An additively separable representation in the Savage framework*

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Abstract

This paper elicits an additively separable representation of preferences in the Savage framework (where the objects of choice are acts: measurable functions from an infinite set of states to a potentially finite set of consequences). A preference relation over acts is represented by the integral over the subset of the product of the state space and the consequence space which corresponds to the act, where this integral is calculated with respect to a "state-dependent utility" measure on this space. The result applies at the stage prior to the separation of probabilities and utilities, and requires neither Savage's P3 (monotonicity) nor his P4 (likelihood ordering). It may thus prove useful for the development of state-dependent utility representation theorems in the Savage framework.

Keywords: Expected utility; additive representation; state-dependent utility; monotonicity.

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1 Introduction

Among the many challenges to the theory of decision under uncertainty proposed by Savage (1954) are worries regarding the state-independence of utilities. Karni (1985); Drèze (1987); Karni and Mongin (2000) have argued that in many applications, and indeed as regards several methodological issues, such as the measurement of subjective probabilities, state-dependent utilities should be used; accordingly, representation theorems have been proposed to elicit (suitably unique) probabilities and state-dependent utilities from preferences.

Simply put, one might think that the elicitation of state-dependent utilities would operate in two stages: firstly, elicit a representation of preferences by a single real-valued function of both states and consequences; then use some mechanism to separate the probability and utility factors of this function – that is, to write it as a product of a (unique) probability function and a (suitably unique) state-dependent utility function. However, so far, such a technique has had differing degrees of success and popularity depending on whether one uses Savage's original framework or the framework proposed by Anscombe and Aumann (1963).

The latter framework employs a (possibly) finite state space with a rich consequence space: traditionally, the consequence space is the set of lotteries over a finite set of outcomes. The set of acts (functions from states to consequences) taking values in such a consequence space is a mixture set:[1] it follows that the von Neumann Morgenstern theorem can be directly applied to obtain an additive representation of preferences by a single function of states and consequences (Fishburn, 1970, p146). This function is sometimes called a "state-dependent utility" (Kreps, 1988, p108),[2] although it is not to be confused with the state-dependent utility function the state-dependent utility theorists aim to elicit (Karni et al., 1983; Karni, 1985; Drèze, 1987): the latter is obtained after appropriate decomposition of the former into the probability and utility factors. To avoid confusion, in this paper, the following terminology will be adopted: the utility function which, along with a (suitably unique) probability function, represents the preferences, will be called the *state-dependent utility*, whereas the function of state–consequence pairs which, taken alone, represents the preference relation will be called *"state-dependent utility"* (with scare quotes); finally, this latter function will be said to yield a *additively separable representation* (here we follow the terminology suggested by Kreps (1988, p108)). The representation theorem in Anscombe and Aumann (1963) operates by adding a monotonicity axiom, allowing one to decompose the "state-dependent utility" into a probability and state-independent utility function. Several representation theorems for state-dependent utility begin with the additively separable representation obtained by the application of the von-Neumann Morgenstern theorem, and employ a different mechanism from Anscombe and Aumann (1963) to separate the "state-dependent utility" into a probability and a state-dependent utility (Karni et al., 1983; Karni and Mongin, 2000).[3] It is thus clear that the Aumann &

[1]Throughout this discussion, the "Reversal of Order" axiom shall be assumed.

[2]Karni et al. (1983, p1022) call this function "NM utility" because of the use of the von Neumann Morgenstern result in its elicitation. The use of this terminology in the Savage framework might however cause confusion.

[3]Drèze (1987) also begins from the application of the von-Neumann Morgenstern theorem, but weakens

Anscombe supports the sort of strategy described above.

By contrast, such a technique is not readily available in the Savage framework, because an additively separable representation of the sort provided by the application of the von-Neumann Morgenstern result is much less simple to obtain. In the Savage (1954) framework, there is a rich (infinite) set of states and a poor (possibly finite) set of consequences. The technique used to elicit probabilities and state-independent utilities proceeds firstly by eliciting probabilities, *relying crucially* on the axioms ensuring state-independence of utilities, and only then eliciting utilities, by the application of the von-Neumann Morgenstern result. There is thus no intermediate stage of the process at which one has constructed a representation of the preferences by a single function of states and consequences (a "state-dependent utility"), but not separated this function into probability and utility components. Accordingly, the main state-dependent utility results which are formulated in the Savage framework do not pass through an intermediate stage where an additively separable representation has been obtained but not yet decomposed; Karni and Schmeidler (1993) and Karni (1993) are examples of such a results.

This fact may be related to a further interesting difference between the two frameworks: namely that, whereas in the Aumann & Anscombe framework there is one axiom for state-independence[4] – Monotonicity – in the Savage framework there are two – P3 and P4. As Karni (1993) has argued, P3, whose statement is similar to the Monotonicity axiom of Anscombe & Aumann, guarantees ordinal state independence, whereas P4 is required for cardinal state independence.[5] Indeed, a significant number of state-dependent theories developed in the Savage framework only weaken P4 without touching P3: Karni and Schmeidler (1993) is an example. So they provide representations of preferences by state-dependent utilities, relying on the supposition that the preferences are ordinally state independent.

