var(\mathbf{V}|H_t, \mathbf{s})] + Var(E[\mathbf{V}|H_t, \mathbf{s}])$. Thus, if $Var(\mathbf{V}|H_t)$ lies in a small neighborhood of 0 then it is also the case for $Var(\mathbf{V}|H_t, \mathbf{s})$. Moreover, as $g(s|V) > 0$ for all $V \in \Omega$ and $s \in \Sigma$, we have that $E[\mathbf{V}|H_t, \mathbf{s} = h] - E[\mathbf{V}|H_t, \mathbf{s} = l]$ is close to 0 when $Var(\mathbf{V}|H_t)$ is sufficiently close to 0. In other words, if the public belief is sufficiently precise, then private signals affect private beliefs just slightly.

Now, consider that market makers cannot infer from a trader's order more than what the trader know, and that a trader's inventory does not provide

²⁹Precisely we use theorem 9.34 page 549 of Schervish (1995). In a related vein we can deduce from the Wald's lemma page 552 of Schervish (1995) that our proxy underestimates the average long term pricing error.

information about the expectation of \mathbf{V} . Thus, from equation (1) we have that at time t the price schedule satisfies

$$E[\mathbf{V}|H_t, \mathbf{s} = l] \leq P_\delta(Q) \leq E[\mathbf{V}|H_t, \mathbf{s} = h], \quad \forall Q \in \mathbb{Z}.$$

Let u_i be a trader i 's utility function that is continuous strictly increasing and concave. Let denote $v_t^l = E[\mathbf{V}|H_t, \mathbf{s} = l]$, and $v_t^h = E[\mathbf{V}|H_t, \mathbf{s} = h]$. Then equations (5) and (6) still hold after substituting u_i, v_t^h, v_t^l to $u, v(\pi_t^h), v(\pi_t^l)$ respectively. Moreover, because of risk aversion $E[u_i(m + xV + (x + \delta Q)\varepsilon)]$ is maximized for $Q^* = -x$. If $Var(V|H_t)$ is sufficiently small, then equations (8) and (9) will still hold after substituting u_i, v_t^h, v_t^l and $-x$ to $u, v(\pi_t^h), v(\pi_t^l)$ and \hat{Q} respectively. Thus for $Var(V|H_t)$ sufficiently close to 0:

$$\arg \max_{Q \in \mathbb{Z}} E[u_i(m + (x + \delta Q)\mathbf{v} - \delta Q P \delta(Q)) | s] = -x \quad \forall s \in \Sigma.$$

That means that when public belief are sufficiently precise, traders' order only reflect their inventory concerns and provides no information about \mathbf{V} .

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Table 1: Long term pricing error for different $\sigma_{\mathbf{v}}/\sigma$

$\sigma_{\mathbf{v}}/\sigma$	minimum <i>LTPE</i>		maximum <i>LTPE</i>		$E[LTPE]$		
	value in \$	% of $E[\mathbf{v}]$	value in \$	% of $E[\mathbf{v}]$	value in \$	% of $E[\mathbf{v}]$	
0.42	1.75	4.99%	4.13	11.61%	2.45	7.01%	33
0.5	1.16	3.31%	5.84	16.93%	1.93	5.52%	3
0.6	0.89	2.56%	7.51	21.44%	1.60	4.57%	30
0.7	0.69	1.97%	9.11	26.03%	1.28	3.67%	30
0.8	0.47	1.39%	10.71	30.61%	0.93	2.66%	29
0.9	0.26	0.74%	12.34	35.26%	0.51	1.45%	29
0.99	0.01	0.04%	13.85	29.56%	0.03	0.07%	28

Comparison of the long term pricing errors for different levels of $\sigma_{\mathbf{v}}/\sigma$. The parameter set is $E[v] = 35\$, \sigma = 7\$, \gamma = 0.08, p = 0.8$. The minimum *LTPE* is defined as $\underline{\pi}(\bar{V} - \underline{V})$, the maximum *LTPE* is defined as $\bar{\pi}(\bar{V} - \underline{V})$ and the expected *LTPE* is defined as $2\bar{\pi}\underline{\pi}(\bar{V} - \underline{V})$. All these measures are given both in absolute value and in percentage of the ex-ante expected value of the asset $E[v]$.

