Dynamic Awareness

Brian Hill
HEC Paris*

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Abstract

In recent years, much work has been dedicated by logicians, computer scientists and economists to understanding awareness, as its importance for human behaviour has become evident. Although logics of awareness have been proposed, little work has been done on change in awareness, despite the importance of awareness change in situations of decision and interaction. The aim of this paper is to make a start on this problem. It proposes, firstly, a versatile model of awareness and awareness change, and secondly, logics for awareness and awareness change developed using this model. The logic of awareness is similar to that proposed by Halpern (Games and Economic Behavior, 37 (2001) 321–339); two logics of awareness change are developed, following the two paradigms in belief revision (Alchourron et al, The Journal of Symbolic Logic 50 (1986) 510–530; van Benthem, Journal of Applied Non-classical Logics 17 (2007) 129–155).

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*GREGHEC, HEC Paris, 1 rue de la Libération, 78351 Jouy-en-Josas, France. hill@hec.fr, www.hec.fr/hill.

People lack awareness of various issues at particular moments. As you read the previous sentence, you were not perhaps aware of the question of whether the capital of Chile is Santiago, or of the sentence “the capital of Chile is Santiago”. Moreover, one’s states of awareness change, and rather often: the awareness of the capital of Chile, which you lacked two sentences ago but have now, is but an example. Finally, awareness (or lack of it) is an important property of people’s epistemic states at particular moments, with significant consequences for the decisions they make, their actions, and their behaviour in situations of interaction. It is this importance which has recently drawn the attention of computer scientists, logicians and game theorists, and has led to several models and logics of awareness being proposed. (See Halpern (2001); Modica and Rustichini (1999); Heifetz et al. (2006) for introductions to the notion of awareness, to the importance and uses of the notion and for logics and semantics of awareness). However, little work has been done to date on *change of awareness*, despite the fact noted above that awareness changes quite frequently, and despite the obvious consequences of such changes for behaviour. The aim of this paper is make a start at filling this gap, by developing a model of awareness and of changes in awareness. That is, a formal way of representing an agent’s epistemic state at a given moment which accurately captures the awareness properties of that state, and a formal set of operations which accurately capture the effect of change in awareness on his state. It is naturally hoped that these models will be of use to decision theorists and game theorists, and perhaps to philosophers looking to get a formal grip on agents’ instantaneous states; in any case, in the development of the model, some philosophical and methodologies issues concerning awareness and the project of understanding epistemic states shall be discussed. Finally, on the basis of the model, logics will be proposed for awareness and awareness change. For the case of awareness, it will be shown how these models provide a new semantics for a widely accepted existing logic of awareness. For the case of awareness change, two sorts of logics will be proposed, mimicking the two paradigms for formal study of belief revision: the AGM-paradigm (Gärdenfors, 1988; Gärdenfors and Rott, 1995; Hansson, 1999) and the Dynamic Epistemic Logic or DEL paradigm (van Benthem, 2007; van Ditmarsch, 2005).

In Section 1, a model of an agent’s state of knowledge, encompassing awareness, shall be proposed, and operators on this model representing change of awareness shall be defined. In Section 2, the logic of awareness and awareness change shall be tackled: both AGM-style postulates for awareness and a DEL-style logic of awareness and awareness change shall be proposed.

1 The model

1.1 A model of knowledge and awareness

We, the theorists, know that he, the agent under study, is unaware of certain sentences.[1] The sentences available to the theorist are not all available to the agent: the languages at their disposal differ. This poses a problem for any theorist wishing the construct a *model* correctly representing the agent's state of knowledge: is this model to be constructed in the agent's language or in the theorist's language? This question is equally important for the project of proposing a *logic* of knowledge and awareness: if, for example, the agent lacks the concepts of knowledge or awareness, the logic of his language will be significantly poorer than that of the theorist wishing to talk about these questions.

In this section, a *model* of the agent's epistemic state which pays close attention to the language the agent has at his disposal at the moment in question, leaving the theorist's language unformalised in the background. In Section 2, *logics* shall be proposed: these logics, and particularly those developed in Sections 2.2 and 2.3 will be expressed in the theorist's language, which will be suitably formalised for the purpose.

Simple agents As stated above, the model of the agent's state of knowledge (and awareness) presented here will *take the agent's point of view*. It will represent his knowledge expressed in his own language, so to speak. This model has already been extensively discussed and defended in Hill (forthcoming) where it was applied to belief revision. Its presentation here shall thus be brief.

The basic notion is the following.

Definition 1 (Interpreted Algebra)**.** An *interpreted algebra* $\mathbf{B}$ is a triple (B_I, B, q), where B_I is the free Boolean algebra generated by a set I (the *interpreting algebra*),[2] B is an atomic Boolean algebra (the *base algebra*),[3] and $q : B_I \to B$ is a surjective Boolean homomorphism.

An *element* of $\mathbf{B}$ is a pair $(\phi, q(\phi))$, $\phi \in B_I$. Elements of an interpreted algebra shall be referred to (without risk of confusion) by the appropriate elements of the interpreting algebra, and shall often be called "sentences". Note that there is a Boolean structure on the set of elements of $\mathbf{B}$, induced by the Boolean structure on

[1] Following recent practice, we shall suppose awareness to apply to sentences.

[2] A Boolean algebra is a distributed complemented lattice; the order will be written as $\leqslant$, meet, join, complementation and residuation as $\wedge, \vee, \neg, \to$. The free Boolean algebra generated by a set X shall be noted as B_X for the rest of the paper; details on this and other notions used in this paper may be found in Koppelberg (1989).

[3] Standard terminology is employed here: an atom of a Boolean algebra is an element $a \in B$, such that, for all $x \in B$ with $\perp \leqslant x \leqslant a$, either $x = \perp$ or $x = a$.

the interpreting algebra B_I; the ordinary Boolean connective symbols $\wedge, \vee, \neg, \rightarrow$ shall be used when speaking of this structure, with $\wedge$ and $\neg$ taken as primitive for the purposes of definitions and proofs.

The *consequence relation* $\Rightarrow_{\mathbf{B}}$ for an interpreted algebra $\mathbf{B}$ is defined as follows: for any elements ϕ, ψ of (B_I, B, q), $\phi \Rightarrow_{\mathbf{B}} \psi$ if and only if $q(\phi) \leqslant q(\psi)$. $\Leftrightarrow_{\mathbf{B}}$ will designate the derived equivalence relation. The subscripts may be dropped if they are evident from the context.

Interpreted algebras will be used to model the local logical structure the agent has at his disposal at a given moment. Given that the agent is aware of only a finite number of sentences at any moment, the interpreted algebra representing the logical structure he is using at that moment can be assumed to be finite; this assumption will hold throughout the paper. The interpreting algebra models the *local language* effective at the moment in question – the language that the agent is using at that moment. I is the set of sentences treated as primitive at that moment. The base algebra is the *local logic* on this language – the logic the agent is presupposing on his language at that moment. This is the part of the structure which provides the consequence relation. Just as the elements of the interpreting algebra may be thought of as the *sentences* of the local logical structure, the elements of the base algebra may be thought of as the (local) *propositions*. Accordingly, q is the map taking sentences to propositions, and may be thought of as the *valuation* of the sentences of the language. Note that the atoms of the base algebra can be thought of as "states" or "*small* worlds": worlds in the sense that every sentence of the local language receives a valuation in each world (thanks to q); *small* in the sense that *only* the sentences of the local language receive any valuation in these worlds. For a more detailed discussion and defence of this model, see Hill (forthcoming).

Interpreted algebras represent the language and underlying logic in which the agent his thinking. As such, they naturally capture the agent's state of awareness, via the fact that they represent explicitly the language he sees himself to be using (at that moment): he is aware of the sentences in that language, and unaware of sentences which do not belong to it. So, for example, the interpreted algebra represening the language you were using just before beginning this paper may have contained several sentences pertaining to logic, knowledge, awareness and so on, but the sentence stating that Santiago is the capital of Chile was absent from that algebra: thus your unawareness of it.

Moreover, interpreted algebras can also capture infractions of logical omniscience, through the fact that the consequence relation is relative to the interpreted algebra – and thus to the agent and the moment in question – and not fixed independently by the theorist. This relativity is worth noting because it will have some important consequences below. The question of logical omniscience, however, is beyond the scope of the paper and shall not be discussed further; interested readers are referred to Hill (forthcoming).

Interpreted algebras model the local logical structure employed by the agent at a given moment, but they do not represent which of the sentences in question he takes himself to know or believe.[4] This can be represented by an element of the algebra. We thus have the following structure:

Definition 2 (Pointed algebra)**.** A *pointed algebra* is a pair $(\mathbf{B}, \chi)$ where $\mathbf{B} = (B_I, B, q)$ is an interpreted algebra and χ is an element of $\mathbf{B}$ such that $\chi \not\Leftrightarrow_{\mathbf{B}} \perp$. χ will be called the knowledge element.

An agent knows ϕ in $(\mathbf{B}, \chi)$, if $\chi \Rightarrow_{\mathbf{B}} \phi$. This model treats the agent's knowledge set as consistent and logically closed – relative to the notion of logical consequence in play in that situation. Thus pointed algebras are defined up to (local) logical equivalence on the knowledge elements. Mapping the element χ to the set of atoms $x \leqslant q(\chi)$, one obtains a standard modal-logic model of knowledge in terms of possible worlds (though *small* possible worlds); taking the filter generated by χ (the set of sentences ϕ such that $\chi \Rightarrow \phi$), one obtains the standard belief-revision model as a consistent set of sentences closed under logical consequence (though *local* logical consequence).

So, to take an example, the pointed algebra modelling your state at the beginning of this paper (which, recall, included several sentences pertaining to logic, knowledge, awareness, and so on), had a knowledge element implying what you took yourself to know (about these issues) at that moment. Quite probably, as you read the second sentence of this article, the knowledge element of the pointed algebra implied the sentence stating that Santiago is the capital of Chile: this captures the fact that you became aware of a sentence you knew to be true.

Reflective agents The agent modelled using the interpreted algebra of Definition 1 has a propositional language. He therefore cannot express sentences regarding (his own) knowledge or awareness; or more precisely, he cannot treat such sentences as if knowledge or awareness were operators. This is because the underlying algebra used is a Boolean algebra, the algebra of propositional logic. To model an agent employing a richer language at a given moment requires a richer algebra: for example, a modal language will require a modal algebra (or Boolean algebra with operators, as it called in (Blackburn et al., 2001, Ch 5)). Given that the theorist has a language with knowledge and awareness (as operators of the language), and given that this is the subject of the paper, it is worth considering the

[4]From an internal point of view, which is the point of view taken at this stage of the paper, the traditional notions of belief and knowledge coincide. See also §2.2.

case of the agent which can talk about his current state of knowledge and awareness at a particular moment. To model this, an algebra with two operators will be required: one, the knowledge operator, will correspond to the logic S5 (the standard logic for knowledge); the other operator representing awareness and being, as we shall see, essentially trivial.[5] Such an algebra shall be called an epistemic algebra.

