Pricing Kernels and Dynamic Portfolios

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Abstract

We investigate the structure of the pricing kernels in a general dynamic investment setting by making use of their duality with the self financing portfolios. We generalize the variance bound on the intertemporal marginal rate of substitution introduced in Hansen and Jagannathan (1991) along two dimensions, first by looking at the variance of the pricing kernels over several trading periods, and second by studying the restrictions imposed by the market prices of a set of securities.

The variance bound is the square of the optimal Sharpe ratio which can be achieved through a dynamic self financing strategy. This Sharpe ratio may be further enhanced by investing dynamically in some additional securities. We exhibit the kernel which yields the smallest possible increase in optimal dynamic Sharpe ratio while agreeing with the current market quotes of the additional instruments.

Keywords: Pricing Kernel, Sharpe Ratio, Self Financing Portfolio, Variance–Optimal Hedging.

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1. Introduction

The duality between pricing kernels and portfolio payoffs is the key to many fundamental results in asset pricing theory. In a one period setting, a pricing kernel is a random variable m_{t+1} which satisfies the equality

$$R^f_{t,t+1} E_t[m_{t+1}] w_t = E_t\left[m_{t+1} w_{t+1}\right] \tag{1}$$

for every portfolio with payoff w_{t+1} at time $(t+1)$ and value w_t at time t, where $R^f_{t,t+1}$ and E_t denote respectively the (gross) risk free rate from t to $(t+1)$ and the conditional expectation operator corresponding to the information available at time t. Harrison and Kreps (1979) show that the existence of a pricing kernel is equivalent to the law of one price while the absence of arbitrage corresponds to the existence of a positive pricing kernel.

If we know the prices today and the payoffs tomorrow of a set of securities, then a positive pricing kernel m_{t+1} consistent with these securities provides an efficient method to produce contingent claim prices in an arbitrage free framework. The kernel m_{t+1} yields an arbitrage free price F_t today for a payoff F_{t+1} tomorrow through the equation

$$R^f_{t,t+1} E_t[m_{t+1}] F_t = E_t\left[m_{t+1} F_{t+1}\right].$$

This technique is especially useful when the market is incomplete and the claim F_{t+1} cannot be obtained as the payoff of a portfolio based on the primitive securities.

Every positive pricing kernel yields however a different arbitrage free price system, and in many situations the resulting range of contingent claim prices is so wide as to be of little practical use. It is then natural to seek a rationale to reduce the set of admissible pricing kernels, and in turn the range of corresponding prices. The quest for such a rationale is a central theme in asset pricing theory. Bernardo and Ledoit (2000) show for instance that setting upper and lower bounds to a pricing kernel in every state of the world controls the maximum gain–loss ratio of every investment strategy. Balduzzi and Kallal (1997) consider the restrictions imposed by the risk premia assigned by the pricing kernels on some arbitrary sources of risk.

The variance bound on the pricing kernels introduced in Hansen and Jagannathan (1991) is another important consequence of the duality between kernels and portfolios. The square of the Sharpe ratio of every portfolio is smaller than the variance of every pricing kernel, once properly normalized, and equality obtains for a unique portfolio whose payoff is also itself a pricing kernel. This result is useful in two ways. On the one hand, the variance of every pricing kernel yields an upper bound to the Sharpe ratios which portfolio managers may expect to obtain in the market. On the other hand the Sharpe ratio of any portfolio is a lower bound to the variance of the pricing kernels, and this allows to reject the asset pricing theories for which the discount factor does not display enough variation across the states of nature. Bekaert and Liu (2001) give an extensive account of the growing use of these bounds in financial economics.

In view of this result, Cochrane and Saá–Requejo (2000) reduce the set of admissible pricing kernels by rejecting candidates with large variance on the ground that they may give rise to abnormal good–deals in the form of investment opportunities with large Sharpe ratios. They reason that although positive pricing kernels with large variance do not create arbitrage opportunities, they are nevertheless suspicious and should be discarded. Cochrane and Saá–Requejo (2000) compute the upper and lower bounds for the price of a contingent claim when a variance bound is imposed on the kernels. Černý (2002) and Longarela (2001) generalize this result. Černý (2002) defines a good–deal in terms of certainty equivalent while Longarela (2001) replaces the variance of the kernel with the measure of model mispecification introduced in Hansen and Jagannathan (1997).

Our contribution is to extend the investigation of the duality between investment strategies and pricing kernels from a single period to several consecutive trading periods. A pricing kernel from time t to horizon T is a random variable m_T which satisfy the equality

$$R_s^f E_s[m_T] w_s = E_s[m_T w_T], \tag{2}$$

for every intermediate period s between t and T and for every self financing portfolio whose value varies from w_s to w_T between time s and horizon T. We denote here R_s^f the gross risk free rate from s to T.

The time dimension of this duality has so far been limited to the description of the information set implicit in the conditional expectation of Equation 1. We generalize the variance bound of Hansen and Jagannathan (1991) to a multiperiod setting by showing that the standard deviation of the intertemporal marginal rate of substitution over a span of trading periods is larger than the optimal Sharpe ratio available over the corresponding investment horizon through dynamic self financing strategies. Every investment span gives rise to a different variance bound, and it is legitimate to expect a sharper restriction on the pricing kernels than the one which results from a single trading period.

The asset pricing results which follow from restrictions on the pricing kernels have so far been obtained through a repeated use of a single period analysis. This is for instance the case in both Bernardo and Ledoit (2000) and Cochrane and Saá–Requejo (2000) who compute contingent claim price bounds recursively. They cannot deal with a constraint on the kernels which is defined over several periods and which cannot be written as a succession of constraints on the one period intertemporal marginal rates of substitution.

An important example of such a constraint is the observation of the current market prices of a set of new securities on top of the original ones. It is then natural to study the set of kernels which agree with these quotes. If the payoffs of the new securities span several trading periods, this constraint cannot be written in a convenient time separable way. Our multiperiod analysis handles these constraints and allows us to exhibit the sharper variance bounds which they generate.

We propose a theory of pricing kernels in a general dynamic investment environment. We describe the structure of the pricing kernels which are consistent with the stochastic evolution of a finite number of securities. Equation 2 highlights the duality between the pricing kernels and the value processes of the self financing portfolios. We show that the pricing kernel with minimum conditional variance over a span of trading periods is the unique kernel which is also the final value of a self financing portfolio. This final value happens to have minimum conditional second moment among the self financing portfolios. We refer to this strategy as the L^2 minimum portfolio. The analysis of this duality yields a number of results, both on the pricing kernels and on the dynamic investment strategies.

As explained above, positive pricing kernels allow to derive the price dynamics of new instruments in an arbitrage free framework. This technique is also often described as the choice of a risk neutral probability distribution in which discounted security prices are martingale. The new instruments may for instance be derivatives written on the original securities. We take a partial equilibrium point of view and we assume that the new securities have no effect on the price dynamics of the original ones. The introduction of additional instruments may therefore only enhance the efficient frontier available through dynamic trading.

This increase in efficiency depends on the price dynamics of the new instruments. We show that if the price process followed by the new instruments is derived from a pricing kernel consistent with the original securities, then the increase in the optimal dynamic Sharpe ratio is a function of the extent to which the new instruments help dynamically replicate the kernel. This suggests that a manager who seeks to maximize the dynamic Sharpe ratio of her fund by increasing her investment scope should consider first the securities which best replicate the kernel.

The maximum gain in efficiency is obtained once the kernel is perfectly replicated with both the original and the additional securities so that it becomes the final value of a self financing strategy. The maximum dynamic Sharpe ratio is then the standard deviation of the pricing kernel. This also proves that the standard deviation of a given pricing kernel is an upper bound to the dynamic Sharpe ratio which can be reached through dynamic self financing strategies which invest in a arbitrarily large number of instruments, provided that the price process of these instruments is derived from the given kernel.

Once the pricing kernel is perfectly replicated, no more mean–variance efficiency gain may be expected from the introduction of new securities and the strategy which replicates the kernel belongs to the enhanced efficient frontier. If we use a pricing kernel which is already the final value of a self financing strategy based on the original securities in the first place, then no efficiency gain is possible right from the start. This means that every new instrument is priced by this kernel in such a way as to be useless for the construction of a dynamically mean–variance efficient strategy. The pricing kernel with minimum–variance is the only kernel enjoying this property. This special kernel corresponds therefore to a min–max in terms of dynamic Sharpe ratio. Cochrane and Saá–Requejo (2000) have proposed to eliminate dynamics which create "good–deals", where they define a good–deal as an investment strategy with a large instantaneous Sharpe ratio. The minimum–variance kernel extends this methodology to an intertemporal Sharpe ratio. It generates conservative dynamics which do not allow any increase in Sharpe ratio, thereby eliminating "good–deals" in a dynamic sense.

Besides its interpretation in terms of portfolio management, the minimum–variance pricing kernel has received attention in the finance literature for another related issue: the variance–optimal hedge of a contingent claim. Schweizer (1995) derives the price of a contingent claim from the cost of its optimal replication by means of self financing strategies. Optimality is measured by a quadratic loss function. This price happens to be identical to the one derived from the minimum–variance pricing kernel. The importance of the variance–optimal hedging strategy is highlighted by the remark that every pricing kernel can be written as the variance–optimal hedge residual of a contingent claim.

We prove that the cost of the variance–optimal hedge of a security does not change as new hedging instruments are introduced, as long as these instruments are themselves priced according to the cost of their variance–optimal hedge, that is if their price dynamics is derived from the minimum–variance pricing kernel.

We next investigate the situation where, on top of the original securities, the current market prices of a set of additional securities are available. These new instruments could typically be a set of actively traded calls and puts written on the original securities. In line with the option pricing literature, we shall sometimes refer to the collection of these prices as a smile. We illustrate the significance of this situation by considering two dynamic investment problems, the dynamic management of a portfolio on the one hand, and the pricing and hedging of a contingent claim on the other hand.

We consider first a fund manager who trades in a finite number of securities and who considers investing in derivative instruments written on them. Markets are frictionless and perfectly competitive and we assume that the manager knows the price dynamics of the underlying securities. Although she observes the prices of all traded securities every period, she does not know the future price dynamics of the derivative instruments. The manager could for instance be an equity portfolio manager who is considering investing in convertible bonds written on the shares in which she is trading. The manager faces several interconnected questions. Which derivatives should she select? Which price dynamics will they follow? How should she optimally manage her portfolio with the new instruments? Which performance gain can she expect from expanding her investment scope?

Consider now an investment banker who is seeking to price and hedge an exotic derivative instrument written on some underlying securities. The banker knows the price process followed by the underlying securities, and he observes the market quotes of a set of actively traded derivatives written on them, for instance vanilla calls and puts, but he does not know their price dynamics. The exotic derivative is not actively traded and no market price is readily available. The banker seeks to use the traded derivatives, together with the underlying securities, in order to hedge the exotic instrument. He is confronted with several questions, echoing the questions raised by the fund manager. Which price dynamics will follow the traded derivatives? At which price should he deal in the exotic instrument? Which is the best hedging strategy using both the underlying securities and the traded derivatives?

In a complete market setting, the questions raised by both the fund manager and the investment banker find immediate answers. For every derivative instrument, only one price dynamics is consistent with absence of arbitrage, and it is given by the value process of its exact replication strategy. No performance gain can be expected in the management of a portfolio by the introduction of new securities since the opportunity set is not modified by the addition of redundant securities. There is no need either for the banker to hedge the exotic instrument with the traded derivatives since it is already perfectly replicated with the underlying securities. In an incomplete market setting however, exact replication is typically not possible and many price dynamics for the new instruments may be consistent with the observed market quotes and the principle of absence of arbitrage. An important question arises as to which rationale allows to reduce the choice among admissible price dynamics. We offer a rationale which answers the concerns of both the fund manager and the investment banker.

Following again the logic of limiting good–deals in a dynamic sense, we characterize the kernel which yields a minimum increase in optimum Sharpe ratio while agreeing with the prices of the instruments for which market quotes are available. Drawing on the duality with the dynamic portfolios, we describe the efficient investment strategies which corresponds to this kernel. They solve a max–min problem in terms of dynamic Sharpe ratio. These strategies have a remarkable feature, they hold fixed quantities of the quoted instruments, on top of an investment in the L^2 minimum portfolio for the original securities.

The constraint of matching the smile reduces the set of admissible pricing kernels and leads to a higher variance bound on the kernels. We describe this set and we show that the increase in the variance bound is given by the distance, in the metric of the variance–optimal hedge residuals, between the observed market quotes of the instruments and the cost of their variance–optimal hedge.

We show that the pricing kernel which limits dynamic good–deals while agreeing with the smile is also optimal in terms of variance–optimal hedge for two reasons. First it prices a contingent claim as close as possible to the cost of its variance–optimal hedge. Second this price is the initial value of a constrained optimal hedging strategy. In both cases, the constrained optimality corresponds to a min–max where we consider the worst possible contingent claim. We show finally that the contingent claim price derived from this kernel is equal to the value of the variance–optimal hedge of the claim, when the dynamic hedging strategy uses both the original securities and the instruments of the smile.

The paper is organized as follows. Sections 2 to 4 describe the self financing portfolios and their mean–variance properties. They draw heavily on Henrotte (2001) which provides an extensive account of the structure of these dynamic investment strategies. Section 5 studies the structure of the pricing kernels and generalizes the Hansen and Jagannathan (1991) variance bound to a multiperiod setting. Section 6 explains how to price additional securities in an incomplete market setting while avoiding mean–variance good–deals in a dynamic sense. It relates the increase in the slope of the efficient frontier with the extent to which the additional securities help replicate the kernel. Section 7 studies the pricing kernels and the price dynamics which are consistent with the constraint of matching the market quotes of a given set of securities. We derive a lower bound to the variance of these kernels and we describe the minimum increase in the optimal dynamic Sharpe ratio implied by this constraint. This lower bound and this minimum are reached for a pricing kernel and an efficient dynamic strategy which we describe in Section 8. We propose this dynamics as a solution to our two investment problems in incomplete markets, the mean–variance management of a portfolio and the optimal hedge of a contingent claim.

2. Dynamic Portfolios

2.1. Initial Market Structure

We consider a finite number n of underlying securities traded in a frictionless and competitive market over a set of discrete times with finite horizon. We index the trading dates by the integers between 0 and a final horizon T. Information is described by a filtration $\mathcal{F} \stackrel{\text{def.}}{=} \{\mathcal{F}_t\}_{0 \leq t \leq T}$ over a probability space $(\Omega, \mathcal{F}_T, P)$.

Throughout the article, equalities and inequalities between random variables are understood to hold P almost surely. We denote respectively $E[F]$ and $E_t[F]$ the expected value and the conditional expectation with respect to $\mathcal{F}_t$ of a random variable F in $L^1(P)$. We let $L^2_t(P)$ be the space of random variables in $L^2(P)$ which are measurable with respect to $\mathcal{F}_t$ and we let $L^2(P; I\!\!R^n)$ be the space of random vectors in $I\!\!R^n$ with components in $L^2(P)$. If f_t is positive and measurable with respect to $\mathcal{F}_t$, we define $L^2_t(P, f_t)$ as the set of random variables F such that $f_t F$ belongs to $L^2_t(P)$. We define in the same way $L^2_t(P, f_t; I\!\!R^n)$ for random vectors in $I\!\!R^n$. We close this list of technical notations by letting $x'y$ denote the usual scalar product of two vectors x and y in $I\!\!R^n$.

An unspecified numeraire is fixed every period and we let p_t be the vector of prices of the n securities in this numeraire at time t. We let d_t be the numeraire dividend distributed by the securities at time t. The owner of one unit of security i at time t is entitled to receive the dividend d^i_{t+1} in numeraire the next period. We let $\phi_t \stackrel{\text{def.}}{=} (p_t + d_t)$ be the cum-dividend price vector of the securities at time t. The vector processes $\{p_t\}_{0 \leq t \leq T}$, $\{d_t\}_{0 \leq t \leq T}$, and $\{\phi_t\}_{0 \leq t \leq T}$ are adapted to the filtration $\mathcal{F}$. We do not limit ourselves to equities and the dividends should be understood as general, and possibly contingent, numeraire distributions.

We do not rule out that some security might be redundant at some trading period and in some state of the world but we do impose that the law of one price holds. For the remainder of the article, we shall assume that the following two assumptions are satisfied.

Assumption 1 *Prices and returns of the securities do not vanish. For every period t between 0 and T and for every period s between 1 and T the price vectors p_t and ϕ_s are P almost surely different from the null vector.*

Assumption 2 *Law of one price. For every period t between 0 and $(T-1)$, and for every random vectors X_t and Y_t in $I\!R^n$ measurable with respect to $\mathcal{F}_t$, the equality $\phi'_{t+1}X_t = \phi'_{t+1}Y_t$ implies $p'_tX_t = p'_tY_t$.*

2.2. Self Financing Portfolios

A dynamic portfolio X starting at time t is a process in $I\!R^n$ adapted to $\mathcal{F}$ and indexed by time s with $t \leq s \leq (T-1)$, where X^i_s represents the number of units of security i held in portfolio X at time s. We let $w(X)$ be the value process of portfolio X, naturally defined by $w_s(X) \stackrel{\text{def.}}{=} p'_sX_s$ for $s \leq (T-1)$ and we let $w_T(X) = \phi'_TX_{T-1}$.

We say that a dynamic portfolio X starting at time t is self financing at time s whenever $w_s(X) = \phi'_sX_{s-1}$ and that it is self financing whenever it is self financing from $(t+1)$ to T. We remark that the definition of the final value of the strategy implies that a dynamic portfolio is always self financing at time T.

It is easily checked that the law of one price implies that two self financing portfolios with identical final values at time T share the same value process. This property will allow us later to identify two such dynamic portfolios.

Henrotte (2001) characterizes the set of self financing dynamic portfolios starting at time t with the property that their final value at time T is in $L^2(P)$. Saving on notation, we denote $\mathcal{X}_t$ this set with no explicit reference to T since which we shall keep this final horizon constant throughout our analysis. We also let $w_T(\mathcal{X}_t) \stackrel{\text{def.}}{=} \{w_T(X)\ ;\ X \in \mathcal{X}_t\}$ be the set in $L^2(P)$ of terminal values of portfolios in $\mathcal{X}_t$. Besides the self financing condition, no restriction is imposed on the value process of the portfolios at periods prior to the final horizon.

Henrotte (2001) builds a positive process h by backward induction from the final value $h_T = 1$ at time T. This process plays a central role in the description of the structure of $\mathcal{X}_t$, and more generally in the mean–variance analysis. It is closely linked to the notion of dynamic Sharpe ratio and it can be interpreted as a correction lens for myopic investors.

We denote N^+ the Moore–Penrose generalized inverse of a symmetric matrix N in $\mathbb{R}^n \times \mathbb{R}^n$. The matrix N^+ is itself symmetric, commutes with N, and satisfies[1]

$$NN^+N = N,$$
$$N^+NN^+ = N^+.$$

If N is a random matrix measurable with respect to $\mathcal{F}_t$, then $N^+(\omega)$ is defined for every ω in Ω and N^+ is also measurable with respect to $\mathcal{F}_t$.

Proposition 1 *The adapted process h defined by $h_T = 1$ at time T and the backward equation*

$$h_t \stackrel{def.}{=} \left(p_t' N_t^+ p_t\right)^{-1} \tag{3}$$

with $N_t \stackrel{def.}{=} E_t\left[h_{t+1}\phi_{t+1}\phi_{t+1}'\right]$ for $0 \le t \le (T-1)$, is well defined, P almost surely positive, and satisfies $\phi_t \in L_t^2(P, \sqrt{h_t}; \mathbb{R}^n)$ for every period t between 0 and T as soon as the following two conditions are met:

(a). $\phi_T \in L^2(P; \mathbb{R}^n)$;

(b). $d_t \in L_t^2(P, \sqrt{h_t}; \mathbb{R}^n)$ for every period t with $0 \le t \le (T-1)$.

The following properties then hold.

(i). For every dynamic portfolio $X \in \mathcal{X}_t$ the process $\{h_s w_s(X)^2\}_{t \le s \le T}$ is a submartingale, that is, for every period s with $t \le s \le (T-1)$ we have

$$h_s w_s(X)^2 \le E_s\left[h_{s+1} w_{s+1}(X)^2\right] \le E_s\left[w_T(X)^2\right].$$

(ii). The set $\mathcal{X}_t$ is the set of self financing dynamic portfolios starting at time t such that $w_s(X) \in L_s^2(P, \sqrt{h_s})$ for every period $t \le s \le T$.

[1] see Theil (1983) for a general description of the Moore–Penrose inverse.

(iii). The set $w_T(\mathcal{X}_t)$ is closed in $L^2(P)$.

