On Some Collusive and Signaling Equilibria in Ascending Auctions for Multiple Objects[1]

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Abstract

We consider two ascending auctions for multiple objects: the SEAMO (simultaneous English auction for multiple objects) and the the JAMO (Japanese auction for multiple objects). We first derive a (competitive) Perfect Bayesian Equilibrium of the JAMO by exploiting the strategic equivalence between the JAMO and the Survival Auction which consists of a finite sequence of sealed-bid auctions. Then, we prove that many of the (unwanted) collusive or signaling equilibria studied in the literature in the framework of the SEAMO do not have a counterpart in the JAMO. However, it is shown that certain collusive equilibria based on retaliatory strategies do exist in both auctions.

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1 Introduction

Since the first series of FCC spectrum auctions held in the US, academics and policymakers alike have recognized almost unanimously at least three main advantages of the openness and simultaneity of the FCC auction rules: They ensure a fully transparent bidding process, that enables extensive information revelation of bidders' valuations, and at the same time allows bidders to build efficient aggregations of licenses.[2] Yet the openness and simultaneity of the FCC auctions also facilitate tacit collusion. Bidders can observe each other's behavior and can thus coordinate on collusive agreements. Cramton and Schwartz (1999, 2000) report on bidding phases of the FCC which illustrate many of the communication and coordination devices tacitly used in practice by bidders; Klemperer (2001) provides further evidence and discussion, also relating to the recent European UMTS auctions.

In this paper we consider two auction mechanisms which are simplified versions of the FCC and of some European UMTS auctions: The SEAMO (simultaneous English auction for multiple objects), which is the version closer to the actual FCC auctions, and the JAMO (Japanese auction for multiple objects), which differs in at least two basic respects. Both auctions are simultaneous ascending auctions. However, unlike the SEAMO and the FCC auctions, in the JAMO, prices are raised directly by the auctioneer, and closing is not simultaneous but rather license-by-license. We show that these two differences are already sufficient to eliminate many (unwanted) collusive or signaling equilibria that are equilibria of the SEAMO.[3] In particular, jump bid equilibria constructed in Gunderson and Wang (1998) and collusive equilibria constructed in Engelbrecht-Wiggans and Kahn (1998) and Brusco and Lopomo (2001) are not equilibria of the JAMO. Nonetheless, we show that equilibria involving retaliatory strategies do exist in both the JAMO and the SEAMO. These equilibria share some features with bidding behavior reported by Cramton and Schwartz (1999, 2000) in the actual FCC auctions.

More generally, our results are inspired by the following questions: Can we establish a link between some well identified auction rules in the FCC auc-

[2]See e.g., McAfee and McMillan (1996), Cramton (1997, 1998), Milgrom (1998), Cramton and Schwartz (1999, 2000), Klemperer (2001).

[3]Albano *et al.* (2001) and Branco (2001) provide evidence that the JAMO may perform well in terms of both efficiency and revenue in certain environments with complementarities.

tions and the emergence of a particular type of collusive equilibria? To what extent are signaling, collusive or retaliatory strategies sensitive to certain modifications of the auction rules? These questions are more than theoretic preoccupations. Economists and market designers weigh off the pros and cons of different auction mechanisms in order to maximize revenue and/or efficiency in the allocation of scarce resources. However, the pursuit of those objectives would be hampered by choosing an auction mechanism which is deemed to facilitate collusion or retaliation. Our analysis suggests that the SEAMO does facilitate (tacit) collusion relative to the JAMO.

The paper is organized as follows. Section 2 contains a description of the framework and of the actual auction rules. In Section 3, which is the main section, we consider a series of signaling equilibria and show that many of the equilibria of the SEAMO have no counterpart in the JAMO. Section 4 indicates some directions for future research. The main proofs are contained in the Appendix.

2 Three Ascending Auctions

2.1 Framework

Throughout the paper we work with a framework close to the one of Krishna and Rosenthal (1996). Two objects are auctioned to a set of participants of two types: M global bidders who are interested in both objects and N_k local bidders who are interested in only one of the two objects, $k = 1, 2$. Both global and local bidders draw their values all independently from some smooth distribution F with positive density f, both defined over $[0, 1]$. Let v_k and u_k denote the stand-alone value of object $k = 1, 2$ to a global and to a local bidder respectively. The value of the bundle v_B to a global bidder is greater or equal to the sum of stand-alone values, that is,

$$v_B = v_1 + v_2 + \alpha,$$

where $\alpha \geq 0$ is commonly known and coincides across all global bidders. The nature of bidders, local and global, is also commonly known.[4]

[4]The fact that a bidder with $v_1 = 0$ and $v_2 > 0$ qualifies as a global bidder when $\alpha = 0$ is a degenerate case; we stress that what distinguishes global from local bidders is that

We further restrict the analysis to the following cases: (i) $v_1 = v_2 = x \in [0, 1]$ and $\alpha \geq 0$. (ii) $v_1, v_2 \in [0, 1]$ and $\alpha = 0$; (iii) $v_1, v_2 \in [0, 1]$ and $\alpha > 1$; Krishna and Rosenthal (1996) consider case (i); Brusco and Lopomo (2001) consider cases (ii) and (iii).

2.2 Auction Rules

The auction mechanisms we consider are more or less simplified versions of the simultaneous ascending auction used by the FCC for the sale of spectrum licenses in the US. The third mechanism (SA) is equivalent to the first (JAMO) and is used mainly to simplify some of the analysis. We briefly describe the rules.

JAMO: Prices start from zero for all objects and are simultaneously and continuously increased on all objects until only one agent is left on a given object, in which case prices on that auction stop and continue to rise on the remaining auctions. Once an agent has dropped from a given auction the exit is irrevocable. The last agent receives the object at the price at which the auction stopped. The number and the identity of agents active on any auction is publicly known at any given time, (including information about the identity of local and global bidders). The overall auction ends when all agents but one have dropped out from all auctions. We refer to this mechanism as the Japanese auction for multiple objects (JAMO); some also refer to it as the English clock auction.

SEAMO: The auction proceeds in rounds. At each round, $t = 1, 2, ..,$ each bidder submits a vector of bids where bids for single objects are taken from the set $\{\emptyset\} \cup (b^k(t-1), +\infty)$, where $\emptyset$ denotes "no bid", and $b^k(t-1)$ is the "current outstanding" bid, that is, the highest submitted bid for object k up to round $t - 1$. Thus for each object k a bidder can either remain silent or raise the high bid of the previous round of at least $\nu > 0$, (we take ν arbitrarily close to zero). All licenses close simultaneously. The auction ends if all bidders remain silent on all objects, and the winners are the "standing high bidders" determined at round $t - 1$ and they pay their last bid. Given the simultaneity of closing, we refer to this mechanism as the simultaneous

global bidders have a potential for obtainig a positive value from each object besides the typically positive complementarity.

English auction for multiple objects (SEAMO).

