A note on the take-it-or-leave-it bargaining procedure with double moral hazard and risk neutrality

A. Citanna
HEC - Paris; and GSB - Columbia University, NY

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In this note we study a take-it-or-leave-it bargaining procedure between two risk neutral individuals engaged in the joint stochastic production of a commodity. Each individual has to exert effort, that is, to provide a one–dimensional input which is unobserved to the other individual. The output–contingent sharing rule is constrained to lead to nonnegative consumption for both individuals, a limited liability constraint. The individuals enter joint production in one of two possible occupations, or tasks, the p-agent and the a-agent, which differ in their incentive intensity. Hence, incentives are asymmetric. The p–agent makes a take-it-or-leave-it offer to the a–agent, and has therefore all the contractual power, modulo providing the a–agent an exogenously given reservation utility.

Although the general characterization of team problems with limited liability is well-known (see, e.g., Holmstrom (1982) and Legros and Matsushima (1991)), as well as for the special case of the principal-agent model (see Holmstrom (1979)), the team problem we study –with asymmetric incentives– has not been fully analyzed. For example, properties of the expected surplus and payoffs critical to our equilibrium analysis in Chakraborty and Citanna (2001, 2002) are not typically spelled out. Hence, the details are provided in this note, with a particular stress on those properties that are key for obtaining wealth effects when individuals can choose endogenously which task to perform, as in Chakraborty and Citanna (2001, 2002).

The contractual situation described here has double, asymmetric moral hazard. The asymmetry of the effort productivity is assumed to be a first order effect. That is, we assume that the a–agent's effort has a greater impact on the probability of success than the p–agent's.

We introduce increasingly restrictive conditions on the probability of success as a function of efforts, on the cost of effort, as well as on the output gains, and study what they yield in terms of properties of the optimal contract.

First, minimal concavity of the success probability in efforts, as well as convexity of the cost of effort provision, is assumed, together with enough regularity to get incentive compatible efforts as continuous functions of the sharing rule.

Second, existence of a solution to the contractual problem and boundedness of the sharing rule are shown by adding the condition that marginal productivity of each effort is less than its marginal cost when the other side totally shirks.

In order to obtain that the net gain to the a–agent is proportional to the bonus offered to him, as well as concavity of the p–agent's gain and of the total (net expected) surplus –both natural and intuitive properties of the contract–, we introduce restrictions on the second partials of the success probability and of the disutility of effort. We focus on production processes which have negligible second order cross–effects, or small cross–partials of the probability of success with respect to efforts. Although some nonseparability may be present, this cannot be too strong to overturn the first order effects. Strategic complementarities that could reverse the relative importance of the effort exerted in production by the a–agent are therefore excluded. These turn out to be the main restrictions imposed in the analysis of the problem leading to a characterization identical to Chakraborty and Citanna's (2001, 2002). These restrictions also imply that the a–agent's incentive compatible effort is nondecreasing and concave in the bonus, while the p–agent's is nonincreasing. Further, they imply that the a–agent's effort reacts more to a bonus change than the p–agent's does. They finally also imply that at the optimum the p–agent must offer a bonus that cannot be lower than what maximizes his own payoff. When it is strictly higher, it is because the participation constraint binds.

It is then shown that the optimal contract gives rise to a hump–shaped expected surplus as a function of the a–agent's reservation utility. This surplus, generated by the optimal contract, is increasing in the a–agent's reservation utility as long as his limited liability binds. It increases at a rate less than one, and then becomes flat until the reservation utility of the p–agent is so high that the p–agent's limited liability binds.

The present note then generalizes the setup used by Chakraborty and Citanna (2001,2002) beyond the linear–quadratic case. The latter is introduced at the end as an example of production problems satisfying the general assumptions.

1 The model

1.0.1 Technology

Our starting point is the production technology. Production yields a stochastic output $X(\omega)$ through the exercise of unobservable effort. Here $\omega \in \mathbf{\Omega}$ represents the states of the world. Denoting efforts by e_p and e_a, for the p-agent and the a-agent respectively, we assume that $e_i \in [0,1]$, for $i = a, p$. The cost of effort e for the individual is $c(e) \geq 0$. We suppose that $\Omega = 2$, and that X is related to efforts as follows:

$$X(\omega) = \begin{cases} X(2) & \text{with probability } f(e_p, e_a) \\ X(1) & \text{with probability } 1 - f(e_p, e_a), \end{cases}$$

where $X(2) > X(1) \geq 0$. Hereafter we use $\Delta X = X(2) - X(1)$. Extensions to $\Omega > 2$ may be accommodated with the usual monotonicity conditions.

We start off with general assumptions on f and c.

A1 a) $c(e)$ is a C^1 function, increasing and strictly convex, with $c(0) = 0$; b) $f(e_p, e_a)$ is a C^1 function, increasing and concave in both of its arguments; c)

$$f_a(e_p, e_a) \geq f_p(e_p, e_a) \geq 0 \text{ and } \max\{f_a(e_p, e_a), f_p(e_p, e_a)\} > 0.$$

where $f_p(e_p, e_a) = \partial f(e_p, e_a)/\partial e_p$ and $f_a(e_p, e_a) = \partial f(e_p, e_a)/\partial e_a$.