Definition 3 (Epistemic algebra)**.** An *epistemic algebra* B is a Boolean algebra with two operators: f, the knowledge operator, which satisfies normality, multiplicativity, T, 4 and 5,[6] and f_a, the awareness operator, with $f_a x = \top$, for all $x \in B$.

This can be used to define the notion of interpreted epistemic algebra, with a richer structure than the Boolean interpreted algebra.

Definition 4 (Interpreted Epistemic Algebra)**.** An *interpreted epistemic algebra* $\mathbf{B}$ is a triple (B_I, B, q), where: B_I is the free epistemic algebra generated by a set I (the *interpreting algebra*), B is an atomic epistemic modal algebra (the *base algebra*), and $q : B_I \to B$ is a surjective epistemic homomorphism.[7]

The knowledge and awareness operators in the interpreting algebra shall be respectively denoted by K and A, following the tradition. Elements and the consequence relation are as in Definition 1.

A *pointed epistemic algebra* is a pair $(\mathbf{B}, \chi)$ where $\mathbf{B}$ is an interpreted epistemic algebra and χ is an element of $\mathbf{B}$, not $\Rightarrow_{\mathbf{B}}$-equivalent to $\bot$, such that $\chi \Rightarrow_{\mathbf{B}} K\chi$ and if $\chi \not\Rightarrow_{\mathbf{B}} \psi$, then $\chi \Rightarrow_{\mathbf{B}} \neg K\psi$.[8]

The remarks following Definition 1 hold here. K is the knowledge operator, satisfying the ordinary S5 axioms;[9] A is the awareness operator. With these operators in the language, the agent can consider – and explicitly know – sentences

[5]For the purposes of this paper, we will only consider the agent's ability to talk about his own knowledge and awareness at that very moment. The case of the agent considering his past or future states of knowledge or awareness, or the knowledge or awareness of others shall not be explicitly considered; in particular, only the single-agent case is considered. Nevertheless the results in this paper do not have nothing to say about this cases: generally speaking, they would be tackled along the same lines as the investigation of the theorist's understanding of the agent's epistemic state in Section 2.2.

[6]The first two conditions correspond to the generalisation rule and the axiom K and are, respectively, $f\top = \top$ and $f(x \wedge y) = fx \wedge fy$. See (Blackburn et al., 2001, Ch 5) (note however that they take the diamond as primitive, whereas the box is taken as primitive here). The latter conditions are the algebraic formulations of the standard axioms: for example, T is $fx \leqslant x$. Analogous points to those made for Boolean algebras in footnote 2 apply to epistemic algebras.

[7]It preserves the Boolean connectives and the operators: for all $\phi \in B_I$, $f(q(\phi)) = q(f(\phi))$.

[8]These conditions ensure that the knowledge set of the agent satisfies the ordinary S5 axioms for knowledge

[9]Evidently, weaker knowledge operators can be dealt with in similar ways.

pertaining to his own state of knowledge and awareness. Since the agent can only consider those sentences of which he is aware, and he is aware of any sentence he considers, the agent's awareness operator in his local language at that moment is trivial: for any sentence in his local language at that moment, just being in that language implies that he is aware of it. Thus the condition on f_a in the definition of epistemic algebras.[10]

The major difference between the simple agent (modelled by an ordinary interpreted algebra) and the reflective agent (modelled by an interpreted epistemic algebra) lies in the fact that the latter can entertain sentences describing his own state of knowledge or ignorance. Such expressive power is known to create complications in the case of belief revision (Fuhrmann, 1989; van Ditmarsch, 2005). In the case of awareness change, it implies (for classic S5 knowledge) that the agent must know his attitude – of knowledge or ignorance – toward any new sentence of which he becomes aware, and thus, in a certain sense, "gain" knowledge on becoming aware of a new sentence. As we shall see, this is not the case for the simple agent: he can become aware of a new sentence without gaining any new knowledge whatsoever, apart from the (local) tautologies expressed using the new sentence.

Despite this difference, all the results and remarks in the paper apply to both cases, unless explicitly stated. There are three reasons for introducing and using both the ordinary and epistemic algebras here. Firstly, it counts as an advantage of this approach that it can deal with richer and poorer agent languages. Secondly, it seems that both types of algebra are useful in different modelling situations (sometimes one considers one's state of knowledge, sometimes one is unaware of it). Finally, different paradigms for modelling change have a preference for different richnesses of language (see Section 2).

1.2 Changes of awareness

The sort of model proposed in Section 1.1 has been used to model belief change (Hill, forthcoming). In this section, operators relevant for changes in awareness shall be introduced.

Given that awareness is treated as a property of the interpreted (epistemic) algebra rather than specifically of the knowledge set expressed in this algebra, only operations on the algebra itself need be considered. Intuitively speaking, there are two sorts of changes involving awareness, each of which can be captured with operations on algebras: becoming aware of a sentence and becoming unaware of a sentence. However, as shall be seen shortly, it shall be useful to factorise the

[10] For lack of space, we shall not use a more general definition, capable of accounting for unawareness at past or future moments, or awareness of others.

first sort of change into two sorts of operation.

Becoming aware If unawareness of a sentence ϕ is represented as the absence of ϕ from the interpreted algebra modelling the agent's state of knowledge, then becoming aware of ϕ should involve the *extension* of the interpreted algebra to include ϕ. However, at first glance, there are several interpreted algebras $\mathbf{B}'$ that contain $\mathbf{B}$ and ϕ, and indeed they correspond to different cases of becoming aware. For example, when you became aware of the question of the capital of Chile whilst reading a logic paper, the two issues were completely independent to each other. On the other hand, when, whilst searching for the proof of a result, you become aware of (or *recall*) a theorem, which allows you to conclude the desired result on the basis of what you have already established, then this theorem does enter into important logical relationships with that of which you were already aware (the result desired and those which you have already established). These differences are reflected in the interpreted algebra in which your knowledge is couched after your becoming aware – in the former case, the sentence about the capital of Chile does not enter into any non-trivial logical relationships with other elements, whereas, in the latter case, the theorem does. This range of ways of becoming aware poses a challenge for a project of modelling and notably giving a logic for awareness change: should they be modelled by a single non-deterministic becoming-aware operation or a range of different becoming-aware operations? Neither of these options is appetising: the first flies in the face of the idea that, just like the case of coming to believe something, there should be one way for the agent to become aware of something; the second keeps the determinism of the operation, but at the price of a multiplication of becoming-aware operations which is difficult comprehend. Fortunately, it shall be possible to avoid both of these options: the range of possibilities shall be captured by *factorising* them into a single operation for becoming aware, which does nothing but add a sentence to the local language, and an operation for establishing logical relationships between sentences. There is thus a determinate operation for becoming aware, but a range of ways of applying the second operation which establishes logical relationships between the new sentences and the old. On the other hand, the understanding of cases of becoming aware is economic insofar as it requires only two operations as opposed to a larger range. The difference between the single deterministic becoming-aware operation and the range of ways of becoming aware corresponding to compositions with an operation of establishing logical relationships shall be brought out in the next section, since logics shall be proposed for both.

The basic becoming-aware operation, called *expansion*, is a sort of "free" extension of the interpreting algebra: free in the sense that the language is simply extended with no non-trivial logical structure added.

Definition 5 (Expansion)**.** The expansion of an interpreted (epistemic) algebra $\mathbf{B} = (B_I, B, q)$ by a sentence ϕ, call it $\mathbf{B} + \phi$, is defined as follows: if $\phi \notin B_I$, $\mathbf{B} + \phi = (B_{I \cup \{\phi\}}, B', q')$, where $B' = B \otimes B_{\{x\}}$[11] and q' is the unique (epistemic) homomorphism such that, for $\psi \in B_I$, $q'(\psi) = q(\psi)$ and $q'(\phi) = x$. If $\phi \in B_I$, $\mathbf{B} + \phi = \mathbf{B}$.

The expansion of the pointed algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi) + \phi$, is $(\mathbf{B} + \phi, \chi)$. Here and throughout the paper, the same symbol (in this case χ) shall be used both for the element of original algebra ($\mathbf{B}$) and its image under the embedding (in this case, the image in $\mathbf{B} + \phi$), since the difference is clear from the context. The expansion of the pointed (epistemic) algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi) + \phi$, is $(\mathbf{B} + \phi, \chi \wedge \bigwedge_{\psi \in \mathbf{B}, \chi \not\Rightarrow \psi} \neg K(\phi \vee \psi) \wedge \neg K(\neg\phi \vee \psi))$.[12]

This may be seen as analogous to belief-revision theory's expansion (Gärdenfors, 1988):[13] it simply adds the sentence ϕ to the local language, without altering any other logical structure; accordingly, it does not alter the knowledge set, except, in the case of pointed epistemic algebras, to take account of the agent's ignorance of the new sentence. However, in many cases, the newly added sentence will enter into logical relationships with existing sentences (in so far as they feature in the local logical structure effective at that moment). To model this, the operation of restriction will be used.

Definition 6 (Restriction)**.** The restriction of an interpreted algebra $\mathbf{B} = (B_I, B, q)$ by ϕ, written $\mathbf{B}/\phi$, is (B_I, B', q'), where $B' = B/q(\phi)$ and q' is the composition of this quotient with q. The restriction of the pointed algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi)/\phi$, is $(\mathbf{B}/\phi, \chi)$, where, as above, χ is used to refer to the image of the knowledge element in $\mathbf{B}$ under the quotient map to $\mathbf{B}/\phi$.

[11] Recall (footnote 2 and 6) that $B_{\{x\}}$ is the free Boolean (respectively epistemic) algebra generated by the single element x. The free product of (atomic) Boolean algebras is the Boolean algebra whose set of atoms is the Cartesian product of the sets of atoms of the original Boolean algebras; the free product of (atomic) epistemic algebras is the epistemic algebra whose set of atoms is a set of copies of subsets of the Cartesian product of the sets of atoms of the original algebras, where each subset contains at least one element corresponding to each of the atoms of the original algebras and the copies exhaust the set of relations on such subsets such that any two atoms are connected if and only if their components are connected as atoms of the original algebras. Several of the operators used in this section are discussed in (Hill, 2006, Ch. 5) and Hill (forthcoming).