Two basic differences distinguish the two mechanisms. First, the JAMO does not allow for rounds of bidding; bidders press buttons corresponding to the objects on which they wish to bid; by releasing a button, a bidder quits that auction irrevocably; thus, bidders have "smaller" strategy spaces than in SEAMO; in particular they have no influence on the pace at which prices rise. Second, closing is not simultaneous in the JAMO but rather license-by-license. We shall highlight the role of these distinguishing features in the emergence of collusive and signaling equilibria.

SA: The auction proceeds in rounds. At each round, $t = 1, 2, ..$, each bidder submits a vector of sealed bids where bids for a single object are taken from the set $\{\emptyset\} \cup (b(t-1), +\infty)$, where $\emptyset$ again denotes "no bid" or "not allowed to bid", and $b(t-1)$ here is the lowest among *all* bids submitted during the previous round. In the following rounds, the bidders who submitted $b(t)$ in the current round, are not allowed to bid on the object where they submitted $b(t)$. At the end of each round the auctioneer only announces $b(t)$, the object for which $b(t)$ was submitted and the identity of the bidders that submitted the bid. An object is attributed to the last bidder having the right to bid on that object and the winner will pay an amount of money equal to the last lowest bid on that object. Note that since at least one bidder exits from some object in any given round, the two objects will be attributed after at most $2(M + N_1 + N_2) - 1$ rounds. Following Fujishima *et al.* (1999), we refer to this auction as the survival auction (SA).

2.3 Some Basic Results

Before turning to the collusive and signaling equilibria, we derives some basic results on the three mechanisms just described. The main result (Prop. 2) characterizes a natural Perfect Bayesian Equilibrium (PBE) of the JAMO, which in turn induces a corresponding equilibrium in the other two mechanisms.

In the JAMO or the SA, the information available to a bidder at any time t, is described by H_t. In the JAMO, t coincides with the current level reached by prices, while, in the SA, t represents the current minimum admissible bid on any of the objects; in both mechanisms, H_t contains, for each object, the set of active bidders on the object as well as, the price at which the other bidders dropped out. The following proposition due to Fujishima *et al.* (1999) is useful in characterizing equilibria in the JAMO; it is used in the proof of Prop. 2.

Proposition 1 *The JAMO and the SA are strategically equivalent.*

Proof. The proof is as in Fujishima *et al.* (1999). It consists in showing isomorphism (identity) of the information sets H_t as well as of their precedence relation in the two auctions; from this one shows isomorphism (again identity) of the action spaces and that corresponding actions induce same payoffs. ■

The proposition implies that the JAMO and the SA are outcome equivalent and that their equilibria coincide. This allows us to use the easier SA to analyze the JAMO.

Next, we derive a PBE of the SA (and JAMO), which is obtained as the solution to certain equations defining the optimal bid (or exiting time). Fix an object, say object 1, and fix a bidder who is active on the object. Let $\Pi(p, H_t)$ denote the expected payoff to the bidder, if, at time t and given the information set H_t, he submits a bid of p on that object, while the other bidders are assumed to play optimal strategies in the current round, and all bidders are assumed to play optimal continuation strategies in the following rounds. We assume these have already been determined appropriately and are known to the bidder. It can be shown that optimal bids (equivalently, optimal exiting times) can be obtained as the smallest solutions to equations of the form:

$$\Pi'(p, H_t) = 0. \tag{1}$$

These conditions can be used to compute, for any bidder and any object, the optimal bid (equivalently, exiting time). Consider a bidder active only on one object, say 1; he will bid $p = v$, where v is his value added from purchasing object 1, (i.e., u_1 or v_1 for locals or globals buying only that object, and $v_1 + \alpha$ for globals who have already bought object 2). This can be obtained from Eq. (1) since it reduces to $(v - p) \cdot g(p) \leq 0$ for some density g, which yields the standard condition, $p = v$, for English or second price auctions on one object. Similarly, in the absence of synergies (case (ii)) a global bidder active on both object will bid v_1 on object 1 and v_2 on object 2. In cases (i) and (iii), if all local bidders have already exited the auction, the game is equivalent to an auction for a single object, the bundle, where bidders pay twice their bid, and so, a global bidder will bid $v_B/2$ on each object.[5] In the presence of both global and local bidders, the characterization of optimal bid in cases (i) and (iii) is more complex: for a global bidder active on both objects, Eq. (1) can be shown to reduce to

$$\left(\frac{N_1}{N_1+N_2}\pi_{L_1}(p,H_t)+\frac{N_2}{N_1+N_2}\pi_{L_2}(p,H_t)\right)f_{N_1+N_2}(p|t)=0, \qquad (2)$$

where $\pi_{L_k}(p,H_t)$ is the continuation payoff if a local bidder on object k and all the other global bidders on the two objects set the lowest bid p in the round corresponding to H_t, and $f_{N_1+N_2}(\cdot|t)$ is another conditional density that is specified in the proof. Notice that as long as local bidders are active, the bid that solves Eq. (2) does not depend on the number of other global bidders active on the two objects. We can state:

Proposition 2 *Suppose bidders bid only on objects they value. Then the exiting times for both local and global bidders conditional on information until time t, $\{\tau^*(H_t)\}$, which are obtained as the smallest solutions to Eq. (1), constitute a symmetric PBE of the JAMO.*

Proof. See the Appendix. ■

Thanks to Prop. 1, we can prove Prop. 2 without using the optimal-stopping machinery that would be necessary to compute the exiting time in the JAMO. Indeed, as the SA ends in a finite number of rounds, we can characterize the equilibrium using backward induction arguments.

A crucial step in the proof of Prop. 2 is that , in cases (i) and (iii), a global bidders active on two objects has a best response to submit the same bid for the two objects. In other words, in the JAMO, a global bidder exits both objects simultaneously. Therefore, his optimal bid will not depend on how the other bidders behave after he has quit the auctions. This simplifies the characterization of the equilibrium. Moreover, in case (i), it can be shown

[5] For example, when the complementarity is very high, say $\alpha > 2$, a global bidder is always willing to pay for the bundle more than any two local bidders, and so all local bidders will exit in early rounds of the auction, and only global bidders will bid for the bundle.

that the equilibrium of Prop. 2 is the unique symmetric PBE in (weakly) undominated strategies when bidders bid only on objects they value.[6]

However, for intermediate values of $\alpha \in (0, 1)$, and if v_1 and v_2 are different, submitting the same bid on the two objects is not always optimal. Then a global bidder's optimal bids will depend on how exiting on one object will affect the other global bidders' behavior on the other object. In this case showing existence of a PBE is already problematic.[7]

Next, we relate equilibria of the SEAMO and the JAMO.

Proposition 3 *Every PBE of the JAMO induces a PBE of the SEAMO.*

Proof. Let $\{t^*(H_t)\}$ be the exiting times constituting a PBE for the JAMO. Then all bidders bidding the standing high bid plus an arbitrarily small bid increment in each round and stopping to bid according to these exiting times (also along out of equilibrium paths) constitutes an (arbitrarily close) PBE of the SEAMO. Because winning bidders pay their own last high bid and because of the simultaneity of the closing, there are no profitable deviations from the above strategies. ■

This also implies that the set of outcomes induced by PBE of the JAMO is contained in the set of outcomes induced by PBE of the SEAMO. The converse of this as well as of Prop. 3 is not true; the SEAMO has many more equilibria. In what follows, we will see examples of equilibria that are PBE of the SEAMO but not otherwise.