A1.c means that jobs have a different impact of effort on output, or incentive intensity. When $f_p = 0$ for all e_p, e_a, we are in the well-known principal–agent case. A1.c is the key assumption on technology in Chakraborty and Citanna (2001, 2002). As for A1.a and A1.b, one can switch from strict convexity of c to strict concavity of f for what follows.

1.0.2 Contracts

In the take-it-or-leave-it procedure, the p-agent offers the a-agent a wage as a function of the output to maximize profits (a sharing rule). The contract is then accepted or rejected by the a-agent. The contract has to guarantee the a-agent at least a reservation utility of U, in expected terms (the individual rationality (IR) constraint).

The p-agent makes an offer to the a-agent, i.e., a compensation scheme $w(\omega)_{\omega \in \mathbf{\Omega}}$. We use the notation $\Delta w \equiv w(2) - w(1)$ for the 'bonus/malus'.

If the individuals receive their wealth at the time of contracting, as opposed to waiting until after production takes place, they have to decide who gets to hold wealth until uncertainty is resolved. Also, if there is a start-up cost $C > 0$ and the firm needs to borrow to start, it must be decided who gets to borrow, whether the individuals or the match, pooling the resources of the partners. Such decisions may have an impact if the cost of borrowing or the return from lending depend on the amount of the loan, that is, if financial markets are imperfect. In what follows, we assume that there is a riskfree asset that can be exchanged by the firm as well as the individuals at a given rate of $r \geq 1$. In this case, we can without loss of generality assume that the p–agent is financially responsible also for the liabilities or assets of the firm. Then, after production the p–agent receives $\widetilde{W}_p^f(r) = W_p^f(r) + W_F^f(r) + X(1)$ no matter what the realized state of the world is, where $W_j^f(r)$ is the final wealth of $j = a, p$ or F, the firm. Since it is assumed that the individuals or the firm can invest and borrow at the riskfree rate r, the final wealth depends on r.

The p-agent thus solves:

$$\begin{array}{lll}
\max\limits_{w(1),\Delta w, e_a, e_p} & \widetilde{W}_p^f(r) + (\Delta X - \Delta w) f(e_p, e_a) - w(1) - c(e_p) \quad s.t. & \\
W_a^f(r) + \Delta w \ f(e_p, e_a) + w(1) - c(e_a) \geq U & (IR) & \\
e_a \in \arg\max_{e' \in [0,1]} \Delta w \ f(e_p, e') - c(e_a) & (IC_a) & (C) \\
e_p \in \arg\max_{e' \in [0,1]} (\Delta X - \Delta w) f(e', e_a) - c(e_p) & (IC_p) & \\
\min[w(1) + \Delta w, w(1)] \geq -W_a^f(r) & (LL_a) & \\
\min[\Delta X - w(1) - \Delta w, -w(1)] \geq -\widetilde{W}_p^f(r) & (LL_p) &
\end{array}$$

The p-agent maximizes expected total wealth minus the cost of effort. Note the separability of utility for consumption and effort, which is used to simplify computations.

2 Analysis

2.0.3 Incentive compatible efforts

The (IC) constraints imply that the agents are playing a Nash equilibrium among themselves. The set of incentive compatible efforts is therefore nonempty given the continuity of the objective functions and the compactness, convexity of the choice sets. It only depends on Δw, and we denote with $f(\Delta w)$ the probability computed at an incentive compatible choice of efforts, $e_p(\Delta w)$ and $e_a(\Delta w)$. If (e_p, e_a) is an incentive compatible choice of efforts, then it must satisfy[1]

$$\begin{array}{c}
\Delta w f_a(e_p, e_a) - c'(e_a) + \mu_a - \nu_a = 0 \\
(\Delta X - \Delta w) f_p(e_p, e_a) - c'(e_p) + \mu_p - \nu_p = 0
\end{array} \tag{1}$$

and

$$\begin{array}{c}
\min\{e_a, \mu_a\} = 0, \min\{1 - e_a, \nu_a\} = 0 \\
\min\{e_p, \mu_p\} = 0, \min\{1 - e_p, \nu_p\} = 0
\end{array} \tag{2}$$

It is immediate to see that if $\Delta w \leq 0$ then it is a dominant strategy to set $e_a = 0$, or $e_a(\Delta w) = 0$ if $\Delta w \leq 0$. Similarly, $e_p(\Delta w) = 0$ if $\Delta w \geq \Delta X$. We will look at problems that have the following additional property.

P1 There exists continuous selections $e_p(\Delta w), e_a(\Delta w)$ of incentive compatible efforts.

Although P1 does not explicitly list conditions on f, c leading to it, many such sufficient conditions exist. P1 amounts to conditions on derivatives of f and c that make the determinant of the derivative of (1) with respect to e_p, e_a nonnull, a regularity condition. A sufficient condition for property P1 is that

[1] That the Kuhn-Tucker conditions are necessary and sufficient here follows from assumption A1. That the FOCs are also sufficient in this case for problem (C) follows from Rogerson (1985). We are of course using differentiability of f and c.

f, c have constant second derivatives. The Nash equilibrium given Δw is then unique.

We will maintain assumption A1 and property P1 hereafter.