Condition (b) of Proposition 1 involves the variable h_t which is derived recursively through Equation 3. The following lemma provides a sufficient condition independent of h.

Lemma 1 *If ϕ_T is an element of $L^2(P; \mathbb{R}^n)$ (Condition (a) of Proposition 1), then d_t belongs to $L^2_t(P, \sqrt{h_t}; \mathbb{R}^n)$ for every period t from 0 to $(T-1)$ (Condition (b) of Proposition 1) if one security, say Security k, pays no dividend and is such that $(p^k_T/p^k_t)d_t$ is an element of $L^2(P; \mathbb{R}^n)$ for every period t from 0 to $(T-1)$.*

For the remainder of the article, we assume that Conditions (a) and (b) of Proposition 1 are satisfied so that the results of this proposition apply.

Assumption 3 *Conditions (a) and (b) of Proposition 1 are satisfied.*

Two equations will prove useful. For every period t between 0 and $(T-1)$,

$$\phi_{t+1} = N_t N_t^+ \phi_{t+1}, \tag{4}$$

and the law of one price implies then that

$$p_t = N_t N_t^+ p_t. \tag{5}$$

The process h acts as a weight which regularizes the prices and the values of the self financing portfolios in $\mathcal{X}_t$ every period. Once we multiply these processes by the square root of h, they all have finite second moments every period. Henrotte (2001) shows that the process h is the largest process with value $h_T = 1$ at horizon T having this regularization property.

3. Optimal Hedge

This section investigates the hedging properties of the self financing dynamic portfolios. We first show how to construct a dynamic strategy which best replicates a payoff F_T at horizon T, starting from a value w_t at time t. The loss function which we choose at horizon T is the norm of $L^2(P)$, which is well defined for the portfolios in $\mathcal{X}_t$. We then study the cost and quality of the optimal hedge and we show that the value process of the optimal solution is unique. When the final payoff F_T is zero, we obtain as a special case the L^2 minimum portfolio which is the hedging numeraire used by Gouriéroux et al. (1998). We show that our analysis can be extended to deal with the optimal replication of securities described by a sequence of contingent cash flows instead of a single final payoff. We introduce interest rates by mean of default free zero coupon bonds and we relate our work with the concept of variance–optimal signed martingale measure introduced in Schweizer (1995).

3.1. Construction of an Optimal Hedge

The optimal L^2 replication of a contingent claim involves a mixture of forward and backward equations. We derive first the cost of the optimal hedge every period in a backward way, and we then use this process in order to construct the optimal hedging strategy through a forward equation.

Proposition 2 *For every period t such that $0 \leq t \leq (T-1)$, for every initial value w_t in $L_t^2(P, \sqrt{h_t})$, and for every payoff F_T in $L^2(P)$,*

$$\operatorname*{essinf}_{\substack{X \in \mathcal{X}_t \\ w_t(X)=w_t}} E_t\left[(F_T - w_T(X))^2\right] = E_t\left[\left(F_T - w_T(X^{t,w_t,F_T})\right)^2\right] = h_t(F_t - w_t)^2 + g_t. \tag{6}$$

F_t and g_t are defined by backward induction by $g_T \overset{def.}{=} 0$ and for $t \leq s \leq (T-1)$,

$$F_s \overset{def.}{=} p_s' N_s^+ E_s\left[h_{s+1} F_{s+1} \phi_{s+1}\right],$$
$$g_s \overset{def.}{=} E_s\left[g_{s+1}\right] + E_s\left[h_{s+1}(F_{s+1})^2\right] - E_s\left[h_{s+1} F_{s+1} \phi_{s+1}'\right] N_s^+ E_s\left[h_{s+1} F_{s+1} \phi_{s+1}\right].$$

For every period s between time t and $(T-1)$, the random variable F_s belongs to $L_s^2(P, \sqrt{h_s})$ and g_s is a nonnegative random variable in $L^1(P)$ which is measurable with respect to $\mathcal{F}_s$. The dynamic portfolio X^{t,w_t,F_T} is defined recursively by

$$X_t^{t,w_t,F_T} \overset{def.}{=} h_t(w_t - F_t) N_t^+ p_t + N_t^+ E_t\left[h_{t+1} F_{t+1} \phi_{t+1}\right], \tag{7}$$

$$X_s^{t,w_t,F_T} \overset{def.}{=} h_s\left(\phi_s' X_{s-1}^{t,w_t,F_T} - F_s\right) N_s^+ p_s + N_s^+ E_s\left[h_{s+1} F_{s+1} \phi_{s+1}\right], \tag{8}$$

for $(t+1) \leq s \leq (T-1)$. The dynamic portfolio X^{t,w_t,F_T} belongs to $\mathcal{X}_t$, starts at time t with initial value w_t, and satisfies

$$E_s\left[\left(F_T - w_T(X^{t,w_t,F_T})\right)^2\right] = h_s\left(F_s - w_s(X^{t,w_t,F_T})\right)^2 + g_s \tag{9}$$

for every period s between t and T.

It is easily checked that if F_T, F_T^a, and F_T^b are in $L^2(P)$, if w_t, w_t^a, and w_t^b are in $L_t^2(P, \sqrt{h_t})$, and if γ_t is measurable with respect to $\mathcal{F}_t$ with $\gamma_t F_T$ in $L^2(P)$ and $\gamma_t w_t$ in $L_t^2(P, \sqrt{h_t})$, then

$$
\begin{aligned}
X^{t,w_t^a,F_T^a} + X^{t,w_t^b,F_T^b} &= X^{t,w_t^a+w_t^b,F_T^a+F_t^b}, \\
\gamma_t X^{t,w_t,F_T} &= X^{t,\gamma_t w_t,\gamma_t F_T}.
\end{aligned} \tag{10}
$$

It is clear from Proposition 2 that the optimization program

$$
\operatorname*{essinf}_{X \in \mathcal{X}_t} E_t\left[(F_T - w_T(X))^2\right]
$$

is solved in X^{t,F_t,F_T} with g_t as optimal value. The variable F_t is therefore the initial cost of the best replication strategy of the payoff F_T, while g_t describes the quality of this optimal hedge.

We remark that the construction of both F_s and g_s from F_T in Proposition 2 is respectively linear and quadratic and does not depend on the starting time t as long as $t \leq s$. This allows us to construct a linear operator Q_t and a quadratic operator G_t for every period t between 0 and T from the space of random variables in $L^2(P)$ to the space of random variables measurable with respect to $\mathcal{F}_t$ such that $Q_t(F_T) \stackrel{\text{def.}}{=} F_t$ and $G_t(F_T) \stackrel{\text{def.}}{=} g_t$ as defined recursively in Proposition 2. This proposition shows that $Q_t(F_T)$ belongs to $L_t^2(P, \sqrt{h_t})$ while $G_T(F_T)$ is an element of $L^1(P)$. At time T, the operators Q_T and G_T are trivially respectively the identity and the null operator. We derive from Equation 6 that

$$
G_t(F_T) = E_t\left[\left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)^2\right]. \tag{11}
$$

We still denote G_t the bilinear operator defined by polarization for two payoffs F_T^a and F_T^b in $L^2(P)$ as

$$
\begin{aligned}
G_t(F_T^a, F_T^b) &= \frac{1}{2}\left(G_t(F_T^a + F_T^b) - G_t(F_T^a) - G_t(F_T^b)\right) \\
&= E_t\left[\left(F_T^a - w_T(X^{t,Q_t(F_T^a),F_T^a})\right)\left(F_T^b - w_T(X^{t,Q_t(F_T^b),F_T^b})\right)\right].
\end{aligned} \tag{12}
$$

The following lemma lists some properties of these operators which will be used throughout our analysis.

Lemma 2 *Let s and t be two periods such that $t \leq s \leq T$, let F_T be a payoff in $L^2(P)$, and let w_t be an initial value in $L_t^2(P, \sqrt{h_t})$.*

(i). For every dynamic portfolio X in $\mathcal{X}_t$, $Q_s(w_T(X)) = w_s(X)$.

(ii). $G_t(F_T) = 0$ *if and only if* F_T *belongs to* $w_T(\mathcal{X}_t)$*. For every dynamic portfolio* X *in* $\mathcal{X}_t$*,* $G_t(w_T(X), F_T) = 0$.

(iii). $h_s w_s(X^{t,w_t,0}) Q_s(F_T) = E_s\left[w_T(X^{t,w_t,0}) F_T\right]$.

(iv). For every dynamic portfolio X *in* $\mathcal{X}_s$*,*

$$h_s\left(Q_s(F_T) - w_s(X^{t,w_t,F_T})\right) w_s(X) = E_s\left[\left(F_T - w_T(X^{t,w_t,F_T})\right) w_T(X)\right].$$

3.2. Uniqueness of the Optimal Hedge

The next result shows that Optimization Problem 6 of Proposition 2 has a unique solution, at least in terms of value at time T, and therefore also in terms of value process. We recall that we cannot expect to obtain a unique portfolio because we do not rule out redundancy between the securities.

Lemma 3 *We consider a period* t *between* 0 *and* $(T-1)$*, an initial value* w_t *in* $L^2_t(P, \sqrt{h_t})$*, and a payoff* F_T *in* $L^2(P)$*. For every dynamic portfolio* Y *in* $\mathcal{X}_t$*, the equality* $w_T(Y) = w_T(X^{t,w_t,F_T})$ *holds* P *almost surely on the set* $A_t(Y)$ *in* $\mathcal{F}_t$ *defined by*

$$A_t(Y) \overset{def.}{=} \left\{\omega \in \Omega \text{ such that: } E_t\left[(F_T - w_T(Y))^2\right] = h_t(Q_t(F_T) - w_t)^2 + G_t(F_T) \right.$$
$$\left. \text{and } w_t(Y) = w_t\right\}.$$

3.3. L^2 Minimum Portfolio

The L^2 minimum portfolio obtains for a special choice of final payoff F_T and initial value w_t. For $F_T = 0$ and $w_t = 1/\sqrt{h_t}$, we introduce the simplified notations $X^t \overset{\text{def.}}{=} X^{t,1/\sqrt{h_t},0}$ and $w^t_s \overset{\text{def.}}{=} w_s(X^t)$ for $t \le s \le T$. Notice that $1/\sqrt{h_t}$ is an element of $L^2_t(P, \sqrt{h_t})$. Equations 7 and 8 of Proposition 2 show that the self financing strategy X^t is obtained by investing every period s between t and $(T-1)$ the value w^t_s in the portfolio $h_s N^+_s p_s$ whose value at time s is $h_s p'_s N^+_s p_s = 1$ and $X^t_s = h_s w^t_s N^+_s p_s$. The self financing condition implies that $w^t_{s+1} = \phi'_{s+1} X^t_s$ so that

$$w^t_{s+1} = w^t_s h_s \phi'_{s+1} N^+_s p_s. \tag{13}$$

Proposition 2 proves that the final value w_T^t of this strategy has minimum conditional second moment among the dynamic portfolios in $\mathcal{X}_t$

$$\operatorname*{essinf}_{\substack{X \in \mathcal{X}_t \\ w_t(X)=1/\sqrt{h_t}}} E_t\left[w_T(X)^2\right] = E_t\left[(w_T^t)^2\right] = h_t (w_t^t)^2 = 1. \tag{14}$$

Statement (iii) of Lemma 2 with $s = t$ and $w_t = 1/\sqrt{h_t}$ implies that

$$Q_t(F_T) = \frac{1}{\sqrt{h_t}} E_t\left[w_T^t F_T\right], \tag{15}$$

which shows that Q_t is a positive operator whenever w_T^t is itself positive. If w_s^t does not vanish at time s between t and T, we also have

$$Q_s(F_T) = \frac{1}{h_s w_s^t} E_s\left[w_T^t F_T\right].$$

3.4. Hedging of a Sequence of Cash Flows

We generalize our analysis from a single payoff at horizon T to a sequence of contingent cash flows every period up to T. This will prove important later when we introduce additional securities with possibly complex distribution schedules and different maturities.

We consider a period t between 0 and $(T-1)$ and we let $f = \{f_s\}_{t+1 \le s \le T}$ be a sequence of cash flows from $(t+1)$ up to T adapted to $\mathcal{F}$. We say that a dynamic portfolio X starting at time t finances the cash flow f_s at time s with $s \le (T-1)$ when $w_s(X) = (\phi_s' X_{s-1} - f_s)$ and that it finances the sequence of cash flows f if it finances the cash flows f_s from $(t+1)$ to $(T-1)$. At the last period, we recall that we have defined the final value of a dynamic portfolio X by the equation $w_T(X) = \phi_T' X_{T-1}$.

We create a one to one operator θ_f on the set of dynamic portfolios starting at time t which transforms the self financing portfolios into strategies which finance the sequence of cash flows f as follows. For every dynamic portfolio X starting at time t, we let $Y = \theta_f(X)$ be the dynamic portfolio starting at time t defined by $Y_t = X_t$ and

$$Y_s = X_s - \left(\textstyle\sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u\right) h_s N_s^+ p_s$$

for $(t+1) \le s \le (T-1)$. The following lemma yields some first properties of this operator.

Lemma 4 *Let X and Y be two dynamic portfolios starting at time t such that $Y = \theta_f(X)$.*

(i). The portfolio X is self financing if and only if the portfolio Y finances the sequence of cash flows f.

(ii). $w_t(Y) = w_t(X)$ *and* $(f_T - w_T(Y)) = (F_T - w_T(X))$ *with*

$$F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s,$$

where we let $w_T^T = 1$.

We remark that the payoff F_T is obtained at time T by investing every cash flow of the sequence f in the L^2 minimum portfolio up to time T.

We let $\mathcal{X}_t(f)$ be the set of dynamic portfolios starting at time t which finance f and which end up at horizon T with a value in $L^2(P)$. The following proposition proves the equivalence between the variance–optimal hedge of F_T through self financing portfolios in $\mathcal{X}_t$ and the L^2 optimal replication of the sequence f by means of dynamic strategies in $\mathcal{X}_t(f)$. Some integrability condition on the sequence f are needed for this result.

Proposition 3 *Let $f = \{f_s\}_{t+1 \leq s \leq T}$ be a sequence of cash flows such that f_s belongs to $L_s^2(P, \sqrt{h_s})$ for every period s from $(t+1)$ to T. The payoff $F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s$ is in $L^2(P)$ and the mapping θ_f is one to one from $\mathcal{X}_t$ to $\mathcal{X}_t(f)$. For every initial value w_t in $L_t^2(P, \sqrt{h_t})$, we have*

$$\begin{aligned}\operatorname*{essinf}_{\substack{Y \in \mathcal{X}_t(f) \\ w_t(Y) = w_t}} E_t\left[(f_T - w_T(Y))^2\right] &= \operatorname*{essinf}_{\substack{X \in \mathcal{X}_t \\ w_t(X) = w_t}} E_t\left[(F_T - w_T(X))^2\right] \\ &= h_t\left(Q_t(F_T) - w_t\right)^2 + G_t(F_T)\end{aligned}$$

and the first program is solved in $Y = \theta_f\left(X^{t, w_t, F_T}\right)$.

The optimal hedging strategies for the two equivalent optimization programs of Proposition 3 start with an identical initial value at time t equal to $Q_t(F_T)$ and lead to the same replication error described by $G_t(F_T)$. The next lemma explains how both the optimal hedging cost $Q_t(F_T)$ and the optimal hedging quality $G_t(F_T)$ can be directly computed from the sequence f.

Lemma 5 *Let $f = \{f_s\}_{t+1 \leq s \leq T}$ be a sequence of cash flows such that f_s is in $L_s^2(P, \sqrt{h_s})$ for every period s from $(t+1)$ to T and let $F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s$. We define the processes*

$\bar{f} = \{\bar{f}_s\}_{t \le s \le T}$ *and* $\bar{g} = \{\bar{g}_s\}_{t \le s \le T}$ *from the sequence* f *by backward induction as follows. We let* $\bar{f}_T = \bar{g}_T \stackrel{def.}{=} 0$*, and*

$$\begin{aligned}
\bar{f}_s &\stackrel{def.}{=} p_s' N_s^+ E_s \left[h_{s+1} (\bar{f}_{s+1} + f_{s+1}) \phi_{s+1} \right], \\
\bar{g}_s &\stackrel{def.}{=} E_s \left[\bar{g}_{s+1} \right] + E_s \left[h_{s+1} (\bar{f}_{s+1} + f_{s+1})^2 \right] \\
&\quad - E_s \left[h_{s+1} (\bar{f}_{s+1} + f_{s+1}) \phi_{s+1}' \right] N_s^+ E_s \left[h_{s+1} (\bar{f}_{s+1} + f_{s+1}) \phi_{s+1} \right],
\end{aligned}$$

for $t \le s \le (T-1)$*. For every period* s *between* t *and* T *we have*

$$\begin{aligned}
Q_s(F_T) &= \sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u + \bar{f}_s, \\
G_s(F_T) &= \bar{g}_s,
\end{aligned}$$

with the convention that $\sum_{u=t+1}^{t} f_u \sqrt{h_u} w_s^u = 0$*. In particular at time* t *we have* $Q_t(F_T) = \bar{f}_t$ *and* $G_t(F_T) = \bar{g}_t$*.*

3.5. Interest Rates

We introduce from now on a money market. For the rest of the article we assume that Security 1 is a risk free zero coupon bond paying a unique dividend of one unit of numeraire at maturity T.

Assumption 4 *For every period* t *between* 0 *and* $(T-1)$ *the price* p_t^1 *of the zero coupon bond is positive.*

We let $R_t^f \stackrel{\text{def.}}{=} 1/p_t^1$ be the nominal risk free return from investing in the zero coupon bond from time t up to horizon T. This buy and hold strategy belongs to $\mathcal{X}_t$, we denote it $\mathbf{1}_t$. We remark that $w_s(\mathbf{1}_t) = Q_s(1) = p_s^1 = 1/R_s^f$ for $s \ge t$ and we learn from Statement (i) of Proposition 1 that $h_t (p_t^1)^2 \le E_t \left[h_{t+1} (p_{t+1}^1)^2 \right] \le 1$. We define $H_t \stackrel{\text{def.}}{=} h_t / (R_t^f)^2$ so that, with this normalization, this last inequality writes $H_t \le E_t \left[H_{t+1} \right] \le 1$ and the normalized process H is a positive submartingale, with $H_T = 1$.

We derive from Statement (iii) of Lemma 2 with $F_T = 1$ and $w_t = 1/\sqrt{h_t}$ that for every period s between t and T we have

$$E_s \left[w_T^t \right] = \frac{h_s w_s^t}{R_s^f}, \tag{16}$$

$$E_t \left[w_T^t \right] = \frac{\sqrt{h_t}}{R_t^f} = \sqrt{H_t}. \tag{17}$$

We remark that the sufficient condition of Lemma 1 which requires that $(p_T^k/p_t^k)d_t$ be in $L^2(P; I\!R^n)$ holds with $k = 1$ as soon as $R_t^f d_t$ is in $L^2(P; I\!R^n)$ for every period t between 0 and $(T-1)$. This is the case for instance if R_t^f is bounded and d_t belongs to $L^2(P; I\!R^n)$.

3.6. Variance–Optimal Martingale Measure

We have seen that the operator Q_t is positive as soon as the final value $w_T^t = w_T(X^t)$ of the L^2 minimum portfolio X^t is itself positive. We show that when this happens, the cost $Q_s(F_T)$ at time s between t and $(T-1)$ of the optimal hedge of a payoff F_T in $L^2(P)$ can be expressed as the discounted conditional expectation of F_T in a probability distribution different from the original probability P. This new probability distribution is called the minimum–variance probability distribution or the variance–optimal martingale probability.

We first notice from Equation 16 that if w_T^t is positive, then the value w_s^t of the strategy X^t at time s is also positive. Statement (iii) of Lemma 2, together with Equation 16, yields the following result

$$Q_s(F_T) = \frac{1}{R_s^f} \frac{E_s\left[w_T^t F_T\right]}{E_s\left[w_T^t\right]}. \tag{18}$$

If z is a positive random variable in $L^1(P)$, we denote P^z and E^z the probability distribution and its corresponding expectation operator obtained from the original probability P by means of the positive Radon-Nikodym derivative $z/E[z]$. For every random variable F such that zF is in $L^1(P)$ we have $E^z[F] = E[zF]/E[z]$ and $E_t^z[F] = E_t[zF]/E_t[z]$.

We use this construct here with $z = w_T^t$ and we obtain

$$Q_s(F_T) = \frac{1}{R_s^f} E_s^{w_T^t}\left[F_T\right],$$

which shows that $Q_s(F_T)$ can indeed be written as a discounted expectation in the modified probability distribution $P^{w_T^t}$.