[12] A (epistemic) homomorphism between interpreted (epistemic) algebras is simply a pair of (epistemic) homomorphisms, between the interpreting and base algebras respectively, commuting accordingly. See also Definition 7. Note that the knowledge elements satisfy the conditions for knowledge elements in Definitions 2 and 4: in particular, the clause for pointed epistemic algebras, which effectively assures that the agent knows that he is ignorant as to the truth of newly available sentences which are not already implied by his knowledge, guarantees negative introspection.

[13] Naturally, there is no equivalent of revision in awareness change, for there is no problem with incoming information "contradicting" the current state of awareness.

For interpreted epistemic algebras, the easiest way to define restriction is in terms of the relational or Kripke structures associated with the algebras.[14] $\mathbf{B}/\phi = (B_I, B', q')$, where B' is the modal algebra corresponding to the restriction, to the $\neg q(\phi)$ states, of the relational structure corresponding to the algebra B, and q' is the modal homomorphism which agrees with q regarding the valuation of the elements of I on the common states. This definition guarantees that there is an embedding, ι of the set of atoms of B' into the set of atoms of B. The restriction of the pointed epistemic algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi)/\phi$, is $(\mathbf{B}/\phi, \chi')$ where, for any atom $x \in B'$, $x \leqslant q'(\chi')$ iff $\iota(x) \leqslant \chi$ in B.

In the pointed algebra after restriction, there is new logical structure. This operator does not alter the local language but only the structure on that language. It thus does not alter awareness, but may lead to a change in the agent's beliefs (or knowledge): $\neg\phi$ may now be known for example.[15] The structural change in the algebra corresponds to the removal of all small worlds where ϕ: indeed, the restriction operation on interpreted epistemic algebras is none other than the operation involved in models of public announcement (see van Ditmarsch and Kooi (2006), for example). For this reason, little specific attention shall be dedicated to this operator in what follows.

Intuitively, these two operators seem to be able to capture the sorts of change in awareness mentioned above. Before the capital of Chile came into play, it was absent from the pointed algebra modelling your knowledge; afterwards, it was in the algebra, but logically independent of the rest of the sentences there, which presumably pertained to logic and logic papers. This is modelled simply by the operation of expansion with the sentence stating that Santiago is the capital of Chile. As for the case of the theorem, before becoming aware of it, it was not in the pointed algebra modelling your state of knowledge; afterwards, not only did it belong to the (new) pointed algebra, it entered into logical relations with other elements of the algebra (given the theorem, an result you have already established implies the one you want to prove). The change is modelled by the composition of an expansion – adding the theorem to the pointed algebra – and a restriction – ruling out those "small worlds" where the theorem and the earlier result hold but the result which is to be proven does not.

The consideration of these cases suggests that the two operations are sufficient to model any shift in interpreted algebra which corresponds to becoming aware, but does not alter the logical structure on the sentences in the initial algebra. Such

[14]This definition thus tacitly uses results showing the relationships between relational structures and modal algebras (Blackburn et al., 2001, Ch 5): for the finite case, which is the main one considered here, not only does each relational structure generate a modal algebra, but each (finite) model algebra generates a corresponding relational structure.

[15]There is certainly change of belief or knowledge, but which change may be a subtle question in the case of pointed epistemic algebras: see van Ditmarsch and Kooi (2006) for example.

shifts can be characterised formally as cases where the initial algebra is *embedded* in the final one, where embedding is understood in the following sense.[16]

Definition 7. An *embedding* of $\mathbf{B} = (B_I, B, q)$ into $\mathbf{B}' = (B_{I'}, B', q')$, with $I \subseteq I'$, consists of a pair of injective (epistemic) homomorphisms $\sigma_i : B_I \to B_{I'}$ and $\sigma_b : B \to B'$, the former generated by the inclusion of I in I', such that $q' \circ \sigma_i = \sigma_b \circ q$. This pair will be called σ. The embedding is said to be non-trivial if $\mathbf{B} \neq \mathbf{B}'$.

Furthermore, an embedding of $(\mathbf{B}, \chi)$ into $(\mathbf{B}', \chi')$ consists of an embedding of the interpreted algebras σ, such that $\chi' = \sigma(\chi)$, in the case of pointed algebras, or $\chi' = \sigma(\chi) \wedge \bigwedge \neg K(\phi \vee \psi)$ for pointed epistemic algebras, where the conjunction is taken over the $\psi \in \mathbf{B}$, such that $\chi \not\Rightarrow_{\mathbf{B}} \psi$ and the non-trivial $\phi \in \mathbf{B}' \setminus \mathbf{B}$.

(Non-trivial) embeddings nicely capture the properties of becoming aware. The fact that the new algebra is larger than the old one represents the agent as becoming aware in the shift from the old to the new algebra; on the other hand, the fact that the old algebra embeds into the new one, with no change in its logical structure (the mapping is an embedding) captures the fact that in becoming aware there is no change to that of which he was already aware. Finally, for a given algebra and any given extension of the local language, there are several algebras with that language in which the original algebra embeds: this captures the fact that they are several ways of becoming aware of a given sentence (or set of sentences), depending on the logical relationships with the elements of which the agent was already aware.

The following proposition shows that the operations of expansion and restriction are indeed sufficient to model any such change.

Proposition 1. *Let σ be an embedding of $\mathbf{B_1}$ into $\mathbf{B_2}$ (both either interpreted or interpreted epistemic algebras). Then there is a sequence of expansions followed by restrictions such that $\mathbf{B_1} + \phi_1 + \cdots + \phi_n / \psi_1 / \ldots / \psi_k = \mathbf{B_2}$. Similarly for pointed (epistemic) algebras.*[17]

This result shows that the factorisation of the process of becoming aware into an expansion and a restriction is "complete", in the sense that it can account for any case of becoming aware.

Finally, for the purposes of the next section, the following definition will be useful.

[16]Embeddings, thus defined, not only preserve the structure of the algebras, but also the sentences taken as primitive – thus the condition that $I \subseteq I'$. This seems natural, given that the cases of interest are those where the agent's awareness has increased, and the mere fact of becoming aware does not seem to effect the logical structure nor the primitive sentences which one had previously. A slightly more complicated version of Proposition 1 still holds for a weakened notion of embedding.

[17]Proofs are to be found in the appendix.

Definition 8. An interpreted (resp. pointed, epistemic) algebra $\mathbf{B_2}$ (resp. $(\mathbf{B_2},\chi_2)$) is said to be an *extension of* $\mathbf{B_1}$ *(resp.* $(\mathbf{B_1},\chi_1)$*) by* $\phi_1,\ldots,\phi_n$ $(n \geqslant 1)$ if and only if there is a sentence $\psi \in \mathbf{B_2} \setminus \mathbf{B_1}$ such that $\mathbf{B_2} = \mathbf{B_1} + \phi_1 + \cdots + \phi_n/\psi$ (resp. $(\mathbf{B_2},\chi_2) = (\mathbf{B_1},\chi_1) + \phi_1 + \cdots + \phi_n/\psi)$).

The extensions correspond to the range of different ways of becoming aware of a sentence discussed above; as already noted, there are several extensions of a given algebra by a given sentence. By Proposition 1, for any embedding, the embedding algebra is an extension of the embedded algebra by the appropriate set of sentences; the converse (every extension is an embedding) is evidently true. Finally, since the expansion and restriction operations commute, any extension by a set of sentences can be thought of as a sequence of extensions, each by a single sentence.

Becoming unaware Not only can one become aware of a sentence, but a sentence can also fall out of awareness. Although you were aware of the capital of Chile when reading the second sentence of this article, you had perhaps forgotten about it by the middle of the first section. If awareness is the presence of a sentence in the interpreted algebra, then becoming unaware involves the loss of the sentence, and indeed the structure associated with it. The simple operation of narrowing shall be used to model this.

Definition 9 (Narrowing)**.** The narrowing of an interpreted (epistemic) algebra (B_I, B, q) by ϕ, written $\mathbf{B}-\phi$, is defined as follows: if $\phi \in B_I$, $\mathbf{B}-\phi = (B_{I\setminus\Phi}, B', q')$, where Φ is the smallest $I' \subseteq I$ such that $\phi \in B_{I'}$, q' is the restriction of q to $B_{I\setminus\Phi}$ and B' is the image of $B_{I\setminus\Phi}$ under q. If $\phi \notin \mathbf{B}$, $\mathbf{B} - \phi = \mathbf{B}$.

The narrowing of the pointed (epistemic) algebra $(\mathbf{B},\chi)$ by ϕ, written as $(\mathbf{B},\chi)-\phi$, is $(\mathbf{B} - \phi, \psi)$, where ψ is the $\Rightarrow_{\mathbf{B}-\phi}$-minimal element such that $\chi \Rightarrow_{\mathbf{B}} \psi$.[18]

To a certain extent, this may be seen as an equivalent to belief revision's contraction: whereas the latter corresponds to situations where one reneges on beliefs, the former allows a modelling of cases when a sentence falls out of awareness. When reading the second sentence of this paper, the sentence regarding the capital of Chile was in the interpreted algebra involved in the modelling of your state of knowledge. Subsequently, this sentence fell out of that algebra. The resulting, smaller, algebra is the sub-algebra of remaining sentences: this is exactly the result of narrowing the original algebra by the sentence in question.

On the other hand, there are important dissimilarities between the narrowing operation and contraction, many of which are due to a difference in the two

[18]Note that this definition is correct, because we are supposing finite algebras (footnote 1.1).This supposition is stronger than required: algebras containing infinima for any set of elements would be sufficient.

phenomena. Whereas the contraction operation seeks to capture those situations where one is forced to give up a belief, one will never find oneself in a position where one accepts to or decides to become unaware of a sentence. Becoming unaware is a gradual, more ephemeral process, for which the question of time is perhaps more crucial. For such reasons, the narrowing operation should not be understood as the same sort of model as those proposed by current theories of belief revision and communication. Whereas the latter models consider the effect on an initial state of incoming information, taken to be delivered to the agent at a given instant, narrowing captures a longer process, where the sentence falling out of awareness does not represent an element delivered to (or extracted from) the agent at any particular moment, but is rather part of the characterisation of the change which the agent undergoes. For a more subtle model of changes in awareness, which accounts for some of these issues, though is less amenable to axiomatisation, see Hill (2006, §2.3).