Lemma 1 *Let A1 and P1 hold. When $f_p(e_p, e_a) \equiv 0$, $e_a(\Delta w)$ is nondecreasing for all e. When $f_p(e_p, e_a) > 0$, $e_p(\Delta w)$ is constant implies $e_a(\Delta w)$ is nondecreasing, and $e_a(\Delta w)$ is constant implies $e_p(\Delta w)$ is nonincreasing.*

Proof. When $f_p(e_p, e_a) \equiv 0$, $e_p(\Delta w) = 0$ for all e, then only first order effects matter. From (1) and since $f_a(e_p, e_a) > 0$, if $\Delta w' > \Delta w$,

$$\Delta w' f_a(0, e_a(\Delta w)) - c'(e_a(\Delta w)) \begin{cases} > 0 & \text{if } e_a(\Delta w) > 0 \\ = -\mu_a & \text{if } e_a(\Delta w) = 0 \end{cases}$$

In the first case, the marginal benefit of the a–agent's effort is greater than its marginal cost. Using A1, i.e., concavity of f and convexity of c, we see that $e_a(\Delta w') \geq e_a(\Delta w)$, i.e., $e_a(\Delta w)$ is nondecreasing (and obviously $e_p(\Delta w)$ nonincreasing).

When $f_p(e_p, e_a) > 0$, then typically both $e_a(.)$ and $e_p(.)$ are not constant at least over some range of Δw. When one of the two functions is constant, then again only first order effects matter. Hence, if $e_p(\Delta w)$ is constant, $e_a(\Delta w)$ is nondecreasing by the previous reasoning.

On the other hand, if $e_a(\Delta w) = e_a(\Delta w') = e_a$, if $\Delta w' > \Delta w$ and if $e_p(\Delta w) < 1$, from (1)

$$\begin{aligned} & (\Delta X - \Delta w') f_p(e_p(\Delta w), e_a) - c'(e_p(\Delta w)) \\ < \quad & (\Delta X - \Delta w) f_p(e_p(\Delta w), e_a) - c'(e_p(\Delta w)) = 0 \end{aligned} \tag{3}$$

so that A1 implies $e_p(\Delta w') \leq e_p(\Delta w)$, i.e., $e_p(\Delta w)$ is nonincreasing. ■

Lemma 1 implies that $e_a(\Delta w)$ is nondecreasing when $\Delta w \geq \Delta X$, and that $e_p(\Delta w)$ is nonincreasing when $\Delta w \leq 0$. This will be useful in proving Lemma 2 below.

Later, it will be useful to have $e_a(\Delta w)$ nondecreasing and $e_p(\Delta w)$ nonincreasing *for all* $\Delta w \in [0, \Delta X]$. This is true if $f_p(e_p, e_a) \equiv 0$ (the principal–agent setup), and by continuity it will be true if second order cross effects are negligible relative to first order ones, i.e., if f_{pa} is sufficiently close to zero relative to f_a and f_p. In this case, even strategic complementarity of efforts (i.e., $f_{pa} > 0$) can be accomodated.[2]

[2] What are sufficient conditions to get that $e_a(\Delta w)$ is nondecreasing in Δw, while $e_p(\Delta w)$ is nonincreasing for all Δw when second order effects are not negligible? Suppose that

$$f_{pa} \leq 0 \text{ and } \frac{f_a(e,e)}{f_p(e,e)} = const \text{ (independent of } e)$$

When this holds, it must be true that for any Δw where neither function is constant and any $\Delta w' > \Delta w$,

$$[e_a(\Delta w') - e_a(\Delta w)] > 0 \text{ and } [e_p(\Delta w') - e_p(\Delta w)] < 0, \text{ or vice versa.}$$

For suppose not, and say both terms are positive. By increasing the bonus/malus to $\Delta w' > \Delta w$, the p–agent's FOC's gives the inequality (3). Now, since $e_a(\Delta w') > e_a(\Delta w)$ and $f_{pa} \leq 0$,

2.0.4 Δw is a bonus to the a–agent

Substituting in (C) these functions, we obtain

$$\begin{array}{ll}
\max_{w(1),\Delta w} \ \widetilde{W}_p^f(r) + (\Delta X - \Delta w) f(\Delta w) - w(1) - c(e_p(\Delta w)) & s.t. \\
W_a^f(r) + \Delta w f(\Delta w) + w(1) - c(e_a(\Delta w)) \geq U & (IR) \\
\min[w(1), w(1) + \Delta w] \geq -W_a^f(r) & (LL_a) \\
\min[\Delta X - w(1) - \Delta w, -w(1)] \geq -\widetilde{W}_p^f(r) & (LL_p)
\end{array} \qquad (C')$$

We now examine the properties of (C'). First, we introduce further assumptions on f, c and ΔX.

A2 We assume that f, c and ΔX are such that

$$\Delta X f_p(e, 0) \leq c'(e_p(0))$$

all $e \geq e_p(0)$, and

$$\Delta X f_a(0, e) \leq c'(e_a(\Delta X))$$

all $e \geq e_a(\Delta X)$.

These conditions essentially say that it does not pay to exert too much effort when the other agent puts in nothing (marginal benefit of effort increase is below its marginal cost). Assumption A2 puts a bound on the bonus/malus use in the contract, as shown in the following lemma.

Lemma 2 *Let A1, A2 and P1 hold. If* $(w^*(1), \Delta w^*)$ *solves* (C'), *then* $\Delta w^* \in [0, \Delta X]$.