One can usually not expect w_T^t to be positive when the cum-dividend prices assume unbounded values. This fact has been noted in Schweizer (1995). When this happens, the minimum–variance probability becomes the variance–optimal signed martingale measure and the operator Q_t, although still well defined, is not positive.

In a continuous time setting, Gouriéroux et al. (1998) shows that w_T^t is always positive as long as prices follow continuous semimartingales with no dividend distribution. They assume a no arbitrage condition which is more strict than the law of one price.

4. Mean–Variance Portfolio Selection

We summarize the mean–variance properties of self financing dynamic portfolios. We consider in this section a time period t between 0 and $(T-1)$ and we study the notions of dynamic Sharpe ratio and efficient frontier conditioned on the information at date t.

For every dynamic portfolio X in $\mathcal{X}_t$, we denote $\mathrm{SR}_t(X)$ the Sharpe ratio conditioned on the information available at time t which results from following the self financing investment strategy X from time t up to horizon T. We let

$$\mathrm{SR}_t(X) \stackrel{\text{def.}}{=} \frac{E_t\left[w_T(X)\right] - R_t^f w_t(X)}{\sqrt{\mathrm{Var}_t\left[w_T(X)\right]}}$$

when $\mathrm{Var}_t\left[w_T(X)\right]$ is non zero and we set $\mathrm{SR}_t(X) \stackrel{\text{def.}}{=} 0$ whenever $\mathrm{Var}_t\left[w_T(X)\right] = 0$.

We denote $R_t(X) \stackrel{\text{def.}}{=} w_T(X)/w_t(X)$ the gross return from period t to horizon T of a dynamic portfolio X in $\mathcal{X}_t$ with non vanishing value $w_t(X)$ at date t. In particular we have $R_t^f = R_t(\mathbf{1}_t)$ when $X = \mathbf{1}_t$ is the strategy which invests without rebalancing in the default free zero coupon bond with maturity T from time t on. If $w_t(X)$ and $\mathrm{Var}_t\left[w_T(X)\right]$ are P almost surely different from zero, we also have

$$\mathrm{SR}_t(X) = \frac{E_t\left[R_t(X)\right] - R_t^f}{\sqrt{\mathrm{Var}_t\left[R_t(X)\right]}},$$

the usual definition of a Sharpe ratio.

Our definition of returns is not innocuous. The choice of non annualized gross returns allows us to bring together in an common framework the theories of dynamic hedging and of dynamic mean–variance analysis. This nice convergence may not hold for other specifications of the returns.

We let the dynamic mean–variance efficient frontier at time t with horizon T, which we denote $\mathcal{E}\mathrm{F}_t$, be the set of portfolios in $\mathcal{X}_t$ which are solution to the optimization program

$$\operatorname*{essinf}_{\substack{X \in \mathcal{X}_t \\ w_t(X) = w_t \\ E_t[R_t(X)] = \mathcal{R}_t}} \quad \mathrm{Var}_t\left[R_t(X)\right]$$

for some expected return target $\mathcal{R}_t$ measurable with respect to $\mathcal{F}_t$ and some positive initial value w_t in $L_t^2(P, \sqrt{h_t})$.

Henrotte (2001) shows that the optimal dynamic Sharpe ratio from time t to horizon T, conditioned on the information available at time t, writes $\mathrm{SR}_t \overset{\text{def.}}{=} \sqrt{1/H_t - 1}$ and

$$\operatorname*{esssup}_{X \in \mathcal{X}_t} \mathrm{SR}_t(X)^2 = \mathrm{SR}_t(X^t)^2 = (\mathrm{SR}_t)^2 \,.$$

The optimal dynamic Sharpe ratio obtains for the portfolios on the efficient frontier $\mathcal{E}\mathrm{F}_t$. Under some regularity condition, every efficient portfolio on $\mathcal{E}\mathrm{F}_t$ can be identified with a combination of the portfolio X^t and the zero-coupon bond with maturity T, where the proportions[2] invested in the two strategies are fixed at time t.

5. Pricing Kernels

We let PK_t be the set of pricing kernels corresponding to the dynamics of the underlying securities from period t up to horizon T. It is defined as the set of random variables m_T in $L^2(P)$ such that

$$R^f_s m_s w_s(X) = E_s \left[m_T w_T(X) \right] , \tag{19}$$

for every period s between t and T and for every dynamic portfolio X in $\mathcal{X}_s$, where we let $m_s \overset{\text{def.}}{=} E_s[m_T]$. This definition highlights the duality between the pricing kernels and the self financing portfolios. We provide an equivalent and more standard definition in terms of security prices.

Lemma 6 *A random variable m_T in $L^2(P)$ is a pricing kernel in PK_t if and only if the following equivalent conditions are satisfied.*

(i). For every period s between t and $(T-1)$, $R^f_s m_s p_s = E_s \left[R^f_{s+1} m_{s+1} \phi_{s+1} \right]$.

(ii). For every period s between t and $(T-1)$, $R^f_s m_s p_s = E_s \left[\sum_{u=s+1}^{T-1} R^f_u m_u d_u + m_T \phi_T \right]$.

We remark that we do not require any positivity condition on the pricing kernels and PK_t is therefore a vector subspace of $L^2(P)$.

[2]The L^2 minimum portfolio X^t lies in the non optimal part of the efficient frontier and $\mathrm{SR}(X^t) = -\,\mathrm{SR}_t$. An optimal dynamic mean–variance strategy should therefore short this portfolio.

5.1. Structure of Pricing Kernels

The next proposition describes the structure of PK_t. We let PK_t^0 be set of kernels in PK_t with zero conditional expectation at time t,

$$\mathrm{PK}_t^0 \stackrel{\text{def.}}{=} \{m_T \in \mathrm{PK}_t \quad \text{such that} \quad m_t = 0\} .$$

We say that two random variables y and z in $L^2(P)$ are conditionally orthogonal at time t if and only if $E_t[yz] = 0$. If Z is a subset of $L^2(P)$, we let $Z^{\perp_t}$ be the set of random variables in $L^2(P)$ conditionally orthogonal at time t to every random variable in Z.

Proposition 4 *Let t and s be two periods such that $0 \leq t \leq s \leq (T-1)$.*

(i). w_T^t *is a pricing kernel in* PK_t *which is therefore not reduced to zero.*

(ii). $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$.

(iii). Every pricing kernel m_T *in* PK_t *satisfies* $m_s/\sqrt{H_s} = \sqrt{h_s} Q_s(m_T)$ *and* m_s *belongs to* $L_s^2(P, 1/\sqrt{H_s})$.

(iv). PK_t *is the set of random variables* m_T *in* $L^2(P)$ *which can be written* $m_T = \xi_t w_T^t + m_T^0$ *for some random variables* ξ_t *in* $L_t^2(P)$ *and* m_T^0 *in* PK_t^0 *so that*

$$\mathrm{PK}_t = \left(L_t^2(P) \times w_T^t\right) + w_T(\mathcal{X}_t)^{\perp_t}$$

where $L_t^2(P) \times w_T^t \stackrel{\text{def.}}{=} \left\{\xi_t w_T^t \quad \text{with} \quad \xi_t \in L_t^2(P)\right\}$.

(v). $\mathrm{PK}_t \cap w_T(\mathcal{X}_t) = L_t^2(P) \times w_T^t$.

(vi). $\mathrm{PK}_t = \left\{\left(F_T - w_T(X^{t,w_t,F_T})\right) \text{ with } F_T \in L^2(P) \text{ and } w_t \in L_t^2(P, \sqrt{h_t})\right\}$ *and*
$\mathrm{PK}_t^0 = \left\{\left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right) \text{ with } F_T \in L^2(P)\right\}$.

(vii). The two sets PK_t^0 *and* PK_t *are closed in* $L^2(P)$ *and* $w_T(\mathcal{X}_t) = \left(\mathrm{PK}_t^0\right)^{\perp_t}$.

Statement (i) proves that w_T^t is a pricing kernel and Statement (v) shows that this is the only kernel which is also the final value of a self financing strategy in $\mathcal{X}_t$. Every other kernel is obtained by adding to it a component which is conditionally orthogonal at time t to the final value of every dynamic portfolio. When the market is complete, $w_T(\mathcal{X}_t) = L^2(P)$ and PK_t reduces to $L_t^2(P) \times w_T^t$, so that, up to a normalization constant ξ_t at time t, the pricing kernel w_T^t is unique. This generalizes to a multiperiod setting the standard results on the structure of the discount factors as exposed for instance in Cochrane (2001).

Statement (vi) highlights the connection between the pricing kernels and the variance–optimal hedging technique described in Section 3. Every pricing kernel can be described as the variance–optimal hedge residual of some contingent claim. Kernels in PK_t^0 obtain for optimal initial values at time t. The next lemma draws on this connection and shows that the operators Q_t and G_t can be used in order to describe how a pricing kernel evaluates a payoff.

Lemma 7 *Let t and s be two periods such that $0 \leq t \leq s \leq T$ and let m_T be a pricing kernel in PK_t. For every payoff F_T in $L^2(P)$ we have*

$$R_s^f m_s Q_s(F_T) + G_s(m_T, F_T) = E_s[m_T F_T]. \tag{20}$$

We remark that Equation 20 is consistent with Equation 19 in the case where there exists a portfolio X in $\mathcal{X}_t$ such that $F_T = w_T(X)$. We derive indeed from Statements (i) and (ii) of Lemma 2 that $Q_s(w_T(X)) = w_s(X)$ and $G_s(m_T, w_T(X)) = 0$.

5.2. Variance Bounds on Pricing Kernels

We have defined in Section 4 the strategy X^t as the L^2 minimum portfolio. It is the dynamic portfolio whose final value w_T^t has minimum conditional second moment at time t within $\mathcal{X}_t$. We show here that w_T^t is also L^2 optimal within PK_t, and we derive from this optimality a set of intertemporal bounds on the variance of the pricing kernels.

Proposition 5 *We consider two periods t and s such that $0 \leq t \leq s \leq T$. Every pricing kernel m_T in PK_t satisfies*

$$E_s\left[m_T^2\right] = \frac{m_s^2}{H_s} + G_s(m_T), \tag{21}$$

$$\mathrm{Var}_s[m_T] = m_s^2(\mathrm{SR}_s)^2 + G_s(m_T), \tag{22}$$

and therefore also

$$\frac{m_s^2}{H_s} \leq E_s\left[m_T^2\right], \tag{23}$$

$$m_s^2(\mathrm{SR}_s)^2 \leq \mathrm{Var}_s[m_T]. \tag{24}$$

Inequalities 23 and 24 become equalities if $m_T = \xi_t w_T^t$ with ξ_t in $L_t^2(P)$. For $s = t$, this last condition is both sufficient and necessary and the pricing kernel $\xi_t w_T^t$ solves

$$\underset{\substack{m_T \in \mathrm{PK}_t \\ m_t = \sqrt{H_t}\xi_t}}{\operatorname{essinf}} E_t\left[m_T^2\right] = \xi_t^2. \tag{25}$$

This proposition proves that w_T^t is the kernel with minimum–variance in PK_t. For every random variable $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t})$, the pricing kernel $(\bar{m}_t/\sqrt{H_t})w_T^t$ solves indeed

$$\underset{\substack{m_T \in \mathrm{PK}_t \\ m_t = \bar{m}_t}}{\operatorname{essinf}} \mathrm{Var}_t[m_T] = \bar{m}_t^2 \, \mathrm{SR}_t^2 \, .$$

For every intermediate period s between t and T, the square of the optimal dynamic Sharpe ratio from s to T yields a lower bound to the conditional variance at time s of the pricing kernels in PK_t. The gap between the variance of a kernel m_T and its lower bound is $G_s(m_T)$, which measures the quality of the replication of the kernel, that is the distance between the kernel and the set of attainable payoffs $w_T(\mathcal{X}_t)$. For $s = t$, we derive from the decomposition of a kernel proposed in Statement (iv) of Proposition 4 that the excess variance of a pricing kernel is due to the component in PK_t^0 which is conditionally orthogonal to $w_T(\mathcal{X}_t)$. When the kernel belongs to $w_T(\mathcal{X}_t)$, the replication is perfect and the inequality becomes an equality. Statement (v) of Proposition 4 shows that w_T^t is the only kernel enjoying this property.

6. Extension of the Investment Scope

The analysis of the self financing portfolios and their pricing kernels which we developed so far will help us now tackle a central issue in incomplete markets. We study the implications of selecting a price process for some derivative instruments in a way which is consistent with the dynamic behavior of their underlying securities. We focus on two related investment problems, the dynamic management of a portfolio on the one hand, and the optimal hedging of a contingent claim on the other hand. This section deals with basic issues, we postpone until the next one the analysis of the constraint imposed by a smile, which we define as the observation of current market quotes for a set of derivatives.

In addition to the original n securities, we consider n_x new securities which distribute some numeraire dividends every period described by the vector process $\{d_t^x\}_{1 \le t \le (T-1)}$. For every period t between 0 and $(T-2)$, the owner of one unit of security j at time t receives the next period the dividend $d_{t+1}^{x,j}$. A time $(T-1)$, one unit of security j gives right to the final payoff $\phi_T^{x,j}$ at time T. One can think of $\phi_T^{x,j}$ as the sum of a dividend and a residual value which describes the market value of security j at time T. This is meant to handle cases where an instrument has a maturity which is longer than the investment horizon which we consider. If the maturity is shorter than T, the dividends vanish once the instrument matures and the final payoff is zero.

We assume that the dividend process $\{d_t^x\}_{1\leq t\leq(T-1)}$ and the final payoff ϕ_T^x are given and known. We further assume that the dividend process is adapted to $\mathcal{F}$, that the final payoff ϕ_T^x is a random vector in $L^2(P;\mathbb{R}^{n_x})$, and that for every period t between 1 and $(T-1)$ and for every index j the dividend $d_t^{x,j}$ belongs to $L_t^2(P,\sqrt{h_t})$. These new instruments may be derivatives written on the original securities, in which case the dividends and the final payoff are functions of the prices of the original securities. We do not however limit ourselves to this situation and we allow for a very general definition of the new instruments.

We consider a period t between 0 and $(T-1)$ and we let the vector processes $\{p_s^x\}_{t\leq s\leq(T-1)}$ and $\{\phi_s^x\}_{t\leq s\leq(T-1)}$ in $\mathbb{R}^{n_x}$ be respectively the ex and cum dividend price dynamics of the new securities between t and $(T-1)$. We say that this price dynamics starting at time t is admissible if it is adapted to $\mathcal{F}$, if $\phi_s^x=(p_s^x+d_s^x)$ every period, and if it satisfies the law of one price together with the prices of the original n securities. In line with Assumption 2, this last requirement means that for every period s between t and $(T-1)$ and for every vector (u,v) in $\mathbb{R}^n\times\mathbb{R}^{n_x}$ measurable with respect to $\mathcal{F}_s$, the equality $\phi_{s+1}'u+(\phi_{s+1}^x)'v=0$ implies $p_s'u+(p_s^x)'v=0$. It is a weak notion of absence of arbitrage, the minimum structure which we need in order to apply our dynamic mean–variance analysis.

We limit our investigations to admissible price dynamics in a partial equilibrium framework where the price process of the original securities is assumed to be known and fixed. We do not study for instance how the introduction of the new securities may modify the prices of the original ones. The denomination "original" and "new" security is therefore somewhat misleading, it is only a convenient way to describe an extension of the investment scope.

6.1. Admissible Price Dynamics

We first study the existence and the construction of an admissible price dynamics for the new securities. The following lemma checks that an admissible price dynamics may be derived from a positive pricing kernel for the original securities. It is well known that a positive kernel prevents the existence of arbitrage opportunities, it precludes therefore also any violation to the law of one price.

Lemma 8 *Let m_T be a positive pricing kernel in* PK_t *and let $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ be defined by the equations*

$$p_s^x = \frac{E_s\left[\sum_{u=s+1}^{T-1} R_u^f m_u d_u^x + m_T \phi_T^x\right]}{R_s^f m_s}, \tag{26}$$

and $\phi_s^x = p_s^x + d_s^x$. These two processes form an admissible price dynamics for the new securities.

Even when no positive kernel is available, and in particular even if we do not know if w_T^t is positive, it is possible to exhibit an admissible price dynamics for the new securities. We define the processes $\{\bar{p}_s^x, \bar{\phi}_s^x\}_{t \le s \le (T-1)}$ in $I\!R^{n_x}$ by the backward equations

$$\begin{aligned} \bar{p}_s^{x,j} &= p_s' N_s^+ E_s\left[h_{s+1} \bar{\phi}_{s+1}^{x,j} \phi_{s+1}\right], \\ \bar{\phi}_s^{x,j} &= \bar{p}_s^{x,j} + d_s^{x,j}, \end{aligned}$$

for every index j from one to n_x.

Lemma 9 *The processes $\{\bar{p}_s^x, \bar{\phi}_s^x\}_{t \le s \le (T-1)}$ form an admissible price dynamics for the new securities. If w_T^t is positive, they coincide with the processes derive in Lemma 8 from the kernel $m_T = w_T^t$.*

Drawing on the analysis of Section 3.4, we remark that if we let $f = \{f_s\}_{t+1 \le s \le T}$ be the sequence of cash flows corresponding to the dividends and the final payoff of new security j with $f_s = d_s^{x,j}$ for s between $(t+1)$ and $(T-1)$ and $f_T = \phi_T^{x,j}$, then the process $\{\bar{p}_s^{x,j}\}_{t \le s \le (T-1)}$ coincides with the process $\{\bar{f}_s\}_{t \le s \le (T-1)}$ defined in Lemma 5. We derive from Proposition 3 and Lemma 5 that if we let

$$F_T^x \overset{\text{def.}}{=} \sum_{s=t+1}^{T-1} \sqrt{h_s} w_T^s d_s^x + \phi_T^x,$$

then $\bar{p}_t^{x,j} = Q_t(F_T^{x,j})$ and $\bar{p}_t^{x,j}$ represents the cost of the variance–optimal hedge of the sequence of cash flows generated by new security j from time $(t+1)$ up to horizon T. Likewise, the price $\bar{p}_s^{x,j}$ corresponds to the cost of the variance–optimal hedge of the remaining cash flows from time $(s+1)$ up to T.

We remark that for every pricing kernel m_T in PK_t we have

$$E_t\left[m_T F_T^x\right] = E_t\left[\sum_{s=t+1}^{T-1} R_s^f m_s d_s^x + m_T \phi_T^x\right] \tag{27}$$

which shows again that F_T^x can be interpreted as a single final payoff equivalent to the sequence of cash flows generated by the new securities from $(t+1)$ up to T.

6.2. *Extended Asset Structure*

We consider now an admissible price dynamics $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ and we study the extended asset structure between time t and horizon T which consists in the n original securities together with the n_x new ones with their proposed price dynamics. We denote p_s^e and ϕ_s^e the ex and cum dividend prices at time s for the extended asset structure. The first n components of the vectors p_s^e and ϕ_s^e are respectively p_s and ϕ_s while their last n_x components are respectively p_s^x and ϕ_s^x.

The extended asset structure satisfies both Assumptions 1 and 2 as well as Conditions (a) and (b) of Proposition 1. The zero coupon bond is one of the original security and it remains traded. The results of Sections 2 to 5 can therefore be brought to bear, with period t corresponding to the initial trading period 0 in these sections. We remark that the fact that we do not impose any absence of redundancy between the securities allows us to consider with very general instruments.

We set $h_T^e = h_T = 1$ and for s between t and $(T-1)$, we let h_s^e, H_s^e, $\mathcal{X}_s^e$, Q_s^e, G_s^e, PK_s^e, $X^{s,e}$, $w_T^{s,e}$, SR_s^e be the counterparts to respectively h_s, H_s, $\mathcal{X}_s$, Q_s, G_s, PK_s, X^s, w_T^s, SR_s for the extended asset structure. Notice that we have $H_s^e \overset{\text{def.}}{=} h_s^e/(R_s^f)^2$ and $\mathrm{SR}_s^e \overset{\text{def.}}{=} \sqrt{1/H_s^e - 1}$.

It is clear that PK_s^e is a subset of PK_s. The next lemma shows that a necessary and sufficient condition for a pricing kernel in PK_t to belong to PK_t^e is to "price" correctly the new securities. It is a direct application of Lemma 6 to the extended structure.

Lemma 10 *A pricing kernel m_T in PK_t belongs to PK_t^e if and only the following equivalent conditions are satisfied.*

(i). For every period s between t and $(T-1)$, $R_s^f m_s p_s^x = E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}^x\right]$.

(ii). For every period s between t and $(T-1)$, $R^f_s m_s p^x_s = E_s\left[\sum_{u=s+1}^{T-1} R^f_u m_u d^x_u + m_T \phi^x_T\right]$.