This has consequences for the meaningful challenges which the phenomenon of becoming unaware poses for the modeler and logician. Whereas, in the case of contraction, a crucial question is that of choosing, for contraction by a disjunction, which of the disjuncts to give up, the analogous question for narrowing – choosing which clause to become unaware of when narrowing by a complex sentence – is effectively meaningless, since such a choice never presents itself. Indeed, generally, cases of becoming unaware involve becoming unaware of sentences – not only a certain number of "basis sentences" but also logical combinations of them (for example, conjunctions with sentence of which the agent stays aware). In terms of the proposed model, where the agent's language (interpreting algebra) is a set of sentences generated by a set of (locally) primitive sentences I, cases of becoming unaware involve sentences falling out of the generating set I (with the ensuing repercussions for awareness of compound sentences). Therefore, it is essentially narrowings by sentences in I which are the important cases for understanding awareness change. Indeed, the above definition, applied to a single element of I, behaves as one would expect: the resulting algebra is the subalgebra generated by the other elements of I. For the sake of completeness, the definition has been extended to other elements of the algebra by supposing that they characterise a change in which the agent becomes unaware of all the primitive sentences they (essentially) contain.

2 The logic

Models have been proposed and motivated for awareness and change of awareness. In this section, the question of the logic of awareness and awareness change shall be treated.

Consider, for the sake of analogy, the logics proposed for changes of belief: notably, the logics of belief revision. There are (at least) two paradigms, each with its own idea of what counts as a "logic" of belief change. The classic AGM paradigm considers a logic to be a collection of rules on how beliefs should change. The project is to propose (formal) models of belief states and operations on these models such that these operations comply to the rules for change and exhaust the set of possible ways in which one can comply with these rules (see Gärdenfors (1988); Gärdenfors and Rott (1995); Hansson (1999)). It is noteworthy that in this paradigm, the logical structures involved – notably the models of agents' belief states – are generally propositional; talk of belief and belief change is only in the metalanguage. On the other hand, the Dynamic Epistemic Logic (DEL) paradigm adopts a richer (object) language, containing not only operators for belief but also operators describing the changes which may occur (generally in the form of incoming information). It imposes axioms and rules on this language, thus forming a logic, and searches for classes of structures which interpret the language and are sound and complete with respect to the logic (see van Benthem (2007); van Ditmarsch (2005)). Note that soundness and completeness is desired not only with respect to the dynamic fragment of the logic (describing change in belief) but also with respect to the static part (describing the beliefs at a given moment).

In Section 2.1, postulates for awareness change, in the AGM-style, shall be proposed. At this stage, the logic of (static) awareness need not be discussed, since the theorist's language – the only language where one can talk non-trivially about the agent's current state of awareness (see Section 1.1) – is not formalised in this paradigm, but left as the meta-language. In Sections 2.2 and 2.3, the DEL-styled approach is adopted: in this case, for the logic to be non-trivial, it is the theorist's language which must be taken as the object language (and thus formalised). This is what is done: in Section 2.2, logics of (static) awareness are given, and a sound and complete semantics is proposed in terms of pointed (epistemic) algebras; in Section 2.3, the logics and the semantics are extended to the dynamic case, to obtain sound and complete logics for awareness change.

2.1 Logic of awareness change, AGM style

The aim of this section is to propose AGM-style postulates for awareness change. Recall that there are two basis sorts of change: becoming aware and becoming unaware. Furthermore, there are many ways of becoming aware of a given sentence; using the terminology of Definition 8, there are several *extensions* of the current state of awareness (modelled by the appropriate algebra) by a given sentence (of which the agent is not already aware). However, as shown in Proposition 1, any change which results in an increase in awareness can be factorised into *expansions* – the "free" addition of sentences to the interpreted algebra – and *restrictions* – which alter the logical structure of the algebra without touching the local language. To the extent that restrictions are operations which end up altering the agent's knowledge without changing his awareness, they are akin to well-understood operators of knowledge or belief change, for which postulates have already been proposed and discussed. For this reason, restriction shall not be considered in this section nor in subsequent sections. Indeed, a set of postulates will be proposed which characterise the operation of expansion: that is, expansion satisfies the postulates and the only operation satisfying the postulates is the expansion operation (this sort of result is analogous to the AGM representation theorems for belief change). Naturally, not all cases of becoming aware will satisfy all of these axioms, because not all cases of becoming aware consist only of expansions; many are followed by restrictions. However, it will be easy to see what distinguishes expansions from other extensions: indeed, it will be shown that the set of extensions is characterised completely by a well-defined and intuitive subset of the postulates characterising expansion. Finally, postulates will be presented for narrowing and a representation theorem proved.

Becoming aware For $\mathbf{B}$ an interpreted (epistemic) algebra and ϕ a sentence, we have the following postulates.

(E 1) $\mathbf{B} + \phi$ is an interpreted (epistemic) algebra

(E 2) $\phi \in \mathbf{B} + \phi$ (Success)

(E 3) if $\phi \in \mathbf{B}$, $\mathbf{B} = \mathbf{B} + \phi$ (Vacuity)

(E 4) for a sentence ψ, $\psi \notin B_{I\cup\{\phi\}}$, then $\psi \notin \mathbf{B} + \phi$ (Minimality)

(E 5) there is an embedding from $\mathbf{B}$ into $\mathbf{B} + \phi$ (Inclusion)

(E 6) if $\phi \notin \mathbf{B}$, then there is no $\psi \in \mathbf{B}$ such that $\psi \Leftrightarrow_{\mathbf{B}+\phi} \phi$ (Freeness)[19]

(E 6') if $\phi \notin \mathbf{B}$, then for all $\psi \in \mathbf{B}$, $K(\psi \wedge \phi)$ and π are non-trivially $\Rightarrow_{\mathbf{B}+\phi}$-equivalent iff $\pi = K(\psi' \wedge \phi)$ with $\psi' \in \mathbf{B}$, $\psi \Leftrightarrow_{\mathbf{B}} \psi'$; similarly for $\neg\phi$ (Modal Freeness)[20]

Freeness and modal freeness are the axioms particular to expansion: freeness guarantees, in the case of ordinary interpreted algebras, that the extension involved is simply an expansion, with no subsequent restriction being applied, whereas

[19]Here ψ denotes the element of $\mathbf{B}$ and its image in $\mathbf{B} + \phi$; see Definition 5.

[20]Non-trivial equivalence occurs between two different but $\Rightarrow$-equivalent elements: ie. they have different elements of the interpreting algebra, but the same element of the base algebra.

modal freeness, which is required in addition in the case of epistemic algebras, is required for the same purpose (to guarantee there are no restrictions which only effect the knowledge operator).

For the case of pointed (epistemic) algebras, postulates are needed regarding the element modelling knowledge. Consider the following postulates, where $(\mathbf{B}, \chi)$ is a pointed (epistemic) algebra, ϕ is a sentence, the interpreted (epistemic) algebra of $(\mathbf{B}, \chi) + \phi$ is $\mathbf{B}'$ and the knowledge element is χ'.

(E 7) For all $\psi \in \mathbf{B}$, $\chi' \Rightarrow_{\mathbf{B}'} \psi$ iff $\chi \Rightarrow_{\mathbf{B}} \psi$ (Conservativity)

(E 8) For all $\psi \in \mathbf{B}' \setminus \mathbf{B}$, $\chi' \Rightarrow_{\mathbf{B}'} \psi$ iff $\psi \Leftrightarrow_{\mathbf{B}'} \top$ (Ignorance)

(E 8') For all $\psi \in \mathbf{B}$ such that $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \top$ and $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \bot$, if $\chi \not\Rightarrow_{\mathbf{B}} \psi$ then $\chi' \Rightarrow_{\mathbf{B}'} \neg K(\phi \vee \psi)$ and $\chi' \Rightarrow_{\mathbf{B}'} \neg K(\neg\phi \vee \psi)$ (Ignorance)

(E 7) guarantees that just by becoming aware the agent does not gain any knowledge regarding that of which he was already aware. Note that, in the case of pointed algebras, (E 8') is vacuous, since there are no modal operators in the algebra; however, it is required in the case of pointed epistemic algebras to guarantee that the knowledge set satisfies negative introspection.[21] On the other hand, (E 8) is too strong in the case of pointed epistemic algebras, because it demands that the agent gains no knowledge, not even regarding his own ignorance, and this conflicts with negative introspection. It is needed in the case of pointed algebras, however, to assure that nothing is learnt in the change.

Remark. There is a simple and evident analogy between the first five postulates and Gärdenfors postulates $(K^{+}1)-(K^{+}4)$, $(K^{+}6)$ for belief expansion (Gärdenfors, 1988, §3.2). The following important difference should nevertheless be emphasized: whilst the AGM framework uses a propositional language, these postulates and the result below also hold for the case where the agent has a knowledge (and awareness) operator in his language – that is, for interpreted epistemic algebras.

Philosophically speaking, the analogy extends into the justification for the postulates: in the belief case each of these postulates have their intuition, though each is debatable, and it seems as if the situation may be similar in the awareness case. Take for example the postulate of inclusion: (in the awareness case) it says that becoming aware of something should not affect the logical relationships one accepts between the sentences of which one was already aware. Although there is a strong intuition about the correctness of this postulate – one often does not and generally should not change one's opinions about the logical relationships between sentences just because something else has come into play – and although

[21] Note that the restriction $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \top$, $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \bot$ in (E 8') is empty in the case of expansion though not necessarily for extension in general.

this intuition is certainly pertinent in many cases (for example, becoming aware of the capital of Chile whilst reading a paper on awareness), one might argue that there are cases where it is justifiably violated. These will generally be cases where becoming aware of something makes one realise an error or false presupposition under which one was harbouring. The analysis of these cases are however complex, for it is not clear whether they are to be understood as cases where it is the fact of becoming aware which leads to a change in the initial logical structure, or whether there is a change of awareness of the sort considered here which is then followed by some sort of belief revision operation, under which the logical structure used changes. Evidently, a combined theory of belief and awareness change is required to clarify the weight of this objection against inclusion, and such a project lies beyond the scope of this paper. We will thus satisfy ourselves with the immediate intuitive appeal of the postulates (it will be shown below that any lack of intuitiveness in the freeness postulates have to do with the fact that they characterise expansion as opposed to any other extension).

For the operation of expansion, there is the following representation theorem.

