Stopping Games - Recent Results

Eilon Solan[*] and Nicolas Vieille[†]

May 13, 2001

Abstract

We survey recent results on the existence of the value in zero-sum stopping games with discrete and continuous time, and on the existence of ε-equilibria in non zero-sum games with discrete time.

[*]Department of Managerial Economics and Decision Sciences, Kellogg Graduate School of Management, Northwestern University, *and* the School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. e-mail: eilons@post.tau.ac.il

[†]Département Finance et Economie, HEC, 1, rue de la Libération, 78 Jouy-en-Josas, France. e-mail: vieille@hec.fr

1 Introduction

Stopping games have been introduced by Dynkin [4] as a generalization of optimal stopping problems, and later used in several models in economics and management science, such as optimal equipment replacement, job search, consumer purchase behavior, research and development (see Mamer [11] and the references therein), and the analysis of strategic exit (see Ghemawat and Nalebuff [6] or Li [10]).

The basic setting is as follows. The game is defined by two processes a and b, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, endowed with a filtration $\mathbf{F}$. Two players are allowed to stop at any time. The payoff to player 1 is given by one of the two processes depending on who stopped first. Formally, the two players choose stopping times σ and τ and player 1 receives from player 2 the amount

$$\mathbf{E}\left[a_\sigma \mathbf{1}_{\sigma<\tau} + b_\tau \mathbf{1}_{\tau\leq\sigma, \tau<+\infty}\right].$$

Much work has been devoted to the study of this zero-sum game, both in a discrete time and in a continuous time framework. In discrete time, Dynkin [4] proved the existence of the value under an assumption that at any stage only one of the two players is allowed to stop, and Neveu [13] proved the existence of the value under the assumption $a \leq b$. After those seminal contributions, most of the literature focused on continuous-time games, in the context of the general theory of stochastic processes. Bismut [1] proved the existence of the value under the assumption $a \leq b$ and an assumption known as Mokobodsky's hypothesis (in addition, several regularity assumptions are needed). The latter assumption was later removed, see e.g. Lepeltier and Maingueneau [9]. Some authors also worked in the diffusion case, see e.g. Cvitanić and Karatzas [3]. Finally, most work involves a symmetrized payoff function

$$\gamma(\sigma, \tau) = \mathbf{E}\left[a_\sigma \mathbf{1}_{\sigma<\tau} + b_\tau \mathbf{1}_{\tau<\sigma} + c_\sigma \mathbf{1}_{\sigma=\tau<+\infty}\right],$$

where c is a third given process, under the assumption $a \leq c \leq b$. This list of references is by no means exhaustive.

Comparatively few studies deal with non-zero-sum (two-player) stopping games. For such games, a, b, c are $\mathbf{R}^2$-valued processes, and the i-th coordinate is the payoff to player i, see e.g. Mamer [11], Morimoto [12], Hideo [7], and Ohtsubo [14],[15].

When the players are restricted to stopping times, the value needs not exist in general, even if the processes are nonrandom and constant. For instance, for the stopping game in discrete time defined by $a_n = b_n = 1$ and $c_n = 0$, one has

$$\sup_\sigma \inf_\tau \gamma(\sigma, \tau) = 0 \text{ while } \inf_\tau \sup_\sigma \gamma(\sigma, \tau) = 1.$$

The purpose of this paper is to survey recent work on stopping games that aim at obtaining the existence of the value under no order conditions on the processes a, b and c, by suitably convexifying the set of strategies of the players.

The paper is organized as follows. Section 2 contains a brief discussion of the appropriate convexification. Sections 3 and 4 present results on zero-sum games, respectively for discrete time and continuous time models. In both cases, the proof is sketched in the simple case of deterministic payoff functions. Finally, Section 5 discusses a result on two-player non-zero-sum stoping games with deterministic payoff functions.

2 Randomized stopping times

This section contains a brief discussion of the proper way of convexifying the set of stopping times. A more extensive treatment can be found in Rosenberg et *al.* [16], or Touzi and Vieille [22]. We follow the logic of behavior strategies and enlarge the set of strategies by allowing a player to stop, at any stage, with positive probability.