The goal of this paper is to provide an equivalent in the Savage framework to the additively separable representation of preferences that is obtained in the Anscombe & Aumann framework by the application of the von Neumann-Morgenstern result. The paper proposes a set of axioms on a preference relation such that, if they are satisfied, there is a suitably unique function of states and consequences representing the preference relation (a "state-dependent utility"). Intuitively, one might expect the relevant axioms to be more or less the same as the Savage axioms except that the state-independence axioms P3 and P4 are excluded; indeed, as suggested above, the interest of such a result would lie in its use as a first stage in state-dependent utility results which do not assume P3 and P4 but employ other means for separating the probability from the utility. In fact, neither P3 (monotonicity) nor P4 shall figure among the axioms proposed here. Moreover, apart from some technical modifications relating to the Savage's P6, the axioms will correspond rather directly to those proposed in Savage (1954).

Apart from the technical difficulty of showing that the sort of additively separa-

the Reversal of Order axiom, rather than just Monotonicity.

[4] See footnote 1.

[5] In the Anscombe and Aumann (1963) setting, because of the linear structure on the consequence space, the difference between ordinal and cardinal state independence collapses. See Karni (1993); Wakker and Zank (1999).

ble representation obtained in the Anscombe & Aumann framework also holds in the Savage framework, there is a conceptual difficulty: namely, that of finding the correct analogy to the Anscombe & Aumann representation. At the appropriate stage of the Anscombe & Aumann elicitation, one obtains a "state-dependent utility" function from state–consequence pairs to the real numbers such that the preference relation is represented by the sum of the values of this function on the state–consequence pairs realised by the acts.[6] However, as Wakker and Zank (1999) note, in the Savage case, where the state space is infinite, the difficulty is that of finding an equivalent to this sum. The solution to the problem adopted here is as follows: take the "state-dependent utility" to be a measure on the product of the state space (which is a measure space in the Savage framework) and the consequence space (since no measure structure is assumed on the set of consequences, this is taken to be the measure space generated by the singleton sets of consequences). As in the Anscombe & Aumann case, the "state-dependent utility" is a function from state–consequence pairs to the real numbers; the difference, given that we are now working in the infinite case, is that the measure structure needs to be respected. Finally, note that acts are measurable subsets of the product space: in particular, an act is the subset of state–consequence pairs it realises. So the expected utility of the act is just the integral of the "state-dependent utility" measure over this set. The result in this paper (Theorem 1 in Section 2) will yield measure of this sort, in such a way that it has appropriate uniqueness properties.

In Section 2, the technical notions shall be introduced and the theorem shall be stated. Section 3 will contain a discussion of the result and comparison with relevant literature. The proofs are to be found in the Appendix.

2 Axioms and theorem

Let S be a set of states, with a σ-algebra of events $\mathcal{F}_S$, and let C be a set of consequences. Note that there is a naturally defined σ-algebra in the algebra generated by $S \times C$: namely the product of the σ-algebra of events $\mathcal{F}_S$ with the "discrete" σ-algebra on C, containing all singletons of C. Let this σ-algebra on $S \times C$ be $\mathcal{F}_{SC}$. Let $\mathcal{A}$ be the set of measurable functions from S to C: they are called the acts. Each act may also be thought of as a subset of $S \times C$: namely the set $\{(s, f(s)) |\ s \in S\}$. This subset shall also be called f. Since an act is a measurable function, the subset of $S \times C$ is also measurable (it belongs to $\mathcal{F}_{SC}$). Finally, let $\preceq$ be a binary relation on $\mathcal{A}$; $\prec$, $\sim$, $\succ$ and $\succeq$ are defined from $\preceq$ in the usual way.

Notation. Let $\mathcal{A}_p$ be the set of partial (measurable) functions[7] from S to C. Write f_A for the partial function which is defined on $A \subseteq S$ and agrees with f on this domain. Note, there exist f, f', A with $f_A = f'_A$. As for acts, to each partial function f_A there corresponds a subset of $S \times C$, which shall also be denoted f_A.

For A and B disjoint, $f_A g_B$ is the partial function taking the values of f on A and the values of g on B.

[6]An act realises a state–consequence pair if and only if it is a function taking the state in the pair to the consequence in the pair.

[7]Henceforth all functions, partial functions and sets of states shall be assumed to be measurable.

Given an event A, A^c is the set $S \setminus A$; since it is measurable, it is also an event.

As is standard, $\preceq_A$ will denote the order $\preceq$ on acts, given the event A. This can also be thought of as an order on the partial functions with domain A: indeed, it is the natural order derived from $\preceq$. $\preceq$ will be extended to partial acts with common domains in this way: for partial acts f_A and g_A defined on an event A, $f_A \preceq g_A$ iff, for any act e, $f_A e_{A^c} \preceq_A g_A e_{A^c}$. Axiom A2 below assures that this extension is not trivial.

Finally, the traditional notion of null event shall be employed: an event A is *null* iff, for any pair of acts $f, g \in \mathcal{A}$, $f \sim_A g$.

We assume three of the basic axioms of Savagean decision theory.