For a pricing kernel m_T in PK^e_t, we derive from Equation 27 that

$$R^f_t m_t p^x_t = E_t\left[m_T F^x_T\right] \tag{28}$$

and the kernel evaluates identically at time t the sequence of cash flows generated by the new securities and the unique final payoff F^x_T.

6.3. Sharpe Ratio Improvement

The optimal dynamic Sharpe ratio may only increase as a result of the extension of the investment scope, which means that for every period s between t and $(T-1)$ we have $\mathrm{SR}_s \leq \mathrm{SR}^e_s$ and $H^e_s \leq H_s$. The following result quantifies this increase in terms of pricing kernels. We recall that $w^{t,e}_T$ is the value at time T of the L^2 minimum portfolio $X^{t,e}$ in the set of self financing strategies $\mathcal{X}^e_t$ for the extended asset structure.

Result 1 *For every pricing kernel m_T in PK^e_t and for every period s between t and $(T-1)$,*

$$m_s^2\left[(\mathrm{SR}^e_s)^2 - (\mathrm{SR}_s)^2\right] = G_s(m_T) - G^e_s(m_T),$$

in particular at time t, $(\mathrm{SR}^e_t)^2 - (\mathrm{SR}_t)^2 = G_t(w^{t,e}_T)/H^e_t$.

Result 1 tells us that the optimal dynamic Sharpe ratio increases inasmuch as the replication of the pricing kernels for the extended asset structure is enhanced by the use of the additional securities. If, as exposed in Lemma 8, a positive pricing kernel m_T is used in order to generate the price dynamics of an increasing number of new instruments, then the Sharpe ratio increases as long as $G^e_s(m_T)$ decreases and the new instruments help replicate the kernel. This suggests that one should consider in priority new securities which best contribute to the replication quality of the kernel.

Once enough instruments have been introduced so that m_T is perfectly replicated, the optimal dynamic Sharpe ratio ceases to increase as new instruments are added. The optimal dynamic Sharpe ratio from s to T reaches then the maximum possible value consistent with the kernel m_T. This maximum is given by the variance of the kernel,

$$(\mathrm{SR}^e_s)^2 = \mathrm{Var}_s[m_T/m_s].$$

With no clear indication on how to choose a pricing kernel, a fund manager runs the risk of picking a kernel with a large variance which induces large potential increases in performance for some carefully selected new instruments. The perceived increase in performance may only be the result of a dubious choice of price dynamics for the additional securities. With this pitfall in mind, we investigate the admissible price dynamics which yields the lowest possible increase in Sharpe ratio for the corresponding optimal dynamic strategy. This situation corresponds to a min–max in terms of dynamic Sharpe ratio. Without any smile constraint, we show that it is possible to avoid any mean–variance abnormal good–deal.

6.4. Absence of Good Deal

We consider an admissible price dynamics $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ and the extended asset structure which it generates from time t up to horizon T. We characterize the situation where no gain in dynamic Sharpe ratio may be expected from trading in the new securities. Absence of dynamic good–deal at time s corresponds to the equality $\mathrm{SR}_s^e = \mathrm{SR}_s$. We first propose two equivalent characterizations of this situation in terms of the minimum–variance kernel.

Lemma 11 *For every period s between t and $(T-1)$, the equality $\mathrm{SR}_s^e = \mathrm{SR}_s$ holds if and only if the following equivalent conditions hold.*

(i). $w_T^{s,e} = w_T^s$.

(ii). w_T^s *belongs to* PK_s^e.

These equivalent conditions imply that $p_s^x = \bar{p}_s^x$.

We next show that absence of dynamic good–deal at every period obtains when the price dynamics of the new instruments corresponds to the cost of their variance–optimal hedge.

Proposition 6 *The following two statements are equivalent.*

(i). $\mathrm{SR}_s^e = \mathrm{SR}_s$ *for every period s between t and $(T-1)$.*

(ii). $p_s^x = \bar{p}_s^x$ *for every period s between t and $(T-1)$.*

If no dynamic good–deal is available at time t, it seems intuitive that no good–deal should exist either at a later trading date s. We only prove this fact for the periods s such that the value w_s^t of the L^2 minimum strategy does not vanish.

Proposition 7 *If* $\mathrm{SR}_t^e = \mathrm{SR}_t$*, then at every period* s *between* t *and* $(T-1)$ *such that* w_s^t *does not vanish we have* $\mathrm{SR}_s^e = \mathrm{SR}_s$*.*

When w_T^t is positive, we know that w_s^t never vanishes and we may further characterize the absence of good–deal at time t in terms of the entire price process of the new instruments.

Result 2 *If* w_T^t *is positive, then* $\mathrm{SR}_t^e = \mathrm{SR}_t$ *if and only if* $p_s^x = \bar{p}_s^x$ *for every period* s *between* t *and* $(T-1)$*. When this happens, we also have* $\mathrm{SR}_s^e = \mathrm{SR}_s$ *for* $t \leq s < T$*.*

The use of the price dynamics $\{\bar{p}_s^x, \bar{\phi}_s^x\}_{t \leq s \leq (T-1)}$ can therefore be justified on two grounds. On the one hand it corresponds to the cost of the variance–optimal hedge of the cash flows generated by the new securities, and on the other hand it prevents any abnormal good–deal at every trading period. The next proposition describe a further interesting property of this price dynamics. The cost of the variance–optimal hedge of any payoff does not change if the new securities are used as additional hedging instruments.

Proposition 8 *If* $p_s^x = \bar{p}_s^x$ *for every period* s *between* t *and* $(T-1)$ *then the operators* Q_s^e *and* Q_s *are identical for every period* s *between* t *and* $(T-1)$*.*

7. Smile Consistent Kernels and Dynamics

We consider again a period t between 0 and $(T-1)$ and the n_x new securities described by their dividends and final payoffs. A smile at time t is a random vector S_t^x in $\mathbb{R}^{n_x}$, measurable with respect to $\mathcal{F}_t$, which describes the prices of the n_x new securities at period t. We start by studying the pricing kernels which are consistent with the smile and we provide a lower bound on the variance of these kernels. We then study the admissible price dynamics for the new securities between period t and horizon T which agree with the smile S_t^x at time t, and we derive a lower bound on the optimal dynamic Sharpe ratio for the corresponding extended market structure.

7.1. Smile Consistent Pricing Kernels

A pricing kernel consistent with the smile S_t^x at time t is a pricing kernel m_T in PK_t which satisfies

$$R_t^f m_t S_t^x = E_t \left[\sum_{s=t+1}^{T-1} R_s^f m_s d_s^x + m_T \phi_T^x \right].$$

According to Equation 27, this is equivalent to the requirement that $R_t^f m_t S_t^x = E_t\left[m_T F_T^x\right]$. We let $\mathrm{PK}_t(S_t^x)$ be the set of pricing kernels consistent with the smile S_t^x at time t. We give conditions for the set $\mathrm{PK}_t(S_t^x)$ to be non empty and we study the properties of the pricing kernels in $\mathrm{PK}_t(S_t^x)$.

We extend the definition of the operators Q_t and G_t from random variables to random vectors. If F_T^a and F_T^b are two random vectors respectively in $L^2(P; I\!R^{n_a})$ and $L^2(P; I\!R^{n_b})$, we let $Q_t(F_T^a)$ be the random vector in $I\!R^{n_a}$ such that $(Q_t(F_T^a))_i \stackrel{\text{def.}}{=} Q_t((F_T^a)_i)$ and we let $G_t(F_T^a, F_T^b)$ be the random matrix of size $(n_a \times n_b)$ such that

$$\left[G_t(F_T^a, F_T^b)\right]_{i,j} \stackrel{\text{def.}}{=} G_t\left((F_T^a)_i, (F_T^b)_j\right).$$

We also denote $G_t(F_T^a)$ the symmetric matrix $G_t(F_T^a, F_T^a)$. We shall need the following inequality.

Lemma 12 *Let F_T^a and F_T^b be respectively a random variable in $L^2(P)$ and a random vector in $L^2(P; I\!R^{n_b})$, then $G_t(F_T^a, F_T^b) G_t(F_T^b)^+ G_t(F_T^b, F_T^a) \leq G_t(F_T^a)$.*

In the same spirit as above, if w_t^a is a random vector in $L_t^2(P, \sqrt{h_t}; I\!R^{n_a})$ and if F_T^a is a vector payoff in $L^2(P; I\!R^{n_a})$, we let X^{t, w_t^a, F_T^a} be the random matrix of size $n \times n_a$ whose ith column is the random vector process $X^{t,(w_t^a)_i,(F_T^a)_i}$ in $I\!R^n$ which describes the variance–optimal hedging strategy of the payoff $(F_T^a)_i$ starting at time t with wealth $(w_t^a)_i$. It is then natural to let $w_s(X^{t,w_t^a,F_T^a})$ represent the value process of these n_a dynamic portfolios, a random vector in $I\!R^{n_a}$ such that $w_s(X^{t,w_t^a,F_T^a})_i \stackrel{\text{def.}}{=} w_s(X^{t,(w_t^a)_i,(F_T^a)_i})$.

We have already introduced the vector payoff $F_T^x \stackrel{\text{def.}}{=} \sum_{s=t+1}^{T-1} \sqrt{h_s} w_T^s d_s^x + \phi_T^x$. We let $F_t^x \stackrel{\text{def.}}{=} Q_t(F_T^x)$ be the cost at time t of the variance–optimal hedge of the component of F_T^x. We have seen that $F_t^x = \bar{p}_t^x$ and F_t^x can also be described as the cost at time t of the optimal replication strategy of the cash flows generated by the new securities from time (t+1) up to horizon T. We let $M_T^{t,x} \stackrel{\text{def.}}{=} F_T^x - w_T(X^{t,F_t^x,F_T^x})$ represent the gap at maturity T between the vector payoff F_T^x and its variance–optimal hedge. Notice that $Q_t(M_T^{t,x}) = 0$ and that Equation 12 implies that $G_t(F_T^x) = E_t\left[M_T^{t,x}(M_T^{t,x})'\right]$.

The next lemma yields some first results on $\mathrm{PK}_t(S_t^x)$. We define the random vector Λ_t^x

and the random variables K_t^x and H_t^x as follows,

$$\Lambda_t^x \stackrel{\text{def.}}{=} \sqrt{h_t} G_t(F_T^x)^+ \left(S_t^x - F_t^x\right),$$
$$K_t^x \stackrel{\text{def.}}{=} h_t \left(S_t^x - F_t^x\right)' G_t(F_T^x)^+ \left(S_t^x - F_t^x\right) = \Lambda_t^{x\prime} G_t(F_T^x) \Lambda_t^x,$$
$$H_t^x \stackrel{\text{def.}}{=} \frac{H_t}{(1+K_t^x)}.$$

Lemma 13 *Let m_T be a pricing kernel in $\mathrm{PK}_t(S_t^x)$. We have*

$$R_t^f m_t \left(S_t^x - F_t^x\right) = G_t\left(m_T, F_T^x\right), \tag{29}$$

$$\frac{m_t^2}{H_t} K_t^x \leq G_t\left(m_T\right), \tag{30}$$

and m_t belongs to $L_t^2(P, 1/\sqrt{H_t^x})$.

For a random variable $\bar{m}_t$ in $\mathcal{F}_t$, we let

$$m_T^{t,x}(\bar{m}_t) \stackrel{\text{def.}}{=} \frac{\bar{m}_t}{\sqrt{H_t}} \left(w_T^t + (\Lambda_t^x)' M_T^{t,x}\right).$$

This kernel will play a central role in our analysis, we list here some basic properties.

Lemma 14 *For every random variable $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t^x})$ the random variable $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t which satisfies $E_t[m_T^{t,x}(\bar{m}_t)] = \bar{m}_t$ and*

$$E_t\left[\left(m_T^{t,x}(\bar{m}_t)\right)^2\right] = \frac{\bar{m}_t^2}{H_t^x}.$$

For $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t^x})$ and $t \leq s \leq T$ we let $m_s^{t,x}(\bar{m}_t) \stackrel{\text{def.}}{=} E_s\left[m_T^{t,x}(\bar{m}_t)\right]$. The next condition will be shown to be necessary and sufficient for the existence of smile consistent pricing kernels.

Condition 1 $\left(S_t^x - F_t^x\right) = G_t(F_T^x) G_t(F_T^x)^+ \left(S_t^x - F_t^x\right)$.

Proposition 9 *(i). If the smile S_t^x satisfies Condition 1, then for every random variable $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t^x})$, the pricing kernel $m_T^{t,x}(\bar{m}_t)$ is in $\mathrm{PK}_t(S_t^x)$. In particular $\mathrm{PK}_t(S_t^x)$ contains pricing kernels m_T such that m_t does not vanish.*

(ii). Reciprocally, if there exists a pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ such that m_t does not vanish, then the smile S_t^x satisfies Condition 1.

We next investigate the L^2 properties of the pricing kernels in $\mathrm{PK}_t(S_t^x)$. This will provide a lower bound to the variance of the kernels which are consistent with the smile.

7.2. Variance Bound with a Smile

The following proposition proves that the kernel $m_T^{t,x}(m_t)$ has minimum L^2 norm within $\mathrm{PK}_t(S_t^x)$.

Proposition 10 *Every pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ satisfies the following two inequalities,*

$$\frac{m_t^2}{H_t^x} \leq E_t\left[m_T^2\right], \tag{31}$$

$$m_t^2(\mathrm{SR}_t)^2 + m_t^2(R_t^f)^2\left(S_t^x - F_t^x\right)' G_t(F_T^x)^+\left(S_t^x - F_t^x\right) \leq \mathrm{Var}_t\left[m_T\right]. \tag{32}$$

These two inequalities become equalities if and only if $m_T = m_T^{t,x}(m_t)$.

When m_t does not vanish, Inequality 32 writes also

$$(\mathrm{SR}_t)^2 + (R_t^f)^2\left(S_t^x - F_t^x\right)' G_t(F_T^x)^+\left(S_t^x - F_t^x\right) \leq \mathrm{Var}_t\left[m_T/m_t\right].$$

This inequality describes how the smile constraint sharpens the variance bound on the marginal rate of substitution which we derived in Proposition 5. The increase in the bound is a function of the distance, in the metric described by the matrix $G_t(F_T^x)^+$, between the observed prices S_t^x of the instruments in the smile and the cost F_t^x of their variance–optimal hedge.

In the simple case where $n_x = 1$ and the smile data is limited to one instrument, the increase in the square of the Sharpe ratio writes $(S_t^x - F_t^x)^2/G_t(F_T^x)$. It is large when the hedging quality is high and the difference between F_t^x and S_t^x is large. Intuitively, this says that it is "costly" for a pricing kernel to produce prices which deviate much from the cost of the optimal hedge when the replication is good, as this would require a kernel with a large variance.

7.3. Optimal Dynamic Sharpe Ratio with a Smile

We recall that an admissible price dynamics starting at time t for the new securities is a couple of vector processes $\{p_s^x, \phi_s^x\}_{t\leq s\leq(T-1)}$ in $\mathbb{R}^{n_x}$ adapted to $\mathcal{F}$ which satisfies $\phi_s^x = p_s^x + d_s^x$ and such that, together with the price processes of the original securities, they satisfy the law of one price. We say that a price dynamics for the new securities is consistent with the smile S_t^x at time t if it is admissible and if it satisfies $p_t^x = S_t^x$. The next proposition studies the existence of smile consistent price dynamics. It shows in particular the necessity of Condition 1.

Proposition 11 *(i). Let m_T be a positive pricing kernel in $\mathrm{PK}_t(S_t^x)$. The price dynamics $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ defined by the backward equations*

$$p_s^x = E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}^x\right] / \left(R_s^f m_s\right)$$

and $\phi_s^x = p_s^x + d_s^x$ is consistent with the smile S_t^x at time t.

(ii). If there exists a price dynamics consistent with the smile S_t^x at time t, then the smile S_t^x satisfies Condition 1.

We derive from Propositions 9 and 11 that a sufficient condition for the existence of a price dynamics consistent with the smile is Condition 1, together with the requirement that the kernel $m_T^{t,x}(\sqrt{H_t^x})$ be positive. The price dynamics generated by this kernel has interesting properties which we investigate in the next section. We consider here the general case.

We assume that there exists a price dynamics consistent with the smile S_t^x at time t. We let $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ be such a consistent price dynamics and we consider the extended asset structure which it generates between time t and horizon T.

We learn from Proposition 11 that Condition 1 is satisfied and we know from Proposition 9 that the pricing kernel $m_T^{t,x}(\bar{m}_t)$ is an element of $\mathrm{PK}_t(S_t^x)$, for every variable $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t^x})$. The set $\mathrm{PK}_t(S_t^x)$ is therefore not trivial.

Statement (ii) of Lemma 10 proves that if m_T is a pricing kernel for the extended asset structure, then

$$R_t^f m_t p_t^x = E_t\left[\sum_{s=t+1}^{T-1} R_s^f m_s d_s^x + m_T \phi_T^x\right].$$

Since $p_t^x = S_t^x$, we obtain that m_T belongs to $\mathrm{PK}_t(S_t^x)$. This proves that the set PK_t^e of pricing kernels for the extended asset structure is a subset of $\mathrm{PK}_t(S_t^x)$.

The next proposition provides a lower bound to the optimal dynamic Sharpe ratio of the extended asset structure. Notice that for every period s between t and $(T-1)$ we have $\mathrm{SR}_s^e = \sqrt{1/H_s^e - 1}$. We define similarly $\mathrm{SR}_s^x \overset{\text{def.}}{=} \sqrt{1/H_s^x - 1}$ and we remark that

$$(\mathrm{SR}_t^x)^2 = (\mathrm{SR}_t)^2 + (R_t^f)^2 \,(S_t^x - F_t^x)'\, G_t(F_T^x)^+ \,(S_t^x - F_t^x)\,.$$

Proposition 12 *Let* $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ *be a price dynamics for the new securities which is consistent with the smile* S_t^x *at time* t*. The corresponding extended asset structure satisfies* $0 < H_t^e \le H_t^x \le H_t \le 1$ *and its optimal dynamic Sharpe ratio from* t *to* T *satisfies* $\mathrm{SR}_t \le \mathrm{SR}_t^x \le \mathrm{SR}_t^e$*. The minimum increase in the square of the optimal dynamic Sharpe ratio from the original asset structure to the extended one is given by*

$$(\mathrm{SR}_t^x)^2 - (\mathrm{SR}_t)^2 = (R_t^f)^2\,(S_t^x - F_t^x)'\,G_t(F_T^x)^+\,(S_t^x - F_t^x)\,.$$

Every pricing kernel m_T *in* PK_t^e *satisfies the inequality* $m_t^2(\mathrm{SR}_t^x)^2 \le \mathrm{Var}_t\,[m_T]$.

The next section investigates situations where the Sharpe ratio reaches its smile constrained lower bound SR_t^x.

8. Two Dynamic Investment Problems

Our analysis will help us answer the questions raised by the fund manager and the investment banker who are seeking a rationale for selecting a price dynamics for some new instruments in an incomplete market setting. A basic requirement is to avoid working with price dynamics which create arbitrage opportunities. A second objective is to be consistent with the market quotes of some liquid derivative instruments. The positive pricing kernels in $\mathrm{PK}_t(S_t^x)$ fulfill these requirements. When markets are incomplete however, these kernels are usually not unique, and an additional rationale is needed in order to pick a "good" candidate. The fund manager is afraid of generating spurious dynamic good–deals, while the banker would like to keep a close link between the price of a security and the cost of its dynamic hedge.

We show in this section that when the kernel $m_T^{t,x}$ is positive, it meets these two concerns. For the fund manager, it generates a smile consistent price dynamics which yields the smallest possible increase in Sharpe ratio. For the investment banker, it produces derivative prices which are as close as possible to the hedging cost under the constraint of the smile.

8.1. Portfolio Management and the Smile

We consider a price dynamics $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ consistent with the smile S_t^x at time t and the corresponding extended asset structure which it generates. We study the situation where the optimal dynamic Sharpe ratio of the extended asset structure reaches its theoretical lower bound, as described in Proposition 12.

The kernel $m_T^{t,x}$ plays here again a crucial role. We know from Proposition 11 that Condition 1 is satisfied and from Proposition 9 that $m_T^{t,x}(\sqrt{H_t^x})$ is a smile consistent pricing kernel in $\mathrm{PK}_t(S_t^x)$. From now on, we shall simply let $m_T^{t,x}$ denote the kernel $m_T^{t,x}(\sqrt{H_t^x})$ with $m_t^{t,x} = \sqrt{H_t^x}$.