Table 2: Long term pricing error for different γ

γ	minimum <i>LTPE</i>		maximum <i>LTPE</i>		$E[LTPE]$	
	value in \$	% of $E[\mathbf{v}]$	value in \$	% of $E[\mathbf{v}]$	value in \$	% of $E[\mathbf{v}]$
0.01	0.06	0.16%	8.34	23.84%	0.11	0.32%
0.02	0.12	0.35%	8.28	23.65%	0.24	0.69%
0.04	0.29	0.84%	8.11	23.14%	0.57	1.62%
0.06	0.53	1.53%	7.86	22.45%	1.00	2.87%
0.08	0.89	2.56%	7.51	21.44%	1.60	4.57%
0.1	1.50	4.28%	6.90	19.72%	2.46	7.03%
0.11	2.05	5.84%	6.35	18.15%	3.10	8.85%

Comparison of the long term pricing errors for different levels of traders' risk aversion γ . The parameter set is $E[\mathbf{v}] = 35\$$, $\sigma = 7\$$, $\sigma_{\mathbf{v}}/\sigma = 0.6$, $p = 0.8$, $\underline{V} = 30.8\$$, $\overline{V} = 39.2\$$. The minimum *LTPE* is defined as $\underline{\pi}(\overline{V} - \underline{V})$, the maximum *LTPE* is defined as $\overline{\pi}(\overline{V} - \underline{V})$ and the expected *LTPE* is defined as $2\overline{\pi}\underline{\pi}(\overline{V} - \underline{V})$. All these measures are given both in absolute value and in percentage of the ex-ante expected value of the asset $E[v]$.

Table 3: Long term pricing after a stock split

new shares	minimum $LTPE$		maximum $LTPE$		$E[LTPE]$	
	value in \$	% of $E[\mathbf{v}]$	value in \$	% of $E[\mathbf{v}]$	value in \$	% of $E[\mathbf{v}]$
1	1.50	4.28%	6.90	19.72%	2.46	7.03%
2	0.40	1.15%	8.00	22.85%	0.77	2.20%
4	0.16	0.46%	8.24	23.54%	0.32	0.90%
6	0.10	0.29%	8.30	23.71%	0.20	0.56%
8	0.07	0.20%	8.33	23.79%	0.14	0.41%
10	0.05	0.16%	8.34	23.84%	0.11	0.32%

Comparison of the long term pricing errors after splitting the stock in to 2 to 10 new shares. The parameter set is $E[\mathbf{v}] = 35\$, \sigma = 7\$, \sigma_{\mathbf{v}}/\sigma = 0.6, p = 0.8, \gamma = 0.1, \underline{V} = 30.8\$, \overline{V} = 39.2\$. The minimum $LTPE$ is defined as $\underline{\pi}(\overline{V} - \underline{V})$, the maximum $LTPE$ is defined as $\overline{\pi}(\overline{V} - \underline{V})$ and the expected $LTPE$ is defined as $2\overline{\pi}\underline{\pi}(\overline{V} - \underline{V})$. All these measures are given both in absolute value and in percentage of the ex-ante expected value of the asset $E[v]$.$

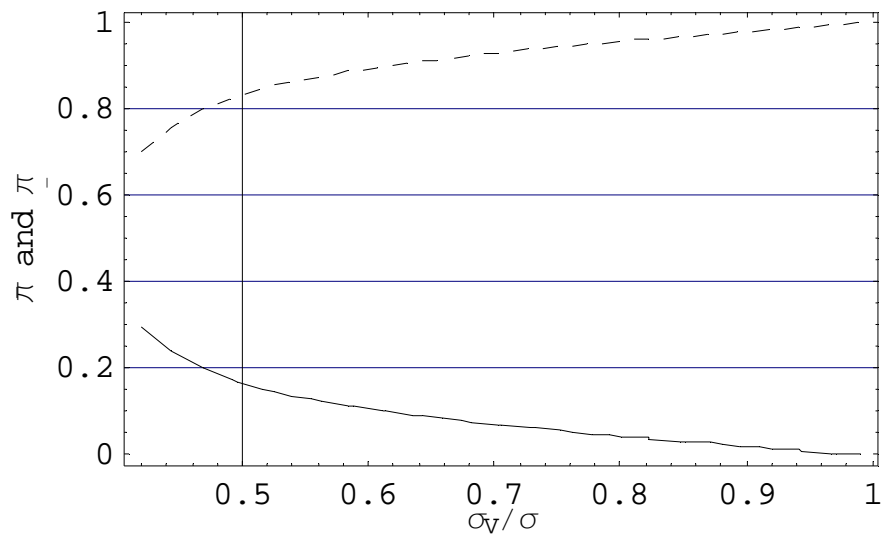


Figure 1: This figure shows $\underline{\pi}$ (—) and $\bar{\pi}$ (- - -) as function of the proportion of fundamental volatility that can be explained with the aggregation of private information. The set of parameters is $E[v] = 35\text{\$}$, $\sigma = 7\text{\$}$, $\gamma = 0.08$, $p = 0.8$.