Axiom A1 (Weak order)**.** $\preceq$ is a weak order: (a) For all f, g in $\mathcal{A}$, $f \preceq g$ or $g \preceq f$. (b) For all f, g and h in $\mathcal{A}$, if $f \preceq g$ and $g \preceq h$, then $f \preceq h$.

Axiom A2 (Sure-thing principle)**.** For any acts f, g, h, h' in $\mathcal{A}$ and any non-null event A, $f_A h_{A^c} \preceq g_A h_{A^c}$ iff $f_A h'_{A^c} \preceq g_A h'_{A^c}$

Axiom A3 (Non-triviality)**.** There are acts f, g in $\mathcal{A}$ such that $f \not\preceq g$.

Following Abdellaoui and Wakker (2005) and Gilboa (1987), we shall factorise Savage's P6 axiom into two elements: solvability (which differs slightly from the solvability axioms proposed in the articles mentioned, in so far as they suppose monotonicity and thus at least ordinal state independence, and this is not to be assumed here) and the standard Archimedean axiom.

Axiom A4 (Solvability)**.** For acts f, g, h in $\mathcal{A}$, $f \prec g \prec h$, there exists an event $A \subseteq S$ such that $f_A h_{A^c} \sim g$.

Axiom A5 (Archimedean)**.** There is no infinite sequence of disjoint non-null events E_i such that there exists a pair of acts f and g with $f \prec_{E_i} g$ for all i.

Remark 1. The solvability axiom (in tandem with non-triviality) implies that all atoms are null.[8] To see why, consider an non-null atom a and two consequences such that $c_1 \prec_a c_2$: any two acts identical except on a, and taking values c_1 and c_2 on a respectively, form a counterexample to Solvability. It follows that the state space is non-atomic, for if not, the preference relation would be trivial. Since all atoms are null, they will be treated under the consideration of null events in subsequent argument; henceforth, no special consideration shall be given to any atoms in the state space.

Theorem 1. *For S, C, $\preceq$ satisfying (A1-A5), there exists a measure U on $(S \times C, \mathcal{F}_{SC})$ such that, for every $f, g \in \mathcal{A}$,*

$$f \preceq g \text{ iff } \int_f dU \leqslant \int_g dU \tag{1}$$

Furthemore, let U' be any other measure satisfying this equation. Then there exists $a > 0$ and a measurable function $b : S \to \Re$ such that $U' = aU + b$.

[8] An atom is an event such that, for every sub-event, either it or its complement is null.

3 Discussion

In order the bring out the relevant points regarding the result in the previous section, it is perhaps instructive to compare it with another result in the literature which deals with essentially the same problem; namely, that proposed by Wakker and Zank (1999). That paper differs from the current one firstly in its presentation of the result, and secondly, and more importantly, in the content of the result (axioms and techniques used). Let us first of all consider the differences in presentation, before turning to the axioms involved and techniques used.

Wakker and Zank (1999) are interested in the stage prior to the separation of probability and utility, or as they put it, to the "identifiability of probability". They are thus dealing with the same question as has been posed above; the reader is referred to that paper for an extended discussion of the importance of the sort of result obtained in Wakker and Zank (1999) and here. The result obtained in their paper is presented in two forms (Theorems 11 and 12). Firstly, they elicit a functional from the set of acts to the reals which represents the preference relation and which, amongst its other properties, is additive: for any finite partition of the state space, there is a set of functionals, one for each element of the partition, which sum to give the original functional. The second form of the result, which, as they emphasise, does not add any empirical content to the first, is in the form of a measure on the state space and a state-dependent utility function such that the preference relation is represented by the integral of the state-dependent utility function over the measure. Although the state-dependent utility function involved in the representation does resemble those discussed by Karni et al. (1983); Karni and Mongin (2000), insofar as it represents the preference relation in tandem with another function (a probability function, or in the Wakker and Zank (1999) case, a countably additive measure on the state space), by contrast to the cases discussed by those theorists, there is no unique separation of probabilities and utilities here (that was not the goal), and so the function can be altered given appropriate modifications of the measure.

The result presented in this paper takes a slightly different form. It proposes a representation of the preference relation by a single ("state-dependent utility") measure on the product of the state space and the consequence space. Since an act is a measurable function, it corresponds to a measurable subset of the product space; namely, the set of state–consequence pairs realised by the act. The expected utility of the act is just integral of the measure over this set, just as, in the Anscombe & Aumann case, it was the sum over the state-consequence pairs realised by the act.

The difference in formulation between this paper and Wakker and Zank (1999) is of no deep significance. In particular, it is possible to retrieve one formulation from the others. For the case of the additive functional, the "state-dependent utility" measure is naturally considered as an additive functional, with the functionals for elements of a partition just being the integrals with respect to the measure over these partitions. For the case of the state-dependent utility and the measure on the state space, it is simple to decompose the "state-dependent utility" measure on the product of the state space and the consequence space into a measure component on the state space and a set of state-dependent utility functions, one for each state: this decomposition is rather arbitrary, hence the weak uniqueness properties of the measure and state-dependent utilities obtained. On the other hand, it should also be possible, by reversing these steps, to retrieve the "state-dependent utility" measure from additive functionals or measure-state-dependent-utility pairs.