Proof. Suppose not. First consider an optimal $(w^*(1), \Delta w^*)$ such that $\Delta w^* < 0$ and such that $e_p(\Delta w)$ is constant in a closed neighborhood $U_{\Delta w^*}$. Then, f is constant on this neighborhood. Let $f(\Delta w^*) = f^*$ and choose

$$\begin{array}{rcl}
w'(1) & = & w^*(1) - f^*(\Delta w' - \Delta w^*) \\
\Delta w' & = & \max\{\Delta w \mid \Delta w \leq 0 \text{ and } \Delta w \in U_{\Delta w^*}\}
\end{array}$$

$$\begin{array}{rcl}
(\Delta X - \Delta w') f_p(e_p(\Delta w), e_a(\Delta w')) - c'(e_p(\Delta w)) & \leq & \\
(\Delta X - \Delta w') f_p(e_p(\Delta w), e_a(\Delta w)) - c'(e_p(\Delta w)) & < & 0
\end{array}$$

Then, the marginal benefit of the p–agent's effort is less than its marginal cost. Using A1, we see that $e_p(\Delta w') < e_p(\Delta w)$, a contradiction. A similar argument shows that the terms cannot be both negative.

Since $e_a(\Delta w) = 0$ if $\Delta w \leq 0$, $e_a(.)$ will have to increase, and $e_p(.)$ decrease, in a right neighborhood of $\Delta w = 0$. Similarly, $e_a(.)$ decreases and $e_p(.)$ increases in a left neighborhood of ΔX. Continuity implies that the two functions must cross. Now the constant rate of substitution f_a/f_p along the diagonal implies that there is a unique Δw for which $e_a(.)$ and $e_a(.)$ will cross.

Finally, the two functions must be globally monotonic. If not, there is going to be a pair of points $\Delta w, \Delta w'$ where $e_j(\Delta w) = e_j(\Delta w')$ for $j = p$ or a, but $e_i(.)$ has the wrong slope. This implies our monotonicity property.

First, since $\Delta w' < 0$, $w'(1) < w^*(1)$ implies (LL_p) is satisfied. Second,

$$w'(1) + \Delta w' = w^*(1) + \Delta w^* + (1 - f^*)(\Delta w' - \Delta w^*) > w^*(1) + \Delta w^*,$$

and (LL_a) is satisfied. Also,

$$\Delta w' f(\Delta w') + w'(1) - c(e_a(\Delta w')) = \Delta w' f^* + w^*(1) - f^*(\Delta w' - \Delta w^*) = w^*(1) + f^* \Delta w^*,$$

so that the (IR) is also satisfied. Hence, $(w'(1), \Delta w')$ is feasible and gives the same payoff to the p–agent, as it is easily checked.

Next, let $(w^*(1), \Delta w^*)$ be such that $\Delta w^* \leq 0$ and, by Lemma 1, $e_p(\Delta w) < e_p(\Delta w^*)$ for $0 \geq \Delta w > \Delta w^*$. Choose

$$\begin{aligned} w'(1) &= w^*(1) + f^* \Delta w^* \\ \Delta w' &= 0 \end{aligned}$$

It is immediately verified that (LL_p), (LL_a) and (IR) are satisfied. Now let $e_p(0) = e'_p$ and $f(0) = f'$. The p–agent makes $\Delta X f' - w^*(1) - f^* \Delta w^* - c(e'_p)$, or

$$\Delta X(f' - f^*) + c(e^*_p) - c(e'_p) + (\Delta X - \Delta w^*) f^* - w^*(1) - c(e^*_p)$$

which is greater than or equal to the payoff at $(w^*(1), \Delta w^*)$ if and only if

$$\Delta X(f' - f^*) + c(e^*_p) - c(e'_p) \geq 0$$

Applying the mean value theorem, this is true if and only if

$$-\Delta X f_p(\widehat{e})(e^*_p - e'_p) + c'(\widehat{e})(e^*_p - e'_p) \geq 0$$

where $\widehat{e} \in [e'_p, e^*_p]$. Equivalently,

$$\Delta X f_p(\widehat{e}, 0) \leq c'(\widehat{e}),$$

which follows from A2.

On the other hand, let $(w^*(1), \Delta w^*)$ with $\Delta w^* > \Delta X$. First assume that $e_a(\Delta w)$ is constant in a closed neighborhood $U_{\Delta w^*}$. Then, f is constant on this neighborhood. Choose

$$\begin{aligned} w'(1) &= w^*(1) + f^*(\Delta w^* - \Delta w') \\ \Delta w' &= \min\{\Delta w \mid \Delta w \geq \Delta X \text{ and } \Delta w \in U_{\Delta w^*}\} \end{aligned}$$

Since $w'(1) > w^*(1)$ and $\Delta X - w'(1) - \Delta w' \geq w^*(1) + \Delta w^*$ (since $(1 - f^*)\Delta w^* \geq (1 - f^*)\Delta w'$), (LL_a), (LL_p) and are satisfied. Also,

$$\Delta w' f' - w'(1) - c(e'_a) = \Delta w^* f^* - w^*(1) - c(e^*_a),$$

so that (IR) is satisfied, while the p–agent's payoff is unchanged. When $\Delta w^* > \Delta X$ and, by Lemma 1, $e_a(\Delta w) < e_a(\Delta w^*)$ for $\Delta X \leq \Delta w < \Delta w^*$, let

$$\begin{aligned} w'(1) &= w^*(1) + f^*(\Delta w^* - \Delta w') \\ \Delta w' &= \Delta X \end{aligned}$$