Theorem 1 (Representation / characterisation theorem)**.** *The operation of expansion of interpreted algebras (respectively interpreted epistemic algebras) satisfies (E 1 – 6) (resp. (E 1 – 6')). The operation of expansion of pointed algebras (resp. pointed epistemic algebras) satisfies (E 1 – 6), (E 7), (E 8) (resp. (E 1 – 6'), (E 7), (E 8')).*

Furthermore, let $\sim$ *be any function taking pairs of interpreted (epistemic) algebras and sentences to interpreted (epistemic) algebras, satisfying (E 1 – 6) (resp. (E 1 – 6')). Then, for all* $\mathbf{B}$ *and* ϕ, $\mathbf{B} \sim \phi = \mathbf{B} + \phi$*. Similarly for pointed (epistemic) algebras and (E 1 – 6), (E 7), (E 8) (resp. (E 1 – 6'), (E 7), (E 8')).*

It was claimed above that the freeness postulates are required to characterise expansion rather than any other extension. This can be shown rigourously by the following representation theorem for extensions.

Theorem 2 (Representation / characterisation theorem)**.** *Extensions of interpreted algebras and of interpreted epistemic algebras satisfy (E 1 – 5). Extensions of pointed algebras (resp. pointed epistemic algebras) satisfy (E 1 – 5), (E 7), (E 8) (resp. (E 1 – 5), (E 7), (E 8')).*

Furthermore, let $\sim$ *be any relation between pairs of interpreted (epistemic) algebras and sentences and interpreted (epistemic) algebras, satisfying (E 1 – 5). Then, for all* $\mathbf{B}$ *and* ϕ, $\mathbf{B} \sim \phi$ *is an extension of* $\mathbf{B}$*. Similarly for pointed (epistemic) algebras and (E 1 – 5), (E 7), (E 8) (resp. (E 1 – 5), (E 7), (E 8')).*

As noted above, there are several possible extensions of a given algebra by a given sentence; this result is thus not so much a characterisation of an operation of extension as a characterisation of the relation, between pairs of algebras and sentences, of ... being an extension of ... by .... It is a comforting fact for the correctness of the model proposed that its characterising properties are all rather intuitive, if not beyond debate.

Becoming unaware For $\mathbf{B}$ an interpreted (epistemic) algebra and ϕ a sentence,

(N 1) $\mathbf{B} - \phi$ is an interpreted (epistemic) algebra

(N 2) if $\psi \in B_\Phi$, then $\psi \notin \mathbf{B} - \phi$ (Success)[22]

(N 3) if $\phi \notin \mathbf{B}$, $\mathbf{B} = \mathbf{B} - \phi$ (Vacuity)

(N 4) for ψ a sentence, $\psi \in B_{I \backslash \Phi}$, then $\psi \in \mathbf{B} - \phi$ (Minimality)

(N 5) there is an embedding from $\mathbf{B} - \phi$ into $\mathbf{B}$ (Inclusion)

Once again, for $(\mathbf{B}, \chi)$ a pointed (epistemic) algebra and ϕ a sentence, the interpreted (epistemic) algebra, call it $\mathbf{B}'$, of $(\mathbf{B}, \chi) - \phi$ satisfies all the postulates above, and the knowledge element, call it χ' satisfies

(N 6) For all $\psi \in \mathbf{B}'$, $\chi' \Rightarrow_{\mathbf{B}'} \psi$ iff $\chi \Rightarrow_{\mathbf{B}} \psi$ (Conservativity)

Remark. There is an analogy between the (N 1) – (N 3) and (N 5) and the postulates $(K^{-}1)$–$(K^{-}4)$ for belief contraction (Gärdenfors (1988, §3.4) and Alchourron et al. (1985)). Belief contraction's recovery axiom $(K^{-}5)$, though loosely related to the minimality axiom (N 4), does not have a strict analogue here, because the expansion and narrowing operations are not exactly "inverses" as in the case of belief change: although the result of narrowing and then expanding by the same sentence will be an interpreting algebra with the same language, it may not have the same logical structure.[23] Moreover, (N 2) has the form it does because of the interpretation of narrowing by a complex sentence, according to which the agent becomes unaware of all the primitive sentences contained in it: see the discussion following Definition 9. It was noted that the important case is that of narrowing by an element of I. In such cases, (N 2) reduces to the statement one naturally expects: $\phi \notin \mathbf{B} - \phi$.

On the philosophical side, similar comments to those made above about becoming aware can be made regarding the intuition of these postulates, with one

[22] Φ is as stated in Definition 9.

[23] However, using the notion of extension (Definition 8, it is possible to replace the minimality axiom with something closer to the recovery axiom of belief contraction, namely: for $\Phi = \{\pi_1, \dots, \pi_n\}$ defined as in Definition 9, $\mathbf{B}$ is an extension of $\mathbf{B} - \phi$ by $\pi_1, \dots, \pi_n$.

exception: the absence of natural instantaneous cases of becoming unaware (see Section 1.2). The discussion up to this point has been neutral on the question of whether postulates and logics of change are to be understood *normatively* – as standards for "rational" change of epistemic state – or *descriptively* – as a collection of properties of our actual changes of epistemic state. It is not the ambition of this paper to embark upon a discussion of this question, and so the comments made above regarding the postulates for becoming aware were formulated so as to apply to both readings. For the case of unawareness, although this neutrality is not impossible to conserve, it is threatened, basically because of the position which states that one can never justifiably become unaware of something. According to this position, it is never rational to become unaware, so there are no rational principles on an operation which makes one unaware (on the other hand, recall from Section 1.2 that the operation can be thought of as describing a change in awareness, so properties of the operation correspond to properties of this change). However, this position may be contestable, on the same sort of grounds used to defend "bounded rationality" behaviour as at least partly rational; namely, the idea that, given the limits on human cognitive capacity (in this case, the limited number of things of which he can be aware at any moment) certain otherwise "irrational" principles are justified (in this case, letting things fall out of awareness). An extended discussion of such issues is beyond the scope of this paper; as in the case of becoming aware, we will satisfy ourselves with the immediate intuitive appeal of the postulates.

Theorem 3 (Representation / characterisation theorem)**.** *The operation of narrowing of interpreted (epistemic) algebras (Definition 9) satisfies (N 1 - 5). The operation of narrowing of pointed (epistemic) algebras satisfies (N 1 - 6).*

Furthermore, let $\sim$ be any function taking pairs of interpreted (epistemic) algebras and sentences to interpreted (epistemic) algebras, satisfying (N 1 – 5). Then, for all $\mathbf{B}$ and ϕ, $\mathbf{B} \sim \phi = \mathbf{B} - \phi$. Similarly for pointed (epistemic) algebras and (N 1 - 6).

2.2 Logic of awareness, Epistemic Logic style

The DEL approach to the likes of belief change begins with a static logic – of belief, or knowledge – and then adds dynamic elements. In this section, the (static) logic of awareness and knowledge shall be considered, as a preliminary to using it in the dynamic logic presented in the Section 2.3.

A logic of awareness and knowledge requires a language, hence the immediate question: whose? The logic of the agent's language, even when the appropriate operators are present, is particularly poor: the awareness operator is trivial, for example (§1.1).[24] The *theorist's* language is more interesting, and that which will be dealt with here; it thus needs to be formalised.

Let the theorist's language be $\mathcal{L}_P^{KA}$, the language generated from a set of propositional letters P by closure under Boolean connectives and two operators K (explicit knowledge) and A (awareness). Pointed (epistemic) algebras are models of the agent's states of knowledge (in the scientific sense of a representation of the target phenomenon); the theorist's language is designed to talk about such states; thus one would expect that the language $\mathcal{L}_P^{KA}$ can be interpreted in pointed (epistemic) algebras. Indeed, it will turn out that pointed (epistemic) algebras are models of the theorist's logic (in the logical sense of a structure where theorems of the logic are satisfied). Sentences of $\mathcal{L}_P^{KA}$ can be evaluated on the pointed (epistemic) algebra $(\mathbf{B}, \chi)$, whose interpreting algebra is generated by a subset of P, in the following way:

$$\begin{array}{rcl} (\mathbf{B},\chi) \vDash A\phi & \text{iff} & \phi \in \mathbf{B} \\ (\mathbf{B},\chi) \vDash K\phi & \text{iff} & \chi \Rightarrow_{\mathbf{B}} \phi \\ (\mathbf{B},\chi) \vDash \neg\phi & \text{iff} & (\mathbf{B},\chi) \nvDash \phi \\ (\mathbf{B},\chi) \vDash \phi \wedge \psi & \text{iff} & (\mathbf{B},\chi) \vDash \phi \textit{ and } (\mathbf{B},\chi) \vDash \psi \end{array}$$

This does not specify the valuation of all sentences in $\mathcal{L}_P^{KA}$: it does not allow valuations of sentences regarding non-epistemic states of affairs. This is natural: pointed (epistemic) algebras are after all models from the agent's point of view, and the "real" state of affairs is external to this point of view, and known only (by supposition) to the theorist. To capture the "real" state of affairs, and to allow valuations of other sentences, exogenous structure needs to be added, namely a maximally consistent set of propositions of the propositional language $\mathcal{L}_P$ generated by P; call it Ξ. Such a structure determines the valuation of sentences which do not necessarily appear in the interpreted algebra, and so of which the agent is not aware, as well as those of which he is aware. The only condition placed on Ξ will guarantee that what is being modelled is knowledge: in the modal logic literature at least, knowledge is distinguished from belief by the fact that what agent purports to know is in fact true. This is represented by the requirement that $\Xi' \Rightarrow_{\mathbf{B}} \chi$, where Ξ' is the projection of Ξ into the algebra $\mathbf{B}$.

Let $\mathcal{B}$ (resp. $\mathcal{B}^e$) be the class of pairs $((\mathbf{B}, \chi), \Xi)$, each consisting, on the one hand, of a pointed algebra (resp. pointed epistemic algebra) $(\mathbf{B}, \chi)$ whose interpreting (epistemic) algebra is generated by a subset of P, and, on the other hand, a maximally consistent set Ξ of sentences of $\mathcal{L}_P$. Valuations of sentences in $\mathcal{L}_P^{KA}$

[24]Evidently, consideration of operators in the agent's language describing the awareness of others is of interest: treatment of such cases may be carried out along the lines of the case of the theorist.

are provided by extending the previous clauses to elements of $\mathcal{B}$ (resp. $\mathcal{B}^e$) in the obvious way, and by adding the following clause:

$$((\mathbf{B}, \chi), \Xi) \vDash \phi \quad \text{iff} \quad \phi \in \Xi$$

The clauses given above provide a valuation of the language $\mathcal{L}_P^{KA}$ over $\mathcal{B}^e$ (the pointed epistemic algebras), but not over $\mathcal{B}$ (the pointed algebras). There is no way to evaluate sentences with embedded operators – sentences pertaining to the agent's knowledge about his knowledge, or his knowledge about his awareness and so on. This is simply because pointed algebras model the agent as not possessing the knowledge and awareness operators in his language, and so not being able to formulate sentences involving them, about which he could have (first-order) knowledge or awareness. Although there is insufficient space to discuss the issue at length, it should be noted that this consequence of the model is not without its intuition: although one can, via theories of behaviour, give an account of knowledge such that an observer can attribute knowledge to an agent who does not possess the concept of knowledge, it is much more difficult to give an account such that the observer can attribute second-order knowledge (knowledge about the agent's own knowledge) to an agent devoid of the concept.[25] Technically, for pointed algebras, it suffices to limit oneself to the sub-language of $\mathcal{L}_P^{KA}$ consisting of formulas of modal depth at most one (ie. no embedding of operators), call it $\overline{\mathcal{L}_P^{KA}}$: the above clauses provide a valuation of $\overline{\mathcal{L}_P^{KA}}$ over $\mathcal{B}$.