It should however be noted that the measure proposed here is a measure on the entire product space, and thus will measure subsets of that space which do not correspond to acts or partial acts; for example it will measure the union of two event–consequence pairs, even when the events are not disjoint and the consequences are not equal. This "extension" is generally unproblematic, because it is entirely determined by the expected utilities of acts or partial acts.[9]

Beyond the differences in presentation, there is a difference in the axioms proposed and the techniques used in this paper and in that of Wakker and Zank (1999).

The major difference in the axioms lies in the use of a monotonicity axiom (closely related to Savage's P3): Wakker and Zank (1999) require such an axiom (precisely, they require "strict monotonicity"), whereas no such axiom is demanded here. As noted in the Introduction, the avoidance of this axiom is attractive given the role that this result is intended to play in state-dependent utility results; nevertheless, as Wakker and Zank point out, this axiom is widely assumed in economics.

Another central difference is the structure of the consequence space. Wakker and Zank (1999) consider only the case where the consequence space is the set of real numbers (in the appendix, they extend the result to connected topological spaces), whereas no particular structure is demanded on the consequence space in this paper. In particular, the result proved here applies to finite, as well as infinite, consequence spaces. This is closer to Savage's original theory, where the structural burden is borne by the state space, allowing very weak constraints on the consequence space. Once again, the use of the real numbers as a consequence space in economics is widespread, although there are certainly cases where one would like to be able to relieve oneself of this assumption.

The difference in consequence spaces is related to a third difference between the two papers: namely, the axioms used to play the role of Savage's P6. Wakker and Zank (1999) use a selection of fairly weak but closely related conditions on the preference relation, which differ between the different theorems. Two of these ("simple continuity" and "supnorm continuity") are defined with reference to the topological structure on the consequence space, and thus cannot be defined in the framework treated in this paper. The third ("pointwise continuity") does not make reference to this structure, but its use in the proof essentially consists in showing that it implies one of the other notions of continuity (Lemma 20), and so it is not clear to what extent it can be applied in the framework assumed here.

Instead of applying continuity constraints of this sort, the current paper borrows a factorisation of Savage's P6 from Abdellaoui and Wakker (2005) and Gilboa (1987). There are two axioms: solvability, which does not quite correspond to the solvability in Abdellaoui and Wakker (2005) and Gilboa (1987) (their definitions rely on monotonicity), and the Archimedean axiom. The latter is as standard; the former is a richness axiom stating that, for any triplet of acts with strict preferences between them, one can obtain an act indifferent from the middle act by "mixing" the most and least preferred

[9] Under the Savage axioms, and most notably the Sure-Thing Principle, an expected utility on acts naturally generates a unique measure on the product space.

acts. This axiom is rather strong, and has several important consequences. Firstly (Remark 1, Section 2), it implies that the state space is effectively atomless, in so far as any atoms are null (Wakker and Zank assume atomless state spaces). Secondly, it drives not only the elicitation of the expected utility function, but also assures that it is a countably-additive measure (this is a role played by the continuity assumptions in Wakker and Zank (1999)).

Given these differences, it is no surprise that the techniques used in the two papers are, at least at first glance, rather different. In a word, the Wakker and Zank (1999) results work by applying the Debreu (1960) result for finite state spaces to partitions of an infinite space and showing that the utility functions obtained from Debreu's theorem can be "stuck" together. On the other hand, the result here operates by showing that an appropriate set of equivalence classes of partial acts is an Archimedean, regular, positive, ordered local semigroup in the sense of Krantz et al. (1971), and then applying their Theorem 4 from Chapter 2 to get a representation (see Definition 1 and Theorem 2 in the Appendix). The advantage of this theorem is the weakness of the conditions required: this is what allows the result shown here to do without specific assumptions regarding the topological structure of the consequence space or the monotonicity of the preference relation.

Appendix: Proof of Theorem 1

The proof relies heavily on Theorem 4 in Chapter 2 of Krantz et al. (1971). It is worth reproducing the essential definition and the statement of the theorem.

Definition 1 (Krantz et al. (1971), p44)**.** Let A be a nonempty set, B and $\succsim$ binary relations on A and $\circ$ a binary operation from B to A. The quadruple $< A, \succsim, B, \circ >$ is an *Archimedean, regular, positive, ordered local semigroup* iff, for all $a,\ b,\ c,\ d \in A$, the following eight axioms are satisfied:

1. $< A, \succsim >$ is a total order: that is, an anti-symmetric weak order (a weak order such that, if $a \succeq b$ and $b \succeq a$, then $a = b$).
2. if $(a, b) \in B$, $a \succsim c$, and $b \succsim d$, then $(c, d) \in B$.
3. if $(c, a) \in B$ and $a \succsim b$, then $c \circ a \succsim c \circ b$.
4. if $(a, c) \in B$ and $a \succsim b$, then $a \circ c \succsim b \circ c$.
5. $(a, b) \in B$ and $(a \circ b, c) \in B$ iff $(b, c) \in B$ and $(a, b \circ c) \in B$; and when both conditions hold $(a \circ b) \circ c = a \circ (b \circ c)$.
6. if $(a, b) \in B$, $a \circ b \succ a$.
7. if $a \succ b$, then there exists $c \in A$ such that $(b, c) \in B$ and $a \succsim b \circ c$.
8. $\{n |\ n \in N_a\ and\ b \succ na\}$ is a finite set.

where N_a, a subset of the positive integers, and na, an element of A for each $n \in N_a$, are defined inductively as follows:

(i) $1 \in N_a$ and $1a = a$;

(ii) if $n-1 \in N_a$ and $((n-1)a, a) \in B$, then $n \in N_a$ and na is defined to be $((n-1)a) \circ a$;

(iii) if $n-1 \in N_a$ and $((n-1)a, a) \notin B$, then for all $m \geqslant n$, $m \notin N_a$.