In discrete time, this leads to the following notion. A strategy (of player 1) is a $\mathbf{F}$-adapted process $\mathbf{x} = (x_n)$ with values in $[0, 1]$. x_n is to be interpreted as the probability that player 1 stops the game at stage n, conditional on the game being still alive at that stage. In computing the payoff induced by a pair of strategies $(\mathbf{x}, \mathbf{y})$, one assumes that the randomizations performed by the players in the various stages are mutually independent, and independent from the payoff processes. Thus, a strategy $\mathbf{x}$ that corresponds to the stopping time σ is

$$x_n = \begin{cases} 0 \text{ on } \sigma > n \\ 1 \text{ on } \sigma \leq n \end{cases}$$

In continuous time, this leads to the following notion. A strategy (of player 1) is a right-continuous, non-decreasing adapted process (F_t) with $F_t \in [0, 1]$ for each t. Here, F_t may be interpreted as the probability that player 1 will have stopped before time t (including t). Thus, the strategy that corresponds to a stopping time σ is the process (F_t) defined as $F_t = \mathbf{1}_{\sigma \leq t}$. The payoff associated with the two strategies (F_t) and (G_t) can be written as

$$\gamma(F, G) = \mathbf{E}\left[\int_{[0,\infty)} a(1-G)dF + \int_{[0,\infty)} b(1-F)dG + \sum_{0\leq t<\infty} c_t \Delta F_t \Delta G_t\right],$$

where $\Delta F_t = F_t - F_{t-}$ is the jump of F at time t.

The alternative standard way of convexifying the set of stopping times is to follow the logic of mixed strategies and to define a strategy as, loosely speaking, a probability distribution over stopping times. The equivalence between mixed strategies and behavior strategies holds under fairly general assumptions, see Kuhn [8]. For a discussion specific to the case of stopping games and to the above notions, see Touzi and Vieille [22].

3 Zero-Sum Games in Discrete Time

We here deal with zero-sum stopping games in discrete time. We first describe the setup and the result in a precise way, then give an overview of the proof.

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $(\mathcal{F}_n)$ be a filtration over $(\Omega, \mathcal{A}, \mathbf{P})$ (the information available at stage n). Let $(a_n), (b_n), (c_n)$ be adapted processes, defined over $(\Omega, \mathcal{A}, \mathbf{P})$. We assume

$$\sup_n |a_n|, \sup_n |b_n|, \sup_n |c_n| \in L^1(\mathbf{P}). \tag{1}$$

By properly enlarging the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, one can assume w.l.o.g. that it supports a double sequence $(X_n, Y_n)_{n=0}^{\infty}$ of *iid* variables, uniformly distributed over $[0,1]$, such that, for each n: (i) (X_n, Y_n) is independent of the process $(a_k, b_k, c_k)_k$; (ii) (X_n, Y_n) is $\mathcal{F}_{n+1}$-measurable, and independent of $\mathcal{F}_n$. X_n and Y_n are used by the players in their randomizations.

Define the stopping game as follows. A *strategy* for player 1 (*resp.* player 2) is a $[0,1]$-valued adapted process $\mathbf{x} = (x_n)$ (*resp.* $\mathbf{y} = (y_n)$). Given strategies $(\mathbf{x}, \mathbf{y})$, define the stopping stages of players 1 and 2 by $\theta_1 = \inf\{n \geq 0, X_n \leq x_n\}$, $\theta_2 = \inf\{n \geq 0, Y_n \leq y_n\}$, and set

$$\theta = \min(\theta_1, \theta_2). \tag{2}$$

Thus, θ is the stage at which the game stops.

We set

$$r(\mathbf{x}, \mathbf{y}) = a_{\theta_1} 1_{\theta_1 < \theta_2} + b_{\theta_2} 1_{\theta_2 < \theta_1} + c_{\theta_1} 1_{\theta_1 = \theta_2 < +\infty}.$$

The payoff of the game is $\gamma(\mathbf{x}, \mathbf{y}) = \mathbf{E}(r(\mathbf{x}, \mathbf{y}))$.