We recall that $F_T^{x,j}$ corresponds to the final payoff of the self financing strategy which holds one unit of new security j from t to T and which reinvests every dividend distributed by this security in the L^2 minimum portfolio for the initial securities up to horizon T. As a result, the random variable

$$m_T^{t,x} = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}\left(w_T^t + (\Lambda_t^x)' M_T^{t,x}\right)$$

is the final value of a self financing dynamic portfolio starting at t which combines on the one hand some constant quantities of the n_x new securities given by the vector $(\sqrt{H_t^x}/\sqrt{H_t})\Lambda_t^x$, and on the other hand a portfolio based on the n original securities. We denote Y^e this self financing portfolio. Since, according to Lemma 14, the payoff $m_T^{t,x}$ is in $L^2(P)$, the portfolio Y^e is in $\mathcal{X}_t^e$. Its value at time t is

$$w_t(Y^e) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}\left(\frac{1}{\sqrt{h_t}} + (\Lambda_t^x)'(S_t^x - F_t^x)\right) = \frac{1}{R_t^f\sqrt{H_t^x}}.$$

Proposition 13 *Let $\{p_s^x, \phi_s^x\}_{t\leq s\leq (T-1)}$ be a price dynamics consistent with the smile S_t^x at time t. The optimal dynamic Sharpe ratio SR_t^e of the extended asset structure reaches its minimum value SR_t^x if and only if one the following equivalent conditions hold.*

(i). $w_T^{t,e} = m_T^{t,x}$.

(ii). The value process of the dynamic portfolio $X^{t,e}$ is identical to the one of a self financing strategy in $\mathcal{X}_t^e$ which holds constant quantities from t to T of the new securities given by the vector $(\sqrt{H_t^x}/\sqrt{H_t})\Lambda_t^x$.

(iii). The pricing kernel $m_T^{t,x}$ belongs to PK_t^e.

(iv). For every period s between t and $(T-1)$,

$$\begin{aligned} R_s^f m_s^{t,x} p_s^x &= E_s\left[R_{s+1}^f m_{s+1}^{t,x}\phi_{s+1}^x\right] \\ &= E_s\left[\sum_{u=s+1}^{T-1} R_u^f m_u^{t,x} d_u^x + m_T^{t,x}\phi_T^x\right]. \end{aligned}$$

Since dynamic mean–variance efficient portfolios for the extended asset structure are fixed combinations from t to T of the strategy $X^{t,e}$ and the risk free bond, Statement (ii) provides a simple characterization of absence of good–deal under the constraint of a smile. The corresponding optimal strategies keep constant quantities through time of the securities which define the smile, and these quantities are proportional to the vector $G_t(F_T^x)^+\,(S_t^x - F_t^x)$.

We next investigate if reaching the lower bound of the Sharpe ratio at time t implies that an equivalent lower bound is reached at a later trading date s, for the smile given by the price vector p_s^x. For every period s between t and $(T-1)$ we let

$$\begin{aligned} K_s^x &\stackrel{\text{def.}}{=} h_s\,(p_s^x - \bar{p}_s^x)'\,G_s(F_T^x)^+\,(p_s^x - \bar{p}_s^x)\,, \\ H_s^x &\stackrel{\text{def.}}{=} \frac{H_s}{(1+K_s^x)}, \\ \mathrm{SR}_s^x &\stackrel{\text{def.}}{=} \sqrt{1/H_s^x - 1}. \end{aligned}$$

Proposition 14 *Let* $\{p_s^x, \phi_s^x\}_{t\le s\le (T-1)}$ *be a price dynamics consistent with the smile* S_t^x *at time* t*. If* $\mathrm{SR}_t^e = \mathrm{SR}_t^x$ *then at every period* s *between* t *and* $(T-1)$ *such that* $m_s^{t,x}$ *does not vanish,* $\mathrm{SR}_s^e = \mathrm{SR}_s^x$*.*

Combining these results with Propositions 9 and 11, we obtain that when the kernel $m_T^{t,x}$ is positive, it generates a smile consistent price dynamics which avoids good–deals every period.

Result 3 *If the pricing kernel* $m_T^{t,x}$ *is positive and if the smile* S_t^x *satisfies Condition 1, then the price dynamics* $\{p_s^x, \phi_s^x\}_{t\le s\le (T-1)}$ *defined by*

$$p_s^x = E_s\left[R_{s+1}^f m_{s+1}^{t,x} \phi_{s+1}^x\right] / \left(R_s^f m_s^{t,x}\right)$$

and $\phi_s^x = p_s^x + d_s^x$ *is consistent with the smile* S_t^x *at time* t*. It generates an extended asset structure such that* $\mathrm{SR}_s^e = \mathrm{SR}_s^x$ *for every period* s *between time* t *and horizon* $(T-1)$*.*

8.2. Hedging and Pricing with the Smile

We now turn our attention to the problem of hedging and pricing derivatives with the constraint of a smile. We assume here that both w_T^t and $m_T^{t,x}$ are positive.

For a pricing kernel m_T in PK_t such that $m_t \stackrel{\text{def.}}{=} E_t[m_T]$ is positive and a payoff F_T in $L^2(P)$, we let $J_t(F_T; m_T)$ represent the quality of the variance–optimal hedge of F_T with the original securities, under the constraint that the replication starts at time t with a value derived from the kernel m_T. Formally we let

$$J_t(F_T; m_T) \stackrel{\text{def.}}{=} \underset{\substack{X \in \mathcal{X}_t \\ w_t(X) = E_t[m_T F_T]/(R_t^f m_t)}}{\text{essinf}} E_t\left[(F_T - w_T(X))^2\right].$$

We know from Proposition 2 that $J_t(F_T; m_T) = h_t(w_t(X) - Q_t(F_T))^2 + G_t(F_T)$. When $m_T = w_T^t$, Equation 18 proves that $w_t(X) = Q_t(F_T)$ and $J_t(F_T; m_T)$ reaches its minimum over the set of pricing kernels with positive conditional expectation at time t with $J_t(F_T; w_T^t) = G_t(F_T)$. The following proposition shows that the kernel $m_T^{t,x}$ solves a min–max problem in terms of hedging quality over all possible normalized payoffs in $L^2(P)$ at horizon T.

Proposition 15 *If w_T^t and $m_T^{t,x}$ are positive and if the smile S_t^x satisfies Condition 1, then the optimization program*

$$\underset{\begin{cases} m_T \in \text{PK}_t(S_t^x) \\ m_t > 0 \end{cases}}{\text{essinf}} \quad \underset{\begin{cases} F_T \in L^2(P) \\ E_t\left[F_T^2\right] = 1 \end{cases}}{\text{esssup}} \quad J_t(F_T; m_T) - J_t(F_T; w_T^t)$$

is solved for the pricing kernel $m_T^{t,x}$ with minimum value K_t^x.

Since $J_t(F_T; m_T) - J_t(F_T; w_T^t) = h_t(w_t(X) - Q_t(F_T))^2$ with $w_t(X) = E_t[m_T F_T]/(R_t^f m_t)$, the min–max problem of Proposition 15 can also be interpreted as selecting the smile consistent kernel which produces contingent claim prices as close as possible to the cost of the optimal unconstrained hedge. This result proves the constrained optimality of the kernel $m_T^{t,x}$, both in terms of hedging and in terms of pricing.

We now study the optimal hedge for the extended asset structure generated by the kernel $m_T^{t,x}$. We show that this kernel generates prices which correspond to the cost of the variance–optimal hedge constructed with both the original and the new securities.

Proposition 16 *Let us assume that $m_T^{t,x}$ is positive and that the smile S_t^x satisfies Condition 1. Let us consider the extended asset structure which this pricing kernel generates through the smile consistent price dynamics $\{p_s^x, \phi_s^x\}_{t \leq s \leq (T-1)}$ defined by*

$$p_s^x = E_s\left[R_{s+1}^f m_{s+1}^{t,x} \phi_{s+1}^x\right] / \left(R_s^f m_s^{t,x}\right)$$

and $\phi_s^x = p_s^x + d_s^x$*. For every payoff* F_T *in* $L^2(P)$*, the price generated by the pricing kernel* $m_T^{t,x}$ *coincides with the cost of the variance–optimal hedge of* F_T *which uses both the original and the new securities, that is*

$$Q_s^e(F_T) = E_s\left[m_T^{t,x} F_T\right] / \left(R_s^f m_s^{t,x}\right) \tag{33}$$

for every period s *between time* t *and horizon* T*. Furthermore at time* t *we have*

$$Q_t^e(F_T) = Q_t(F_T) + (S_t^x - F_t^x)' \, G_t\,(F_T^x)^+ \, G_t\,(F_T^x, F_T)\,. \tag{34}$$

We check that for $F_T = F_T^x$ in Equation 34, Condition 1 implies that $Q_t^e(F_T^x) = S_t^x$. Equation 34 explains how to extrapolate the quotes of the smile to any additional contingent claim.

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Appendix to Section 3

Proof of Lemma 2. We only prove the second part of Statement (ii) which is not in Henrotte (2001). We use Equation 12 with $F_T^a = w_T(X)$ and $F_T^b = F_T$. The equality $G_t(w_T(X), F_T) = 0$ results from the fact that $Q_t(w_T(X)) = w_t(X)$ and $w_T(X) = w_T(X^{t,w_t(X),w_T(X)})$. *Q.E.D.*

Proof of Lemma 4. Statement (i). Let X be a self financing dynamic strategy starting at time t and let $Y = \theta_f(X)$. For every period s from $(t+1)$ up to $(T-1)$ we have

$$w_s(Y) = p_s' Y_s = p_s' X_s - \left(\sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u \right) h_s p_s' N_s^+ p_s.$$

We know from Equation 3 that $h_s p_s' N_s^+ p_s = 1$. The fact that X is self financed at time s implies that $p_s' X_s = \phi_s' X_{s-1}$. Since furthermore $\sqrt{h_s} w_s^s = 1$, we derive

$$w_s(Y) = \phi_s' X_{s-1} - \left(\sum_{u=t+1}^{s-1} f_u \sqrt{h_u} w_s^u \right) - f_s.$$

For $u \leq (s-1)$ we know from Equation 13 that $w_s^u = w_{s-1}^u h_{s-1} \phi_s' N_{s-1}^+ p_{s-1}$ and therefore

$$\begin{aligned} w_s(Y) &= \phi_s' X_{s-1} - \left(\sum_{u=t+1}^{s-1} f_u \sqrt{h_u} w_{s-1}^u h_{s-1} \phi_s' N_{s-1}^+ p_{s-1} \right) - f_s \\ &= \phi_s' \left[X_{s-1} - \left(\sum_{u=t+1}^{s-1} f_u \sqrt{h_u} w_{s-1}^u \right) h_{s-1} N_{s-1}^+ p_{s-1} \right] - f_s \\ &= \phi_s' Y_{s-1} - f_s, \end{aligned}$$

which proves that Y finances the sequence of cash flows f. Reciprocally, the same equations proves that X is self financing as soon as Y finances f.

Statement (ii). The equality $w_t(Y) = w_t(X)$ results from the fact that $Y_t = X_t$. At time T, and since Equation 13 implies that $w_T^u = w_{T-1}^u h_{T-1} \phi_T' N_{T-1}^+ p_{T-1}$ for $u \leq (T-1)$, an analysis similar to the one developed above yields

$$\begin{aligned} w_T(Y) - f_T &= \phi_T' Y_{T-1} - f_T \\ &= \phi_T' X_{T-1} - \left(\sum_{u=t+1}^{T-1} f_u \sqrt{h_u} w_{T-1}^u h_{T-1} \phi_T' N_{T-1}^+ p_{T-1} \right) - f_T \\ &= w_T(X) - \sum_{u=t+1}^{T-1} f_u \sqrt{h_u} w_T^u - f_T \\ &= w_T(X) - \sum_{u=t+1}^{T} f_u \sqrt{h_u} w_T^u, \end{aligned}$$

and we obtain that $(w_T(Y) - f_T) = (w_T(X) - F_T)$. *Q.E.D.*

Proof of Proposition 3. The fact that the payoff F_T belongs to $L^2(P)$ results from the equalities

$$E\left[(f_s \sqrt{h_s} w_T^s)^2\right] = E\left[h_s f_s^2 E_s\left[(w_T^s)^2\right]\right] = E\left[h_s f_s^2\right],$$

and the fact that every cash flow f_s belongs to $L_s^2(P, \sqrt{h_s})$. Since both f_T and F_T are in $L^2(P)$ and since, according to Statement (ii) of Lemma 4, $(f_T - w_T(Y)) = (F_T - w_T(X))$, $w_T(Y)$ is in $L^2(P)$ if and only if $w_T(X)$ is itself in $L^2(P)$. We conclude with Statement (i) of Lemma 4 that the mapping θ_f is one to one from $\mathcal{X}_t$ to $\mathcal{X}_t(f)$. The equivalence between the two optimization programs is then a direct consequence of the properties of the mapping θ_f. *Q.E.D.*

Proof of Lemma 5. We prove these results by backward induction. We first deal with the hedging cost. At time T, we check that $Q_T(F_T) = F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s$. Let us assume that $Q_{s+1}(F_T) = \sum_{u=t+1}^{s+1} f_u \sqrt{h_u} w_{s+1}^u + \bar{f}_{s+1}$ for $t \leq s \leq (T-1)$. Since $\sqrt{h_{s+1}} w_{s+1}^{s+1} = 1$, we also have $Q_{s+1}(F_T) = \sum_{u=t+1}^{s} f_u \sqrt{h_u} w_{s+1}^u + \bar{f}_{s+1} + f_{s+1}$. Using the equations $w^u_{s+1} = w^u_s h_s \phi'_{s+1} N^+_s p_s$, $N^+_s N_s N^+_s = N^+_s$, and the definition of $\bar{f}_s$, we derive

$$\begin{aligned}
Q_s(F_T) &= p'_s N^+_s E_s\left[h_{s+1} Q_{s+1}(F_T)\phi_{s+1}\right] \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} p'_s N^+_s E_s\left[h_{s+1} w^u_{s+1}\phi_{s+1}\right] + p'_s N^+_s E_s\left[h_{s+1}(\bar{f}_{s+1} + f_{s+1})\phi_{s+1}\right] \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} p'_s N^+_s E_s\left[h_{s+1}\phi_{s+1}\phi'_{s+1}\right] N^+_s p_s h_s w^u_s + \bar{f}_s \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} p'_s N^+_s p_s h_s w^u_s + \bar{f}_s \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} w^u_s + \bar{f}_s,
\end{aligned}$$

and this proves the desired backward induction.

For the hedging quality, $\bar{g}_T = G_T(F_T) = 0$ at time T and we assume that $\bar{g}_{s+1} = G_{s+1}(F_T)$ for $t \leq s \leq (T-1)$. We know from Proposition 2 that

$$\begin{aligned}
G_s(F_T) &= E_s\left[G_{s+1}(F_T)\right] + E_s\left[h_{s+1}F^2_{s+1}\right] - E_s\left[h_{s+1}F_{s+1}\phi'_{s+1}\right] N^+_s E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right] \\
&= E_s\left[G_{s+1}(F_T)\right] + E_s\left[h_{s+1}\left(F_{s+1} - \phi'_{s+1}N^+_s E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right]\right)^2\right],
\end{aligned}$$

where $F_{s+1} = Q_{s+1}(F_T)$. According to our previous result,

$$\begin{aligned}
Q_{s+1}(F_T) &= \sum_{u=t+1}^{s} f_u \sqrt{h_u} w^u_{s+1} + \bar{f}_{s+1} + f_{s+1} \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} h_s w^u_s \phi'_{s+1} N^+_s p_s + \bar{f}_{s+1} + f_{s+1} \\
&= \phi'_{s+1} Z_s + \bar{f}_{s+1} + f_{s+1},
\end{aligned}$$

where $Z_s = \sum_{u=t+1}^{s} f_u \sqrt{h_u} h_s w^u_s N^+_s p_s$ is a vector measurable with respect to $\mathcal{F}_s$. We compute

$$\begin{aligned}
F_{s+1} - \phi'_{s+1}N^+_s E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right] &= \phi'_{s+1}Z_s - \phi'_{s+1}N^+_s N_s Z_s \\
&\quad + \bar{f}_{s+1} + f_{s+1} - \phi'_{s+1}N^+_s E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}\right]
\end{aligned}$$

and since, according to Equation 4, $N_s N_s^+ \phi_{s+1} = \phi_{s+1}$, we obtain that

$$\begin{aligned} G_s(F_T) &= E_s\left[G_{s+1}(F_T)\right] \\ &\quad + E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1} - \phi'_{s+1} N_s^+ E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}\right]\right)^2\right] \\ &= E_s\left[\bar{g}_{s+1}\right] + E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)^2\right] \\ &\quad - E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi'_{s+1}\right] N_s^+ E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}\right]. \end{aligned}$$

This proves that $G_s(F_T) = \bar{g}_s$ and concludes the backward induction proof. *Q.E.D.*

Appendix to Section 5

Proof of Lemma 6. Let m_T be a pricing kernel in PK_t and let s be a trading period between t and $(T-1)$. For every index i from 1 to n, there exists a self financing portfolio Y^i in $\mathcal{X}_s$ such that $w_s(Y^i) = p_s^i$ and $w_{s+1}(Y^i) = \phi_{s+1}^i$. We create indeed this portfolio by holding one unit of security i at time s, and by investing the value ϕ_{s+1}^i of the portfolio at time $(s+1)$ in the L^2 minimum portfolio X^{s+1} up to horizon T. This strategy is obviously self financing. Since the portfolio X^{s+1} is worth $1/\sqrt{h_{s+1}}$ at time $(s+1)$, the value at time T of the strategy writes $w_T(Y^i) = \phi_{s+1}^i \sqrt{h_{s+1}} w_T^{s+1}$ and satisfies

$$E\left[w_T(Y^i)^2\right] = E\left[h_{s+1}(\phi_{s+1}^i)^2 E_{s+1}\left[(w_T^{s+1})^2\right]\right] = E\left[h_{s+1}(\phi_{s+1}^i)^2\right],$$

which is finite according to Proposition 1. We conclude that Y^i is indeed in $\mathcal{X}_s$. Since m_T is in PK_t, we have

$$\begin{aligned} R_{s+1}^f m_{s+1} \phi_{s+1}^i &= E_{s+1}[m_T w_T(Y^i)], \\ R_s^f m_s p_s^i &= E_s[m_T w_T(Y^i)], \end{aligned}$$

and we conclude that $R_s^f m_s p_s = E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}\right]$, which proves Statement (i).

The reciprocal is easily obtained by backward induction on s, making use of the self financing condition at every trading period. The equivalence between Statements (i) and (ii) is straightforward. *Q.E.D.*

Proof of Proposition 4. Statement (i). The payoff w_T^t is an element of $w_T(\mathcal{X}_t)$ and is therefore in $L^2(P)$ so that w_T^t is in $L^2(P)$. Consider a period s between t and T and a dynamic portfolio X in $\mathcal{X}_s$. We know from Statement (iii) of Lemma 2 with $w_t = 1/\sqrt{h_t}$ and

$F_T = w_T(X)$ that $h_s w_s^t Q_s(w_T(X)) = E_s\left[w_T^t w_T(X)\right]$ and we conclude with Statement (i) of Lemma 2 and Equation 16 that $R_s^f E_s\left[w_T^t\right] w_s(X) = E_s\left[w_T^t w_T(X)\right]$ which proves that w_T^t is a pricing kernel in PK_t.

Statement (ii). If m_T is an element of PK_t^0 then for every portfolio X in $\mathcal{X}_t$

$$R_t^f m_t w_t(X) = E_t\left[m_T w_T(X)\right] = 0$$

since $m_t = 0$ and m_T belongs to $w_T(\mathcal{X}_t)^{\perp_t}$.

Reciprocally, consider a payoff m_T in $w_T(\mathcal{X}_t)^{\perp_t}$, a period s between t and T, and a portfolio X in $\mathcal{X}_s$. For every event A_s in $\mathcal{F}_s$, we create a dynamic portfolio Y starting at time t with zero wealth in the following way. We do not invest until time s. At time s we do nothing until the horizon T in case the event A_s does not occur. If the event A_s occurs at time s, we purchase the portfolio $\sqrt{H_s} X_s$ and we borrow its cost $\sqrt{H_s} w_s(X)$ by selling $R_s^f \sqrt{H_s} w_s(X)$ units of zero coupon bonds. We then follow the self financing strategy of $\sqrt{H_s} X$ until time T when we redeem the bond.