The importance of this definition is expressed by the following theorem (Theorem 4 in Krantz et al. (1971)).

Theorem 2 (Krantz et al. (1971), p45-6)**.** *Let $< A, \succsim, B, \circ >$ be an Archimedean, regular, positive, ordered local semigroup. Then there is a function ϕ from A to $\Re_+$ such that for all $a, b \in A$,*

(i) $a \succsim b$ iff $\phi(a) \geqslant \phi(b)$;

(ii) if $(a, b) \in B$, then $\phi(a \circ b) = \phi(a) + \phi(b)$.

Moreover, if ϕ and ϕ' are two functions from A to $\Re_+$ satisfying conditions (i) and (ii), then there exists $\alpha > 0$ such that, for any nonmaximal $a \in A$, $\phi'(a) = \alpha\phi(a)$.

Let us now begin the proof of Theorem 1. Pick any act $e \in \mathcal{A}$, which shall remain fixed throughout the proof. Let $\mathcal{A}^+ = \{f_A \in \mathcal{A}_p \mid A \textit{ nonnull event and for any nonnull event } A' \subseteq A, f_{A'} \succ e_{A'}\}$ and $\mathcal{A}^- = \{f_A \in \mathcal{A}_p \mid A \textit{ nonnull event and for any nonnull event } A' \subseteq A, e_{A'} \succ f_{A'}\}$. At least one of these sets is non-empty, by Axioms A2 and A3. Furthermore, let $\mathcal{A}^+_\sim$ be the set of equivalence classes of $\mathcal{A}^+$ under $\sim$; similarly for $\mathcal{A}^-_\sim$. $[f_A]$ shall denote the equivalence class (element of $\mathcal{A}^+_\sim$) containing f_A. Note that there is the following order $\precsim^+_\sim$ on $\mathcal{A}^+_\sim$: $[f_A] \precsim^+_\sim [g_B]$ iff, for $f_A \in [f_A]$ and $g_B \in [g_B]$, $f_A e_{A^c} \preceq g_B e_{B^c}$. Furthermore, there is a relation on $\mathcal{A}^+_\sim$, $\mathcal{B}^+$, and an operation $\circ$ defined as follows. $([f_A], [g_B]) \in \mathcal{B}^+$ iff there are $f_A \in [f_A]$ and $g_B \in [g_B]$ such that A and B are disjoint. For $([f_A], [g_B]) \in \mathcal{B}^+$, $[f_A] \circ [g_B] = [f_A g_B]$ where $f_A \in [f_A]$, $g_B \in [g_B]$ and A and B are disjoint. It is straightforward to check that these relations and operations are well-defined. Similar relations and operations can be defined on $\mathcal{A}^-_\sim$. We begin by proving the following claim.

Claim $< \mathcal{A}^+_\sim, \precsim^+_\sim, \mathcal{B}^+, \circ >$ and $< \mathcal{A}^-_\sim, \precsim^-_\sim, \mathcal{B}^-, \circ >$ are Archimedean regular positive ordered local semigroups (Definition 1).

The importance of the claim should be clear: it will allow us to apply Theorem 2.

Proof of claim. Since the cases are similar, we shall only treat the case of $< \mathcal{A}^+_\sim, \precsim^+_\sim, \mathcal{B}^+, \circ >$ here. Many stages of the proof will rely on the following lemma.

Lemma 1. *For partial acts f_A and g_B, if $f_A e_{A^c} \succeq g_B e_{B^c}$, there exists an event $A' \subseteq A$, such that $f_A e_{A^c} \succeq f_{A'} e_{A'^c} \sim g_B e_{B^c}$.*

Proof. If $f_A e_{A^c} \sim g_B e_{B^c}$ let $A' = A$. If not, $f_A e_{A^c} \succ g_B e_{B^c} \succ e$, so applying solvability, there is an event $A'' \subseteq A$ such that $f_{A \setminus A''} e_{A^c \cup A''} \sim g_B e_{B^c}$. Setting $A' = A^c \setminus A''$ yields the required result. $\square$

Remark 2. Note that an equivalent formulation of this lemma is: For partial acts f_A and g_B, if $[f_A] \succsim^+ [g_B]$, then there exists an event $A' \subseteq A$, such that $f_{A'} \in [g_B]$.

Let us now show that the clauses of Definition 1 are satisfied.