Observe that $w'(1) > w^*(1)$ so that (LL_a) is satisfied. Also,

$$\Delta X - w'(1) - \Delta w' = -w^*(1) - f^*(\Delta w^* - \Delta X) \geq \Delta X - w^*(1) - \Delta w^*$$

if and only if $(1-f^*)\Delta w^* \geq (1-f^*)\Delta X$, which is verifed, and (LL_p) also holds. As for the p–agent's payoff, this is identical at $(w^*(1), \Delta w^*)$ to $-w'$. Finally, the (IR) is satisfied if and only if

$$\Delta X f_a(0, \widehat{e}) < c'(\widehat{e})$$

for $\widehat{\widehat{e}} \in [e_a', e_a^*]$, again true if A2 holds. ■

Since Δw is nonnegative in the optimal contract, it is a bonus given to the a–agent. As a by–product, existence of a solution to (C') if the constraint set is nonempty is now obvious since the p-agent's payoff function is continuous, and the (LL) constraints imply $\widetilde{W}_p^f(r) \geq w(1) \geq -W_a^f(r)$ and $\Delta X \geq \Delta w \geq 0$, defining a compact set for $(w(1), \Delta w)$.

2.0.5 Gains and surplus

We introduce the notation:

$$\begin{aligned} &g_a(\Delta w) \equiv \Delta w f(\Delta w) - c(e_a(\Delta w)) \\ &g_p(\Delta w) \equiv (\Delta X - \Delta w) f(\Delta w) - c(e_p(\Delta w)) \\ &g(\Delta w) \equiv g_a(\Delta w) + g_p(\Delta w) \end{aligned}$$

where g_p is the gain to the p-agent, g_a the gain to the a-agent, and g is the surplus from the match, all net of wealth and of the low state output, and as functions of the bonus Δw. We add assumptions on f and c, as follows.

A3 a) f and c are at least C^2, and f is h.d.1; b) small second order cross effects relative to first order ones, e.g., $f_{pa} \approx 0$; and c) for all e_p, e_a,

$$\frac{f_a(e_p, e_a)}{f_p(e_p, e_a)} \geq \frac{\Delta w f_{aa}(e_p, e_a) - c''(e_a)}{(\Delta X - \Delta w) f_{pp}(e_p, e_a) - c''(e_p)}.$$

Assumption A3.b amounts to essentially requiring the problem to have little strategic complementarity effects across tasks. In other words, the problem must be effectively separable in e_p and e_a. This does not mean that only one effort is used in the optimal contract. The following (small) curvature assumption is also made on the now twice–differentiable incentive compatible efforts $e_p(.)$ and $e_a(.)$.

P2 $e_p''(\Delta w)$, $e_a''(\Delta w)$ exist and are small, with $e_a''(\Delta w) \leq 0$.

Without assumptions A3 and P2, the exogenous productivity assumption can be reversed by strategic complementarities (or substitution) of efforts between agents, and the analysis of the contractual game becomes considerably more subtle and complicated. We can now prove fundamental properties for the gains, in line with the intuitive view of the team problem where the a–agent exerts the more productive effort. Also, let Δw_p, Δw_g be maximizers of g_p and g, respectively.

Lemma 3 *Under A1-A3, P1 and P2, when $\Delta X > \Delta w > 0$: a) g_a is strictly increasing; b) g_p and g are strictly concave; c) $\Delta w_p < \Delta w_g$.*

Proof. a) Since under the maintained assumptions (in particular, A3.a) $e_p(.)$ and $e_a(.)$ are differentiable, for $\Delta X > \Delta w > 0$, using (1) and neglecting terms with $f_{pa}(.,.)$ by A3.b, we have

$$g_a'(\Delta w) = f(\Delta w) + \Delta w f_p(e_p, e_a) e_p'(\Delta w)$$

where again, $e_j = e_j(\Delta w)$, for $j = p, a$. Given A3.b, e_a is increasing and e_p decreasing in Δw (see footnote 2). Now, using P1 and (1) one can show that, using A3.b and neglecting terms containing f_{pa} ,

$$|e_p'(\Delta w)| \leq |e_a'(\Delta w)|$$

because of A3.c. Since f is h.d. 1, and using also $f_a \geq f_p$,

$$\begin{aligned} g_a'(\Delta w) &= f_a e_a(\Delta w) + f_p e_p(\Delta w) - \Delta w f_p(e_p, e_a) \left| e_p'(\Delta w) \right| \\ &\geq f_a e_a(\Delta w) + f_p e_p(\Delta w) - \Delta w f_a(e_p, e_a) \left| e_a'(\Delta w) \right| \\ &= f_a [e_a(\Delta w) - \Delta w e_a'(\Delta w)] + f_p e_p(\Delta w). \end{aligned}$$

Concavity of e_a (i.e., P2) implies that the first term is positive, and $g_a' > 0$.