The game has a value $v \in \mathbf{R}$ if

$$v = \sup_{\mathbf{x}} \inf_{\mathbf{y}} \gamma(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{y}} \sup_{\mathbf{x}} \gamma(\mathbf{x}, \mathbf{y}).$$

It is convenient to introduce discounted payoffs, as in Yasuda [23]. For $\lambda \in (0,1]$, the λ-discounted evaluation is given by

$$\gamma_\lambda(\mathbf{x}, \mathbf{y}) = \lambda \mathbf{E}\left[(1-\lambda)^{\theta+1} r(\mathbf{x}, \mathbf{y})\right]. \tag{3}$$

The game has a λ-discounted value $v_\lambda \in \mathbf{R}$ if

$$v_\lambda = \sup_{\mathbf{x}} \inf_{\mathbf{y}} \gamma_\lambda(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{y}} \sup_{\mathbf{x}} \gamma_\lambda(\mathbf{x}, \mathbf{y}).$$

Yasuda [23] proved the existence of the discounted value v_λ, by adapting Shapley's [17] argument.

Theorem 1 *(Rosenberg, Solan and Vieille, [16]) Every stopping game such that (1) holds has a value v. Moreover, $v = \lim_{\lambda \to 0} v_\lambda$.*

We sketch below the proof in the deterministic case; that is, the payoff at each stage n depends only on n. Thus, a_n, b_n, c_n are real numbers. The proof for the general case builds upon the ideas described below.

We denote by G_n the game that starts at stage n. G_n is similar to the original game, but players are restricted to use strategies that stop before stage n with probability 0. In particular, G_0 coincides with G.

We denote by $v_n(\lambda)$ the λ-discounted value of G_n, for each $n \in \mathbf{N}$ and $\lambda \in (0, 1]$. Define $v_n(0) = \limsup_{\lambda \to 0} v_n(\lambda)$. We shall prove that $\sup_{\mathbf{x}} \inf_{\mathbf{y}} \gamma(\mathbf{x}, \mathbf{y}) \geq v_0(0)$. By exchanging the roles of the two players, one immediately deduces

$$\inf_{\mathbf{y}} \sup_{\mathbf{x}} \gamma(\mathbf{x}, \mathbf{y}) \leq \liminf_{\lambda \to 0} v_0(\lambda),$$

which implies both conclusions of the Theorem, since $\sup_{\mathbf{x}} \inf_{\mathbf{y}} \gamma(\mathbf{x}, \mathbf{y}) \leq \inf_{\mathbf{y}} \sup_{\mathbf{x}} \gamma(\mathbf{x}, \mathbf{y})$.

As usual, $v_n(\lambda)$ satisfies a recursive equation (dynamic programming principle), that here takes the form

$$v_n(\lambda) = (1-\lambda) \sup_{x \in [0,1]} \inf_{y \in [0,1]} \{(1-x)(1-y)\, v_{n+1}(\lambda) + x(1-y)\, a_n + y(1-x)\, b_n + xy\, c_n\} \tag{4}$$

By possibly taking a subsequence (λ_p) that converges to zero, we may assume that $v_n(0) = \lim_{\lambda \to 0} v_n(\lambda)$. We let $x_n(\lambda)$ achieve the supremum in (4), and let $x_n(0)$ denote any limit point of $(x_n(\lambda))$ as λ goes to zero. Thus, one has for each $y \in [0, 1]$ and every $\lambda \geq 0$,

$$\begin{aligned}(1-\lambda)(x_n(\lambda) y\, v_{n+1}(\lambda) \quad + \quad & x_n(\lambda)(1-y)\, a_n + (1-y) x_n(\lambda)\, b_n \\ & +(1 - x_n(\lambda))(1-y)\, c_n) \geq v_n(\lambda). \end{aligned} \tag{5}$$

This can be rephrased by introducing the process $(\widetilde{v}_n(\lambda))_n$ defined, for $\lambda \geq 0$, by

$$\widetilde{v}_n(\lambda) = \begin{cases} v_n(\lambda) & n \leq \theta \\ r_\theta & n > \theta, \end{cases}$$

where r_θ is equal to a_θ, b_θ or c_θ depending on whether $\theta_1 < \theta_2$, $\theta_1 > \theta_2$ or $\theta_1 = \theta_2 < +\infty$. Note that $\widetilde{v}_n(\lambda)$ depends on $(\mathbf{x}, \mathbf{y})$, through the value of θ. Though $(v_n(\lambda))$ is a sequence of numbers, $(\widetilde{v}_n(\lambda))$ is a process, the randomness being caused by the random choices of the players. Whenever useful, we shall write $\widetilde{v}_n^{\mathbf{x},\mathbf{y}}(\lambda)$ to emphasize which strategies are used.