Lemma 2 (Clause 1.). *$\precsim^+$ is a total order.*

Proof. The order $\precsim^+$ on $\mathcal{A}^+_\sim$ inherits the properties of connectedness and transitivity from the order $\preceq$ on $\mathcal{A}$: so axiom A1 guarantees that $\precsim^+$ is a weak order. Furthermore, since $\mathcal{A}^+_\sim$ is obtained by quotienting on $\sim$, $\precsim^+$ is anti-symmetric. It is thus a total order. □

Lemma 3 (Clause 2.). *If $([f_A], [g_B]) \in \mathcal{B}^+$, $[f_A] \succsim^+ [f'_{A'}]$ and $[g_B] \succsim^+ [g'_{B'}]$, then $([f'_{A'}], [g'_{B'}]) \in \mathcal{B}^+$.*

Proof. Since $([f_A], [g_B]) \in \mathcal{B}^+$, there are elements $f_A \in [f_A]$, $g_B \in [g_B]$, with A and B disjoint. Using Lemma 1, choose events $A'' \subseteq A$ and $B'' \subseteq B$ such that $f_{A''}e_{A''^c} \sim f'_{A'}e_{A'^c}$ and $g_{B''}e_{B''^c} \sim g'_{B'}e_{B'^c}$. So $f_{A''} \in [f'_{A'}]$, $g_{B''} \in [g'_{B'}]$; since A and B are disjoint, so are A'' and B'', and hence $([f'_{A'}], [g'_{B'}]) \in \mathcal{B}^+$. □

Note that, since $\circ$ is commutative, Clause 3. is satisfied if and only if Clause 4. is.

Lemma 4 (Clauses 3. and 4.). *If $([f_A], [g_B]) \in \mathcal{B}^+$ and $[f_A] \succsim^+ [h_C]$, then $[f_A] \circ [g_B] \succsim^+ [h_C] \circ [g_B]$*

Proof. Since $([f_A], [g_B]) \in \mathcal{B}^+$, there are elements $f_A \in [f_A]$, $g_B \in [g_B]$, with A and B disjoint. Using Lemma 1, choose an event $A' \subseteq A$ such that $f_{A'} \in [h_C]$. By definition of $\mathcal{A}^+_\sim$, $f_{A'}e_{A'^c} \preceq f_A e_{A^c}$; by Axiom A2, it follows that $f_{A'}g_B e_{(A'\cup B)^c} \preceq f_A g_B e_{(A\cup B)^c}$. Hence $[h_C] \circ [g_B] \precsim^+ [f_A] \circ [g_B]$. □

Lemma 5 (Clause 5.). *$([f_A], [g_B]) \in \mathcal{B}^+$ and $([f_A]\circ[g_B], [h_C]) \in \mathcal{B}^+$ iff $([g_B], [h_C]) \in \mathcal{B}^+$ and $([f_A], [g_B] \circ [h_C]) \in \mathcal{B}^+$; and when both conditions hold $([f_A] \circ [g_B]_\circ[h_C] = [f_A] \circ ([g_B] \circ [h_C])$.*

Proof. If $([f_A], [g_B]) \in \mathcal{B}^+$ and $([f_A]\circ[g_B], [h_C]) \in \mathcal{B}^+$, then there are f_A, g_B and h_C, members of $[f_A]$, $[g_B]$ and h_C respectively, such that A, B and C are disjoint. It follows that $([g_B], [h_C]) \in \mathcal{B}^+$ and $([f_A], [g_B] \circ [h_C]) \in \mathcal{B}^+$; furthermore $([f_A] \circ [g_B]_\circ[h_C] = [f_A g_B h_C] = [f_A] \circ ([g_B] \circ [h_C])$. The same argument works in the other direction. □

Lemma 6 (Clause 6.). *If $([f_A], [g_B]) \in \mathcal{B}^+$, then $[f_A] \circ [g_B] \succ^+_\sim [f_A]$.*

Proof. Since $([f_A], [g_B]) \in \mathcal{B}^+$, there are elements $f_A \in [f_A]$, $g_B \in [g_B]$, with A and B disjoint. $f_A g_B e_{(A\cup B)^c}$ and $f_A e_{A^c}$ differ solely on B; by A2, it is their comparison on this set that decides the preference ordering between them. Furthermore, by definition of $\mathcal{A}^+$, $g_B \succ e_B$; hence the required result.

□

Lemma 7 (Clause 7.). *If $[f_A] \succ^+_\sim [g_B]$, then there exists a $[h_C] \in \mathcal{A}^+_\sim$ with $([g_B], [h_C]) \in \mathcal{B}^+$ and $[f_A] \succsim^+ [g_B] \circ [h_C]$.*

Proof. Using Lemma 1, find $A' \subseteq A$ such that $f_{A'} \in [g_B]$. Since $f_A e_{A^c} \succ f_{A'} e_{A'^c}$, $A \setminus A'$ is a non-null event; take C to be any non-null event which is a subset of $A \setminus A'$. Since $f_A \in \mathcal{A}^+$ and C is non-null, $f_C \in \mathcal{A}$; since $A' \cup C \subseteq A$, $f_{A' \cup C} e_{(A' \cup C)^c} \preceq f_A e_{A^c}$. Taking $[h_C] = [f_C]$ gives the result. □

Lemma 8 (Clause 8.). *Defining $N_{[f_A]}$ and $n[f_A]$ as in Definition 1: for all $[f_A], [g_B] \in \mathcal{A}$, $\{n \mid n \in N$ and $[g_B] \succ n[f_A]\}$ is finite.*

Proof. Suppose not. Then there exists an infinite sequence $f^i_{A_i} \in \mathcal{A}^+$ such that $f^i_{A_i} e_{A_i^c} \sim f^j_{A_j} e_{A_i^c}$ and A_i and A_j are disjoint for each $i \neq j$ $f = f^i_{A_i} e_{(\bigcup A_i)^c}$, e and A_i violate A5.