This dynamic portfolio Y is clearly self financing and starts indeed in t with zero wealth. Its final value $w_T(Y)$ is given by $w_T(Y) = 1_{A_s}\sqrt{H_s}\left(w_T(X) - R_s^f w_s(X)\right)$. It is an element of $L^2(P)$ since $\sqrt{H_s}$ is bounded, $w_T(X)$ is in $L^2(P)$, and $\sqrt{H_s} R_s^f w_s(X) = \sqrt{h_s} w_s(X)$ is in $L^2(P)$.

according to Statement (ii) of Proposition 1. We conclude that Y is an element of $\mathcal{X}_t$ and since m_T is in $w_T(\mathcal{X}_t)^{\perp_t}$, we obtain

$$E_t\left[1_{A_s}\sqrt{H_s} m_T\left(w_T(X) - R_s^f w_s(X)\right)\right] = 0.$$

This equality holds for every event A_s in $\mathcal{F}_s$ and therefore

$$E_s\left[\sqrt{H_s} m_T\left(w_T(X) - R_s^f w_s(X)\right)\right] = 0$$

and since $\sqrt{H_s} > 0$ we conclude that $R_s^f E_s\left[m_T\right] w_s(X) = E_s\left[m_T w_T(X)\right]$ and m_T is in PK_t.

The strategy $\mathbf{1}_t$ which consists in holding the zero coupon bond from time t on is an element of $\mathcal{X}_t$ with final payoff 1. Since m_T is conditionally orthogonal to this strategy, we derive that $E_t\left[m_T\right] = 0$ and m_T is indeed an element of PK_t^0.

Statement (iii). First notice that if m_T is in PK_t, m_T belongs to $L^2(P)$ so that m_T is in $L^2(P)$. Equation 15 shows that $Q_s\left(m_T\right) = (1/\sqrt{h_s})E_s\left[w_T^s m_T\right]$. Since m_T is in PK_t and w_T^s is in $w_T(\mathcal{X}_s)$ with $w_s^s = 1/\sqrt{h_s}$, we compute

$$Q_s\left(m_T\right) = \frac{1}{\sqrt{h_s}} R_s^f m_s w_s^s = \frac{1}{h_s} R_s^f m_s$$

and eventually $m_s/\sqrt{H_s} = \sqrt{h_s}Q_s(m_T)$ which is in $L^2(P)$ according to Proposition 2.

Statement (iv). We check first that if ξ_t is an element of $L^2_t(P)$, the product $\xi_t w^t_T$ is in $L^2(P)$ as required. Indeed

$$E\left[\left(\xi_t w^t_T\right)^2\right] = E\left[\left(\xi_t w^t_T\right)^2\right] = E\left[{\xi_t}^2 E_t\left[(w^t_T)^2\right]\right] = E\left[{\xi_t}^2\right] < \infty$$

since $E_t\left[(w^t_T)^2\right] = 1$. It is now clear that $\xi_t w^t_T$ and the sum $m^0_T + \xi_t w^t_T$ is in PK_t for every variable m^0_T in PK^0_t.

Reciprocally, if m_T is a pricing kernel in PK_t, we let $\xi_t = m_t/\sqrt{H_t}$. We know from Statement (iii) that ξ_t is in $L^2_t(P)$. As seen above, the product $\xi_t w^t_T$ is therefore in PK_t and so is the difference $m^0_T = m_T - \xi_t w^t_T$. We check that

$$E_t\left[m^0_T\right] = E_t\left[m_T - \xi_t w^t_T\right] = m_t - \xi_t E_t\left[w^t_T\right] = m_t - \frac{m_t}{\sqrt{H_t}}\sqrt{H_t} = 0,$$

since $E_t\left[w^t_T\right] = \sqrt{H_t}$ from Equation 17. This proves that m^0_T is a element of PK^0_t and that m_T is in $\mathrm{PK}^0_t + \left(L^2_t(P) \times w^t_T\right)$. We conclude with Statement (ii) that $\mathrm{PK}_t = w_T(\mathcal{X}_t)^{\perp_t} + \left(L^2_t(P) \times w^t_T\right)$.

Statement (v). We have already seen that $L^2_t(P) \times w^t_T$ is a subset of PK_t. It is also a subset of $w_T(\mathcal{X}_t)$ since for every element ξ_t of $L^2_t(P)$, the product $\xi_t w^t_T$ is in $L^2(P)$ and corresponds to the value at time T of the self financing portfolio $\xi_t X^t$.

Reciprocally, if m_T is an element of $\mathrm{PK}_t \bigcap w_T(\mathcal{X}_t)$, we know that, as an element of PK_t, it writes $m_T = m^0_T + \xi_t w^t_T$, with m^0_T in $w_T(\mathcal{X}_t)^{\perp_t}$ and ξ_t in $L^2_t(P)$. Since both $\xi_t w^t_T$ and m_T are in $w_T(\mathcal{X}_t)$, so is m^0_T and we obtain that $E_t\left[(m^0_T)^2\right] = 0$. We conclude that $m^0_T = 0$ and $m_T = \xi_t w^t_T$ is an element of $L^2_t(P) \times w^t_T$.

Statement (vi). We consider a payoff F_T in $L^2(P)$, an initial value w_t at time t in $L^2_t(P, \sqrt{h_t})$ and a period s between t and T. We let $m_T = \left(F_T - w_T(X^{t,w_t,F_T})\right)$, obviously an element of $L^2(P)$. We apply Statement (iv) of Lemma 2 successively with $X = \mathbf{1}_s$, the strategy in $\mathcal{X}_s$ which buys and holds one unit of the zero coupon from time s until maturity T, and with X any dynamic portfolio in $\mathcal{X}_s$. We obtain

$$m_s = E_s\left[m_T\right] = \frac{h_s}{R^f_s}\left(Q_s(F_T) - w_s(X^{t,w_t,F_T}))\right)$$

and $R^f_s m_s w_s(X) = E_s\left[m_T w_T(X)\right]$. This proves that m_T is a pricing kernel in PK_t. If w_t is chosen equal to $Q_t(F_T)$, then

$$m_t = \frac{h_t}{R^f_t}\left(Q_t(F_T) - w_t(X^{t,Q_t(F_T),F_T})\right) = 0$$

and m_T belongs to PK_t^0.

Reciprocally, if m_T^0 is a pricing kernel in PK_t^0, then m_T^0 is in $L^2(P)$ and the final value $w_T(X^{t,Q_t(m_T^0),m_T^0})$ is an element of $w_T(\mathcal{X}_t)$. We have obtained above that the variable $\left(m_T^0 - w_T(X^{t,Q_t(m_T^0),m_T^0})\right)$ is an element in PK_t^0 and this proves that $w_T(X^{t,Q_t(m_T^0),m_T^0})$ is also in PK_t^0. Since $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$ we conclude that $w_T(X^{t,Q_t(m_T^0),m_T^0}) = 0$ and if we choose $F_T = m_T^0$, then $m_T^0 = \left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)$.

We consider now a pricing kernel m_T in PK_t. From Statement (iv) we write $m_T = m_T^0 + \xi_t w_T^t$ with m_T^0 in PK_t^0 and ξ_t in $L_t^2(P)$. We let $F_T = m_T^0$ and we have seen that $m_T^0 = \left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)$. We compute $-\xi_t w_T^t = -\xi_t w_T(X^{t,1/\sqrt{h_t},0}) = w_T(X^{t,-\xi_t/\sqrt{h_t},0})$ so that, following Equation 10,

$$m_T = F_T - w_T\left(X^{t,Q_t(F_T),F_T} + X^{t,-\xi_t/\sqrt{h_t},0}\right) = F_T - w_T\left(X^{t,Q_t(F_T)-\xi_t/\sqrt{h_t},F_T}\right).$$

We conclude that $m_T = \left(F_T - w_T(X^{t,w_t,F_T})\right)$ with $F_T = m_T^0$ and $w_t = Q_t(F_T) - \xi_t/\sqrt{h_t}$.

Statement (vii). We first prove that $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$ is closed in $L^2(P)$. Consider a sequence $\{m_T^{0,n}\}_{n\geq 0}$ in $w_T(\mathcal{X}_t)^{\perp_t}$ which converges in $L^2(P)$ to m_T^0. We prove that m_T^0 belongs to $w_T(\mathcal{X}_t)^{\perp_t}$. Let X be any dynamic portfolio in $\mathcal{X}_t$. The sequence $\{E_t[m_T^{0,n} w_T(X)]\}_{n\geq 0}$ is null for every n and converges in $L^1(P)$ to $E_t\left[m_T^0 w_T(X)\right]$ since

$$E\left[\left|E_t\left[m_T^{0,n} w_T(X)\right] - E_t\left[m_T^0 w_T(X)\right]\right|\right] \leq \|w_T(X)\|_{L^2(P)} \| \left(m_T^{0,n} - m_T^0\right) \|_{L^2(P)}.$$

Therefore $E_t\left[m_T^0 w_T(X)\right] = 0$ and m_T^0 belongs to $w_T(\mathcal{X}_t)^{\perp_t}$ which is closed in $L^2(P)$.

We consider next a sequence $\{m_T^n\}_{n\geq 0}$ in PK_t which converges in $L^2(P)$ to m_T. From Statement (iv) we find two sequences $\{m_T^{0,n}\}_{n\geq 0}$ and $\{\xi_t^n\}_{n\geq 0}$ respectively in $w_T(\mathcal{X}_t)^{\perp_t}$ and in $L_t^2(P)$ such that $m_T^n = m_T^{0,n} + \xi_t^n w_T^t$ for every $n \geq 0$. The sequence $\{m_T^n\}_{n\geq 0}$ is a Cauchy sequence in $L^2(P)$ and we compute

$$E_t\left[(m_T^n - m_T^m)^2\right] = E_t\left[\left(m_T^{0,n} - m_T^{0,m} + (\xi_t^n - \xi_t^m) w_T^t\right)^2\right]$$
$$= E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right)^2\right] + (\xi_t^n - \xi_t^m)^2 E_t\left[\left(w_T^t\right)^2\right] + 2(\xi_t^n - \xi_t^m) E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right) w_T^t\right]$$
$$= E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right)^2\right] + (\xi_t^n - \xi_t^m)^2,$$

since $E_t\left[\left(w_T^t\right)^2\right] = 1$ and $E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right) w_T^t\right] = 0$. We obtain

$$E\left[(m_T^n - m_T^m)^2\right] = E\left[\left(m_T^{0,n} - m_T^{0,m}\right)^2\right] + E\left[(\xi_t^n - \xi_t^m)^2\right],$$

which shows that both $\{m_T^{0,n}\}_{n\geq 0}$ and $\{\xi_t^n\}_{n\geq 0}$ are Cauchy sequences which converge respectively to m_T^0 and ξ_t in $L^2(P)$.

On the one hand we know that m_T^0 belongs to $w_T(\mathcal{X}_t)^{\perp_t}$ because this set is closed in $L^2(P)$. On the other hand ξ_t is measurable with respect to $\mathcal{F}_t$ and belongs to $L_t^2(P)$. It is easily checked that the sequence $\{\xi_t^n w_T^t\}_{n\geq 0}$ converges in $L^2(P)$ to $\xi_t w_T^t$ so that $m_T = m_T^0 + \xi_t w_T^t$ and, according to Statement (iv), m_T belongs to PK_t. We conclude that PK_t is closed in $L^2(P)$.

Since $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$, it is clear that $w_T(\mathcal{X}_t)$ is a subset of $\left(\mathrm{PK}_t^0\right)^{\perp_t}$. Let us prove the reverse inclusion by considering a random variable F_T in $\left(\mathrm{PK}_t^0\right)^{\perp_t}$. We let $F_t = Q_t(F_T)$. The final value $w_T(X^{t,F_t,F_T})$ is in $w_T(\mathcal{X}_t)$ and therefore $\left(F_T - w_T(X^{t,F_t,F_T})\right)$ is in $\left(\mathrm{PK}_t^0\right)^{\perp_t}$. We know from Statement (vi) that $\left(F_T - w_T(X^{t,F_t,F_T})\right)$ is in PK_t^0 and we conclude that $E_t\left[\left(F_T - w_T(X^{t,F_t,F_T})\right)^2\right] = 0$, which proves that $F_T = w_T(X^{t,F_t,F_T})$ and that F_T belongs to $w_T(\mathcal{X}_t)$. *Q.E.D.*

Proof of Lemma 7. Equation 12 yields

$$\begin{aligned} G_s\left(m_T, F_T\right) &= E_s\left[\left(m_T - w_T(X^{s,Q_s(m_T),m_T})\right)\left(F_T - w_T(X^{s,Q_s(F_T),F_T})\right)\right] \\ &= E_s\left[m_T F_T\right] - E_s\left[m_T w_T(X^{s,Q_s(F_T),F_T})\right] \\ &\quad - E_s\left[\left(F_T - w_T(X^{s,Q_s(F_T),F_T})\right) w_T(X^{s,Q_s(m_T),m_T})\right]. \end{aligned}$$

The last term vanishes since $\left(F_T - w_T(X^{s,Q_s(F_T),F_T})\right)$ belongs to PK_s^0 as seen in Statement (vi) of Proposition 4 and $w_T(X^{s,Q_s(m_T),m_T})$ is a payoff in $w_T(\mathcal{X}_s)$. Eventually we obtain

$$G_s(m_T, F_T) = E_s\left[m_T F_T\right] - R_s^f m_s w_s(X^{s,Q_s(F_T),F_T}) = E_s\left[m_T F_T\right] - R_s^f m_s Q_s(F_T)$$

which is Equation 20. *Q.E.D.*

Proof of Proposition 5. If we set $F_T = m_T$ in Equation 20, we obtain Equation 21 since $\sqrt{h_s}Q_s(m_T) = m_s/\sqrt{H_s}$, according to Statement (iii) of Proposition 4. Equation 22 results then from Equation 21 and the definition of SR_s, we have indeed

$$\mathrm{Var}_s\left[m_T\right] = E_s\left[m_T^2\right] - \left(E_s[m_T]\right)^2 = m_s^2\left(\frac{1}{H_s} - 1\right) + G_s\left[m_T\right] = m_s^2(\mathrm{SR}_s)^2 + G_s\left(m_T\right).$$

The two inequalities are a direct consequence of Equations 21 and 22 and the fact that $G_s(m_T)$ is nonnegative. If $m_T = \xi_t w_T^t$ with ξ_t in $L_t^2(P)$, then $m_T = w_T(\xi_t X^t)$ and m_T

belongs to $w_T(\mathcal{X}_t)$, and therefore also to $w_T(\mathcal{X}_s)$. According to Statement (ii) of Lemma 2, $G_s(m_T) = 0$ and both inequalities are equalities.

For $s = t$, equality obtains in both cases if and only if $G_t(m_T) = 0$. Statement (ii) of Lemma 2 proves that this happens if and only if m_T belongs to $w_T(\mathcal{X}_t)$, or, according to Statement (v) of Proposition 4, if and only if m_T belongs to the set $L^2_t(P) \times w^t_T$. *Q.E.D.*

Appendix to Section 6

Proof of Lemma 8. We show that the law of one price holds from t to T. We consider a period s between t and $(T-1)$ and a vector (u, v) in $\mathbb{R}^n \times \mathbb{R}^{n_x}$ measurable with respect to $\mathcal{F}_s$ such that $\phi'_{s+1}u + (\phi^x_{s+1})'v = 0$. It results from Lemma 6 and from Equation 26 that

$$R^f_s m_s p_s = E_s\left[R^f_{s+1} m_{s+1} \phi_{s+1}\right],$$
$$R^f_s m_s p^x_s = E_s\left[R^f_{s+1} m_{s+1} \phi^x_{s+1}\right].$$

The equality $\phi'_{s+1}u + (\phi^x_{s+1})'v = 0$ implies that

$$R^f_s m_s \left(p'_s u + (p^x_s)'v\right) = E_s\left[R^f_{s+1} m_{s+1}\left(\phi'_{s+1}u + (\phi^x_{s+1})'v\right)\right] = 0$$

and we conclude that $p'_s u + (p^x_s)'v = 0$ since m_s is positive. *Q.E.D.*

Proof of Lemma 9. We consider a period s between t and $(T-1)$ and a vector (u, v) in $\mathbb{R}^n \times \mathbb{R}^{n_x}$ measurable with respect to $\mathcal{F}_s$ such that $\phi'_{s+1}u + (\bar{\phi}^x_{s+1})'v = 0$. We know from the definition of $\bar{p}^x_s$ and from Equation 5 that

$$\bar{p}^x_s = E_s\left[h_{s+1}\bar{\phi}^x_{s+1}\phi'_{s+1}\right] N^+_s p_s,$$
$$p_s = E_s\left[h_{s+1}\phi_{s+1}\phi'_{s+1}\right] N^+_s p_s.$$

The equality $\phi'_{s+1}u + (\bar{\phi}^x_{s+1})'v = 0$ implies that

$$u' p_s + v'\bar{p}^x_s = E_s\left[h_{s+1}\left(u'\phi_{s+1} + v'\bar{\phi}^x_{s+1}\right)\phi'_{s+1}\right] = 0$$

and we conclude that $p'_s u + (\bar{p}^x_s)'v = 0$.

Let us consider the kernel $m_T = w^t_T$. We learn respectively from Equations 13 and 16 that $w^t_{s+1} = w^t_s h_s \phi'_{s+1} N^+_s p_s$ and $R^f_s m_s = h_s w^t_s$ for every period s between t and $(T-1)$ and we derive that

$$R^f_s m_s \bar{p}^x_s = h_s w^t_s E_s\left[h_{s+1}\bar{\phi}^x_{s+1}\phi'_{s+1}\right] N^+_s p_s = E_s\left[h_{s+1}\bar{\phi}^x_{s+1} w^t_{s+1}\right] = E_s\left[R^f_{s+1} m_{s+1}\bar{\phi}^x_{s+1}\right].$$

A simple induction yields $R^f_s m_s p^x_s = E_s\left[\sum_{u=s+1}^{T-1} R^f_u m_u d^x_u + m_T \phi^x_T\right]$. If w^t_T is positive, m_t does not vanish and we obtain Equation 26. *Q.E.D.*

Proof of Proposition 11. Without loss of generality, we consider the case $s = t$. We assume that $\mathrm{SR}^e_t = \mathrm{SR}_t$ and we show that $w^{t,e}_T = w^t_T$ by proving that $E_t\left[(w^{t,e}_T - w^t_T)^2\right] = 0$. We know from Equation 14 that $E_t\left[(w^t_T)^2\right] = E_t\left[(w^{t,e}_T)^2\right] = 1$. According to Statement (i) of Proposition 4, $w^{t,e}_T$ is in PK^e_t, and therefore also in PK_t and Equation 19 yields $E_t\left[w^{t,e}_T w^t_T\right] = R^f_t E_t\left[w^{t,e}_T\right] w^t_t$. It results from Equation 17 that $E_t\left[w^{t,e}_T\right] = \sqrt{H^e_t}$ and since $R^f_t w^t_t = R^f_t/\sqrt{h_t} = 1/\sqrt{H_t}$, we derive that $E_t\left[w^{t,e}_T w^t_T\right] = \sqrt{H^e_t}/\sqrt{H_t}$. Now $\mathrm{SR}^e_t = \mathrm{SR}_t$ implies that $H^e_t = H_t$ and $E_t\left[w^{t,e}_T w^t_T\right] = 1$ and we conclude that

$$E_t\left[(w^{t,e}_T - w^t_T)^2\right] = E_t\left[(w^{t,e}_T)^2\right] + E_t\left[(w^t_T)^2\right] - 2E_t\left[w^{t,e}_T w^t_T\right] = 0.$$

If $w^{t,e}_T = w^t_T$, then w^t_T belongs to PK^e_t since, according to Statement (i) of Proposition 4, $w^{t,e}_T$ is an element of PK^e_t,

Let us now assume that w^t_T belongs to PK^e_t. The self financing strategy w^t_T is an element of $w_T(\mathcal{X}^e_t)$ since it is in $w_T(\mathcal{X}_t)$. According to Statement (v) of Proposition 4, there exists ξ_t in $L^2_t(P)$ such that $w^t_T = \xi_t w^{t,e}_T$. We know however that $E_t\left[(w^t_T)^2\right] = E_t\left[(w^{t,e}_T)^2\right] = 1$, and we conclude that $\xi_t = 1$ and that $w^{t,e}_T = w^t_T$. Since, according to Equation 17, $E_t\left[w^t_T\right] = \sqrt{H_t}$ and $E_t\left[w^{t,e}_T\right] = \sqrt{H^e_t}$, we conclude that $H^e_t = H_t$ and $\mathrm{SR}^e_t = \mathrm{SR}_t$.