Inequality (5) can be rephrased as follows: for every choice of strategy $\mathbf{y}$, and for each $\lambda \geq 0$, the process $((1-\lambda)^{\min(n,\theta)}\widetilde{v}_n(\lambda))$ is a submartingale, provided that player 1 uses the strategy $\mathbf{x}_\lambda := (x_n(\lambda))$.

In particular, the process $(\widetilde{v}_n(0))$ is a submartingale under the pair of strategies $(\mathbf{x}_0, \mathbf{0})$, where $\mathbf{0}$ is the strategy of player 2 that never stops. Thus, it converges, $\mathbf{P}$-a.s., to some random variable $\widetilde{v}_\infty$.

We now split the discussion in two parts. Assume first that, under the pair $(\mathbf{x}_0, \mathbf{0})$, θ is $\mathbf{P}$-a.s. finite. In that case, θ is also $\mathbf{P}$-a.s. finite for $(\mathbf{x}_0, \mathbf{y})$, whatever be $\mathbf{y}$. Thus, for each $\mathbf{y}$, the limit $\widetilde{v}_\infty$ coincides with r_θ. The submartingale property of $\widetilde{v}_n(0)$ then implies that $\gamma(\mathbf{x}_0, \mathbf{y}) = \mathbf{E}\left[\widetilde{v}_\infty^{\mathbf{x}_0,\mathbf{y}}\right] \geq v_0(0)$. Since $\mathbf{y}$ is arbitrary, $\inf_{\mathbf{y}} \gamma(\mathbf{x}_0, \mathbf{y}) \geq v_0(0)$.

Assume now that $\theta = +\infty$ with positive probability, and let $\varepsilon > 0$ be given. On the event $\{\theta = +\infty\}$, $\widetilde{v}_n(0) = v_n(0)$ for each $n \in \mathbf{N}$. Therefore, the sequence of numbers $v_n(0)$ is convergent, say to v_∞. If $v_\infty \leq 2\varepsilon$,

$$\gamma(\mathbf{x}_0, \mathbf{y}) = \mathbf{E}\left[r_\theta \mathbf{1}_{\theta<+\infty}\right] \geq \mathbf{E}\left[\widetilde{v}_\infty^{\mathbf{x}_0,\mathbf{y}}\right] - \varepsilon \geq v_0(0) - 2\varepsilon,$$

hence $\inf_{\mathbf{y}} \gamma(\mathbf{x}_0, \mathbf{y}) \geq v_0(0) - 2\varepsilon$.

The tricky case is when $v_\infty > 2\varepsilon$. We choose $N \in \mathbf{N}$ such that $|v_n(0) - v_\infty| \leq \varepsilon$ for each $n \geq N$. We define a strategy of player 1 as follows. We first construct two sequences (λ_p, s_p) (possibly of finite length) by the following recursive device:

- Choose $\lambda_1 \in (0,1]$ such that $v_N(\lambda_1) > v_N(0) - \varepsilon^2$; set

$$s_1 = \inf\left\{n \geq N, v_n(\lambda_1) \leq \varepsilon^2\right\}.$$

- For $p \geq 1$, choose $\lambda_{p+1} \in (0,1]$ such that $v_{s_p}(\lambda_{p+1}) \geq v_{s_p}(0) - \varepsilon^2$ and set $s_{p+1} = \inf\{n \geq s_p, v_n(\lambda_{p+1}) \leq \varepsilon^2\}$.

We let $\overline{\mathbf{x}}$ be the strategy that coincides with $\mathbf{x}_0$ up to stage $s_0 = N$, and with $\mathbf{x}_{\lambda_{p+1}}$ from stage s_p up to stage s_{p+1}. Consider what may happen between the two stages s_p and s_{p+1}, assuming that the game was not stopped before stage s_p. Whatever be the strategy $\mathbf{y}$ used by player 2, the process $(1-\lambda_{p+1})^{\min(n,\theta)}\widetilde{v}_n(\lambda_{p+1})$ is a submartingale between s_p and s_{p+1}. Thus, $\widetilde{v}_n(\lambda_{p+1})$ increases on average by a factor of $\frac{1}{1-\lambda_{p+1}}$ from one stage to the following, prior to $\min(\theta, s_{p+1})$. Since $v_n(\lambda_{p+1}) \geq \varepsilon - \varepsilon^2$ for $n \leq \min(\theta, s_{p+1})$ and all quantities are bounded, it must be that $\min(\theta, s_{p+1})$ is finite.