b) Now, using again (1) and neglecting terms with f_{pa}, we have

$$g_p'' = (\Delta X - \Delta w)[f_{aa} \left(e_a'(\Delta w)\right)^2 + f_a e_a''(\Delta w)] - [f_a 2 e_a'(\Delta w) + f_p e_p'(\Delta w)]$$

is negative, because of P2 and what observed above about e_a' and e_p'. Analogous procedure shows that

$$g'' = \Delta w f_{pp} (e_p'(\Delta w))^2 + (\Delta X - \Delta w) f_{aa} \left(e_a'(\Delta w)\right)^2 - [f_a 2 e_a'(\Delta w) - f_p e_p'(\Delta w)]$$

is negative. c) now follows immediately. ■

Using the gain functions, for any choice $(w(1), \Delta w)$ the p-agent's payoff is

$$\widetilde{W}_p^f(r) + g_p(\Delta w) - w(1)$$

while the (IR) constraint is

$$W_a^f(r) + g_a(\Delta w) + w(1) \geq U.$$

Let $(w^*(1), \Delta w^*)(U, W, r)$ be the optimal sharing rule, i.e., a solution to (C'), expressed as a function of U, W and r. We immediately have the following result.

Lemma 4 *Under A1-A3, P1 and P2:*
a) $\Delta w^*(U, W, r) \geq \Delta w_p$.
b) Either the (IR) *constraint binds or* $\Delta w^*(U, W, r) = \Delta w_p$.

Proof. a) If not, then by Lemma 3 increasing Δw and keeping $w(1)$ constant increases the p-agent's objective function without violating any of the constraints.

b) We drop reference to U, W, r for the optimal choice. Suppose not. Then

$$W_a^f(r) + g_a(\Delta w^*) + w^*(1) > U \text{ and } \Delta w^* > \Delta w_p.$$

If $w^*(1) > -W_a^f(r)$, simply decrease $w(1)$, increasing the p-agent's payoff, a contradiction to the optimality of $(w^*(1), \Delta w^*)$. If $w^*(1) = -W_a^f(r)$, and if $\Delta w^* \in (\Delta w_p, \Delta X]$, then for $\Delta w_p < \Delta w' < \Delta w^*$, $g_p(\Delta w') - g_p(\Delta w^*) > 0$, and we can decrease Δw to increase payoffs while satisfying the (IR) constraint, by its continuity, again contradicting the optimality of $(w^*(1), \Delta w^*)$. ■

Our problem (C') is then equivalent to

$$\begin{array}{ll}
\max_{w(1),\Delta w} \ \widetilde{W}_p^f(r) + g_p(\Delta w) - w(1) & s.t. \\
W_a + g_a(\Delta w) + w(1) \geq U & (IR) \\
w(1) \geq -W_a^f(r) & (LL_a) \\
-w(1) \geq -\widetilde{W}_p^f(r) & (LL_p) \\
\Delta X \geq \Delta w \geq \Delta w_p &
\end{array}$$

Note that if $(w^*(1), \Delta w^*)(U, W, r)$ is a solution with $\Delta w^*(U, W, r) > \Delta w_p$, the (IR) constraint must hold with equality. Then we can substitute $w(1) = U - W_a^f(r) - g_a(\Delta w)$ from the (IR) constraint into the (LL) constraints and into the objective function. We then eliminate $w(1)$ from the problem, which becomes

$$\begin{array}{ll}
\max_{\Delta w} \ \widetilde{W}_p^f(r) + W_a^f(r) + g(\Delta w) - U & s.t. \\
g_a(\Delta w) \leq U & (LL_a) \\
g_a(\Delta w) \geq U - W_a^f(r) - \widetilde{W}_p^f(r) & (LL_p) \\
\Delta X \geq \Delta w > \Delta w_p &
\end{array} \qquad (P)$$

In this case, and because g_a is increasing and g is concave, a solution to $(P1)$ which we know exists, is also unique. Note that in $(P1)$ at most one of the (LL) constraints binds at a solution. Also note that a solution $(w^*(1), \Delta w^*)(U, W, r)$ to problem (C') either has $\Delta w^*(U, W, r) = \Delta w_p$ or solves (P).

If the optimum for (P) is at the boundaries (LL), it depends on the reservation utility U, and possibly on W_p, W_a and r. We denote it by $\Delta w^*(U, W_p, W_a, r)$. Let $S(U) \equiv \tilde{S}(\Delta w^*(U, W_p, W_a))$ be the optimal surplus, as a function of U only. Let $\Delta w_i(U, W, r)$ denote Δw which solves (LL_i) in $(P1)$, for $i = p, a$. We are finally ready to state the main property of the solution to the bargaining problem, regarding the derivative of the surplus with respect to the reservation utility.

Proposition 5 *i) Suppose that $\Delta w_g > \Delta w_a(U, W, r)$. Then Δw_g is not feasible for (P), (LL_a) binds and*

$$\Delta w^*(U, W, r) = \Delta w_a(U, W, r)$$

if $\Delta w_a(U, W, r) \geq \Delta w_p$, or

$$\Delta w^*(U, W, r) = \Delta w_p.$$

ii) Suppose that $\Delta w_g < \Delta w_p(U, W, r)$. Then Δw_g is not feasible for (P), (LL_p) binds and

$$\Delta w^*(U, W, r) = \Delta w_p(U, W, r).$$

Furthermore, in case (i),

$$\partial S(U, W, r)/\partial U \in (0, 1]$$

and $\partial S(U)/\partial U < 1$ if $\Delta w_a(U, W, r) > \Delta w_p$; while in case (ii),

$$\partial S(U)/\partial U < 0$$

In all other cases, $\partial S(U)/\partial U = 0$ when S is differentiable.

