Competitive Equilibrium with Moral Hazard in Economies With Multiple Commodities.

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Abstract

We study an economy with competitive commodity markets and exclusive pairwise contractual relations with moral hazard, where both the principal and the agent can be risk averse. We show existence of equilibria and their generic constrained suboptimality, by means of a change in the compensation schemes. Such suboptimality occurs provided the number of commodities is sufficiently large relative to the number of states and pair types, and there are at least three future states of the world.

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1. Introduction

In this paper we embed the principal-agent model into the Arrow-Debreu framework of uncertainty and perfect competition, and show existence of equilibria. While previous attempts, for instance Prescott and Townsend (1984) and Bennardo (1996), consider contractual pairs in which risk-neutral principals design the contract but have no bargaining power, we study principal-agent pairs in which the principal has all the bargaining power, following the partial equilibrium tradition (see Grossman and Hart (1983), e.g.). Our assumption can be justified for example if, when contractual pairs are formed, it is too costly for the agents to switch to another principal if the contract conditions are judged to be unfavorable. In fact, it can be considered that in our economies contractual pairs are fixed from the beginning, and we study the effects of their interaction through competitive commodity markets. The case analyzed in the literature is polar opposite to ours, assuming that competition across principals leaves them with no surplus, i.e., zero expected profits. While their assumption translates into a zero expected profit condition for the principals, in our model principals offer utility-maximizing contracts, and both the principal and the agent can be risk averse: if the principal acts in her own interest (as an individually-owned firm), the assumption of risk neutrality is clearly restrictive in the space of preferences. As a consequence of our modeling strategy, the incentive and participation conditions must be directly incorporated into the principal maximization problem, and nonconvexities arise. We overcome these difficulties in a way that does not prevent the use of smooth analysis for the assessment of the welfare properties of equilibria, essentially extending the approach of Grossman and Hart (1983) to the general equilibrium setting. It should be noted that risk neutrality of the principal together with the zero profit condition substantially simplifies the existence problem, in essence reducing the search for competitive equilibria when some states are unobservable to appending some conditions to the usual incomplete markets equilibrium, and therefore totally bypasses the nonconvexity problems which arise instead in our framework.

As in the relevant literature, our economy exhibits a finite number of types of contractual pairs, but infinitely many ex ante identical individuals for each type. However, in each pair optimal contracts are deterministic, and there is no consumption randomization. The bilateral contract concerns sharing an uncertain output, and this uncertainty is idiosyncratic to the contractual pair; that is, output generates individual risk. To focus on the agency problem, we assume no risk in the aggregate. This assumption is totally innocuous, as it will be seen, and it is made only to simplify the exposition: provided the additional aggregate risk is independent from individual effort, it could be added to our economy. Although our model encompasses several examples of contractual relations under incomplete information (insurer-insuree, security designer-trader, lender-investor), we will focus on the traditional case of firms and workers, paired in independent production units. Output is then exchanged on the commodity markets, which are competitive in the sense that individuals take commodity prices as given. The price taking behavior depends on the assumption of atomistic individuals, whose strategic choice of effort does not affect aggregate statistics such as prices.

In our model, principals do not trade linear financial contracts. Introducing asset (insurance) markets would generate the same existence problems first highlighted by Helpman and Laffont (1975). Linear (and nonexclusive) contracts of this form are compatible with competitive equilibrium if asset trading constraints are imposed, as shown by Bisin and Gottardi (1999). Whether financial contracts are nonlinear or with trading constraints, the bottom line is that with moral hazard, asset markets will be incomplete. Hence we can expect the welfare consequences of the presence of asymmetric information to be complicated by the price effects in a model with multiple commodities and incomplete financial markets. We carry out the study of the welfare properties of our competitive equilibria under the assumption of no financial markets.