Since $v_{s_{p+1}}(\lambda_{p+1}) \leq \varepsilon^2$, the probability that $\theta < s_{p+1}$ (conditioned on $\theta \geq s_p$) is bounded away from zero, and the payoff to player 1, conditioned on $\theta \leq s_{p+1}$, is at least $v_{s_p}(\lambda_{p+1}) \geq v_\infty - \varepsilon - \varepsilon^2$.

These observations imply that $\theta < +\infty$ $\mathbf{P}$-a.s. under $(\overline{\mathbf{x}}, \mathbf{y})$, for each $\mathbf{y}$, and $\gamma(\overline{\mathbf{x}}, \mathbf{y}) \geq v_\infty - 2\varepsilon \geq v_N - 2\varepsilon$. Since $\widetilde{v}_n(0)$ is a submartingale under $(\mathbf{x}_0, \mathbf{y})$, this implies that $\gamma(\overline{\mathbf{x}}, \mathbf{y}) \geq v_0(0) - 2\varepsilon$.

4 Zero-Sum Games in Continuous Time

We deal here with zero-sum stopping games in continuous time and with finite horizon. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, and $T > 0$ a fixed terminal time. Let $a = \{a_t,\ 0 \leq t \leq T\}$, $b = \{b_t,\ 0 \leq t \leq T\}$ and $c = \{c_t,\ 0 \leq t \leq T\}$ be real-valued processes, satisfying the integrability condition

$$E\left[\sup_t |a_t| + \sup_t |b_t| + \sup_t |c_t|\right] < +\infty\ , \tag{6}$$

and we denote by $\mathbf{F}$ the $\mathbf{P}$-augmentation of the filtration generated by the processes a, b and c.

We denote by $\mathcal{V}^+$ the set of adapted, right-continuous, non-decreasing processes, with values in $[0,1]$. For $F, G \in \mathcal{V}^+$, we set

$$\gamma(F,G) = \mathbf{E}\left[\int_{[0,\infty)} a(1-G)dF + \int_{[0,\infty)} b(1-F)dG + \sum_{0 \leq t < \infty} c_t \Delta F_t \Delta G_t\right].$$

Theorem 2 *(Touzi-Vieille [22]) Assume that: (i) (a,b,c) satisfy the integrability condition (6), (ii) the game has a finite horizon T, (iii) a and b are (càdlàg) semimartingales with trajectories continuous at T, (iv) $c \leq b$. Then*

$$\sup_{F \in \mathcal{V}^+} \inf_{G \in \mathcal{V}^+} \gamma(F,G) = \inf_{G \in \mathcal{V}^+} \sup_{F \in \mathcal{V}^+} \gamma(F,G),$$

i.e., the stopping game has a value .

The basic idea is to apply Sion [19] minmax Theorem to the payoff function $\gamma : \mathcal{V}^+ \times \mathcal{V}^+ \to R$. Define

$$\mathcal{S} = \left\{(F_t), \mathbf{E}\left[\int_0^T F_t^2 dt\right] < +\infty\right\}.$$

The set $\mathcal{S}$ is a Hilbert space when endowed with the scalar product $\mathbf{E}\left[\int_0^T F_t G_t dt\right]$, and $\mathcal{V}^+$ is a subset of $\mathcal{S}$, compact for the weak topology $\sigma(\mathcal{S},\mathcal{S})$. However, Sion Theorem does not apply directly since the payoff function γ does not have enough continuity properties.

This difficulty is circumvented by applying Sion Theorem to restricted strategy spaces. Define

$$\mathcal{V}_1 = \left\{(F_t) \in \mathcal{V}^+,\ (F_t) \text{ has continuous trajectories, } \mathbf{P}\text{-a.s.}\right\}$$

and

$$\mathcal{V}_2 = \left\{(G_t) \in \mathcal{V}^+,\ G_T = 1 \text{ on } \{b_T < 0 < a_T\} \text{ and } \Delta G_T = 0 \text{ on } \{b_T > 0\}\ \right\}.$$

and let $\overline{\mathcal{S}} = \left\{(F_t), \mathbf{E}\left[\int_0^T F_t^2 dt + F_T^2\right] < +\infty\right\}$. The space $\overline{\mathcal{S}}$ is a Hilbert space when endowed with the scalar product $\mathbf{E}\left[\int_0^T F_t G_t dt + F_T G_T\right]$. One can check that $\mathcal{V}_2$ is compact for the weak topology $\sigma(\overline{\mathcal{S}},\overline{\mathcal{S}})$. Moreover, γ is separately continuous on $\mathcal{V}_1 \times \mathcal{V}_2$ for the strong topology. Hence, by Sion's Theorem

$$\sup_{\mathcal{V}_1}\inf_{\mathcal{V}_2}\gamma(F,G) = \inf_{\mathcal{V}_2}\sup_{\mathcal{V}_1}\gamma(F,G).$$