Proof. i) Clearly Δw_g is not feasible for (P) because g_a is increasing in Δw and (LL_a) is already binding at $\Delta w_a(U, W, r)$. Since $g(\Delta w)$ is increasing before Δw_g, $\Delta w_a(U, W, r)$ is the maximum for (P). If $\Delta w_a(U, W, r) < \Delta w_p$, then $\Delta w_a(U, W, r)$ is worse than Δw_p, and $\Delta w^*(U, W, r) = \Delta w_p$. ii) The proof of this statement is similar to i), and therefore omitted. As for the derivative, consider case (i). Now $g_a'(\Delta w) > 0$ implies that, as U increases, Δw_a must increase as well. So $\partial \Delta w_a/\partial U > 0$. Then $\partial S(U, W, r)/\partial U = (dg/dw)(\partial \Delta w_a/\partial U) > 0$, since g is increasing when $\Delta w_a(U, W, r) < \Delta w_g$. If $\Delta w_a(U, W, r)$ solves (P), and if $\Delta w_a(U, W, r) > \Delta w_p$, g_p concave implies that as U increases g_p decreases. But since $g = g_p + g_a = g_p + U$, we have $\partial S(U, W, r)/\partial U < 1$. Now the other cases are obvious. ■

Let $U_p(\mathcal{B}^*, U, W, r)$, $U_a(\mathcal{B}^*, U, W, r)$ be the p-agent's and a-agent's payoffs, respectively.

We summarize the properties of the solution to problem (C').

Proposition 6 *a) A solution to* (C') *exists if* $W_a^f(r) + \widetilde{W}_p^f(r) + g_a(\Delta X) \geq U$, *and entails the surplus*

$$S(U,W,r) = \begin{cases} g(\Delta w_p) & \text{if } U < g_a(\Delta w_p) & (S.1) \\ g(\Delta w_a(U)) & g_a(\Delta w_p) \leq U < g_a(\Delta w_g) & (S.2) \\ g(\Delta w_g) & g_a(\Delta w_g) \leq U, \text{ and } U < \widetilde{W}_p^f(r) + W_a^f(r) + g_a(\Delta w_g) & (S.3) \\ g(\Delta w_p(U, \widetilde{W}_p^f(r), W_a^f(r))) & \widetilde{W}_p^f(r) + W_a^f(r) + g_a(\Delta w_g) \leq U & (S.4) \end{cases}$$

Moreover, in case (S.2), $\frac{\partial}{\partial U}[S(U,W,r)] \in (0,1)$, *this derivative being zero in (S.1) and (S.3) and negative in (S.4).*

We interpret the statement of the proposition.

(S.1): here U is so low that (IR) does not bind, although (LL_a) does. In this situation we have 'efficiency wages', and the a-agent gets more than his outside option.

(S.2): here (IR) binds and so does (LL_a), the intuitive case. The monotonicity of surplus $S(U,W,r)$ in U in case (S.2) is explained by the fact that, since (LL_a) and (IR) are binding, an increase in U implies that the a-agent must be paid more in the high state, increasing his incentives to exert effort. While the p-agent's incentives will be so reduced, the net result on surplus will be nonnegative as $f_a \geq f_p$.

(S.3): no (LL) constraint binds, and maximum incentive compatible surplus is achieved.

(S.4): only (LL_p) binds. In this last case, an increase in the a–agent's reservation utility must be paid with an increase in bonus but since the effort exerted is too high already, surplus will decrease.

Note also that $S(U,W,r)$ is independent of W (and so is U_p) unless (LL_p) binds, and it is a constant in cases $(S.1)$ and $(S.3)$. Unless the limited liability constraint binds for the p-agent, net (expected) surplus in a match depends on wealth only if the outside option of the a–agent does.

The optimal solution yields the following (expected) payoffs to the p-agent and the a-agent:

$$U_a(\mathcal{B}^*, U, W, r) = \begin{cases} g_a(\Delta w_p) & \text{if } U < g_a(\Delta w_p) \\ U & \text{otherwise} \end{cases} \tag{4}$$

$$U_p(\mathcal{B}^*, U, W, r) = \widetilde{W}_p^f(r) + W_a^f(r) + S(U,W,r) - U_a(\mathcal{B}^*, U, W, r) \tag{5}$$

It is immediately seen that $\mathcal{B}^*$ is constrained efficient.

2.0.6 Summary

In conclusion, the assumptions we have made to get to Proposition 6 (A1-A3, P1-P3) are summarized as:

–. c is at least C^2, increasing, convex with $c(e) \geq 0$ and $c(0) = 0$.

– f is at least C^2, increasing, concave, and h.d. 1, with

i. $f_a(e_p, e_a) \geq f_p(e_p, e_a) \geq 0$ and $\max\{f_a(e_p, e_a), f_p(e_p, e_a)\} > 0$.

ii. $f_{pa} \approx 0$

– $\Delta X f_p(e,0) \leq c'(e_p(0))$ all $e \geq e_p(0)$, and $\Delta X f_a(0,e) \leq c'(e_a(\Delta X))$ all $e \geq e_a(\Delta X)$.