We study the welfare effects caused by changes in the principal's direct decision variable (e.g.: the compensation scheme, the insurance coverage, the asset payoffs), while leaving endowments fixed. In other words, we examine the welfare consequences directly attributable to the contractual imperfections. We show that the conclusions derived in a partial equilibrium framework, or with a single commodity (i.e. constrained optimality, see Prescott and Townsend (1984)), can be reversed with multiple goods and individuals' risk aversion, provided the price effects are predominant (that is, when the number of commodities and of states is sufficiently large). The intuition behind the result is all contained in the price effects that arise with incomplete financial markets when there are multiple commodities (see Geanakoplos and Polemarchakis (1986) and, for an extension of their results, Citanna, Kajii and Villanacci (1998)). In a competitive economy, the principal designs a contract without taking into account the equilibrium repercussions on prices. Hence there is room to gain efficiency by changing the compensation scheme. However, and as a difference with Geanakoplos and Polemarchakis, in our case constrained suboptimality arises even if traders are not able to transfer resources in any future state of the world, but only if the number of commodities is sufficiently large and if there are at least three future states of the world. This last condition is derived from counting the difference between controls (the compensation schemes) and objectives (the principals' and agents' utilities), once the further implicit incentive and rationality constraints are taken into account. Details are provided in Section 5. Our suboptimality result confirms previous findings of Bennardo (1996)[1] in the economies with zero-profit for the principals. While the intuition is common to both results (that is, price effects with multiple commodities are not taken into account by the principals), it should be noted that our notion of suboptimality cannot be applied in his zero-profit, risk-neutral scenario. That is, with zero-profit and risk neutral principals the suboptimality of equilibrium has to go through an *ad hoc* construction: the planner uses endowment redistributions with an additional sector immune from moral hazard and made up of risk averse individuals. Our intervention plan is more intuitive because based on the direct pegging of the contractual scheme.

2. Set-up of the model

There are H types of production units denoted by subscript h and a continuum of units for each type of Lebesgue measure normalized to one.[2] A production unit is a pair of a principal-firm and an agent-worker. There are C physical commodities, with $C > 1$, and the commodity space is $\mathbb{R}^C_{++}$.

If firm h and worker h engage in production activity, a vector of net output $y^{s_h}_h \in \mathbb{R}^C_{++}$ results, where $s_h = 1, ..., S_h < \infty$. In what follows, for simplicity of notation we will write $y^{s_h}_h = y^{s_h}$, all h. Let $S = \times_{h=1}^{H} S_h$. Production output is a publicly observable outcome[3].

Each worker can put in the production an unobservable effort $a_h \in A_h$ (a finite set of cardinality K), which influences the probability of different outcomes: $\pi_h^{s_h k}$ is the probability of output y^{s_h} if effort a^k_h is chosen by the worker. Worker h is endowed with one indivisible unit of labor, which can only be sold to firm h or consumed at home, obtaining a (reservation) utility level $\underline{V_h} \in \mathbb{R}$.

[1] Lisboa (1996) also analyzes a general equilibrium model with moral hazard in production along the lines of Bennardo, and shows suboptimality through endowment reallocations in equilibria when low effort is provided.

[2] Alternatively, one could think of our economies as the limit of finite economies as the number of individuals in each group tends to infinity. See comments below.

[3] For simplicity of interpretation, the reader may assume that $s_h \geq s'_h$ implies $y^{s_h} \geq y^{s'_h}$.

Each firm proposes a contract to the worker, that is a vector $w_h = (w_h^{s_h})_{s_h=1}^{S_h}$, such that if the individual state s_h is observed the wage $w_h^{s_h}$ is paid. We assume that wages are offered as baskets of commodities, instead of being expressed in nominal terms. As it is customary with the principal-agent literature, the labor market is modeled as a take-it-or-leave-it exchange of one unit of labor.

We are going to model output uncertainty at the firm level as individual risk, and no uncertainty will be derived at the aggregate level. This is accomplished as follows. All units of each type are assumed to be different only with respect to the output realization and the chosen level of effort. Hence they are ex ante identical. Therefore, aggregate states (as functions from $[0,1]^H$ into S) are equivalent if they correspond to the same frequencies of output levels for each type (an argument similar to Malinvaud's (1973))

Once we fix the proportion of units of type h who choose effort level k, denoted by θ_h^k, the frequency of output level s_h, $f_h^{s_h}$, is determined by

$$f_h^{s_h} = \sum_k \theta_h^k f_h^{s_h k}$$

where $f_h^{s_h k}$ is the frequency of output level s_h given effort level a_h^k. Note that for given effort level a_h^k, the probability $\pi_h^{s_h k}$ is also the frequency $f_h^{s_h k}$ as a consequence of the presence of a continuum of individuals for each type and of the Law of Large Numbers (see Uhlig (1996)). Therefore $f_h^{s_h k}$ is given as a primitive of the economy.[4]

Although θ_h^k will be determined in equilibrium, individuals take it as given, as they do with future spot commodity prices. This entails a stronger notion of rational expectations. Therefore, from the individual viewpoint the frequency $f_h^{s_h}$ is also given and unique. This is equivalent to the absence of aggregate risk in this economy. Although independent aggregate risk could be added to the specification of uncertainty faced by individuals, this would only complicate the notation without adding anything substantial to the analysis. Hence we will assume no further independent aggregate risk in the economy.