The appropriate soundness and completeness results are as follows (the rules and axioms are given in Figure 1).

Theorem 4. *The system* $\mathbf{Aw^p} = \{Prop, MP, Gen, K, T, A0, A1, A2\}$ *is a sound and complete axiomatisation with respect to* $\mathcal{B}$.

The system $\mathbf{Aw^r} = \{Prop, MP, Gen, K, T, 4, 5, A0 - A5\}$ *is a sound and complete axiomatisation with respect to* $\mathcal{B}^e$.

Note that $\mathbf{Aw^p}$ is the restriction of $\mathbf{Aw^r}$ to the language $\overline{\mathcal{L}_P^{KA}}$; such a restriction was motivated above.

Furthermore, it is worth remarking that the system $\mathbf{Aw^r}$ is similar to the axiom system proposed by Halpern (2001). That system is complete over several important semantics for awareness that have been recently proposed, notably Halpern's propositionally determined awareness structures (Halpern, 2001) and the generalised standard models of Modica and Rustichini (Modica and Rustichini, 1999); to this extent, the logic can be thought of as the "standard" logic of awareness in the literature. The only difference between the logic presented above and that of

[25] Indeed, one only gets a valuation on pointed epistemic algebras by confounding the knowledge operator in the theorist's language with the knowledge operator in the agent's language.

Figure 1: Static axioms and rules

Prop Axioms of propositional logic

MP Modus ponens

Gen From ϕ infer $A\phi \rightarrow K\phi$

K $K\phi \wedge K(\phi \rightarrow \psi) \rightarrow K\psi$

T $K\phi \rightarrow \phi$;

4 $K\phi \rightarrow KK\phi$;

5 $\neg K\phi \wedge A\neg K\phi \rightarrow K\neg K\phi$

A0 $K\phi \rightarrow A\phi$

A1 $A(\psi \wedge \phi) \leftrightarrow A\phi \wedge A\psi$

A2 $A\phi \leftrightarrow A\neg\phi$

A3 $AK\phi \leftrightarrow A\phi$

A4 $A\phi \leftrightarrow AA\phi$

A5 $A\phi \rightarrow KA\phi$

Halpern is Halpern's irrelevance rule (Irr). This rule is not required here, whereas the completeness proofs that Halpern has proposed rely on it. Nevertheless, he expresses doubts about the necessity of this rule to his results; if he is correct that the rule (Irr) is superfluous, then the model proposed here and the models of Halpern and Modica and Ristichini are essentially equivalent.

Remark. A noteworthy property of Theorem 4 is the dependance on the stipulation that the interpreting algebras of the pointed algebras in $\mathcal{B}$ and $\mathcal{B}^e$ are generated by subsets of P, the set of primitive sentences of $\mathcal{L}_P^{KA}$. In a word, this condition demands that the agent's language always agrees with the theorist's regarding the sentences taken as primitive; it follows that the consequence relation he is using can always be taken to agree with the theorist's. If this condition is dropped, the agent's local consequence relation, under which his knowledge set is closed, may not agree with the theorist's, so his state of knowledge may cease to satisfy some of the axioms, since they are expressed in the theorist's language. The rule of generalisation and the axiom K, for example, may suffer. If the agent takes the sentences ϕ and $\phi \rightarrow \psi$ to be primitive, then it need not be the case that $\phi \wedge (\phi \rightarrow \psi) \Rightarrow \psi$ in his interpreted algebra (even if ψ does belong to this algebra); in such algebras, there are knowledge elements according to which he knows ϕ and $\phi \rightarrow \psi$ but not ψ, or, to put it another way, in which he does not know $(\phi \wedge (\phi \rightarrow \psi)) \rightarrow \psi$, although he is aware of it; so generalisation and K fail. See Hill (forthcoming) for a fuller discussion of this point, and its relation to the problem of logical omniscience.

It is doubtless possible to weaken this condition, and the logic, by adding specific "syntactic" requirements on the forms of sentences in the interpreted (epistemic) algebras. However, it is perhaps more interesting to consider the general question posed by such a situation, a question which concerns the relationship between giving a logic for a phenomenon (such as knowledge or awareness) and (formally) modelling the phenomenon. The problem of primitive sentences concerns the relationship between the theorist's and the agent's languages, and thus only arises when one tries to develop a logic of (the agent's) awareness which (for fear of triviality) needs to be expressed in the theorist's language. This is not to say that such a project should not be pursued – quite the contrary, logics of awareness are important – but rather that when it comes to *modelling* awareness, where the aim is to capture as accurately as possible the agent's state of knowledge and awareness, use of the theorist's language should perhaps be avoided. This seems to argue in favour of a sharp distinction between models of awareness and logics of awareness, and for the development of models which rely as little as possible on the theorist's language; in other words, it argues in favour of the strategy of the present paper, and the model proposed in Section 1.

2.3 Logic of awareness change, DEL style

Dynamic Epistemic Logic extends ordinary Epistemic Logic by adding operators for change; see van Benthem (2007); van Ditmarsch (2005), for example, for recent applications to belief revision. The final task in this paper is to do the same for awareness: extend the logic of knowledge and awareness presented in the previous section by adding operators describing changes in awareness. A logic shall be first be proposed for expansion and narrowing; as in §2.1, to bring out the difference between expansion and extensions in general, a (weaker) logic, for extensions and narrowing, shall then be presented. Furthermore, given that DEL approaches normally use the whole epistemic language, rather than just the fragment of modal depth one, only the case of interpreted epistemic algebras shall be discussed.[26]

The appropriate theorist's language will be an extension of the language $\mathcal{L}_P^{KA}$ containing operators for talking about becoming aware of ϕ and becoming unaware of ϕ. However, as for the case of pointed algebras in Section 2.2, the agent's expressive capacities places restrictions on the things which the theorist can meaningfully say about his changes of awareness. Most notably, the agent's languages do not contain operators for talking about change of awareness. Although this restriction may be unnecessary for the case of becoming unaware – after all, one can consider what one would think had one not been aware of cer-

[26]Dynamic logics of awareness which are sound and complete over $\mathcal{B}$ can be given for a language extending $\overline{\mathcal{L}_P^{KA}}$, in generally the same manner as that described here, with a few added complications.

tain issues – it is certainly natural for the case of becoming aware – to consider what one would think had one been aware of a particular issue, one must already be aware of it. Given this assumption above the agent's languages, it makes no sense for the theorist to consider sentences such as "after becoming aware of 'after becoming aware of ϕ, ψ', χ" or "the agent knows that, after becoming aware of ϕ, ψ". For this reason, the language used will not contain (1) sentences where there is an operator $[+\phi]$ or $[-\phi]$ with ϕ having a subformula $[+\psi]\psi'$ or $[-\psi]\psi'$; and (2) sentences where there is an operator $[+\phi]$ or $[-\phi]$ in the scope of an operator K or A. Let $\mathcal{L}_P^{KA+-}$ be the extension of the language $\mathcal{L}_P^{KA}$ by operators $[+\phi]$, $[-\phi]$ for all $\phi \in \mathcal{L}_P^{KA}$ which contains only sentences where these operators are not in the scope of operators K and A.

For the remainder of this section, for any sentence $\phi \in \mathcal{L}_P^{KA}$, let $\Phi = \{p_1, \dots, p_n\}$ be the set of elements of P featuring in ϕ. For the logic of expansion and narrowing, the valuation of the $\mathcal{L}_P^{KA+-}$ is defined by adding the following clauses to those presented in the previous section.

$$((\mathbf{B},\chi),\Xi) \models [+\phi]\psi \quad \text{iff} \quad ((\mathbf{B},\chi),\Xi) + p_1 + \cdots + p_n \models \psi$$
$$((\mathbf{B},\chi),\Xi) \models [-\phi]\psi \quad \text{iff} \quad ((\mathbf{B},\chi),\Xi) - p_1 - \cdots - p_n \models \psi$$

where the definition of expansion (resp. narrowing) is extended to pairs consisting of pointed epistemic algebras and maximally consistent sets of sentences of $\mathcal{L}_P$ by stating that it does not alter the set of sentences Ξ.

Remark. As opposed to the AGM-style case (especially for the case of expansion),[27] all changes of awareness expressed with the operators $[+\phi]$ and $[-\phi]$ have to be considered to boil down to changes in awareness of the primitive sentences P. This is related to the restrictions on interpreted algebras placed in the static logic and discussed at the end of section §2.2: if changes of awareness were not translatable in terms of changes of awareness of elements of P, then it would be possible to attain, by applying change operations, interpreted algebras whose interpreting algebras were not generated by elements of P, and the static logic proposed in the previous section (which, as noted, is more or less standard in the domain) would no longer be valid. This does have the (perhaps unwelcome) consequence that expansion by $\phi \wedge \psi$ and expansion by $\phi \rightarrow \psi$ have the same result, though it could be argued to be a consequence of the discrepancy between a logic and a model mentioned above, rather than a fault of this logic in particular.

The axioms for dynamic operators are given in Figure 2 (note that, in + K, J

[27] The valuation and the axioms for narrowing take the form they do largely due to the interpretation of narrowing by a complex sentence discussed after Definition 9. Note however that $(\mathbf{B},\chi) - p_1 - \cdots - p_n$ does not necessarily coincide with $(\mathbf{B},\chi) - \phi$ for all ϕ: $p \vee \neg p \vee q$ is a case in point (both p and q are absent from $\mathbf{B} - p - q$ but p is not necessarily absent from $\mathbf{B} - \phi$).