– $\frac{f_a(e_p,e_a)}{f_p(e_p,e_a)} \geq \frac{\Delta w f_{aa}(e_p,e_a) - c''(e_a)}{(\Delta X - \Delta w) f_{pp}(e_p,e_a) - c''(e_p)}$

In addition, we have assumed sufficient regularity for f and c so that

– $e_p(\Delta w), e_a(\Delta w)$ are differentiable functions on $[0, \Delta X]$, with $e_p''(\Delta w), e_a''(\Delta w) \approx 0$ and $e_a''(\Delta w) \leq 0$.

Proposition 6 summarizes an important property for the analysis in Chakraborty and Citanna (2001, 2002). They also considered contractual problems where $g_p(\Delta w_g) \leq g_a(\Delta w_g)$ for all f, with equality only if $f_a = f_p$.

As far as the shape of surplus is concerned, this property is irrelevant. As for equilibrium effects with endogenous matching, if $g_p(\Delta w_g) > g_a(\Delta w_g)$, then no wealth effects would appear in equilibrium. This is because maximum incentive compatible surplus would be achievable with equal division irrespective of the individual wealth level. This follows from the definition of the unrestricted set **R** in Chakraborty and Citanna (2001, 2002). In this sense it is economically more interesting to focus on the opposite case.[3]

It remains to be seen if the set of such f, c and ΔX satisfying all of our assumptions is nonempty.

An example Consider a quadratic cost of effort, i.e.

$$c(e) = c\,\frac{e^2}{2} \quad \text{with } c > 0$$

and f linear, i.e.,

$$f(e_p, e_a, \alpha) = \alpha e_a + (1-\alpha) e_p$$

where $\alpha \in [1/2, 1]$ and $\Delta X < 2c$.

In this case $f_p = 1 - \alpha$, $f_a = \alpha$ and $0 \leq f_p \leq f_a \leq 1$, with $\max\{f_a, f_p\} > 0$. Also, $f_{pa} = 0$.

From the (IC) constraints, we derive the optimal efforts $e_p(\Delta w, \alpha)$ and $e_a(\Delta w, \alpha)$ as continuous functions of Δw and α. They are

$$e_a(\Delta w, \alpha) = \max\{0, \min[\tfrac{\alpha \Delta w}{c}, 1]\}$$
$$e_p(\Delta w, \alpha) = \max\{0, \min\{\tfrac{(1-\alpha)(\Delta X - \Delta w)}{c}, 1]\}$$

Notice that $e_p(\Delta w), e_a(\Delta w)$ are differentiable functions on $[0, \Delta X]$, with $e_p''(\Delta w) = e_a''(\Delta w) = 0$. Now,

$$\begin{aligned} \Delta X f_p(e,0) - c'(e_p(0)) &= \Delta X(1-\alpha) - c(1-\alpha)\Delta X/c = 0 \leq 0 \text{ all } e \geq e_p(0), \\ \Delta X f_a(0,e) - c'(e_a(\Delta X)) &= \Delta X \alpha - c\alpha \Delta X/c = 0 \leq 0 \text{ all } e \geq e_a(\Delta X). \end{aligned}$$

[3] The other property used in Chakraborty and Citanna (2001, 2002) is $g_p(\Delta w_p) > g_a(\Delta w_p)$. However, this is inessential, and everything goes through in that paper even if it does not hold. In the proof of Proposition 1, Step 2, one needs to evaluate Φ at $g_p(\Delta w_p)$, as opposed to at $g_a(\Delta w_p)$, to apply the intermediate value theorem.

Next,

$$\frac{f_a(e_p,e_a)}{f_p(e_p,e_a)} = \frac{\alpha}{1-\alpha} \geq 1 = \frac{-c}{-c} = \frac{\Delta w f_{aa}(e_p,e_a) - c''(e_a)}{(\Delta X - \Delta w) f_{pp}(e_p,e_a) - c''(e_p)}.$$

Then we derive an expression for the probability of success as a continuous function of Δw and α, $f(\Delta w, \alpha)$. This is

$$f(\Delta w,\alpha) = \begin{cases} 1-\alpha & \text{if } \Delta w \leq \Delta X - \frac{c}{1-\alpha} \\ (1-\alpha)^2(\Delta X - \Delta w)/c & \Delta X - \frac{c}{1-\alpha} < \Delta w \leq 0 \\ [\alpha^2 \Delta w + (1-\alpha)^2(\Delta X - \Delta w)]/c & 0 < \Delta w \leq \Delta X \\ \alpha^2 \Delta w / c & \Delta X < \Delta w \leq c/\alpha \\ \alpha & c/\alpha < \Delta w \end{cases}$$

Note that g_a is strictly increasing, and g_p and g are globally concave reaching interior maximums at

$$\Delta w_p(\alpha) = \Delta X \; \frac{\alpha^2 - (1-\alpha)^2}{2\alpha^2 - (1-\alpha)^2}$$

and

$$\Delta w_g(\alpha) = \Delta X \; \frac{\alpha^2}{\alpha^2 + (1-\alpha)^2} \leq \Delta X$$

respectively. Observe that here $g_p(\Delta w_p(\alpha),\alpha) > g_a(\Delta w_p(\alpha),\alpha)$ and that $g_p(\Delta w_g(\alpha),\alpha) \leq g_a(\Delta w_g(\alpha),\alpha)$ for all $\alpha \in [\frac{1}{2},1]$, with the second inequality holding with equality if and only if $\alpha = \frac{1}{2}$.

Hence all the results of this note apply, and in particular Proposition 6.

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