[4]Note also that using a continuum of agents in each group guarantees consistency of price-taking behavior on the commodity markets, but does not yield proper market clearing conditions, effective at every realization of uncertainty, but only on (L^2-) average. The interpretation à la Malinvaud based on limit economies would result in effective market clearing, since in that case we could use Kolmogorov's Strong Law of Large Numbers. However, in either way any large but finite economy would only have approximate market clearing, so we use the continuum of agents to make our model similar to others used in the literature.

For simplicity of exposition we assume $K = 2$, so that $A_h = \{a_h^1, a_h^2\} \subseteq \mathbb{R}^2_{++}$. However, in what follows we will sometimes keep the use of K to denote the number of efforts, and this to help the reader identify where the dimensionality of vectors comes from. Define $A \equiv \times_{h=1}^H A_h$, with generic element a, and $\left(\underline{V_h}\right)_{h=1}^H \equiv V$. Firm h is endowed with a sure vector of goods $e_h \in \mathbb{R}^C_+$ and an uncertain vector of production $y_h \in \mathbb{R}^{CS_h}_{++}$.

Let $\pi_h^k = \left(\pi_h^{s_h k}\right)_{s_h=1}^{S_h}$, and $\pi_h = \left(\pi_h{}^k\right)_{k=1}^K$. Moreover, for $h = 1, ..., H$, denote by $x_{ih}^{s_h c} \in \mathbb{R}_{++}$ the consumption in individual state s_h of good c by firm h (when $i = f$) and worker h (when $i = w$). Define also $(x_{ih}^{s_h c})_{c=1}^C = x_{ih}^{s_h}$, $(x_{ih}^{s_h})_{s_h} = x_{ih}$, for $i = f, w$ and $(x_{fh}, x_{wh})_{h=1}^H = x$. Finally, the price of good c is denoted by $p^c \in \mathbb{R}_{++}$ and $(p^c)_{c=1}^C = p$.

We introduce the basic assumptions of the model. First, we impose standard restrictions on probabilities, that is, first order stochastic dominance of the high-level effort over the low-level.

Assumption 1 (*stochastic dominance*) For any $k = 1, 2$, π_h^k belongs to the open $S_h - 1$-dimensional simplex $\mathcal{S}_h = \left\{\pi \in \mathbb{R}^{S_h}_{++} : \sum_{s=1}^{S_h} \pi^s = 1\right\}$ and $\pi_h \in \mathcal{S}_h \times \mathcal{S}_h$ is such that, if $a^{k'} \geq a^k$,

$$\sum_{s_h \leq \overline{s}} \pi_h^{s_h k} \geq \sum_{s_h \leq \overline{s}} \pi_h^{s_h k'}$$

for all $\overline{s} \in S_h$, with strict inequality for some $\overline{s}$.

Let Π_h be the set of such vectors π_h. Define $\Pi = \times_{h=1}^H \Pi_h$. We then assume that firms and workers' utility functions are constant across all output realizations, and satisfy standard smooth assumptions.

Assumption 2 (*risk-averse principal*)[5] The utility function for firm h is $U_h : A_h \times \mathbb{R}^{S_h C}_{++} \to \mathbb{R}$, where

[5] With only minor changes in the model (namely, allowing for $\mathbb{R}^C_+$ as the consumption space and assuming simple differential concavity of the utility functions, defined without the boundary condition and dropping the unboundedness-from-below assumption) we could accomodate for risk neutrality and still show existence. The test economy that we selected is still valid. Since the extension is trivial and only makes the notation cumbersome, we prefer to concentrate on the case of strict convexity, i.e., of risk aversion for both the principal and the agent. However, the reader can safely apply the existence results to the risk-neutral case.

$$U_h : \left(a_h^k, x_{fh}\right) \mapsto \sum_{s_h} \pi_h^{s_h k} \cdot u_h\left(x_{fh}^{s_h}\right) \equiv U_h^k,$$

and where $u_h : \mathbb{R}_{++}^C \to \mathbb{R}$ is $\mathcal{C}^2$, differentially strictly increasing, differentially strictly concave, with closure of indifference surfaces contained in $\mathbb{R}_{++}^C$. In addition, $\lim_{x\to 0} u_h(x) = -\infty$.