The restriction on player 1's strategies is imposed in order to have continuity of γ. The restriction on player 2's strategies is imposed in such a way that restricting the strategy spaces to $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively entail no loss for the players:

$$\sup_{\mathcal{V}_1}\inf_{\mathcal{V}_2}\gamma(F,G) = \sup_{\mathcal{V}_1}\inf_{\mathcal{V}^+}\gamma(F,G), \text{ and } \inf_{\mathcal{V}_2}\sup_{\mathcal{V}_1}\gamma(F,G) = \inf_{\mathcal{V}_2}\sup_{\mathcal{V}^+}\gamma(F,G). \tag{7}$$

Let us discuss these two equalities in the non-random case. Thus, a, b and c are right-continuous functions defined on $[0,T]$, and continuous at T. The proof in the general case is obtained by elaborating upon the ideas that follow.

The first equality is immediate: let $(F_t) \in \mathcal{V}_1$, and $(G_t) \in \mathcal{V}^+$ be given. Thus, $F : [0,T] \to [0,1]$ is a continuous, non-decreasing function, while $G : [0,T] \to [0,1]$ is a right-continuous, non-decreasing function. Let $\widetilde{G} : [0,T] \to [0,1]$ be the function that agrees with G on $[0,T)$, and where $\widetilde{G}_T$ is equal to G_{T-} or to 1 depending whether $b_T > 0$ or $b_T \leq 0$. Plainly, $\widetilde{G}$ belongs to $\mathcal{V}_2$. On the other hand,

$$\gamma(F,\widetilde{G}) - \gamma(F,G) = \begin{cases} (1-F_T)b_T(1-G_T) \text{ if } Y_T \leq 0 \\ -(1-F_T)b_T\Delta G_T \text{ if } Y_T > 0 \end{cases}$$

In both cases, it is non-positive, which establishes the first equality.

As for the second equality in (7), let $G \in \mathcal{V}_2$ and $F \in \mathcal{V}^+$ be given. By using an argument similar to the one of the previous paragraph, we may assume that F is continuous at T in case $a_T > 0, b_T \geq 0$. Let (F^n) be a sequence of continuous non-decreasing functions such that $F^n_t \to F_{t-}$ for each $t \in [0,T]$. Using the assumption $c \leq b$, one can check that $\limsup \gamma(F^n, G) \geq \gamma(F, G)$, which yields the second equality.

5 Non Zero-Sum Games in Discrete Time

We conclude by presenting a result on two-player non zero-sum stopping games in discrete time. We show how a simple application of Ramsey Theorem, combined with a result of Flesch et al [5], imply the existence of an ε-equilibrium in the *deterministic* case.

We let here $(a_n), (b_n)$ and (c_n) be three bounded sequences in $\mathbf{R}^2$, and let ρ be a uniform bound on the payoffs. The setup of Section 3 then reduces to the following. A strategy of player 1 is a sequence $\mathbf{x} = (x_n)$ in $[0,1]$, where x_n is the probability that player 1 will choose to stop in stage n, if the game was not stopped before.

The payoff of the game is defined to be

$$\gamma(\mathbf{x}, \mathbf{y}) = \mathbf{E}\left[r(\mathbf{x}, \mathbf{y})\right],$$

as in Section 3, except that here $\gamma(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^2$.