□

The claim has thus been proved.

It follows from Theorem 2 that there is a function $U^+ : \mathcal{A}^+_\sim \rightarrow \Re^+$ which respects order, and which maps $\circ$ to addition; moreover, this function is unique up to a positive multiplicative factor. Similarly there is a function $U^- : \mathcal{A}^-_\sim \rightarrow \Re^+$ inversing order (for $[f_A], [g_B] \in \mathcal{A}^-_\sim$, $[f_A] \precsim^-_\sim [g_B]$ iff $U^-([f_A]) \geqslant U^-([g_B])$), and sending $\circ$ to addition, which is unique up to a positive multiplicative factor. Evidently, these naturally induce real-valued functions on $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) sharing the same properties. To ease notation, these functions will also be called U^+ and U^-.

This establishes a representation of the orders on $\mathcal{A}^+_\sim$ and $\mathcal{A}^-_\sim$ by finitely-additive functions U^+ and U^-. It remains to be shown that these functions are countably-additive. This can be shown without any supplementary axioms, thanks largely to the strength of the solvability axiom (this is similar to the situation in Abdellaoui and Wakker (2005)). As above, only the case of U^+ will be considered; the case of U^- is similar.

Proposition 1. *U^+ is countably additive.*

Proof. Consider a countable set $[f_{E_i}] \in \mathcal{A}^+_\sim$, $i \in \mathbb{N}$ such that, for all i, j, $([f_{E_i}], [f_{E_j}]) \in \mathcal{B}^+$ and let $[f_E] = \bigcirc_{i=1}^\infty [f_{E_i}]$. It needs to be shown that $\sum_{i=1}^\infty U^+([f_{E_i}]) = U^+([f_E])$. Suppose not. Since $\bigcirc_{i=1}^n [f_{E_i}] \precsim^+_\sim [f_E]$ for any n, and since U^+ is order preserving, it must hold that $\sum_{i=1}^n U^+([f_{E_i}]) \leqslant U^+([f_E])$ for any n. Hence $U^+([f_E]) \geqslant \sum_{i=1}^\infty U^+([f_{E_i}])$: given the assumption that $U^+([f_E]) \neq \sum_{i=1}^\infty U^+([f_{E_i}])$, we have that $U^+([f_E]) > \sum_{i=1}^\infty U^+([f_{E_i}])$.

Pick a $f_E \in [f_E]$. As was noted in Remark 1, the solvability axiom implies that the state space may be assumed to be atomless (all atoms are null). Accordingly, for any positive real number ϵ, there is an $\epsilon' \leqslant \epsilon$ and $[g_B] \in \mathcal{A}^+_\sim$ such that $U^+([g_B]) = \epsilon'$. So there exists an $0 < \epsilon \leqslant U^+([f_E]) - \sum_{i=1}^\infty U^+([f_{E_i}])$ and a $[g_B] \in \mathcal{A}^+_\sim$ with $U^+([g_B]) = \epsilon$. Using Lemma 1, pick an event $E' \subseteq E$ such that $f_{E'} \in [g_B]$. By construction, for any n, $\sum_{i=1}^n U^+([f_{E_i}]) \leqslant U^+_\sim([f_E]) - U^+([g_B])$, so $\bigcirc_{i=1}^n [f_{E_i}] \precsim^+_\sim [f_{E \setminus E'}]$. It follows that, by a countable number of applications of Lemma 1, it is possible to generate a sequence of disjoint non-null events $E_i \subseteq E \setminus E'$ such that $f_{E_i} \in [f_{E_i}]$ for all i. Thus $\bigcirc_{i=1}^\infty [f_{E_i}] \precsim^+_\sim [f_{E \setminus E'}] \prec^+_\sim [f_E] = \bigcirc_{i=1}^\infty [f_{E_i}]$ (the last equation, by definition of $[f_E]$), which is a contradiction. The assumption that $U^+([f_E]) \neq \sum_{i=1}^\infty U^+([f_{E_i}])$ is thus false.

□

It remains to "calibrate" the functions U^+ and U^-; that is, to assure that the positive and negative utilities add correctly. This is done as follows.

Definition 2. Say that $[f_A] \in \mathcal{A}^+_\sim$ and $[g_B] \in \mathcal{A}^-_\sim$ *cancel* if there is a $f_A \in [f_A]$, $g_B \in [g_B]$, such that A and B disjoint and $f_A g_B e_{(A\cup B)^c} \sim e$.