Figure 2: Dynamic axioms and rules

+ prop for $p \in P$, $[+\phi]p \leftrightarrow p$

+ ∧ $[+\phi](\psi \wedge \chi) \leftrightarrow [+\phi]\psi \wedge [+\phi]\chi$

+ ¬ $[+\phi]\neg\psi \leftrightarrow \neg[+\phi]\psi$

+ A $[+\phi]A\phi$

+ K for $\pi = \bigvee_{j \in J} p_j \vee \bigvee_{k \in K} \neg p_k$ for p_j, p_k distinct elements of Φ, $A\psi \wedge \bigwedge_{l \in J \cup K} \neg A p_l \rightarrow ([+\phi]K(\psi \vee \pi) \leftrightarrow K\psi)$

– prop for $p \in P$, $[-\phi]p \leftrightarrow p$

– ∧ $[-\phi](\psi \wedge \chi) \leftrightarrow [-\phi]\psi \wedge [-\phi]\chi$

– ¬ $[-\phi]\neg\psi \leftrightarrow \neg[-\phi]\psi$

– A for $p \in \Phi$, $[-\phi]\neg Ap$

– K $[-\phi]K\psi \leftrightarrow K\psi \wedge [-\phi]A\psi$

and K may be empty, and ψ may simply be $\bot$). Let $\mathbf{Aw^{rd}}$ be the result of extending $\mathbf{Aw^{r}}$ by these axioms. We have the following theorem.

Theorem 5. *The system* $\mathbf{Aw^{rd}}$ *is sound and complete with respect to* $\mathcal{B}^e$.

As in the AGM case (Section 2.1), a specific axiom is required to guarantee that the operator involved in becoming aware is expansion rather than any other extension. This axiom is + K, which states, more or less, that as little is learnt as possible on becoming aware. Once again, as in the AGM case, the axiom system can be weakened to a system which is sound and complete, but with respect to extensions in general (Definition 8) rather than expansions in particular. Let $\mathbf{Aw^{rd_{ext}}}$ be the system $\mathbf{Aw^{rd}}$ with + K replaced by the weaker axiom

$$+_{\mathbf{ext}}\mathbf{K} \qquad K\psi \rightarrow [+\phi]K\psi$$

Consider a function f_{ext} which associates to every pointed epistemic algebra $(\mathbf{B}, \chi)$ and every sentence ϕ an extension of $(\mathbf{B}, \chi)$ by the primitive sentences $p_1, \ldots, p_n \in \Phi$ (Definition 8). f_{ext} specifies one among the many extensions of $(\mathbf{B}, \chi)$ which contain ϕ and still respect the theorist's language. Satisfaction of $\mathcal{L}_P^{KA+-}$ will be defined relative to a pointed epistemic algebra and a function f_{ext}: all the satisfaction clauses given above are essentially unchanged, except the satisfaction clause for $[+\phi]$, which becomes

$$((\mathbf{B}, \chi), \Xi), f_{ext} \vDash [+\phi]\psi \quad \text{iff} \quad (f_{ext}((\mathbf{B}, \chi), \phi), \Xi) \vDash \psi$$

This says that the ψ will be true when the agent becomes aware of ϕ if and only if it is true in the pointed epistemic algebra which f_{ext} singles out as the appropriate extension of the original algebra to contain ϕ. Let $\mathcal{B}^e_{ext}$ be the set of pairs of pointed epistemic algebras in $\mathcal{B}^e$ and functions f_{ext} of the sort described above. Note that this setup and this satisfaction clause are a generalisation of the case of expansion presented above: this satisfaction clause reduces to the previous one on elements $((\mathbf{B},\chi),\Xi), f_{ext}$ where f_{ext} is an expansion; furthermore, the axiom + K, although not valid on $\mathcal{B}^e_{ext}$, characterises the subset of elements where f_{ext} is an expansion. The generalisation supports the following soundness and completeness result.

Theorem 6. *The system* $\mathbf{Aw}^{\mathbf{rd}_{\mathbf{ext}}}$ *is sound and complete with respect to* $\mathcal{B}^e_{ext}$.

As in Section 2.1, this result brings out the difference between expansion and extensions in general. The strong + K axiom characterises expansions, but does not apply to all extensions. Indeed, the much weaker axiom $+_{ext}$ K characterises all extensions, to the extent that, if it is satisfied in a pointed epistemic algebra $((\mathbf{B},\chi)$, then for any ϕ, there exists an extension by the primitive sentences in ϕ such that the "becoming aware" operator is faithfully interpreted by this extension; the f_{ext} singles out the appropriate extension in such cases.

Awareness is an important phenomenon for human knowledge and action. Moreover, it poses some interesting logical and philosophical questions. This paper offers a preliminary grip on the notion of awareness, and the important issue of awareness change; in the process some of the deeper questions have been touched upon. The central aims were to propose a model of awareness and awareness change that supports a rigourous understanding of the phenomenon and to use this model to develop logics both of awareness and awareness change. The first goal as accomplished in Section 1, with the introduction of the notion of interpreted algebra – a representation of the logical structure the agent is using at a given moment – which allowed a representation of the agent's epistemic state as understood from the agent's point of view, as well as a representation of changes of awareness in terms of operations on interpreted algebras. The second goal was accomplished in Section 2, where logics of awareness and awareness change were developed, both in the AGM-style and the DEL-style. Although more work remains to be done – notably, the results need to be extended to the multi-agent case – this paper offers a sold base for future development.

Appendix: proofs

Several of the proofs given here will rely on the assumption that the algebras are finite; as noted in Section 1.1, this assumption is well motivated in discussions of awareness, notably by the finiteness of human agents at particular moments. Moreover, recall from Definition 5 that, in the case of mappings between algebras, the same symbol shall be used both for an element of original algebra and its image under the mapping.

Proof of Proposition 1. Let $\phi_1, \ldots, \phi_n$ be the sequence of elements in $I_2 \setminus I_1$, and consider $\mathbf{B}_+ = \mathbf{B_1} + \phi_1 + \cdots + \phi_n$. (Since + is commutative, the order of the elements is of no importance.) It follows immediately from Definitions 5 and 7 that the interpreting algebras of $\mathbf{B}_+$ and $\mathbf{B_2}$ – B_{I_+} and B_{I_2} – are isomorphic.

The isomorphism between B_{I_+} and B_{I_2} and the homomorphism σ_B from B_1 to B_2 generate a homomorphism σ_+ from B_+ to B_2 which is injective on $B_1 \subseteq B^+$. Let y be the maximal element of $\sigma_+^{-1}(\perp)$ and pick $\psi \in \mathbf{B}_+$ with $q_+(\psi) = y$. By construction, $\mathbf{B}_+/\psi$ is isomorphic to $\mathbf{B_2}$ (as well as their interpreting algebras, their base algebras are isomorphic). (Note that, since σ_+ is an epistemic homomorphism, the set $\{x \in B_+ | \; y \leqslant x\}$ is a modal filter, so there are none of the well-known complications discussed in van Ditmarsch and Kooi (2006).) It is straightforward to check that the case of pointed (epistemic) goes through.[28] □

Proof of Theorems 1 and 2. It is straightforward to check that the operation of expansion satisfies the postulates.

Consider the other direction. Let $\sim$ be an operation on interpreted (epistemic) algebras satisfying (E 1 – 6) (resp. (E 1 – 6')) and consider an arbitrary algebra $\mathbf{B}$ and an arbitrary sentence ϕ. Let $\mathbf{B}' = \mathbf{B} \sim \phi$. If $\phi \in \mathbf{B}$, by (E 3) $\mathbf{B}' = \mathbf{B} = \mathbf{B} + \phi$. Consider now the case where $\phi \notin \mathbf{B}$. By (E 5), $I \subseteq I'$, and by (E 2), $\phi \in B_{I'}$, so $B_{I\cup\{\phi\}}$ is a subalgebra of $B_{I'}$. However, by (E 4), $B_{I'}$ is a subalgebra of $B_{I\cup\{\phi\}}$; so $B_{I'} = B_{I\cup\{\phi\}}$. By (E 5) again, there is an embedding σ_B of B into B'. Given Proposition 1 and Definition 8, this proves the interpreted algebra part of Theorem 2: $\mathbf{B}'$ is an extension of $\mathbf{B}$ by ϕ. For Theorem 1, by (E 6), there are no non-trivial relations between ϕ and the elements of $\mathbf{B}$ (this is guaranteed by (E 6) and (E 6') in the epistemic case); thus, there is an embedding from the free product $B \otimes B_{\{x\}}$ into B', which acts like σ_B on B and takes x to $q'(\phi)$. Since q' is surjective, $B' = B \otimes B_{\{x\}}$. By the commutation properties of the embedding, $\mathbf{B}' = \mathbf{B} + \phi$; the embedding in (E 5) is the canonical homomorphism between $\mathbf{B}$ and $\mathbf{B} + \phi$.

Consider finally an operation $\sim$ on pointed (epistemic) algebras which satisfies the appropriate postulates. By the reasoning above, the interpreting (epistemic) algebra of $(\mathbf{B}, \chi) \sim \phi$ is $\mathbf{B} + \phi$, for all $(\mathbf{B}, \chi)$ and ϕ. Let χ' be the knowledge element of $(\mathbf{B}, \chi) \sim \phi$. In the case of pointed algebras, by (E 7) and (E 8), χ' is the smallest element of $\mathbf{B} + \phi$ which is below χ, and this is none other than χ itself; so $(\mathbf{B}, \chi) \sim \phi = (\mathbf{B}, \chi) + \phi$. In the case of pointed epistemic algebras, (E 7) implies that $\chi' \Rightarrow_{\mathbf{B}+\phi} \chi$, and (E 8') implies that, for each $\psi \in \mathbf{B}$ such that $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \top$, $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \perp$ and $\chi \not\Rightarrow_{\mathbf{B}} \psi$, $\chi' \Rightarrow_{\mathbf{B}+\phi} \neg K(\psi \vee \phi)$ and $\chi' \Rightarrow_{\mathbf{B}+\phi} \neg K(\psi \vee \neg\phi)$. Since

[28]This proof assumes that the algebras are finite; however, an infinite version may be obtained by restricting by the set $\sigma_+^{-1}(\perp)$ rather than assuming a maximal element.

$\chi' \Rightarrow_{\mathbf{B}_\phi} K\chi'$, by Definition 4, it follows that χ' is neither below $\psi \vee \phi$ not $\psi \vee \neg\phi$ for such ψ. These properties determine the value of χ' in $\mathbf{B} + \phi$: it is the conjunction of χ with all $\neg K(\psi \vee \phi)$ and $\neg K(\psi \vee \neg\phi)$ for $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \top$, $K(\phi \vee \psi) \not\Leftrightarrow_{\mathbf{B}'} \bot$ and $\chi \not\Rightarrow_{\mathbf{B}} \psi$; so $(\mathbf{B}, \chi) \sim \phi = (\mathbf{B}, \chi) + \phi$. □

Proof of Theorem 3. It is straightforward to check that the operation of narrowing satisfies the postulates.