Theorem 3 *(Shmaya, Solan and Vieille, [18]) For each $\varepsilon > 0$, the stopping game has an ε-equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$; that is, a pair of strategies that satisfies:*

$$\gamma^1(\mathbf{x}, \mathbf{y}^*) \leq \gamma^1(\mathbf{x}^*, \mathbf{y}^*) + \varepsilon \text{ and } \gamma^2(\mathbf{x}^*, \mathbf{y}) \leq \gamma^2(\mathbf{x}^*, \mathbf{y}^*) + \varepsilon \text{ for each } \mathbf{x}, \mathbf{y}.$$

Note that if payoffs are not uniformly bounded, an ε-equilibrium needs not exist. (as a counter example, take the game defined by $a^i_n = b^i_n = n-1, c^i_n = n-2$ for every $i = 1, 2$, and every $n \in \mathbf{N}$). Moreover, even when payoffs are bounded a 0-equilibrium needs not exist (e.g. $a^i_n = b^i_n = n/(n+1)$, $c^i_n = (n-1)/n$ for every $i = 1, 2$, and every $n \in \mathbf{N}$).

Fix $\varepsilon > 0$ once and for all, and an ε-discretization Z of the set $[-\rho, \rho]^2$; that is, Z is a finite set such that for every $u \in [-\rho, \rho]^2$ there is $z \in Z$ with $\|z - u\| < \varepsilon$.

For every two positive integers $k < l$ we define a *periodic* stopping game $G(k, l)$ as follows:

$$a_n(k,l) = a_{k+(n \bmod l)},\ b_n(k,l) = b_{k+(n \bmod l)} \text{ and } c_n(k,l) = c_{k+(n \bmod l)}$$

This is "the game that starts at stage k and restarts at stage l." We denote by $\gamma_{k,l}(\mathbf{x},\mathbf{y})$ the payoff function in the game $G(k,l)$.

The game $G(k,l)$ may be analyzed as a particular stochastic game $\Gamma(k,l)$ with absorbing states. To see this, assume for example $k=0$ and $l=2$, and consider the stochastic game $\Gamma(0,2)$ described by the matrix

	b_0 *	b_1 *
a_0 *	c_0 *	a_0 *
a_1 *	b_0 *	c_1 *

In this game player 1 chooses a row and player 2 chooses a column. The first line corresponds to the pure strategy *never stop*, the second and third ones to the pure strategies *stop in stage 0* and *stop in stage 1*. The columns are to be interpreted symmetrically for player 2. The meaning of an asterisked entry is that the game moves to an absorbing state (i.e., ends) as soon as such an entry is played.

Thus, each stage of $\Gamma(0,2)$ corresponds to two stages (a period) of $G(0,2)$.

Using Flesch et al. [5], the game $\Gamma(k,l)$ has a stationary ε-equilibrium, or equivalently, for each $\varepsilon>0$, the game $G(k,l)$ has a periodic ε-equilibrium $(\mathbf{x}(k,l),\mathbf{y}(k,l))$, with period $l-k$.

For each $k<l$, we choose $z(k,l)\in Z$ such that

$$\|\gamma(\mathbf{x}(k,l),\mathbf{y}(k,l))-z(k,l)\|<\varepsilon.$$

For every pair of non-negative integers we attached an element in Z – a color. By Ramsey Theorem (see, e.g., Bollobás [2]) there is an infinite set $K\subseteq\mathbf{N}\cup\{0\}$ and $z\in Z$ such that $z(k,l)=z$ for every $k,l\in K$, $k<l$.

In particular, there exists an increasing sequence of non-negative integers $k_1<k_2<\cdots$ such that for every $j\in\mathbf{N}$, $z(k_j,k_{j+1})=z$.

We define a profile $(\mathbf{x},\mathbf{y})$ from stage k_1 on by concatenating the profiles $(\mathbf{x}(k_i,k_{i+1}),\mathbf{y}(k_i,k_{i+1}))$: $(\mathbf{x},\mathbf{y})$ coincides with $(\mathbf{x}(k_i,k_{i+1}),\mathbf{y}(k_i,k_{i+1}))$ from stage k_i up to stage $k_{i+1}-1$. To complete the construction before stage k_1, we recall that every finite-stage game has an equilibrium. The profile $(\mathbf{x},\mathbf{y})$ coincides between stages 0 and k_1-1 with an equilibrium in the k_1-stage game, whose payoffs are $(a_n,b_n,c_n)_{n<k_1}$ if the play is stopped prior to stage k_1, and is z otherwise. There is no difficulty in proving that $(\mathbf{x},\mathbf{y})$ is an ε-equilibrium of the stopping game, starting from stage k_1.[1] Moreover, the expected payoff, conditioned that the game was not stopped before stage k_1, is, up to ε, z. It follows that $(\mathbf{x},\mathbf{y})$ is an ε-equilibrium of the stopping game.