In particular, Prop. 3 implies that the equilibrium of Prop. 2 has a counterpart in the SEAMO. Albano *et al.* (2001), within an example with 2 objects and 4 bidders, argue that the JAMO obtains close to ex-post efficiency with higher revenues than the revenue-maximizing ex-post efficient mechanism, and that it dominates both the sequential and the one-shot simultaneous auctions in terms of ex-ante efficiency. Branco (2001) obtains similar results in a somewhat different framework. Given the above proposition, these results immediately extend to corresponding equilibria of the SEAMO.

[6] See Section 3.3 for more discussion on the latter assumption. Essentially, it says that local bidders do not bid on both objects.

[7] For example, Athey's (2001) theorem does not apply due to the presence of non unidimensional signals; McAdams' (2001) theorem does not apply because of the modularity condition on the payoffs.

3 Collusive and Signaling Equilibria

In this section, we consider certain collusive and signaling devices and equilibria that have been studied in the literature, typically in the framework of the SEAMO, and show that they are not viable in the JAMO, due to the more restrictive nature of the strategy spaces. We also construct equilibria involving retaliatory strategies for both the JAMO and the SEAMO.

3.1 Some Signaling Devices

Bidders in the FCC auctions attempted to communicate in a variety of ways. Since there is no way of proving any private exchange of information among bidders, we are bound to analyze communication arising through the exploitation of the auction rules themselves. This section analyzes some common communication devices also apparently used in the actual FCC auctions, namely code and jump bidding, and withdrawal bids, from the viewpoint of the JAMO.

Code Bidding: Code bidding is one of the more obvious forms of signaling. Since bids are expressed in dollars and since, at least in the FCC auctions, most licenses displayed six-digit prices, bidders could use the last three digits to encode messages. Code bids had different natures. Some bidders used the last three digits to "disclose" their identities. For example, in the AB auction (Auction 4), GTE frequently used "483" as the last three digits; this number corresponds to "GTE" on the telephone keypad. In other circumstances code bidding had a *reflexive* nature. The last three digits were used by a bidder both to signal a license of special interest to her and the license on which the same bidder was punishing competitors for not bumping the first market.[8]

In the JAMO mechanism, bidders are obliged to use code bidding in very specific way: to stop bidding on a given license as soon as the price encodes "meaningful" digits. However, this strategy would irrevocably exclude that bidder from competing for that license, and with two objects, would therefore also exclude her from bidding for the bundle; moreover it would also exclude her from performing any retaliation, since she should presumably be

[8]See Cramton (1997) and Cramton and Schwartz (1999, 2000) for detailed accounts of collusive behavior in the actual FCC auctions.

interested in purchasing the only remaining object. It follows that:

Proposition 4 *Code bidding is ineffective in any PBE of the JAMO (with two objects).*

While this excludes signaling equilibria that rely on code bidding when two objects are auctioned, the result may not extend to more than two objects. For example, suppose three licenses are being auctioned, suppose a bidder is interested in purchasing license, say 1, and that she is active on all licenses at an early stage of the auction. Then she can stop bidding on, say, license 3 at a price whose digits encode a message similar to the one used by GTE, while remaining active on the other two licenses. This allows her to use license 2 as a potential threat for retaliation. The extent to which retaliation will be successful or credible so as to eventually constitute a PBE of the corresponding game, is something that is explored further below (in the context of two objects). But in principle, a higher number of licenses for sale (without restrictions on the number of licenses bidders are allowed to bid on) makes for more possibilities of sending messages or code bids even in the JAMO auction. Clearly, such a signaling device becomes more difficult and costly to use if prices are raised not continuously but in predetermined finite amounts.

Jump Bidding: It need not always be in the interest of the bidders to increase prices at the minimum pace required by the auction rules. In fact, Gunderson and Wang (1998) show how a bidder in a SEAMO can benefit by using jump bids as a signal of a high valuation, possibly causing other bidders to drop out earlier; this may lead to lower revenues for the seller.[9] While jump bids are possible in the SEAMO they are obviously not in the JAMO. The FCC's recent decision to limit the amount by which bids can be raised e.g., in the LMDS auction (Auction 17), may suggest a change in this direction, see also Cramton and Schwartz (2000).

Bid Withdrawals: While the FCC had originally allowed unlimited number of bid withdrawals in order to allow bidders to make more efficient aggre-

[9] A crucial assumption for the existence of these equilibria is that the bidder making the jump bids have discontinuous support for valuations. See also Avery (1998) for further equilibria involving jump bids in the context of one-object English auctions with affiliated values.

gations of licenses, it was soon noticed that they could be used as signaling devices. As Cramton and Schwartz (2000) report, withdrawal bids were apparently used in FCC auctions as part of a warning or of retaliatory strategies, as well as part of cooperative strategies, where bidders attempted to split licenses among themselves. Neither the JAMO nor the SEAMO versions described above allow for withdrawal bids. Again, the FCC's recent decision to limit their number to two, e.g., in the LMDS auction (Auction 17), suggests another change in this direction.

3.2 Closing Rules

Milgrom (2000) contains a description of the tâtonnement logic that inspired most of the FCC auction rules. In particular, the rules specified that bidding would remain open on all licenses until there were no new bids on *any* license. This simultaneous closing rule allows each losing bidder to switch at any time from the lost license to a substitute or to stop bidding on a complement. However, as Milgrom points out, it is also vulnerable to collusion.

Milgrom's Example: Consider the following example from Milgrom (2000). Two bidders bid for two objects 1 and 2, which are each worth 1 to both bidders. Milgrom shows that there exists a sequential equilibrium of the SEAMO (with complete information) such that the selling price for both objects is ν, i.e., the smallest possible bid, and the bidders realize the highest collusive payoff of $2 \cdot (1 - \nu)$, (see Theorem 8, p. 264).

The logic of the equilibrium is that both players buy one object each at the lowest possible price by using a simple threatening strategy: Bidder 1 bids ν on auction 1 if bidder 2 has never bid on 1; otherwise he does not bid. If bidder 2 has bid on 1, then bidder 1 reverts to a "competitive" bidding strategy, that is to keep bidding on each object until a price of 1 is reached; bidder 2 plays symmetrically.

As Milgrom suggests, such a low revenue equilibrium is avoided if closing is not simultaneous but rather license-by-license. According to such closing rule, bidding would stop on a license if at any round there is no new bid on that license. The JAMO provides an example of license-by-license closing. Indeed, once all bidders but one drop from one license and remain active on the other licenses, the first license closes irrevocably. The result of Theorem 9 in Milgrom (2000), which states that at each (trembling-hand) perfect equilibrium with license-by-license closing the price of each license is at least $1 - \nu$ carries over to the JAMO (also with complete information), where in fact the price of each license is exactly 1. By applying Prop. 2 to the example described above where as in our usual framework the bidders' values are private information, the following result can be shown to follow:

Corollary 1 *Suppose that bidders 1 and 2 have (private) values of 1 for both objects, and $\alpha = 0$, then, in the PBE of the JAMO, the selling price is 1 for each object.*

Such a selling price of 1 (or $1 - \nu$) is also not guaranteed in the SEAMO with incomplete information as the equilibria constructed in Engelbrecht-Wiggans and Kahn (1998) and Brusco and Lopomo (2001) show.