Call $\mathcal{U}_h$ the set of the above defined utility functions u_h and also $\mathcal{U} = \times_h \mathcal{U}_h$. Endow it with the $\mathcal{C}^2$ compact-open topology.

Assumption 3 (*risk-averse agent*) The utility function for worker h is $V_h : A_h \times \mathbb{R}_{++}^{S_h C} \to \mathbb{R}$, where

$$V_h : \left(a_h^k, x_{wh}\right) \mapsto \sum_{s_h} \pi_h^{s_h k} \cdot v_h\left(x_{wh}^{s_h}\right) - a_h^k \equiv V_h^k,$$

and where $v_h : \mathbb{R}_{++}^C \to \mathbb{R}$ has the same properties as u_h.

Call $\mathcal{V}_h$ the set of the above defined utility functions v_h and also $\mathcal{V} = \times_h \mathcal{V}_h$. Following the partial equilibrium literature, we assume absence of risk sharing opportunities other than the bilateral contract.

Assumption 4 (*no assets*) There are no financial instruments.[6]

The compensation scheme is real, in the sense of specifying the right to a proportion of the commodity bundle owned by the firm. It is based on verifiable information (the output states), and we require that it satisfy limited liability.

Assumption 5 (*limited liability*) The wage offered by each firm is a proportion of the value of the firm's assets, i.e.

$$w_h^{s_h} = \omega_h^{s_h} p\left(e_h + y^{s_h}\right)$$

with $0 \le \omega_h^{s_h} \le 1$, all s_h.

[6] This assumption means absence of linear financial contracts, and it is not in contrast with the insurance interpretation of the main contract. It could be disposed of in a model where bounds are imposed on asset trading (see Bisin and Gottardi (1999)), and where only principals trade assets. Of course, additional nonlinear contracts could easily be added to the main contractual relation.

The timing of the model is simple. First, each pair h of firm and worker exchange a contract (w_h, a_h). Then the worker chooses a level of effort a_h, production takes place and an individual state s_h for each individual arises. Owners of firms and workers go to the market with their share of production and/or endowments and exchange them at the prevailing price.

The objective of each worker is to choose an effort, and then buy goods in order to maximize his utility. The objective of each firm is to choose a contract and the effort to be given by the worker, and then buy goods in order to maximize its utility.

Worker h 's maximization problem is

$$\begin{array}{l} \max_{a_h^k, x_{wh}} V_h\left(a_h^k, x_{wh}\right) \ s.t. \\ -px_{wh}^{s_h} + \omega_h^{s_h} p\left(e_h + y^{s_h}\right) \geq 0 \ \text{ for all } s_h \end{array} \tag{2.1}$$

for given p, y, e_h and ω_h. Firm h ' s maximization problem is

$$\begin{array}{ll} & \max_{a_h^k, x_{wh}, \omega_h, x_{fh}} U_h\left(a_h^k, x_{fh}\right) \ s.t. \\ (1) & -px_{fh}^{s_h} + \left(1-\omega_h^{s_h}\right) p\left(y^{s_h} + e_h\right) \geq 0\ , \quad \text{for all } s_h, \\ (2) & 0 \leq \omega_h^{s_h} \leq 1, \ \text{for all } s_h \\ (3) & \left(a_h^k, x_{wh}\right) = \arg\max \ V_h\left(\widetilde{a}_h, \widetilde{x}_{wh}\right) \ \ s.t. \ -p\widetilde{x}_{wh}^{s_h} + \omega_h^{s_h} p\left(e_h + y^{s_h}\right) \geq 0 \\ & \text{for all } s_h \\ (4) & V_h\left(a_h^k, x_{wh}\right) - \underline{V}_h \geq 0. \end{array} \tag{2.2}$$

for given p, y, e_h. Constraints (3) and (4) are the incentive and participation constraint, respectively. Observe that at this stage the problem of the worker is solved also by the firm.