Consider the other direction. Let $\sim$ be an operation on interpreted (epistemic) algebras satisfying (N 1 - 5) and consider an arbitrary algebra $\mathbf{B}$ and an arbitrary sentence ϕ. Let $\mathbf{B}' = \mathbf{B} \sim \phi$. If $\phi \notin \mathbf{B}$, by (N 3) $\mathbf{B}' = \mathbf{B} = \mathbf{B} - \phi$. Consider now the case where $\phi \in \mathbf{B}$. By (N 5), $I' \subseteq I$, by (N 2), for all $\psi \in \Phi$, $\psi \notin B_{I'}$, so $B_{I'}$ is a subalgebra of $B_{I\backslash\Phi}$, where Φ is as stated in Definition 9. However, by (N 4), $B_{I\backslash\Phi}$ is a subalgebra of $B_{I'}$; so $B_{I'} = B_{I\backslash\Phi}$. By (N 5), and by the commutation properties of embeddings, B' is isomorphic to the image of $B_{I\backslash\Phi}$ in B under q and q' is the restriction of q to $B_{I\backslash\Phi}$. Hence $\mathbf{B}' = \mathbf{B} - \phi$.

Consider finally an operation $\sim$ on pointed (epistemic) algebras which satisfies, in addition to the other postulates, (N 6). By the reasoning above, the interpreting algebra of $(\mathbf{B}, \chi) \sim \phi$ is $\mathbf{B} - \phi$, for all $(\mathbf{B}, \chi)$ and ϕ. By (N 6), the knowledge element of $(\mathbf{B}, \chi) \sim \phi$ is the infimum, in $\mathbf{B} - \phi$, of $\{\psi \in \mathbf{B}' \mid \chi \Rightarrow_{\mathbf{B}} \psi\}$; this is just the minimal element ψ' of $\mathbf{B} - \phi$ such that $\chi \Rightarrow_{\mathbf{B}} \psi'$. Thus $(\mathbf{B}, \chi) \sim \phi = (\mathbf{B}, \chi) - \phi$. □

Proof of Theorem 4. Soundness, in both cases, is straightforward.

For the completeness of $\mathbf{Aw^p}$, consider a $\mathbf{Aw^p}$-consistent set of sentences of $\overline{\mathcal{L}_P^{KA}}$, Σ. Using traditional methods ("Lindenbaum's Lemma"), extend it to a maximal consistent set (of sentences of $\overline{\mathcal{L}_P^{KA}}$), Σ^m. Take Ξ to be the set of propositional elements of Σ^m; since the latter is maximally consistent, the former is as well. Let $I = \{p \in P \mid Ap \in \Sigma^m\}$, and $\mathbf{B} = (B_I, B, q)$, where B and B_I are Boolean algebras, B is isomorphic to B_I and q is the isomorphism. By axioms A1 and A2, for any $\phi \in \mathcal{L}_P$, $A\phi \in \Sigma^m$ iff $\phi \in \mathbf{B}$. Furthermore, A0 guarantees that the elements of $\{\psi \mid K\psi \in \Sigma^m\}$ are all elements of $\mathbf{B}$; K guarantees that this set is closed under logical consequence both of $\overline{\mathcal{L}_P^{KA}}$ and of $\mathbf{B}$ (by construction, the two logical consequences coincide, on the propositional language). Take χ to be the infimum of this set; it is thus an element of $\mathbf{B}$,[29] and indeed a non-$\bot$ element, since Σ^m is consistent. T guarantees that $\Xi' \Rightarrow_{\mathbf{B}} \chi$, for Ξ' the projection of Ξ into $\mathbf{B}$. $((\mathbf{B}, \chi), \Xi)$ satisfies Σ^m, and thus Σ.

For the completeness of $\mathbf{Aw^r}$, proceed in a similar way. Take a $\mathbf{Aw^r}$-consistent set of sentences of $\mathcal{L}_P^{KA}$, Σ, and a maximal consistent set extending it, Σ^m. Take

[29] χ evidently exists and is an element of $\mathbf{B}$ in the finite case, which as noted previously, is the pertinent one here. χ will actually exist even if the algebras are infinite, since B is freely generated. Similar points apply to the case of epistemic algebras considered below.

Ξ and I as above, and $\mathbf{B} = (B_I, B, q)$, where B and B_I are isomorphic epistemic algebras. In this case, A1-A4 guarantee that, for any $\phi \in \mathcal{L}_P^{KA}$, $A\phi \in \Sigma^m$ iff $\phi \in \mathbf{B}$; A5 guarantees that one can take $A\phi \Leftrightarrow_{\mathbf{B}} \top$ for all $\phi \in \mathbf{B}$. As above, take χ to be the infimum of $\{\psi \mid K\psi \in \Sigma^m\}$; A0 guarantees that this is an element of $\mathbf{B}$, 4 guarantees that $\chi \Rightarrow K\chi$, 5 guarantees that if $\chi \not\Rightarrow \psi$, then $\chi \Rightarrow \neg K\psi$, and the consistency of Σ^m combined with the coincidence between the consequence relations in $\mathbf{B}$ and $\mathcal{L}_P^{KA}$ guarantee that it is not equivalent to $\perp$. T assures the condition on the relation between χ and Ξ. So $((\mathbf{B},\chi),\Xi)$ is a pair, consisting of an epistemic interpreted algebra and a maximal consistent set of propositional sentences, that satisfies Σ^m and thus Σ.

□

Proof of Theorem 5. Soundness is straightforward. For completeness, consider Σ, a consistent set of sentences $\mathcal{L}_P^{KA+-}$. Take a maximal consistent extension Σ^m, and construct $((\mathbf{B},\chi),\Xi)$ using the $\mathcal{L}_P^{KA}$-fragment of Σ^m as in the proof of Theorem 4. By considering the set $\{\psi \mid [+\phi]\psi \in \Sigma^m\}$ for each ϕ, and using the same technique as in the proof of Theorem 4, construct pairs of interpreted epistemic algebras and maximal consistent sets of propositional letters for each $\phi \in \mathcal{L}_P^{KA}$; similarly for narrowing. It remains to show that, for any ϕ, the pair constructed is the result of applying the appropriate expansion (resp. narrowing) operations.

Consider just expansion; the case of narrowing is similar and simpler. Take an arbitrary ϕ, and let $((\mathbf{B}_\phi,\chi_\phi),\Xi_\phi)$ be the pair constructed, as described above, using $\{\psi \mid [+\phi]\psi \in \Sigma^m\}$. + prop, + $\wedge$ and + $\neg$ guarantee that $\Xi_\phi = \Xi$. It follows from + A and A1 – A4 that $\mathbf{B}_\phi = \mathbf{B} + p_1 + \cdots + p_n$, for $\Phi = \{p_1, \ldots, p_n\}$. + K implies that $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} \chi$ and that, for each $\psi \in \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}} \psi$, $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} \neg K(\psi \vee \pi)$, for π any Boolean combination of elements of P in ϕ but not in $\mathbf{B}$. Since $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} K\chi_\phi$, it follows that $\chi_\phi \not\Rightarrow_{\mathbf{B}_\phi} \psi \vee \pi$ for such ψ and π. So χ_ϕ is the conjunction of χ with all such $\neg K(\psi \vee \pi)$, and thus $((\mathbf{B}_\phi,\chi_\phi),\Xi_\phi) = ((\mathbf{B},\chi),\Xi) + p_1 + \cdots + p_n$.

□

Proof of Theorem 6. Only the case of extension need be considered, since the others have been dealt with above. Soundness is straightforward. For completeness, consider Σ, a consistent set of sentences $\mathcal{L}_P^{KA+-}$; take a maximal consistent extension Σ^m, and construct $((\mathbf{B},\chi),\Xi)$ using the $\mathcal{L}_P^{KA}$-fragment of Σ^m as in the proof of Theorem 4. For an arbitrary ϕ, consider the set $\{\psi \mid [+\phi]\psi \in \Sigma^m\}$ and construct a pair $((\mathbf{B}_\phi,\chi_\phi),\Xi_\phi)$ consisting of an interpreted epistemic algebra and a maximal consistent set of propositional letters as follows. Take Ξ_ϕ to be the set of propositional elements ψ such that $[+\phi]\psi \in \Sigma^m$; since the latter is maximally consistent, the former is as well. Let $I_\phi = \{p \in P \mid [+\phi]Ap \in \Sigma^m\}$, and π be the supremum of $\{\psi \in B_{I_\phi} \setminus B_I \mid [+\phi]K\neg\psi \in \Sigma^m\}$. Let $\mathbf{B}_\phi = (B_{I_\phi}, B_\phi, q_\phi)$, where B_{I_ϕ} is the epistemic algebra generated by I_ϕ and B_ϕ is the quotient by π (with q_ϕ the quotient map).[30] Finally, let χ_ϕ be the infimum of $\{\psi |\ [+\phi]K\psi \in \Sigma^m\}$. By the reasoning in the proof of Theorem 4, this is a well-defined knowledge element.

Note the following properties of $((\mathbf{B}_\phi, \chi_\phi), \Xi_\phi)$. By + prop, + $\wedge$ and + $\neg$, $\Xi_\phi = \Xi$. It follows from + A and A1 – A4 and the construction of B_ϕ that there is an embedding of $\mathbf{B}$ into $\mathbf{B}_\phi$; in fact, it follows that $\mathbf{B}_\phi = \mathbf{B} + p_1 + \cdots + p_n/\pi$ (recall that $\Phi = \{p_1, \ldots, p_n\}$). Finally, it follows from $+_{ext}$ K that $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} \chi$. However, by construction, $\chi \Rightarrow_{\mathbf{B}_\phi} \chi_\phi$, so χ_ϕ is the image of χ in the embedding. Thus $(\mathbf{B}_\phi, \chi_\phi)$ is an extension of $(\mathbf{B}, \chi)$ by $p_1, \ldots, p_n$ and Σ is satisfied in $((\mathbf{B}, \chi), \Xi), f_{ext})$ where $f_{ext}((\mathbf{B}, \chi), \phi) = (\mathbf{B}_\phi, \chi_\phi)$.

□

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[30]That is, the result of removing off the $q(\pi)$-worlds. Note that there are no Moore-style complications in this case, for much the same reason as in the proof of Proposition 1.

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