The Collusive Equilibria of Brusco and Lopomo: Brusco and Lopomo (2001) (henceforth BL) construct several kinds of PBE in undominated strategies of the SEAMO (in our usual framework), some of which are very similar to the ones constructed by Milgrom under complete information. Kwasnica and Sherstyuk (2002) provide some experimental evidence for such equilibria when there are few players and with small complementarities. We shall see that none of BL's equilibria are possible in the JAMO.

The logic of their collusive equilibria is as follows: Consider two global bidders and, for simplicity, take $\alpha = 0$. The bidders use the first round to signal to each other which of the two objects they value the most. If they rank the objects differently, bidders confirm their initial bids in all subsequent rounds and obtain their most preferred object at the minimum price; otherwise they revert to the "competitive" strategy of raising prices on both objects up to their private values. BL then go on to refine this type of collusive equilibrium by allowing bidders to signal more than just the identity of the higher valued object. This allows them to obtain collusive equilibria even more favorable to the bidders. In particular, they show that a collusive equilibrium may also arise when bidders have the same ranking for the objects, also if there are more than two bidders as well as if there are positive complementarities ($\alpha > 1$); they also show, however, that the scope for collusion diminishes as the number of bidders increases and the number of objects is fixed at two; and the possibility of collusion is lowered if the complementarities are large and variable.

Again, the rule driving the presence of such equilibria is the simultaneous closing. The JAMO mechanism instead is built around the irrevocable exit and induces license-by-license closing, which makes the rounds of signaling necessary in the above equilibria impossible. In these examples bidders always have an incentive to bid for any object for which they have positive value. In particular, it follows:

Corollary 2 *The collusive equilibria constructed as PBE of the SEAMO in Brusco and Lopomo (2001) are never PBE of the JAMO.*

Note also that these collusive equilibria are not PBE of the JAMO even if one allows for rounds of cheap talk between the bidders prior to the auction.

As it has often been pointed out, simultaneous closing has the advantage of being more flexible in allowing bidders to revise and update their bidding behavior in forming aggregates, (see e.g., Cramton (1997, 1998), Milgrom (1998, 2000), Cramton and Schwartz (1999, 2000)). Moreover, Kagel and Levin (2000) point out that, especially for intermediate values of the complementarities, ascending auctions may suffer from the *exposure problem* by which global bidders may drop out too early from individual licenses thus reducing efficiency. Although their comparison is with one-shot sealed bid auctions, it seems plausible the exposure problem would be even more pronounced in auctions with license-by-license closing than in ones with simultaneous closing. This is something that needs to be further investigated, also in connection with the rules for withdrawing bids.

3.3 Retaliatory Equilibria and Withdrawal Rules

We have seen examples of collusive equilibria that are equilibria of the SEAMO but are ruled out as equilibria of the JAMO. In this section, we show that equilibria involving certain retaliatory strategies may nonetheless exist in both the SEAMO and the JAMO.

Retaliatory Equilibria: The logic of these retaliatory equilibria is straightforward. Suppose that two objects are put for sale to two bidders, one global bidder who is interested in both objects, and one local bidder who wishes to buy only object 1. Assume all this to be common knowledge. The two bidders have overlapping interests on object 1, and the local bidder wants the global bidder to exit early from object 1. In order to achieve this, the local bidder actively bids on object 2, although the object has no value to him. Such a strategy is potentially costly to both the local and the global bidder; we refer to it as a *retaliatory strategy.* The extent to which the local bidder is successful in inducing the global bidder to drop out early from object 1 depends on whether he succeeds in making his threat credible. We show that the JAMO is not immune to equilibria that effectively involve such strategies.

Proposition 5 *There exist PBE of both the JAMO and the SEAMO where bidders use retaliatory strategies effectively.*

Proof. In Example 2 below, we construct a family of such equilibria in the context of the JAMO and the usual framework of Section 2.1; for the SEAMO there will be corresponding equilibria as in Prop. 2. ■

Before presenting the mentioned family of retaliatory equilibria, we first consider a simpler and more intuitive type of retaliatory equilibrium within a slightly more special framework.

Example 1. Consider our usual framework with two objects and two bidders; one local bidder interested in object 1 and one global bidder interested in both objects 1 and 2, with the same value for the two objects, $v_1 = v_2 = x$, and $\alpha = 0$; assume also all values are drawn according to the uniform distribution on $[0, 1]$. It is easy to see that the following is a PBE of the JAMO:

— all types of the local bidder bid on *both* objects and stay on object 1 until u_1 and on object 2 until $\min(u_1, t_1^2 + \nu)$, where t_1^2 is the global bidder's exiting time from object 1, and $\nu > 0$ arbitrarily small;

— all types of the global bidder exit from object 1 at $t + \nu$ if at t the local bidder is on object 2; otherwise all types of the global bidder stay on both objects until x.

In equilibrium, the global bidder immediately drops out of object 1 inducing the local bidder to also immediately drop out of object 2. As is often typical in such retaliatory equilibria, the retaliating bidder (here the local bidder) obtains a higher ex ante payoff than in the standard equilibrium of Prop. 2, while the other agents (here the global bidder and the auctioneer) are both worse off. ■

To sustain these equilibria, the local bidder threatens the global bidder with a harsh punishment if the latter does not drop out of object 1.

It can be checked that our assumption of uniformly distributed private values, which guarantees the optimality of the global bidder's strategy, turns out to be a special case of the assumption made by BL to sustain their "most collusive" equilibrium, namely, that the expected private value for each good is no less than 1/2.[10]

The above example relies on the fact that the local bidder has some extra information about the global bidder's valuation of object 1 relative to object 2. Without this information he needs to resort to a more refined form of signaling.

Example 2. Consider our usual framework of Section 2.1 with two objects and two bidders; one local bidder interested in object 1 and one global bidder interested in both objects 1 and 2, and suppose for simplicity $\alpha = 0$; assume again all values are drawn according to the uniform distribution on $[0, 1]$. Then, for any $l \in (0, 1]$, the following is a PBE of the JAMO:

— all types of local bidder with $u_1 \leq l$ bid only on object 1 and stay until u_1; all types of local bidder with $u_1 > l$ bid on *both* objects and stay on object 1 until u_1 and on object 2 until $c = l(\sqrt{2} - 1) < l$;

— all types of global bidder with $v_1 < l$ bid on both objects and stay on object 1 until c and on object 2 until v_2 whenever the local bidder is active on both objects, staying until v_1, v_2 respectively otherwise; all types of global bidder with $v_1 \geq l$ bid on both objects always staying until v_1, v_2 respectively.