We need to make the individual effort choice consistent with the definition of θ_h^k. First, let $\theta_h = (\theta_h^k)_{k=1}^K$ and let $\theta = (\theta_h)_{h=1}^H$. Since θ_h^k represents the proportion of individuals choosing effort k, and it is an endogenous quantity, an equilibrium must require that θ_h^k be equal to one (zero) if the corresponding effort dominates (is dominated by) the other at given prices, and that θ_h^k be in between only if both efforts are utility maximizers for each firm in unit h.

Hence for each type h we first split problems 2.1 and 2.2 into K maximization problems

$$\max_{\left(x_{wh}^k\right)} \ \sum_{s_h} \pi_h^{s_h k} \cdot v_h\left(x_{wh}^{s_h k}\right) \tag{2.3}$$

subject to the constraint in problem 2.1 where choice variables are indexed by k, for given p, y_h, e_h and ω_h^k, and similarly we write

$$\max_{\left(x_{fh}^k, \omega_h^k, x_{wh}^k\right)} \quad \sum_{s_h} \pi_h^{s_h k} \cdot u_h\left(x_{fh}^{s_h k}\right) \tag{2.4}$$

subject to the constraints (2.2.1) through (2.2.4) with choice variables again indexed by k and for given p, y_h, e_h, where the superscript k denotes the choice conditional on the effort level $k = 1, 2$. This is coherent with the standard natural way of solving problem 2.2, that is (i) to solve it with respect to (ω_h, x_{fh}) *for fixed* a_h^k, and (ii) to choose the vector $\left(a_h^k, \omega_h^k, x_{fh}^k\right)$ which gives the highest value of the objective function. Note that the principal offers only one contract, contingent on the observable state s_h, and that the solution technique does not correspond to any substantial sequential decision of the principal.

Letting $x_h^{s_h k} = x_{fh}^{s_h k} + x_{wh}^{s_h k}$, the market clearing condition is

$$\sum_h \sum_k \theta_h^k \left(\sum_{s_h} \pi_h^{s_h k} \left(x_h^{s_h k} - y^{s_h} - e_h \right) \right) = 0 \tag{2.5}$$

Because of homogeneity of degree zero of the budget constraints, we can normalize prices setting the price of the last good equal to one. An economy will be identified by an n-tuple

$$(e, y, a, \pi, u, v, \underline{V}) \in \mathbb{R}_+^{HC} \times \mathbb{R}_{++}^{(\sum_h S_h)C} \times A \times \Pi \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}^H$$

of endowments, outputs, effort levels, probabilities, utilities and reservation levels.

Definition 2.1. $(x, \omega, \theta, p) \in \mathbb{R}_{++}^{2(\sum_h S_h)CK} \times \mathbb{R}^{(\sum_h S_h)K} \times \mathbb{R}^{HK} \times \mathbb{R}_{++}^{C-1}$ *is an equilibrium associated to the economy* $(e, y, a, \pi, u, v, \underline{V})$ *if and only if*

1) (optimization and market clearing) for all h, $\left(x_{wh}^k, \omega_h^k, x_{fh}^k\right)$ *solves problems (2.3) and (2.4) for each* $k = 1, 2$ *and* x *satisfies (2.5), and;*

2) (rational expectations)

$$\sum_{s_h} \pi_h^{s_h 2} \cdot u_h\left(x_{fh}^{s_h 2}\right) > (<) \sum_{s_h} \pi_h^{s_h 1} \cdot u_h\left(x_{fh}^{s_h 1}\right) \Rightarrow \theta_h^2 = 1 (= 0)$$

and $0 \leq \theta_h^2 \leq 1$ *otherwise, with* $\sum_k \theta_h^k = 1$, *for all* h.

Before proving existence, we need to add assumptions that make the economic problem interesting, by ruling out the case of empty constraint sets. In particular, we want to make sure that the agent can be asked to participate. Given that we are using a differentiable approach to existence, we will make assumptions to further require that this can be done by staying in the interior of the constraint set. This is so accomplished.