If w^t_T is a pricing kernel in PK^e_t, then we learn from Equation 28 that $E_t\left[w^t_T F^x_T\right] = R^f_t E_t\left[w^t_T\right] p^x_t = \sqrt{h_t} p^x_t$. According to Equation 15, this implies that for every security j we have $p^{x,j}_t = Q_t(F^{x,j}_T)$ and since $Q_t(F^{x,j}_T) = \bar{p}^{x,j}_t$, we conclude that $p^x_t = \bar{p}^x_t$. *Q.E.D.*

Proof of Proposition 6. The fact that Statement (i) implies Statement (ii) results directly from Lemma 11. Let us assume that $p^x_s = \bar{p}^x_s$ and that $\phi^x_{s+1} = \bar{\phi}^x_{s+1}$ for every period s between t and $(T-1)$. We check by backward induction that $h^e_s = h_s$ for every period s between t and $(T-1)$. Let us assume therefore that $h^e_{s+1} = h_{s+1}$. We seek to prove that $h^e_s = h_s$.

We know from Equation 3 of Proposition 1 that $1/h^e_s = (p^e_s)' E_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]^+ p^e_s$. We consider the following matrix A_s, measurable with respect to $\mathcal{F}_s$,

$$A_s = \begin{pmatrix} Id & 0 \\ B_s & Id \end{pmatrix},$$

with $B_s = -E_s\left[h_{s+1}\bar{\phi}^x_{s+1}\phi'_{s+1}\right]N_s^+$. Since A_s is invertible, we remark that

$$(A'_s)^{-1}E_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]^+A_s^{-1} = \left(A_sE_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]A'_s\right)^+$$

and we have

$$\begin{aligned}
\frac{1}{h^e_s} &= (p^e_s)'E_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]^+p^e_s \\
&= (p^e_s)'A'_s(A'_s)^{-1}E_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]^+A_s^{-1}A_sp^e_s \\
&= (A_sp^e_s)'\left(A_sE_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]A'_s\right)^+A_sp^e_s \\
&= (A_sp^e_s)'\left(E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]\right)^+A_sp^e_s.
\end{aligned}$$

We notice that $\bar{p}^x_s = E_s\left[h_{s+1}\bar{\phi}^x_{s+1}\phi'_{s+1}\right]N_s^+p_s = -B_sp_s$ and we have

$$A_sp^e_s = \begin{pmatrix} p_s \\ \bar{p}^x_s + B_sp_s \end{pmatrix} = \begin{pmatrix} p_s \\ 0 \end{pmatrix}$$

and

$$A_s\phi^e_{s+1} = \begin{pmatrix} \phi_{s+1} \\ \bar{\phi}^x_{s+1} + B_s\phi_{s+1} \end{pmatrix}.$$

The block diagonal terms of the matrix $E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]$ vanish since

$$E_s\left[h_{s+1}\phi_{s+1}(\bar{\phi}^x_{s+1} + B_s\phi_{s+1})'\right] = E_s\left[h_{s+1}\phi_{s+1}(\bar{\phi}^x_{s+1})'\right] - E_s\left[h_{s+1}N_sN_s^+\phi_{s+1}(\bar{\phi}^x_{s+1})'\right]$$

and $N_sN_s^+\phi_{s+1} = \phi_{s+1}$, as seen in Equation 4. As a result, we have

$$\left(E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]\right)^+ = \begin{pmatrix} N_s^+ & 0 \\ 0 & C_s \end{pmatrix},$$

for some matrix C_s, and

$$\frac{1}{h^e_s} = (A_sp^e_s)'\left(E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]\right)^+A_sp^e_s = p_sN_s^+p_s.$$

Since $1/h_s = p_sN_s^+p_s$, we obtain that $h^e_s = h_s$, which concludes the backward induction proof. *Q.E.D.*

Proof of Proposition 7. We assume that $\mathrm{SR}^e_t = \mathrm{SR}_t$ and we consider a period s such that w^t_s does not vanish. We know from Lemma 11 that the payoff w^t_T is in PK^e_t, and therefore also in PK_s^e. Since it is also an element of $w_T(\mathcal{X}_s^e)$, we conclude with Statement (v) of Proposition 4 that there exists a variable ξ_s in $L_s^2(P)$ such that $w_T^t = \xi_s w_T^{s,e}$. From the construction of X^t and X^s and the fact that w_s^t does not vanish, we derive that

$$w_T^s = \frac{w_s^s}{w_s^t} w_T^t = \frac{1}{\sqrt{h_s} w_s^t} w_T^t = \frac{\xi_s}{\sqrt{h_s} w_s^t} w_T^{s,e}.$$

Since, according to Equation 14, $E_s\left[(w_T^s)^2\right] = E_s\left[(w_T^{s,e})^2\right] = 1$, we obtain that $w_T^s = w_T^{s,e}$ and we conclude with Lemma 11 that $\mathrm{SR}_s^e = \mathrm{SR}_s$. *Q.E.D.*

Proof of Proposition 8. We us assume that $p_s^x = \bar{p}_s^x$ for every period s between t and $(T-1)$ and we check by backward induction that for every random variable F_T in $L^2(P)$ we have $Q_s^e(F_T) = Q_s(F_T)$ for every period s between t and T. At time T we easily have $Q_T^e(F_T) = Q_T(F_T) = F_T$. Let us assume that $Q_{s+1}^e(F_T) = Q_{s+1}(F_T)$ and let F_{s+1} be this common value. We know from Proposition 2 that

$$\begin{aligned} Q_s(F_T) &= p_s' N_s^+ E_s\left[h_{s+1} F_{s+1} \phi_{s+1}\right], \\ Q_s^e(F_T) &= (p_s^e)' E_s\left[h_{s+1} \phi_{s+1}^e (\phi_{s+1}^e)'\right]^+ E_s\left[h_{s+1} F_{s+1} \phi_{s+1}^e\right], \end{aligned}$$

since, according to Proposition 6, $\mathrm{SR}_s^e = \mathrm{SR}_s$ and $h_{s+1}^e = h_{s+1}$. Drawing from the same analysis and from the same notations as in the proof of Proposition 6, we write

$$\begin{aligned} Q_s^e(F_T) &= (A_s p_s^e)' E_s\left[h_{s+1} (A_s \phi_{s+1}^e)(A_s \phi_{s+1}^e)'\right]^+ E_s\left[h_{s+1} F_{s+1} (A_s \phi_{s+1}^e)\right] \\ &= \begin{pmatrix} p_s' & 0 \end{pmatrix} \begin{pmatrix} N_s^+ & 0 \\ 0 & C_s \end{pmatrix} \begin{pmatrix} E_s\left[h_{s+1} F_{s+1} \phi_{s+1}\right] \\ D_s \end{pmatrix}, \end{aligned}$$

for some random vector D_s in $I\!R^{n_x}$. We conclude that $Q_s^e(F_T) = p_s' N_s^+ E_s\left[h_{s+1} F_{s+1} \phi_{s+1}\right] = Q_s(F_T)$ which concludes the proof by backward induction. *Q.E.D.*

Appendix to Section 7

Proof of Lemma 12. We shall use the fact that if Σ is a σ-algebra on the probability space (Ω, P) and if X is a random variable in $L^2(P)$ and Y is a random vector in $L^2(P; I\!R^n)$, then if we let $N = E\left[YY'|\Sigma\right]$ and $Z = E\left[XY'|\Sigma\right] N^+ Y$, then Z is a random variable in $L^2(P)$ and $E\left[Z^2|\Sigma\right] \leq E\left[X^2|\Sigma\right]$. This fact results from the following two equations:

$$E\left[X^2|\Sigma\right] - E\left[XY'|\Sigma\right] N^+ E\left[YX|\Sigma\right] = E\left[(X-Z)^2|\Sigma\right] \geq 0,$$

and

$$E\left[Z^2|\Sigma\right] = E\left[XY'|\Sigma\right] N^+ E\left[YX|\Sigma\right] \leq E\left[X^2|\Sigma\right].$$

We apply this result with $\Sigma = \mathcal{F}_t$ and

$$X = M_T^{t,a} = F_T^a - w_T\left(X^{t,Q_t(F_T^a),F_T^a}\right),$$
$$Y = M_T^{t,b} = F_T^b - w_T\left(X^{t,Q_t(F_T^b),F_T^b}\right).$$

Equations 11 and 12 show that

$$G_t(F_T^a) = E_t\left[(M_T^{t,a})^2\right],$$
$$G_t(F_T^a, F_T^b) = E_t\left[M_T^{t,a}(M_T^{t,b})'\right],$$

and we obtain that $Z = G_t(F_T^a, F_T^b)G_t(F_T^b)^+ M_T^b$ is an element of $L^2(P)$ and that

$$G_t(F_T^a, F_T^b)G_t(F_T^b)^+ G_t(F_T^b, F_T^a) \leq G_t(F_T^a),$$

which proves Lemma 12. *Q.E.D.*

Proof of Lemma 13. From the definition of $\mathrm{PK}_t(S_t^x)$ and Equation 20 we derive respectively $R_t^f m_t S_t^x = E_t\left[m_T F_T^x\right]$ and $R_t^f m_t F_t^x + G_t\left(m_T, F_T^x\right) = E_t\left[m_T F_T^x\right]$, which yields Equation 29.

We now apply Lemma 12 with $F_T^a = m_T$ and $F_T^b = F_T^x$ and we obtain

$$\left(R_t^f m_t\right)^2 \left(S_t^x - F_t^x\right)' G_t(F_T^x)^+ \left(S_t^x - F_t^x\right) \leq G_t\left(m_T\right)$$

which writes $(m_t^2/H_t)K_t^x \leq G_t(m_T)$ and yields Inequality 30. Since $G_t(m_T)$ is in $L^1(P)$, this inequality shows that $(m_t^2/H_t)K_t^x$ is also in $L^1(P)$. Statement (iii) of Proposition 4 proves that m_t^2/H_t belongs to $L^1(P)$ and therefore so does $(m_t^2/H_t)(1 + K_t^x) = m_t^2/H_t^x$. *Q.E.D.*

Proof of Lemma 14. We start by showing that $m_T^{t,x}(\bar{m}_t)$ is a random variable in $L^2(P)$. According to Statement (vi) of Proposition 4, every component of $M_T^{t,x}$, which is also $w_T(\mathcal{X}_t)^{\perp_t}$ according to Statement (ii) of the same proposition. This implies that $E_t\left[M_T^{t,x} w_T^t\right] = 0$ and

$$\begin{aligned}
E_t\left[\left(m_T^{t,x}(\bar{m}_t)\right)^2\right] &= E_t\left[\left(\frac{\bar{m}_t}{\sqrt{H_t}} w_T^t\right)^2\right] + E_t\left[\left(\frac{\bar{m}_t}{\sqrt{H_t}}(\Lambda_t^x)' M_T^{t,x}\right)^2\right] \\
&= \frac{\bar{m}_t^2}{H_t} E_t\left[\left(w_T^t\right)^2\right] + \frac{\bar{m}_t^2}{H_t}(\Lambda_t^x)' G_t(F_T^x)\Lambda_t^x \\
&= \frac{\bar{m}_t^2}{H_t}(1 + K_t^x) = \frac{\bar{m}_t^2}{H_t^x}.
\end{aligned}$$

Since $\bar{m}_t$ is in $L_t^2(P, 1/\sqrt{H_t^x})$, we conclude that $m_T^{t,x}(\bar{m}_t)$ is indeed in $L^2(P)$. Notice also that $E_t\left[m_T^{t,x}(\bar{m}_t)\right] = \bar{m}_t$ since $E_t\left[M_T^{t,x}\right] = 0$ and $E_t\left[w_T^t\right] = \sqrt{H_t}$.

Next we show that $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t. We derive from above that the random variable $(\bar{m}_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x}$ is in $L^2(P)$ and in $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$. Since H_t^x is smaller than H_t, the ratio $(\bar{m}_t/\sqrt{H_t})$ is in $L_t^2(P)$ and we conclude with Statement (iv) of Proposition 4 that $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t. *Q.E.D.*

Proof of Proposition 9. Statement (i). We know from Lemma 14 that $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t and we check that $R_t^f \bar{m}_t S_t^x = E_t\left[m_T^{t,x}(\bar{m}_t) F_T^x\right]$. We compute

$$
\begin{aligned}
E_t\left[m_T^{t,x}(\bar{m}_t) M_T^{t,x}\right] &= \frac{\bar{m}_t}{\sqrt{H_t}} E_t\left[w_T^t M_T^{t,x}\right] + \frac{\bar{m}_t}{\sqrt{H_t}} E_t\left[M_T^{t,x}\left(M_T^{t,x}\right)'\right] \Lambda_t^x \\
&= \frac{\bar{m}_t}{\sqrt{H_t}} \sqrt{h_t} Q_t(M_T^{t,x}) + \frac{\bar{m}_t}{\sqrt{H_t}} \sqrt{h_t} G_t(F_T^x) G_t(F_T^x)^+ \left(S_t^x - F_t^x\right) \\
&= R_t^f \bar{m}_t \left(S_t^x - F_t^x\right).
\end{aligned}
$$

The last equation results from Condition 1, and the fact that $Q_t(M_T^{t,x}) = 0$. Since $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t, we know that

$$
E_t\left[m_T^{t,x}(\bar{m}_t) w_T\left(X^{t,F_t^x,F_T^x}\right)\right] = R_t^f \bar{m}_t F_t^x
$$

and we obtain that $E_t\left[m_T^{t,x}(\bar{m}_t) F_T^x\right] = R_t^f \bar{m}_t S_t^x$, which proves that $m_T^{t,x}(\bar{m}_t)$ is an element of $\mathrm{PK}_t(S_t^x)$.

Statement (ii). We consider a pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ such that $m_t = E_t[m_T]$ does not vanish. We know from Equation 29 of Lemma 13 that

$$
R_t^f m_t \left(S_t^x - F_t^x\right) = G_t\left(m_T, F_T^x\right).
$$

Since the variables m_t and R_t^f do not vanish, Condition 1 holds if we show that

$$
G_t(F_T^x) G_t(F_T^x)^+ G_t\left(F_T^x, m_T\right) = G_t\left(F_T^x, m_T\right),
$$

or equivalently, if we prove that $G_t\left(F_T^x, m_T\right) = G_t\left(\bar{F}_T^x, m_T\right)$, where we let

$$
\bar{F}_T^x = G_t(F_T^x) G_t(F_T^x)^+ F_T^x.
$$

We check that $G_t(F_T^x) = G_t(\bar{F}_T^x) = G_t(F_T^x, \bar{F}_T^x)$, and therefore that $G_t(F_T^x - \bar{F}_T^x) = 0$. We derive from Statement (ii) of Lemma 2 that every component of the random vector $(F_T^x - \bar{F}_T^x)$

is a payoff in $w_T(\mathcal{X}_t)$. We conclude with the same statement that $G_t\left(\left(F_T^x - \bar{F}_T^x\right), m_T\right) = 0$, and this proves that Condition 1 is satisfied. *Q.E.D.*

Proof of Proposition 10. Inequalities 31 and 32 result respectively from Equations 21 and 22 of Proposition 5, together with Inequality 30 of Lemma 13. It is clear that these two inequalities become jointly equalities if and only if $G_t(m_T) = (m_t^2/H_t)K_t^x$. We show that this happens if and only if $m_T = m_T^{t,x}(m_t)$.

We know from Lemma 13 that m_t belongs to $L_t^2(P, 1/\sqrt{H_t^x})$ and the proof of Lemma 14 shows that the variable $(m_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x}$ is in $L^2(P)$. The variable

$$y_T \stackrel{\text{def.}}{=} m_T - \frac{m_t}{\sqrt{H_t}}(\Lambda_t^x)' M_T^{t,x}$$

is therefore also in $L^2(P)$ and we check that Equality 29 of Lemma 13 implies that $G_t(y_T) = G_t(m_T) - (m_t^2/H_t)K_t^x$. According to Statement (ii) of Lemma 2, the equality $G_t(m_T) = (m_t^2/H_t)K_t^x$ is therefore equivalent to the fact that y_T belongs to $w_T(\mathcal{X}_t)$. Since both m_T and $\left(m_t/\sqrt{H_t}\right)(\Lambda_t^x)' M_T^{t,x}$ are in PK_t, this equivalent condition says that y_T is an element of $\mathrm{PK}_t \bigcap w_T(\mathcal{X}_t)$ which, according to Statement (v) of Proposition 4, is also $L_t^2(P) \times w_T^t$. Therefore the equivalent condition states that the kernel m_T can be written as $m_T = \xi_t w_T^t + (m_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x}$ for some variable ξ_t in $L_t^2(P)$. Since then $E_t\left[m_T\right] = \xi_t\sqrt{H_t}$, we conclude that $\xi_t = \left(m_t/\sqrt{H_t}\right)$ and the equivalent condition writes $m_T = (m_t/\sqrt{H_t})\left(w_T^t + (\Lambda_t^x)' M_T^{t,x}\right) = m_T^{t,x}(m_t)$ as desired. *Q.E.D.*

Proof of Proposition 11. Statement (i). Lemma 8 has established the admissibility of the proposed price dynamics. Lemma 10 shows that $R_t^f m_t p_t^x = E_t\left[m_T F_T^x\right]$ and since m_T belongs to $\mathrm{PK}_t(S_t^x)$, we obtain that $R_t^f m_t p_t^x = R_t^f m_t S_t^x$. Since m_T is positive, we conclude that $p_t^x = S_t^x$.

Statement (ii). We first show that for every random vectors X_t in $I\!R^{n_x}$ measurable with respect to $\mathcal{F}_t$, the equality $X_t' M_T^{t,x} = 0$ implies $X_t'(S_t^x - F_t^x) = 0$. We do this by proving by backward induction that the equality

$$X_t'\left(\phi_s^x + \sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u - w_s\left(X^{t,F_t^x,F_T^x}\right)\right) = 0$$

holds for every period s between $(t+1)$ and T, where we set the sum $\sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u$ to zero for $s = (t+1)$. The equality holds for $s = T$ since $M_T^{t,x} = F_T^x - w_T\left(X^{t,F_t^x,F_T^x}\right)$. Let us assume that

$$X_t'\left(\phi_{s+1}^x + \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} w_{s+1}^u - w_{s+1}\left(X^{t,F_t^x,F_T^x}\right)\right) = 0.$$

The self financing condition at time $(s+1)$ of the n_x portfolios described by the matrix X^{t,F_t^x,F_T^x} implies that $w_{s+1}\left(X^{t,F_t^x,F_T^x}\right) = \phi_{s+1}' X_s^{t,F_t^x,F_T^x}$. We also know that $w_{s+1}^u = h_s w_s^u p_s' N_s^+ \phi_{s+1}$ and we obtain

$$X_t'\left(\phi_{s+1}^x + \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} h_s w_s^u p_s' N_s^+ \phi_{s+1} - \phi_{s+1}' X_s^{t,F_t^x,F_T^x}\right) = 0.$$

The law of one price for the extended asset structure implies that

$$X_t'\left(p_s^x + \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} h_s w_s^u p_s' N_s^+ p_s - p_s' X_s^{t,F_t^x,F_T^x}\right) = 0$$

which also writes

$$X_t'\left(p_s^x + \sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u + d_s^x - w_s\left(X^{t,F_t^x,F_T^x}\right)\right) = 0.$$

Finally, since $\phi_s^x = p_s^x + d_s^x$, we obtain

$$X_t'\left(\phi_s^x + \sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u - w_s\left(X^{t,F_t^x,F_T^x}\right)\right) = 0$$

as desired. For $s = (t+1)$ this equation writes $X_t'\left(\phi_{t+1}^x - w_{t+1}\left(X^{t,F_t^x,F_T^x}\right)\right) = 0$. The law of one price from t to $(t+1)$ implies that $X_t'\left(p_t^x - w_t\left(X^{t,F_t^x,F_T^x}\right)\right) = 0$ and since the price dynamics is consistent with the smile, $p_t^x = S_t^x$ and we conclude that $X_t'(S_t^x - F_t^x) = 0$ as claimed.