This characterizes a family of retaliatory equilibria indexed by the parameter l that are PBE of the JAMO, (see Section 4 for a proof). Note that the equilibria are *not* in undominated strategies, since the local bidder always has a (weakly) dominant strategy to drop from object 2 whenever it is the only object he is bidding on. If the local bidder is active on both auctions this signals that his valuation is above the threshold l, i.e., $u_1 > l$; if he bids only on object 1, then $u_1 \leq l$, and both bidders bid up to their valuations

[10]We are grateful to a referee for pointing this out to us.

and only on the objects they value.

When $l = 1$ we get the standard, non–retaliatory equilibrium of Prop. 2, since with probability one the local bidder will not be active on object 1. When $l \to 0$ we almost get the standard equilibrium, since $c \to 0$, i.e., the local bidder enters both auctions but almost immediately exits object 2.

Unlike the equilibrium of Example 1, here, to ensure incentive compatibility for the local bidder, the bidding threshold c is such that he only weakly prefers the retaliatory equilibrium, his ex ante payoff is the same as in the standard equilibrium, i.e., 1/6; the global bidder continues to be worse off than in the standard equilibrium, her ex ante expected payoff being

$$\frac{2}{3} + \frac{l}{2} - \sqrt{2}l + \frac{3l^2}{2} - \frac{13l^3}{6} + \sqrt{2}l^3 + \frac{l^4}{6} \leq 2/3 \ \ \forall l,$$

while due to the extra bidding on object 2, the auctioneer actually earns higher ex ante revenues than in the previous example and in the standard equilibrium

$$\frac{1}{3} - l + \sqrt{2}l - 2l^2 + \sqrt{2}l^2 + 3l^3 - 2\sqrt{2}l^3 \geq 1/3 \ \ \forall l. \quad \blacksquare$$

Example 2, continued. It is worth pointing out that Example 2 can be extended to a local bidder competing against an arbitrary number M of global bidders with preferences as in cases (i) and (ii) of Section 2.1, and with general distribution function F defined on $[0, 1]$ for values. The PBE described above is still a PBE for any $l \in (\alpha, 1]$, where the parameter value $c(< l)$ is now chosen as the unique solution to the equation

$$\int_c^{l-\alpha} z \cdot G'(z)dz = (c - \alpha) \cdot G(l - \alpha), \tag{3}$$

where $G = F^M$ is the distribution function for the global bidders' highest valuation for object 1 (or 2). In presence of complementarities, the only difference is that a global bidder exits object 1 at c if $v_1 + \alpha < l$, whereas if $v_1 + \alpha > l$ he behaves as described in Section 2.3.[11] In order to ensure incentive compatibility for the local bidder, as M increases, the parameter c also increases. $\blacksquare$

[11] This suggests that such retalatory equilibria may disappear when $\alpha > 1$.

Examples 1 and 2 are also close in spirit to the collusive equilibria of BL: global bidders and the local bidder have overlapping interests on object 1; the local bidder threatens to retaliate (i.e., to be active) on object 2 if the global bidders do not exit object 1. This signaling device is effective since it is common knowledge that the retaliatory bidder is interested in one object only. Thus by not "turning the light off" on object 2 when the price is zero, the local bidder triggers the beliefs that sustain the collusive equilibrium.

JAMO vs. Sequential Auction: A necessary condition for triggering collusive equilibria in the JAMO is simultaneous bidding on both objects. The auctioneer could prevent bidders from adopting retaliatory strategy by selling the objects sequentially[12] rather than simultaneously. Indeed, it is easy to see that, at least with two objects, retaliatory strategies are ineffective, since local bidders cannot gain by bidding on an object they do not value.

However, despite its proofness against retaliatory equilibria, the sequential auction becomes *less* attractive than other mechanisms under the assumption that bidders play a "competitive" equilibrium. Indeed, Albano *et al.* (2001) and Branco (2001) show that the sequential auction performs rather poorly in terms of both efficiency and seller's revenue as compared to the JAMO and two one-shot sealed-bid auctions. In particular, the results confirm the intuition that simultaneous bidding is mainly responsible for the (good) performance of the JAMO both in terms of efficiency and seller's revenue. However, the existence of collusive equilibria in the JAMO suggests that there does exist a trade-off between designing auction rules in order to increase efficiency and revenue, and allowing bidders to exploit the same set of rules for reaching a certain degree of co-ordination.

Finally, we see what happens to the equilibria constructed in Examples 1 and 2 if one allows for withdrawal rules.

Withdrawal Rules: Withdrawal rules in the FCC auctions were originally designed to allow for a more efficient aggregation of licenses, and, until Auction 16, the FCC allowed an unlimited number of withdrawals. If a bidder decides to withdraw her bid from a license, the FCC becomes the standing high bidder, and the withdrawing bidder is charged a penalty equal to the difference between the withdrawn bid and the selling price after the with-

[12] We are grateful to another referee for suggesting this point.

drawal. However, if the penalty is sufficiently low, bidders might use bid withdrawals as a signaling device (as mentioned above) but also as part of a retaliatory strategy.

Consider first the equilibrium of Example 1. If bidders are allowed one bid withdrawal, then as long as the local bidder does not withdraw his bid for object 2 with probability greater than 1/2, this still leads to a PBE without really affecting the equilibrium outcome. It will still be optimal for the global bidder to immediately exit from object 1, and both objects are sold at zero prices in equilibrium. The only difference is that the out-of-equilibrium belief that the local will continue to bid on object 2 if the global continues bidding on object 1 is slightly more credible since the penalty to the local bidder is reduced.

While the possibility of withdrawing bids makes for cheaper retaliatory strategies, thus increasing the credibility that a bidder will continue to bid on an object he does not value, at the same time, it also takes away the commitment value that the retaliating bidder will buy the object he does not value. It is easy to see that introducing the possibility of one bid withdrawal destroys the equilibrium of Example 2, since on one hand, given the global bidder's strategy, the local bidder now has a strictly dominant (continuation) strategy to withdraw all bids where he ends up having to buy the object he does not value; unlike Example 1, this happens with positive probability in equilibrium. On the other hand, if the global bidder assumes that the local bidder will always withdraw his bid for an unwanted object, then she has a best response to exit from the local bidder's unwanted object, object 2, at any $\nu > 0$ and the standard equilibrium follows.

4 Conclusion

Recent research on multi-unit ascending auctions has highlighted the existence of two potentially conflicting features of the auction rules adopted by the FCC and subsequently in some of the European UMTS auctions. On one hand, the transparency and flexibility of the bidding process eases an efficient aggregation of licenses; on the other, the amount of information available to bidders together with the strategic possibilities allowed by the rules may be used to implement tacit collusive agreements, see Cramton and Schwartz (1999, 2000) and Klemperer (2001).

By not allowing bidders to set the pace at which prices rise on individual licenses, the auctioneer can make bidders' signaling devices blunt without losing the information revelation feature of the ascending mechanism. In this sense we have maintained that the SEAMO facilitates tacit collusion relative to the JAMO and have shown that several collusive equilibria, which appear in the SEAMO, do not have a counterpart in the JAMO.

We have also shown that certain retaliatory equilibria are possible in both the JAMO and the SEAMO. Again, it is evident from the construction of such equilibria that they are "harder" to implement in a JAMO than in a SEAMO. A more complete assessment of the relative performance of the two auctions certainly requires further study. We outline some directions for future research.