[1]This is true, provided the periodic profiles in $\Gamma(k,l)$ are chosen to satisfy an additionnal property: the probability of absorption in each period is bounded away from 0. We do not elaborate here on this point.

Extensions to more than two players are limited. We actually proved that if every periodic deterministic game admits an ε-equilibrium, then every non-periodic deterministic game admits an ε-equilibrium. Using the technique of Solan [20] instead of that of Flesch et al [5], one can prove that every *three*-player periodic deterministic game admits an ε-equilibrium (though it needs not be periodic). One can then prove that every three-player deterministic stopping game admits an ε-equilibrium.

Unfortunately, it is currently not known whether every n-player periodic deterministic stopping game admits an ε-equilibrium, for $n \geq 4$. For more details, see Solan and Vieille [21].

To generalize this proof to general two-player stopping games, one needs to generalize Ramsey Theorem to a stochastic setup, and to show that a concatenation of ε-equilibria in periodic non deterministic games yields an ε'-equilibrium, for some $\varepsilon' > 0$ that goes to 0 as ε goes to 0. Whereas Ramsey Theorem can be generalized to a stochastic setup, it is not clear yet how to achieve the second goal.

References

[1] J.M. Bismut. Sur un problème de Dynkin. *Z. Warsch. V. Geb.*, 39:31–53, 1977.

[2] B. Bollobás. Modern graph theory. Springer, 1998.

[3] J. Cvitanić and I. Karatzas. Backward stochastic differential equations with reflection and Dynkin games. *Ann. Prob.*, 24:2024–2056, 1996.

[4] E.B. Dynkin. Game variant of a problem on optimal stopping. *Soviet Math. Dokl.*, 10:270–274, 1969.

[5] J. Flesch, F. Thuijsman, and O.J. Vrieze. Recursive repeated games with absorbing states. *Math. Op. Res.*, 21:1016–1022, 1996.

[6] P. Ghemawat and B. Nalebuff. Exit. *RAND J. Econ.*, 16:184–194, 1985.

[7] N. Hideo. Non-zero-sum stopping games of symmetric Markov processes. *Prob. Th. Rel. Fields*, 75:487–497, 1987.

[8] H.W. Kuhn. Extensive games and the problem of information. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games*. Annals of Mathematics Study 28, Princeton University Press, 1953.

[9] J.P. Lepeltier and M.A. Mainguenau. Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodsky. *Stochastics*, 13:25–44, 1984.

[10] L. Li. Equilibrium exit in stochastically declining industries. *Games and Economic Behavior*, 1:40–59, 1989.

[11] J.W. Mamer. Monotone stopping games. *J. Appl. Prob.*, 24:386–401, 1987.

[12] H. Morimoto. Non zero-sum discrete parameter stochastic games with stopping times. *Prob. Th. Rel. Fields.*, 72:155–160, 1986.

[13] J. Neveu. *Discrete-Parameter Martingales*. North-Holland, Amsterdam, 1975.

[14] Y. Ohtsubo. A non zero-sum extension of Dynkin's stopping problem. *Math. Op. Res.*, 12:277–296, 1987.

[15] Y. Ohtsubo. On a discrete-time non-zero-sum Dynkin problem with monotonicity. *J. Appl. Prob.*, 28:466–472, 1991.

[16] D. Rosenberg, E. Solan, and N. Vieille. Stopping games with randomized strategies. *Prob. Th. Rel. Fields.*, 119:433–451, 2001.

[17] L.S. Shapley. Stochastic games. *Proceedings of the National Academy of Sciences of the U.S.A.*, 39:1095–1100, 1953.

[18] E. Shmaya, E. Solan, and N. Vieille. An application of Ramsey theorem to stopping games. preprint, 2001.

[19] M. Sion. On general minimax theorems. *Pacific J. of Math.*, 8:171–176, 1958.

[20] E. Solan. Three-player absorbing games. *Math. Op. Res.*, 24:669–698, 1999.

[21] E. Solan and N. Vieille. Quitting games. *Math. Op. Res.*, to appear, 2000.

[22] N. Touzi and N. Vieille. Continuous-time Dynkin games with mixed strategies. *SIAM J. Cont. Opt.*, 2000, to appear.

[23] Yasuda. On a randomized strategy in Neveu's stopping problem. *Stoc. Proc. App.*, 21:159–166, 1987.