We consider now a random vectors X_t in $\mathbb{R}^{n_x}$ measurable with respect to $\mathcal{F}_t$. Since $G_t(F_T^x) = E_t\left[M_T^{t,x}(M_T^{t,x})'\right]$ and $G_t(F_T^x)G_t(F_T^x)^+G_t(F_T^x) = G_t(F_T^x)$, some simple algebra shows that

$$E_t\left[\left(X_t' M_T^{t,x} - X_t' G_t(F_T^x) G_t(F_T^x)^+ M_T^{t,x}\right)^2\right] = 0,$$

which proves that $\left(X_t - G_t(F_T^x)^+ G_t(F_T^x) X_t\right)' M_T^{t,x} = 0$. According to our first result, we obtain that $\left(X_t - G_t(F_T^x)^+ G_t(F_T^x) X_t\right)' (S_t^x - F_t^x) = 0$, which also writes

$$X_t'\left((S_t^x - F_t^x) - G_t(F_T^x) G_t(F_T^x)^+ (S_t^x - F_t^x)\right) = 0.$$

Since this last equation is true for every random vectors X_t in $\mathbb{R}^{n_x}$ measurable with respect to $\mathcal{F}_t$, we conclude that $(S_t^x - F_t^x) = G_t(F_T^x)G_t(F_T^x)^+ (S_t^x - F_t^x)$ and the smile satisfies Condition 1. *Q.E.D.*

Proof of Proposition 12. We let $\bar{m}_t = \inf(\sqrt{H_t^x}, \sqrt{H_t^e})$ so that both $(\bar{m}_t/\sqrt{H_t^x})$ and $(\bar{m}_t/\sqrt{H_t^e})$ are in $L_t^2(P)$, and we let $\xi_t = \bar{m}_t/\sqrt{H_t^e}$. Proposition 5 applied to the extended asset structure proves that the pricing kernel $m_T = \xi_t w_T^{t,e}$ is in PK_t^e and satisfies $m_t = \sqrt{H_t^e}\xi_t = \bar{m}_t$ and $E_t[m_T^2] = \xi_t^2$. Since PK_t^e is a subset of $\mathrm{PK}_t(S_t^x)$, the pricing kernel m_T is also in $\mathrm{PK}_t(S_t^x)$. We learn from Equation 31 of Proposition 10 that $m_t^2/H_t^x \le E_t[m_T^2]$, and therefore $(\bar{m}_t^2/H_t^x) \le \xi_t^2 = (\bar{m}_t^2/H_t^e)$. Since $\bar{m}_t$ is positive, we conclude that $H_t^e \le H_t^x \le H_t$ and

$$\mathrm{SR}_t = \sqrt{(1/H_t - 1)} \le \mathrm{SR}_t^x = \sqrt{(1/H_t^x - 1)} \le \mathrm{SR}_t^e = \sqrt{(1/H_t^e - 1)}.$$

We now compute

$$(\mathrm{SR}_t^x)^2 - (\mathrm{SR}_t)^2 = \frac{1}{H_t^x} - \frac{1}{H_t} = \frac{K_t^x}{H_t} = (R_t^f)^2 \left(S_t^x - F_t^x\right)' G_t(F_T^x)^+ \left(S_t^x - F_t^x\right),$$

which yields the desired result. The last statement results directly from Equation 32 of Proposition 10 and the fact that PK_t^e is a subset of $\mathrm{PK}_t(S_t^x)$. *Q.E.D.*

Appendix to Section 8

Proof of Proposition 13. We first remark that the equality between SR_t^x and SR_t^e is equivalent to the fact that H_t^e and H_t^x are themselves identical.

($H_t^e = H_t^x$) implies (i). We apply Lemma 3 to the extended asset structure with $F_T = 0$, $w_t = w_t(X^{t,e}) = (1/\sqrt{h_t^e})$, and $Y = Y^e$ in $\mathcal{X}_t^e$. Since

$$E_t\left[(F_T - w_T(Y))^2\right] = E_t\left[(w_T(Y^e))^2\right] = 1 = h_t^e w_t^2$$

and

$$w_t(Y) = w_t(Y^e) = \frac{1}{R_t^f\sqrt{H_t^x}} = \frac{1}{R_t^f\sqrt{H_t^e}} = \frac{1}{\sqrt{h_t^e}} = w_t,$$

the set $A_t(Y^e)$ has probability one and $m_T^{t,x} = w_T(Y^e) = w_T^{t,e}$.

(i) implies (ii). If $w_T(X^{t,e}) = m_T^{t,x}$ then $w_T(X^{t,e}) = w_T(Y^e)$ and an iterated use of the law of one price and the self financing condition proves by backward induction that $w_s(X^{t,e}) = w_s(Y^e)$ for every period s between t and T.

(i) implies (iii). Statement (i) of Proposition 4 applied to the extended asset structure proves that $w_T^{t,e}$ is an element in PK_t^e. If $m_T^{t,x} = w_T^{t,e}$, then $m_T^{t,x}$ is in PK_t^e.

(iii) implies $(H_t^e = H_t^x)$**.** On the one hand we have proved in Proposition 12 that $H_t^e \leq H_t^x$. On the other hand, if $m_T^{t,x}$ belongs to PK_t^e, Optimization Program 25 applied to the extended asset structure yields

$$\frac{H_t^x}{H_t^e} \leq E_t\left[\left(m_T^{t,x}\right)^2\right].$$

Since $E_t\left[(m_T^{t,x})^2\right] = 1$, we conclude that $H_t^x \leq H_t^e$ and $H_t^e = H_t^x$ as desired.

(ii) implies $(H_t^e = H_t^x)$**.** We let $\Lambda_t \stackrel{\text{def.}}{=} (\sqrt{H_t^x}/\sqrt{H_t})\Lambda_t^x$ and we first show that there exists a portfolio X in $\mathcal{X}_t$ such that $w_T(X^{t,e}) = w_T(X) + \Lambda_t' M_T^{t,x}$.

The strategy $X^{t,e}$ has a value process which is identical to the one of a self financing strategy in $\mathcal{X}_t^e$ which holds the constant quantities Λ_t of the new securities from time t up to horizon T. Every period s between $(t+1)$ and $(T-1)$, this self financing strategy reinvests the dividend $\Lambda_t' d_s^x$ distributed by the new securities in the original securities. If we define the sequence of cash flows $f = \{f_s\}_{t+1 \leq s \leq T}$ by $f_s = -\Lambda_t' d_s^x$ for s between $(t+1)$ and $(T-1)$ and $f_T = 0$, then there exists a dynamic portfolio Z starting at time t which only invests in the original securities, which finances f, and such that $w_T(X^{t,e}) = \Lambda_t' \phi_T^x + w_T(Z)$.

Let Y be the dynamic portfolio starting at time t such that $\theta_f(Y) = Z$. We learn from Lemma 4 that Y is a self financing portfolio which only invests in the original securities and that $w_T(Z) = (w_T(Y) - F_T)$ with

$$F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s = -\Lambda_t' \left(\sum_{s=t+1}^{T-1} d_s^x \sqrt{h_s} w_T^s \right) = \Lambda_t' \phi_T^x - \Lambda_t' F_T^x.$$

We obtain that $w_T(X^{t,e}) = w_T(Y) + \Lambda_t' F_T^x$. If we let $X = Y + X^{t,F_t^x,F_T^x}\Lambda_t$ then X is also a self financing portfolio which only invests in the original securities and $w_T(X^{t,e}) = w_T(X) + \Lambda_t' M_T^{t,x}$. The equality

$$E_t\left[\left(\Lambda_t' M_T^{t,x}\right)^2\right] = \frac{(\Lambda_t^x)' G_t(F_T^x) \Lambda_t^x}{1 + (\Lambda_t^x)' G_t(F_T^x) \Lambda_t^x}$$

proves that $\Lambda_t' M_T^{t,x}$ is in $L^2(P)$. Since $w_T(X^{t,e})$ is also in $L^2(P)$, we conclude that $w_T(X)$ is in $L^2(P)$ and X is a portfolio in $\mathcal{X}_t$ which satisfies $w_T(X^{t,e}) = w_T(X) + \Lambda_t' M_T^{t,x}$ as desired.

According to Statement (i) of Proposition 4, $w_T(X^{t,e})$ is a pricing kernel in PK_t^e and therefore also in PK_t. Statement (iv) of the same proposition proves that there exists ξ_t

in $L^2_t(P)$ and m^0_T in PK^0_t such that $w_T(X^{t,e}) = \xi_t w^t_T + m^0_T$. Since $\Lambda'_t M^{t,x}_T$ is in PK^0_t, the conditional orthogonality between $w_T(\mathcal{X}_t)$ and PK^0_t implies that $w_T(X) = \xi_t w^t_T$ and $m^0_T = \Lambda'_t M^{t,x}_T$. We also learn from Equation 17 that $E_t[w_T(X^{t,e})] = \sqrt{H^e_t}$, which implies that $\xi_t = (\sqrt{H^e_t}/\sqrt{H_t})$. Eventually we obtain that $w_T(X^{t,e}) = (\sqrt{H^e_t}/\sqrt{H_t})w^t_T + \Lambda'_t M^{t,x}_T$.

Since $w^{t,e}_T = w_T(X^{t,e})$ is in PK^e_t, it is consistent with the smile and we have

$$\begin{aligned} S^x_t = \frac{1}{\sqrt{h^e_t}} E_t\left[w^{t,e}_T F^x_T\right] &= \frac{1}{\sqrt{h^e_t}} \frac{\sqrt{H^e_t}}{\sqrt{H_t}} E_t\left[w^t_T F^x_T\right] + \frac{1}{\sqrt{h^e_t}} E_t\left[F^x_T (M^{t,x}_T)'\right]\Lambda_t \\ &= \frac{\sqrt{h_t}}{\sqrt{h^e_t}} \frac{\sqrt{H^e_t}}{\sqrt{H_t}} F^x_t + \frac{1}{\sqrt{h^e_t}} E_t\left[M^{t,x}_T (M^{t,x}_T)'\right]\Lambda_t \\ &= F^x_t + \frac{1}{\sqrt{h^e_t}} G_t(F^x_T)\Lambda_t \end{aligned}$$

and $(S^x_t - F^x_t) = (1/\sqrt{h^e_t})G_t(F^x_T)\Lambda_t$. Condition 1 implies that we also have $(S^x_t - F^x_t) = (1/\sqrt{h_t})G_t(F^x_T)\Lambda^x_t$ and we obtain that $G_t(F^x_T)\sqrt{H^e_t}\Lambda^x_t = G_t(F^x_T)\sqrt{H_t}\Lambda_t$.

Let $\epsilon_T = (\sqrt{H^x_t}/\sqrt{H^e_t})w^{t,e}_T - m^{t,x}_T$. We notice that

$$\epsilon_T = \frac{\sqrt{H^x_t}}{\sqrt{H^e_t}\sqrt{H_t}} \left(\sqrt{H_t}\Lambda_t - \sqrt{H^e_t}\Lambda^x_t\right)' M^{t,x}_T$$

so that

$$E_t\left[\epsilon^2_T\right] = \frac{H^x_t}{H^e_t H_t} \left(\sqrt{H_t}\Lambda_t - \sqrt{H^e_t}\Lambda^x_t\right)' G_t(F^x_T) \left(\sqrt{H_t}\Lambda_t - \sqrt{H^e_t}\Lambda^x_t\right) = 0$$

since $G_t(F^x_T)\left(\sqrt{H_t}\Lambda_t - \sqrt{H^e_t}\Lambda^x_t\right) = 0$. This proves that $\epsilon_T = 0$ and we obtain that $m^{t,x}_T = (\sqrt{H^x_t}/\sqrt{H^e_t})w^{t,e}_T$. On the one hand, Lemma 14 shows that $E_t\left[\left(m^{t,x}_T\right)^2\right] = 1$, and on the other hand we know that $E_t\left[(w^{t,e}_T)^2\right] = 1$. We conclude that $H^e_t = H^x_t$.

(iii) is equivalent to (iv). This equivalence results directly from Lemma 10. *Q.E.D.*

Proof of Proposition 14. We assume that $H^e_t = H^x_t$ and we consider a period s such that $m^{t,x}_s$ does not vanish. We know from Statement (iii) of Proposition 13 that $m^{t,x}_T$ is in PK^e_t, and therefore also in PK^e_s. We have already seen that the payoff $m^{t,x}_T$ is in $w_T(\mathcal{X}^e_t)$, it is therefore also in $w_T(\mathcal{X}^e_s)$. We obtain that $m^{t,x}_T$ is in $\mathrm{PK}^e_s \bigcap w_T(\mathcal{X}^e_s)$ and, according to Statement (v) of Proposition 4,

$$m^{t,x}_T = \frac{m^{t,x}_s}{\sqrt{H^e_s}} w^{s,e}_T, \qquad (35)$$

where we recall that $m_s^{t,x} \overset{\text{def.}}{=} E_s\left[m_T^{t,x}\right]$.

We let $F_T^{s,x} = \sum_{u=s+1}^{T-1} \sqrt{h_u} w_T^u d_u^x + \phi_T^x$. Since $m_T^{t,x}$ is in PK_t^e, Statement (ii) of Lemma 10 shows that

$$R_s^f m_s^{t,x} p_s^x = E_s\left[\sum_{u=s+1}^{T-1} R_u^f m_u^{t,x} d_u^x + m_T^{t,x}\phi_T^x\right] = E_s\left[m_T^{t,x} F_T^{s,x}\right],$$

and since $m_T^{t,x}$ is in PK_t and $Q_s(F_T^{s,x}) = \bar{p}_s^x$, we derive from Equation 20 of Lemma 7 that

$$R_s^f m_s^{t,x} \bar{p}_s^x + G_s\left(m_T^{t,x}, F_T^{s,x}\right) = E_s\left[m_T^{t,x} F_T^{s,x}\right].$$

We also compute

$$G_s\left(m_T^{t,x}, F_T^{s,x}\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}} G_s\left(w_T^t + (\Lambda_t^x)' M_T^{t,x}, F_T^{s,x}\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}} G_s(F_T^x)\Lambda_t^x,$$

since

$$G_s\left((\Lambda_t^x)' M_T^{t,x}, F_T^{s,x}\right) = G_s\left((\Lambda_t^x)' M_T^{t,x}, F_T^x - \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} w_T^u\right) = G_s(F_T^x)\Lambda_t^x.$$

Combining these results we obtain

$$R_s^f m_s^{t,x}\left(p_s^x - \bar{p}_s^x\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}} G_s(F_T^x)\Lambda_t^x,$$

from which we derive

$$\left(R_s^f m_s^{t,x}\right)^2 K_s^x = \frac{h_s H_t^x}{H_t} (\Lambda_t^x)' G_s(F_T^x)\Lambda_t^x.$$

We compute

$$G_s\left(m_T^{t,x}\right) = \frac{H_t^x}{H_t} G_s\left(w_T^t + (\Lambda_t^x)' M_T^{t,x}\right) = \frac{H_t^x}{H_t} (\Lambda_t^x)' G_s(F_T^x)\Lambda_t^x,$$

and we obtain that

$$G_s\left(m_T^{t,x}\right) = \frac{\left(R_s^f m_s^{t,x}\right)^2}{h_s} K_s^x = \frac{\left(m_s^{t,x}\right)^2}{H_s} K_s^x,$$

which, together with Equation 21, implies that

$$E_s\left[\left(m_T^{t,x}\right)^2\right] = \frac{\left(m_s^{t,x}\right)^2}{H_s} + G_s\left(m_T^{t,x}\right) = \frac{\left(m_s^{t,x}\right)^2}{H_s^x}. \tag{36}$$

We wish now to apply the result of Proposition 10 at time s for the smile p_s^x. Notice that the kernel $m_T^{t,x}$ is in PK_s^e and therefore also in $\mathrm{PK}_s(p_s^x)$. The price dynamics $\{p_u^x, \phi_u^x\}_{s \leq u \leq (T-1)}$ is of course consistent with the smile p_s^x and Statement (ii) of Proposition 11 proves that p_s^x satisfies Condition 1 at time s, that is

$$(p_s^x - \bar{p}_s^x) = G_s\left(F_T^{s,x}\right) G_s\left(F_T^{s,x}\right)^+ (p_s^x - \bar{p}_s^x).$$

Equation 36 proves that $m_T^{t,x} = m_T^{s,x}(m_s^{t,x})$. Since $m_s^{t,x}$ is P almost surely different from zero, we also have $m_T^{s,x}(\sqrt{H_s^x}) = (\sqrt{H_s^x}/m_s^{t,x}) m_T^{s,x}(m_s^{t,x})$ and therefore $m_T^{s,x}(\sqrt{H_s^x}) = (\sqrt{H_s^x}/m_s^{t,x}) m_T^{t,x}$. We derive from Equation 35 that $m_T^{s,x}(\sqrt{H_s^x}) = (\sqrt{H_s^x}/\sqrt{H_s^e}) w_T^{s,e}$. Since $E_s\left[(w_T^{s,e})^2\right] = 1$ and, according to Lemma 14, $E_s\left[\left(m_T^{s,x}(\sqrt{H_s^x})\right)^2\right] = 1$, we conclude that $H_s^e = H_s^x$. *Q.E.D.*

Proof of Proposition 15. Let us consider a pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ such that $m_t > 0$ and a payoff F_T in $L^2(P)$ such that $E_t\left[F_T^2\right] = 1$. Let $w_t = E_t\left[m_T F_T\right]/(R_t^f m_t)$ be the value of the payoff F_T derived from the kernel m_T. According to Equation 20 of Lemma 7, $(w_t - Q_t(F_T)) = G_t(m_T, F_T)/(R_t^f m_t)$, and since $J_t(F_T; m_T) - J_t(F_T; w_T^t) = h_t\left(w_t - Q_t(F_T)\right)^2$, we obtain that

$$J_t(F_T; m_T) - J_t(F_T; w_T^t) = \frac{h_t}{\left(R_t^f m_t\right)^2} G_t(m_T, F_T)^2 = \frac{H_t}{m_t^2} G_t(m_T, F_T)^2.$$

We know from Statement (iv) of Proposition 4 that the kernel m_T can be written $m_T = (m_t/\sqrt{H_t}) w_T^t + m_T^0$, where m_T^0 is a kernel in PK_t^0. According to Statement (ii) of Lemma 2 and Equation 12, we have

$$G_t(m_T, F_T) = G_t(m_T^0, F_T) = E_t\left[m_T^0\left(F_T - w_T(X^{t, Q_t(F_T), F_T})\right)\right] = E_t\left[m_T^0 F_T\right].$$

The maximum value of $G_t(m_T, F_T)^2$ over random variables F_T such that $E_t\left[F_T^2\right] = 1$ is $E_t[(m_T^0)^2]$ and we obtain that

$$\operatorname*{esssup}_{\substack{F_T \in L^2(P) \\ E_t[F_T^2]=1}} J_t(F_T; m_T) - J_t(F_T; w_T^t) = (H_t/m_t^2) E_t\left[\left(m_T^0\right)^2\right].$$

Since $E_t[(m_T)^2] = E_t[(m_T^0)^2] + (m_t^2/H_t)$, we have

$$(H_t/m_t^2) E_t\left[\left(m_T^0\right)^2\right] = (H_t/m_t^2) E_t\left[(m_T)^2\right] - 1$$

and we conclude with Inequality 31 of Proposition 10 that the program

$$\underset{\substack{m_T \in \mathrm{PK}_t(S_t^x)\\ E_t[m_T]>0}}{\text{essinf}} (H_t/m_t^2) E_t \left[(m_T)^2 \right] - 1$$

is solved for the pricing kernel $m_T^{t,x}$ with minimum value $(H_t/H_t^x) - 1 = K_t^x$. *Q.E.D.*

Proof of Proposition 16. Since the smile satisfies Condition 1, Proposition 9 shows that the kernel $m_T^{t,x}$ is in $\mathrm{PK}_t(S_t^x)$ and since it is positive, we learn from Proposition 11 that the proposed price dynamics is consistent with the smile. We derive from Proposition 13 that $m_T^{t,x}$ is equal to $w_T^{t,e}$. Equation 33 results then directly from Equation 18 applied to the extended asset structure.

According to Equation 20 and since $m_t^{t,x} = \sqrt{H_t^x}$, we compute

$$Q_t^e(F_T) = \frac{E_t\left[m_T^{t,x} F_T\right]}{R_t^f m_t^{t,x}} = Q_t(F_T) + \frac{1}{R_t^f \sqrt{H_t^x}} G_t\left(m_T^{t,x}, F_T\right).$$

We also derive from Statement (ii) of Lemma 2 that

$$G_t\left(m_T^{t,x}, F_T\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}} G_t\left(w_T^t + (\Lambda_t^x)' M_T^{t,x}, F_T\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}} (\Lambda_t^x)' G_t\left(F_T^x, F_T\right)$$

and we conclude that

$$\begin{aligned} Q_t^e(F_T) &= Q_t(F_T) + \frac{1}{R_t^f \sqrt{H_t^x}} \frac{\sqrt{H_t^x}}{\sqrt{H_t}} (\Lambda_t^x)' G_t\left(F_T^x, F_T\right) \\ &= Q_t(F_T) + \frac{\sqrt{h_t}}{R_t^f \sqrt{H_t}} \left(S_t^x - F_t^x\right)' G_t\left(F_T^x\right)^+ G_t\left(F_T^x, F_T\right) \\ &= Q_t(F_T) + \left(S_t^x - F_t^x\right)' G_t\left(F_T^x\right)^+ G_t\left(F_T^x, F_T\right), \end{aligned}$$

which is Equation 34. *Q.E.D.*

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