Suppose that there exist $[f_A] \in \mathcal{A}^+_\sim$ such that there are no $[g_B] \in \mathcal{A}^-_\sim$ which cancel $[f_A]$ (the case where all elements of $\mathcal{A}^+_\sim$ cancel, and the case where all elements of both $\mathcal{A}^+_\sim$ and $\mathcal{A}^+_\sim$ cancel, are dealt with similarly). Let $\mathcal{I} = \{[f_A] \in \mathcal{A}^+_\sim \mid there\ is\ [g_B] \in \mathcal{A}^-_\sim, [f_A]\ and\ [g_B]\ cancel\}$. There is a natural mapping $\sigma : \mathcal{I} \to \mathcal{A}^-_\sim$, taking $[f_A]$ to the $[g_B]$ such that $[f_A]$ and $[g_B]$ cancel. This mapping is well-defined because of A2: if not, use Lemma 1 to take two $g_{B'}$, $g_{B''}$, $B' \subset B''$ (B' and B'' events) such that $f_A g_{B'} e_{(A\cup B')^c} \sim f_A g_{B''} e_{(A\cup B'')^c} \sim e$: by A2 it follows that $g_{B''\setminus B'} \sim e_{B''\setminus B'}$ contradicting $g_{B''} \in \mathcal{A}^-$. Using a similar technique, it is easy to show that $\mathcal{I}$ is closed under $\preceq^+_\sim$ (if $[f_A] \in \mathcal{I}$ and $[f'_{A'}] \preceq^+_\sim [f_A]$, then $[f'_{A'}] \in \mathcal{I}$) and that σ is surjective.

The following lemma will yield directly the required result.

Lemma 9. *For $[f_A], [f'_{A'}] \in \mathcal{I}$, $U^+([f_A]) \geqslant U^+([f'_{A'}])$ iff $\sigma([f_A]) \preceq^-_\sim \sigma([f'_{A'}])$.*

Proof. Suppose not: there are $[f_A]$, $[f'_{A'}]$ with $U^+([f_A]) \geqslant U^+([f'_{A'}])$ but $\sigma([f_A]) \succ \sigma([f'_{A'}])$. Pick $f_A \in [f_A]$ and let $A'' \subseteq A$ be such that $f_{A''} \in [f'_{A'}]$ (such an A'' exists by Lemma 1). Similarly, pick $g_B \in \sigma([f'_{A'}])$ with B disjoint from A, and let $B' \subseteq B$ be such that $g_{B'} \in \sigma([f_A])$.[10] By definition $f_A g_{B'} e_{(A\cup B')^c} \sim e \sim f_{A''} g_B e_{(A''\cup B)^c}$, and $f_A \succeq f_{A''}$. By A2, it follows that $g_{B'} \preceq g_B$, contrary to the supposition. This proves the claim. $\square$

Lemma 9 states that $-U^+$ represents $\preceq^-_\sim$: by the unicity properties of Theorem 2, there exists an $\alpha > 0$ such that, for all $[f_A] \in \mathcal{I}$, $U^-(\sigma([f_A])) = -\alpha U^+([f_A])$.

Define U, a measure on $\mathcal{F}_{SC}$, as follows. For any element $c \in C$, for any event A, let

$$(2) \qquad U(A,c) = \begin{cases} U^+(f^c_A) & \text{if } f^c_{A'} \succ e_{A'} \text{ for all non-null events } A' \subseteq A \\ -\frac{1}{\alpha}U^-(f^c_A) & \text{if } f^c_{A'} \prec e_{A'} \text{ for all non-null events } A' \subseteq A \\ 0 & \text{if } f^c_{A'} \sim e_{A'} \text{ for all non-null events } A' \subseteq A \end{cases}$$

where f^c_A is the constant partial act taking the value c on A. This definition extends naturally to $\mathcal{F}_S \times \{c\}$ for each $c \in C$: for any $A \in \mathcal{F}_S$, there is a set of subsets $\{A_j | \, j \in J\}$, $\bigcup_J A_j = A$, unique up to addition or removal of null events, such that, for each j, A_j satisfies one of the conditions in (2). In such a case, $U(A,c) = \sum_J U(A_j, c)$.

Finally, U is extended to a measure on $\mathcal{F}_{SC}$, by defining, for each $\chi \in \mathcal{F}_{SC}$,

$$U(\chi) = \sum_C U(\chi_c, c)$$

where χ_c is the projection of χ onto S for consequence $c \in C$.

It follows from the construction that U is a measure on $(S \times C, \mathcal{F}_{SC})$; it follows from the construction and axiom A2 that it represents $\preceq$ according to (1).

[10] Note that the signs are reversed, because the elements of $\mathcal{A}^-_\sim$ are "negative".

Uniqueness Let U' be any other expected utility measure representing $\preceq$. Take $b(A) = \int_{e_A} dU'$, where e is the act used in the construction of U (e is such that, for any event A, $\int_{e_A} dU = 0$). By the additivity properties of U', b is a measure on S. Consider the measure $U'(A, c) - b(A)$: this function is such that, for any event A, $\int_{e_A} d(U' - b) = 0$ – it agrees with U on the zero (partial) acts. Therefore U and $U' - b$ both represent $\mathcal{A}^+$ and $\mathcal{A}^-$, and by the uniqueness clause in Theorem 2, $U' - b = aU$ for some $a > 0$.

This completes the proof of the theorem.

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