Assumption 6 For any h, let $\Gamma_h \subset \mathbb{R}_{+}^{C} \times \mathbb{R}_{++}^{CS_h} \times A_h \times \Pi_h \times \mathcal{V}_h \times \mathbb{R}$ be the set of $\gamma_h \equiv (e_h, y, a_h, \pi_h, v_h, \underline{V}_h)$ such that

$$v_h\left(y^{s_h} + e_h\right) > \frac{1}{\min_{s:\left|\pi_h^{s1} - \pi_h^{s2}\right| > 0} \left|\pi_h^{s1} - \pi_h^{s2}\right|} \max\left\{a_h^1 + \underline{V}_h, a_h^2 + \underline{V}_h, \left|a_h^2 - a_h^1\right|\right\},$$

all s_h.

Assumption 6 in essence says that in any state s_h the agent's utility can be set high enough with the given resources of the firm, yet without fully utilizing them.

We will use $\Gamma = \times_h \Gamma_h$ and $\mathcal{U}$ as our parameter spaces. Define it as $\quad = \Gamma \times \mathcal{U}$. In order to apply degree theory, we will need the following property for the set Γ_h.

Lemma 2.2. Γ_h *is nonempty and path-connected, all* h.

This lemma is technical, although straightforward, and its proof notationally cumbersome, hence it is omitted and is available from the authors upon request. An immediate consequence of Lemma 2.2 is the following result.

Corollary 2.3. *is nonempty and path-connected.*

Proof. $\mathcal{U}$ is path-connected (see Smale (1974)). Path-connectedness of Γ follows from Lemma 2.2. Nonemptiness also is either trivial or follows from the same lemma. ■

We move now to the analysis of the maximization problems in greater detail.

3. The two maximization problems

It is pretty straightforward to see that problems 2.1 and 2.2 can be solved using the method first suggested by Grossman and Hart (1983). Consider the problem, for given p, y, e_h and ω_h,

$$\begin{array}{l} \max_{x_{wh}^{s_h}} v_h\left(x_{wh}^{s_h}\right) \ s.t. \\ -px_{wh}^{s_h} + \omega_h^{s_h} p\left(y^{s_h} + e_h\right) \geq 0. \end{array} \tag{3.1}$$

Define the smooth value function to the above problem as $\widetilde{v}_h\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right)$. It is obvious that $\left(a_h^{k*}, x_{wh}^*\right)$ solves 2.1 if and only if $x_{wh}^{s_h*}$ solves 3.1 for all s_h, and a_h^{k*} solves

$$\max_{a_h^k} \sum_{s_h} \pi^{s_h k} \cdot \widetilde{v}_h\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right) - a_h^k.$$

Also from basic consumer theory we know that, for any $\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right)$, $D_{\omega^{s_h}}\widetilde{v}_h > 0$ and $D^2_{\omega^{s_h}}\widetilde{v}_h < 0$. Fix a vector y. Because of strict monotonicity of $\widetilde{v}_h$, for any given p the restriction of $\widetilde{v}_h\left(\omega_h^{s_h}; p, y^{s_h}, e_h\right)$ at this commodity price vector admits (smooth) inverse, say $\phi_h\left(.; p, y^{s_h}, e_h\right)$. Define $z_h^{s_h} = \widetilde{v}_h\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right)$; then $\phi_h\left(z_h^{s_h}; p, y^{s_h}, e_h\right) = \omega_h^{s_h}$. Since $D_{\omega^{s_h}}\widetilde{v}_h > 0$ and $D^2_{\omega^{s_h}}\widetilde{v}_h < 0$, then $D_{z^{s_h}}\phi_h = \frac{1}{D_{\omega^{s_h}}\widetilde{v}_h} > 0$ and $D^2_{z^{s_h}}\phi_h = -\frac{D^2_{\omega^{s_h}}\widetilde{v}_h}{D_{\omega^{s_h}}\widetilde{v}_h} > 0$. Here we are heavily exploiting both utility state separability and the exclusive nature of the contract.