First, the framework is admittedly restrictive. For example, if the number of licenses is greater than two, the set of equilibria is likely to depend on the composition of the bundles that global bidders are interested in acquiring. That is, with more than two objects there are several ways preferences over bundles can overlap. It is also possible that code-bidding may reappear even in the JAMO. But even with only two objects, the case of mild synergies may already pose non-trivial existence problems.

Second, an issue that has not been addressed is the rationale of having prices rise simultaneously (i.e., at the same "speed") in the JAMO. We have imposed the same "speed" on both objects, being aware that there is no theoretical or empirical justification for this assumption.

Third, other aspects of the FCC auctions such as activity rules, the number of allowable bid withdrawals, and the simultaneity of closing deserve further investigation. Although some modifications of the standard SEAMO undertaken by the FCC may be seen as changes in direction of the JAMO, there seems to be no general agreement on e.g., whether closing should be simultaneous or not. Albano *et al.* (2001) and Branco (2001) show that under certain conditions, license-by-license closing may perform rather well theoretically. Kagel and Levin (2000) on the other hand provide experimental evidence indicating that, at least within certain ranges of bidders' valuations, inefficiencies may arise due to what they call the "exposure problem". Clearly, more needs to be done to better assess the theoretical and empirical performance of the "Japanese" vs. "English" design of the auction and the simultaneous vs. license-by-license closing, as well as of other rules mentioned. Also, while the JAMO and the SA are theoretically equivalent it would be useful to obtain further experimental evidence contrasting their relative performance.

Finally, motivated by considerations of market structure and bidder asymmetries, Klemperer (1998, 2001) suggests an auction format he calls "Anglo-Dutch" that combines an ascending or "English" auction with a first-price sealed-bid or "Dutch" auction. Our results suggest that an alternative that may be worth considering in similar environments is a combination of a "Japanese" with a first-price sealed-bid auction. Similarly, Ausubel and Milgrom (2002) suggest an English ascending auction that allows for package bidding in order to improve efficiency while avoiding some of the problems arising for example from Vickrey-Clarke-Groves mechanisms. Also here it may be worthwhile to consider a "Japanese" design while keeping the remaining features that allow bidders to bid on packages; of course, here the question of how to increase the prices of the items for sale becomes even more pressing.

Appendix

Proof of Proposition 2: If bidders, upon observing a local bidder bidding on an object objects they value, then the exiting times obtained from Eq. (1) are clearly (weakly) dominant strategies for the local bidders, and they are also (weakly) dominant (continuation) strategies for global bidders currently bidding on only one object.[13] Given these exiting times, in what follows, we show that Eq. (1) also yields globally optimal exiting times for global bidders bidding on two objects. We will argue using both the JAMO and the SA.

Case (ii) $v_1, v_2 \in [0,1]$ and $\alpha = 0$, follows from the fact that, since $\alpha = 0$, no global bidder has any incentive to exit on object k after $v_k, k = 1, 2$. On the other hand, there is no incentive to exit before v_k either. As mentioned in the text, these exiting times are also obtained by differentiating $\Pi(p; H_t)$ with respect to p and setting equal to zero.

The following lemmas cover the remaining cases (i) $v_1 = v_2 = x \in [0,1]$ and $\alpha \geq 0$, and (iii) $v_1, v_2 \in [0,1]$ and $\alpha > 1$. Fix a global bidder h and let y_k denote the highest exiting time on object k among all other bidders. For any

[13] In the auction for object k, a local bidder will bid u_k, and a global bidder will bid v_k or $v_k + \alpha$ depending on whether he has already dropped or won the other object.

given strategy profile of the other bidders, denote by $G_k(\cdot|H_t)$ the cumulative distribution function of y_k given the information H_t. Let further H_t^k denote the information set H_t, where the global bidder h has exited object k at t. Then we can state:

Lemma 1 *If $G_k(\cdot|H_t) \leq G_k(\cdot|H_t^{3-k})$, for $k = 1, 2$, then in the JAMO global bidders exit both objects simultaneously.*

The condition on G_k implies that if global bidder h exits one object, the probability of winning the other object will not increase. Therefore, it is always optimal to stay on object k at least until v_k. We need to show that for both objects the optimal exiting time $\tau \geq Max\{v_1, v_2\}$. Case (i) is clear, since $\tau \geq v_1 = v_2 = x$. To see case (iii), assume without loss of generality that $v_2 \geq v_1$. If a global bidder wins one object he will necessarily win both objects, since we are assuming (weakly) undominated continuation strategies, which would lead all other remaining bidders to exit at the latest at 1 from the second object. Therefore, a global bidder who is active on two objects has nothing to lose if he stays on the two objects (at least) until v_2. Indeed, if he exits object 1 before v_2, he does not increase the probability of winning object 2 and he loses the opportunity of winning the bundle. More formally, at $t \in (v_1, v_2)$, the global bidder's expected profit from exiting object 1 is $E[(v_2 - y_2)\mathbf{1}_{\{y_2 \leq v_2\}}|H_t^1]$. If instead, he stays on the two objects until v_2 and then exit the two objects only if he wins no object before v_2, then his expected payoff is

$$\begin{aligned} E[(v_1 + v_2 + \alpha \quad &- \quad (y_1 + y_2))\mathbf{1}_{\{\min\{y_1,\, y_2\} \leq v_2\}}|H_t] \\ &\geq \quad E[(v_1 + \alpha - \max\{y_1, y_2\})\mathbf{1}_{\{\min\{y_1,\, y_2\} \leq v_2\}}|H_t] \\ &\quad\quad +E[(v_2 - \min\{y_1, y_2\})\mathbf{1}_{\{\min\{y_1,\, y_2\} \leq v_2\}}|H_t^1] \\ &> \quad E[(v_2 - y_2)\mathbf{1}_{\{y_2 \leq v_2\}}|H_t^1] \end{aligned}$$

where the first inequality follows from $G_k(\cdot|H_t) \leq G_k(\cdot|H_t^{3-k})$, $k = 1, 2$ and the second from $y_1, y_2 \leq 1 < \alpha$. Thus, it is suboptimal to exit object 1 before v_2 and therefore the bundle bidder will exit simultaneously the two objects at a price no smaller than v_2.[14]

[14] To prove uniqueness of a symmetric PBE, in case (i), we note the assumption on G_k can be dispensed with, since it follows directly from symmetry of the equilibrium strategies.