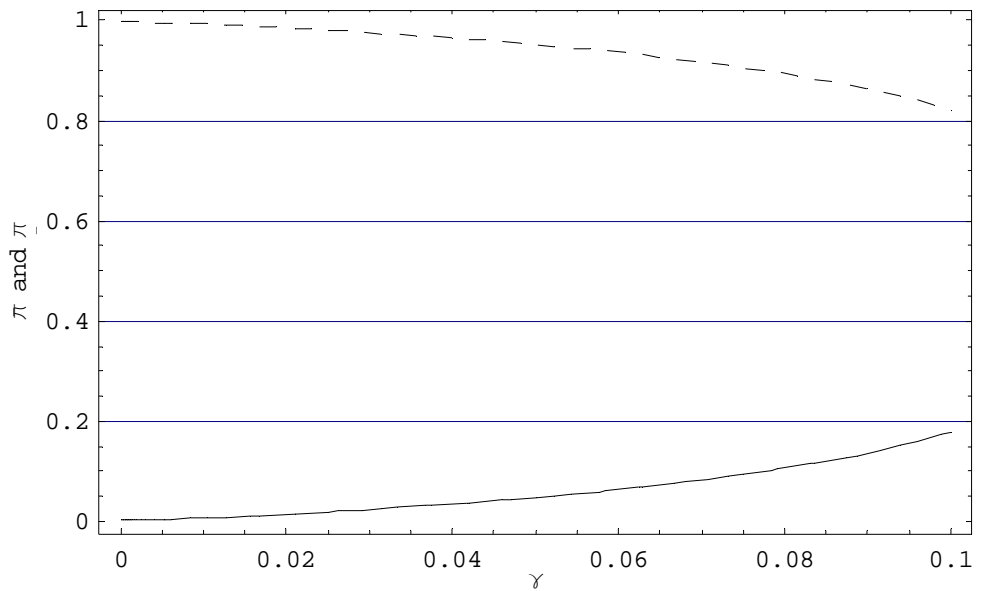


Figure 2: This figure shows $\underline{\pi}$ (—) and $\bar{\pi}$ (- - -) as function of the traders' risk aversion coefficient γ . The set of parameters is $E[\mathbf{v}] = 35\text{\$}$, $\sigma = 7\text{\$}$, $\sigma_{\mathbf{V}}/\sigma = 0.6$, $p = 0.8$, $\underline{V} = 30.8\text{\$}$, $\bar{V} = 39.2\text{\$}$.

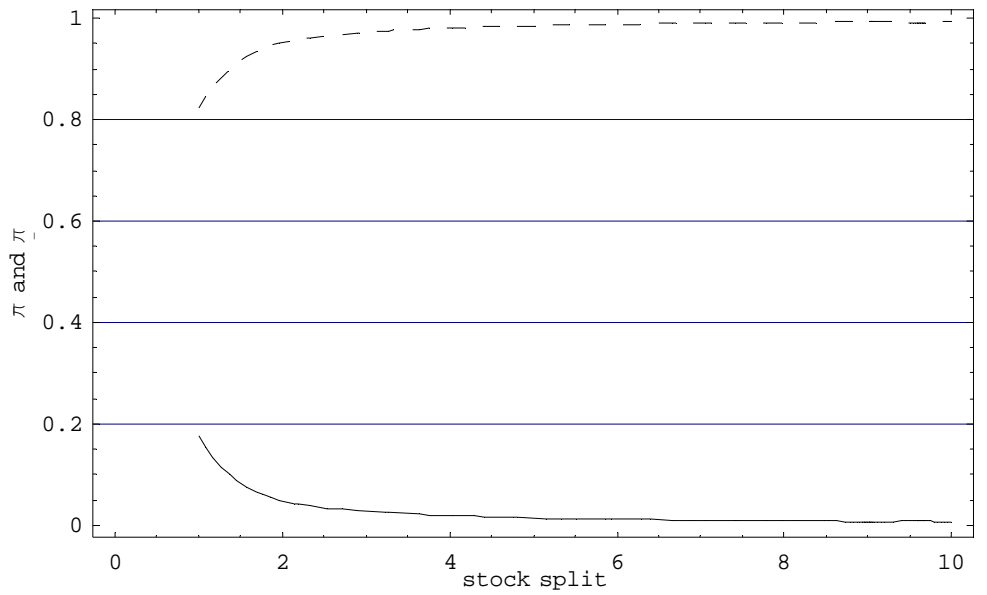


Figure 3: This figure shows $\underline{\pi}$ (—) and $\bar{\pi}$ (- - -) as function of the number of new share obtained from one old share after the stock split. The set of parameters is $E[\mathbf{v}] = 35\$, \sigma = 7\$, \sigma_{\mathbf{v}}/\sigma = 0.6, p = 0.8, \gamma = 0.1, \underline{V} = 30.8\$, \bar{V} = 39.2\$$.