Lemma 2 *Eq. (1) is a necessary and sufficient condition for a globally optimal exiting time whenever $M = 1$ and N_1, N_2 are arbitrary.*

We prove the lemma for a global bidder active on the two objects. The local bidders' strategies is not affected by H_t, thus when a global bidder faces only local bidders, we have $G_k(\cdot|H_t) = G_k(\cdot|H_t^{3-k})$. From Lemma 1 and Prop. 1, in the SA, for the global bidder it is optimal to submit the same bid p on the two objects. Then, three possible outcomes arise with probability 1: (i) p is the lowest bid and the global bidder exits the auction; (ii) a local bidder sets the lowest bid on object 1; (iii) a local bidder sets the lowest bid on object 2. Let $F_N(\cdot|t)$ denote the cumulative distribution function of the lowest valuation among N local bidders given that their valuation for the object is at least t, and let $f_N(\cdot|t) = F_N'(\cdot|t)$. More specifically we can write $F_N(x|t) = 1 - (1 - F(x|t))^N$, where $F(x|t) = \frac{F(x)-F(t)}{1-F(t)}$ is the cumulative distribution of a local bidder's valuation given that his valuation is at least t. Therefore, $\Pi(p; H_t)$ can be rewritten as

$$\begin{aligned} \Pi(p; H_t) &= \int_t^p \pi_{L_1}(s, H_t)(1 - F_{N_2}(s|t))f_{N_1}(s|t)ds \\ &+ \int_t^p \pi_{L_2}(s, H_t)(1 - F_{N_1}(s|t))f_{N_2}(s|t)ds. \end{aligned}$$

Consequently, the necessary condition is obtained deriving with respect to p:

$$\begin{aligned} \Pi'(p; H_t) &= \pi_{L_1}(p, H_t)(1 - F_{N_2}(p|t))f_{N_1}(p|t) \\ &\quad + \pi_{L_2}(p, H_t)(1 - F_{N_1}(p|t))f_{N_2}(p|t) \\ &= \left(\frac{N_1}{N_1 + N_2}\pi_{L_1}(p, H_t) + \frac{N_2}{N_1 + N_2}\pi_{L_2}(p, H_t) \right) f_{N_1+N_2}(p|t), \end{aligned}$$

since $(1 - F_{N_k}(s|t))f_{N_{3-k}}(s|t) = N_{3-k}(1 - F(p|t))^{N_1+N_2-1}f(p|t)$, and hence Eq. (2). To check sufficiency note that

$$\begin{aligned} \Pi''(p; H_t) &= \left(\frac{N_1}{N_1 + N_2}\pi'_{L_1}(p, H_t) + \frac{N_2}{N_1 + N_2}\pi'_{L_2}(p, H_t) \right) f_{N_1+N_2}(p|t) \\ &+ \left(\frac{N_1}{N_1 + N_2}\pi_{L_1}(p, H_t) + \frac{N_2}{N_1 + N_2}\pi_{L_2}(p, H_t) \right) f'_{N_1+N_2}(p|t) \end{aligned}$$

In order to verify that $\pi''(\tau; H_t) \leq 0$, it is sufficient to that observe that the second term vanishes and that $\pi_{L_1}(p, H_t)$ and $\pi_{L_2}(p, H_t)$ are not increasing in p. Indeed, $\pi_{L1}(p, H_t)$ is the global bidder's continuation payoff given that all remaining local bidders' valuation for the objects is at least p that is also the minimum admissible bid in the next round. Clearly, an increase in p cannot improve $\pi_{L_1}(p, H_t)$.

Lemma 3 *The exiting times defined by Eq. (2) are nondecreasing functions of v_B.*

By Lemma 1 the global bidder drops out simultaneously from both objects, which implies that in case (iii), since $\alpha \geq 1$, either they win both objects or none; this implies that their expected payoff functions and hence their exiting times will depend only on the value of the bundle v_B. Since $x = (v_B - \alpha)/2$ this is clearly true also in case (i).

Suppose $\tau = \tau(v_B; H_t)$ is a global bidder's equilibrium bid at information set H_t when his value for the bundle is v_B. Then it must be the case that $\Pi'(\tau; H_t) = 0$, where as above we can write

$$\Pi'(p; H_t) = \left(\frac{N_1}{N_1 + N_2} \pi_{L_1}(p, H_t) + \frac{N_2}{N_1 + N_2} \pi_{L_2}(p, H_t) \right) f_{N_1+N_2}(p|t) = 0$$

for $p = \tau$. Now, the probability of winning at any given price depends only on the other bidder's valuation, and the global bidder's ex-post payoff is non-decreasing in v_B. Thus, if $\bar{v}_B > v_B$ is the global bidder's value for the bundle, then the corresponding expressions $\bar{\pi}_{L_1}(p, H_t)$ and $\bar{\pi}_{L_2}(p, H_t)$, computed with the higher value $\bar{v}_B$, are not smaller than the counterparts $\pi_{L_1}(p, H_t)$ and $\pi_{L_1}(p, H_t)$ computed with v_B. Finally, considering that $\Pi''(\tau; H_t) \leq 0$ we have that $\overline{\Pi}'(p; H_t) \geq \Pi'(p; H_t)$ implies that if τ is optimal for v_B at H_t, submitting $\tau' \geq \tau$ is better for the global bidder with $\bar{v}_B$.

Lemma 4 *Eq. (1) is a necessary and sufficient condition for a globally optimal exiting time for M, N_1, and N_2 arbitrary.*

The proof is by (backward) induction. Consider a global bidder who after some sufficiently large number of stages of the SA is still active on the two objects such that either there is no other global bidder active on the two objects, in which case Lemma 2 applies, or all active bidders are global bidders, in which case, it is a (weakly) dominant strategy to bid $v_B/2$ on both objects.

Let H_t be such that there are two global bidders and N_k local bidders on objects $k = 1, 2$. Fix global bidder h, and let v_1 and v_2 be his valuation for the two objects. Let $\tau(v_B; H_t)$ be a global bidder's strategy in a symmetric equilibrium. The first step is to prove that the global bidders exit simultaneously from the two objects and that the equilibrium strategy $\tau(v_B; H_t)$ is not decreasing in v_B. In order to prove the first property it is sufficient to choose appropriately the out-of-equilibrium-path beliefs for the global bidders. Indeed, if each global bidder believes that exiting only on one object will not increase the probability of winning the other object, then, by Lemma 1, it will be optimal to exit simultaneously the two objects. For the second property, as customary in auction theory, we assume that $\tau(v_B; H_t)$ is not decreasing in v_B then we show that the equilibrium strategy $\tau(\cdot)$ satisfies this restrictions.

If at the information set H_t, global bidder h bids p on the two objects, his expected payoff function is

$$\begin{aligned}\Pi(p; H_t) &= \int_t^p \pi_L(s, H_t)(1 - F_B(s|H_t))f_{N_1+N_2}(s|H_t)ds \\ &+ \int_t^p \pi_B(s, H_t)(1 - F_{N_1+N_2}(s|H_t))f_B(s|H_t)ds \qquad (4)\end{aligned}$$

and hence

$$\begin{aligned}\Pi'(p; H_t) &= \pi_L(p, H_t)(1 - F_B(p|H_t))f_{N_1+N_2}(p|H_t) \\ &+ \pi_B(p, H_t)(1 - F_{N_1+N_2}(p|H_t))f_B(p|H_t), \qquad (5)\end{aligned}$$

where $\pi_L(p, H_t)$ and $\pi_B(p, H_t)$ are the continuation payoffs if respectively a local bidder or another global bidder sets the lowest bid $p \geq t$; and $F_B(\cdot|H_t)$ is the cumulative distribution of the other global bidders' lowest bid, and $f_B(\cdot|H_t) = F'_B(\cdot|H_t)$.

If $\tau(\cdot; H_t)$ is nondecreasing in the value of the bundle, $\pi_L(p; H_t) = 0$ for all $p \geq \tau(v_B; H_t)$. Indeed, $\pi_L(p; H_t)$ is the continuation payoff when the other global bidder bids more than p. However, if $p \geq \tau(v_B; H_t)$, then the other global bidder will value the bundle at least as much as global bidder h. This implies that bidder h cannot expect a positive profit in the following rounds.

Thus, at a symmetric equilibrium, the optimal bid will be the smallest p that solves

$$\pi_B(p; H_t)(1 - F_{N_1+N_2}(p|H_t))f_B(p|H_t) = 0. \tag{6}$$

Since π_B corresponds to a situation where the lowest bid is made by a global bidder, this means that the only remaining active global bidder is bidder h, so we can apply Lemma 2 to get

$$\pi_B(p; H_t) = \int_p^\tau \left(\frac{N_1}{N_1 + N_2} \pi_{L_1}(s; H_p) + \frac{N_2}{N_1 + N_2} \pi_{L_2}(s; H_p) \right) f_{N_1+N_2}(s|H_p^B)ds$$

where τ is the smallest s such that the argument of the integral is zero. Therefore, $\tau(v_B; H_t) = \tau = \tau^*(v_B; H_t)$ that is not decreasing in v_B for Lemma 3. This proves the lemma for the case of two global bidders and $N_1 + N_2$ local bidders.

Suppose now that the number of global bidders is $M > 2$. Observe that at a symmetric equilibrium where $\tau(v_B; H_t)$ is increasing in v_B, global bidder h's continuation payoff is positive only if in the current round there is no other global bidder submitting a higher price than his own. Thus, if with $M - 1$ global bidders the optimal bid (or exiting time) is $\tau(v_B; H_t) = \tau = \tau^*(v_B, H_t)$ obtained as the smallest solution to Eq.(2), then we have

$$\pi_B(p; H_t) = \int_p^\tau \pi_B(s; H_p^B) f_B(s|H_p^B)ds.$$

where, by some abuse of notation, H_p^B is the information set H_t updated by the fact that a bundle bidder exited with a bid of p. This implies that also with M global bidders the optimal bid is $\tau(v_B; H_t) = \tau = \tau^*(v_B; H_t)$ determined as the smallest p solving Eq. (2) that is increasing in v_B. ■

Proof of Example 2: We directly prove the general case mentioned at the end of the example. There are M global bidders and one local bidder on object 1; preferences are according to cases (i) or (ii); values are drawn from a smooth distribution F defined on $[0, 1]$.

We first check optimality for any global bidder, then we check it for the local bidder. If the local bidder is active on both objects, global bidders infer that $u_1 > l$. Hence, a global bidder with $v_1 + \alpha \leq l$ knows that he will not win object 1 even if he wins object 2 and then continues optimally until $v_1 + \alpha$

on object 1. Therefore, exiting object 1 at time c and exiting object 2 at v_2 is a (weak) best reply for such a global bidder. If, however, $v_1 + \alpha > l$, then a global bidder is better off remaining on each object so long as his expected continuation payoff remains strictly positive. Namely, if $v_1, v_2 \in [0, 1]$ and $\alpha = 0$, (case (ii)), then a global bidder remains on objects 1 and 2 until v_1 and v_2 respectively. On the other hand, if $v_1 = v_2 = x$ and $\alpha \geq 0$, (case (i)), then a global bidder's strategy will be of the type described in Section 2.3: there is an optimal time τ that depends on v_B and H_t such that, if he does not win any object before τ he exits both objects, and he continue optimally until $x + \alpha$ otherwise. This proves that the global bidders' strategy is a best reply to the local bidder's strategy.

To prove optimality for the local bidder, we need to show that the local bidder's strategy is a best reply and that it is profitable for the local bidder to bid on both objects if and only if $u_1 > l$, i.e., that the equilibrium is incentive compatible, so that being active on both objects gives a credible signal that $u_1 > l$. Let $y_k \in [0, 1]$ denote the global bidders' highest valuation for object k, $k = 1, 2$, and let $G = F^M$ denote the corresponding distribution function. Keep in mind that the global bidder with the highest valuation for object 2 will exit auction 2 before c only if $y_2 < c$. When the local bidder is not active on object 2 and still active on object 1, the bundle bidder with the highest valuation for object 1 will exit auction 1 at $y_1 + \alpha$. When the local bidder is active on object 2, if $y_1 \leq l - \alpha$, then the global bidder with the highest valuation for object 1 will exit auction 1 at c; whereas if $y_1 > l - \alpha$ and the local bidder is still active on object 1, then the global bidder with the highest valuation for object 1 will exit auction 1 at $y_1 + \alpha$.

If $u_1 < c$, then it is clearly not optimal for the local bidder to bid on both objects since he will have to pay at least c for object 1. Hence we assume $u_1 \geq c$. Suppose that $u_1 \leq l$. If the local bidder decides to implement the retaliatory strategy, then his expected payoff is $\int_0^{l-\alpha} (u_1 - c) \cdot G'(y_1) dy_1 - \int_0^c y_2 \cdot G'(y_2) dy_2$. The first integral is the local bidder's payoff from object 1: the local bidder wins object 1 only if $y_1 < l - \alpha$, (recall that $l \in (\alpha, 1]$) and he pays c. The second integral is the expected payoff from object 2: if $y_2 < c$, then he has to buy object 2 at a price y_2. If, however, at time 0 the local bidder decides to bid only on object 1, his expected payoff is $\int_0^{u_1-\alpha} (u_1 - y_1 - \alpha) G'(y_1) dy_1$.

At equilibrium we want the local bidder to bid only on object 1 when

$u_1 \leq l$, i.e., the following needs to be satisfied

$$\int_0^{l-\alpha} (u_1 - c) \cdot G'(y_1) dy_1 - \int_0^c y_2 \cdot (y_2) dy_2 \leq \int_0^{u_1-\alpha} (u_1 - y_1 - \alpha) \cdot G'(y_1) dy_1,$$

which is satisfied for c solving Eq. (3).

Suppose now that $u_1 > l$. Then, at $t = 0$, the local bidder's expected payoff from adopting the retaliatory strategy must be greater or equal than the payoff from bidding only on object 1, i.e.,

$$\begin{aligned} \int_0^{l-\alpha} (u_1 - c) \cdot G'(y_1) dy_1 \quad &+ \quad \int_{l-\alpha}^{u_1-\alpha} (u_1 - y_1 - \alpha) \cdot G'(y_1) dy_1 \\ - \int_t^c y_2 \cdot G'(y_2) dy_2 \quad &\geq \quad \int_0^{u_1-\alpha} (u_1 - y_1 - \alpha) \cdot G'(y_1) dy_1 \end{aligned}$$

It is easy to check that the above inequality is satisfied for any $t < c$ and for any $l \in (\alpha, 1]$. Finally an appropriate choice of out of equilibrium path belief guarantees that at any $t < c$ the local bidder's expected payoff by insisting with the retaliatory strategy is greater or equal than the payoff of exiting object 2 and continuing on object 1. ■

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