Using the above remarks, we have that $\left(a_h^k, x_{wh}, \omega_h, x_{fh}\right)$ is a solution to problems 2.1 and 2.2 iff for any s_h, $x_{wh}^{s_h}$ solves problem 3.1, and $\left(a_h^k, \omega_h, x_{fh}\right)$ solves

$$\begin{array}{l} \max_{a_h^k, \omega_h, x_{fh}} U_h\left(a_h^k, x_{fh}\right) \ s.t. \ (1), \ (2) \text{ and} \\ (3) \ \sum_{s_h} \pi_h^{s_h k} \cdot \widetilde{v}_h\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right) - a_h^k \geq \\ \sum_{s_h} \pi_h^{s_h k'} \cdot \widetilde{v}_h\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right) - a_h^{k'}, \text{ all } k, k' \in K \\ (4) \ \sum_{s_h} \pi^{s_h k} \cdot \widetilde{v}_h\left(\omega_h^{s_h}, p, y^{s_h}, e_h\right) - a_h^k - \underline{V}_h \geq 0. \end{array} \tag{3.2}$$

for given p, y, e_h and $\tilde{v}_h$. Observe that (3) implies that the firm's choice of a_h is the one carried out by the worker.

After substituting everywhere $\phi_h\left(z_h^{s_h}; p, y^{s_h}, e_h\right)$ for $\omega_h^{s_h}$ in (1) through (4), and after splitting problem 3.2 according to what previously done in Section 2, we can rewrite problems 3.1 and 3.2 as

$$\begin{array}{l} \max_{x_{wh}^{s_h k}} v_h\left(x_{wh}^{s_h k}\right) \ s.t. \\ -px_{wh}^{s_h k} + \phi_h\left(z_h^{s_h k}; p, y^{s_h}, e\right) p\left(e_h + y^{s_h}\right) \geq 0. \end{array} \tag{3.3}$$

for a given p, y, e_h and z_h^k, $k = 1, 2$, and

$$\begin{array}{l} \max_{k\in\{1,2\}} \max_{z_h^k, x_{fh}^k} \; U_h\left(a_h^k, x_{fh}^k\right) \; s.t. \\ (1) \; -px_{fh}^{s_h k} + \left[1 - \phi_h\left(z_h^{s_h k}; p, y^{s_h}, e\right)\right] p\left(e_h + y^{s_h}\right) \geq 0 \;\text{ for any } s_h, \\ (2) \; 0 \leq \phi_h\left(z_h^{s_h k}; p, y^{s_h}, e\right) \leq 1, \;\text{ all } s_h \\ (3) \; \sum_{s_h}\left[\pi_h^{s_h k} - \pi_h^{s_h k'}\right] \cdot z_h^{s_h k} - a_h^k + a_h^{k'} \geq 0, \;\text{ all } k' \neq k \\ (4) \; \sum_{s_h} \pi_h^{s_h k} \cdot z_h^{s_h k} - a_h^k - \underline{V}_h \geq 0 \end{array} \tag{3.4}$$

for given p, y, e_h and ϕ_h. Let $B_h(p, e_h, y, \phi_h, a_h^k)$ be firm h's constraint set given that a_h^k is the chosen effort, for $k = 1, 2$, that is constraints (1) through (4), given a_h^k. We consider for given a_h^k, p and ϕ_h

$$\begin{array}{l} \max_{z_h^k, x_{fh}^k} \; U_h\left(a_h^k, x_{fh}^k\right) \; s.t. \\ z_h^k, x_{fh}^k \in B_h(p, e_h, y, \phi_h, a_h^k) \end{array} \tag{3.5}$$

A first intermediate step in proving our existence theorem is to show that $B_h(.)$ has a nonempty interior, for all k. This is established in the following lemma, which heavily relies upon Assumption 6.

Lemma 3.1. *Under the maintained assumptions, $B_h(p, e_h, y, \phi_h, a_h^k)$ has a nonempty interior, for all k.*

Proof. See the Appendix.

A consequence of Lemma 3.1 is that for any $p, y^{s_h}, e_h, \phi_h, a_h^k$, there exists $\left(\underline{x}_{fh}^k, \underline{z}_h^k\right)$ in $B_h(p, e, y, \phi_h, a_h^k)$ such that $\underline{x}_{fh}^k \gg 0$ (with corresponding $\underline{\omega}^{s_h k} \in (0, 1)$). Therefore, if a solution $\left(x_{fh}^{k*}, z_h^{k*}\right)$ to 3.5 exists, it must be the case that for $k = 1, 2$, and for any s_h, $u_h\left(x_{fh}^{s_h k*}\right) > u_h\left(\underline{x}_{fh}^{s_h k}\right) > -\infty$. Define $\underline{u}_{fh}\left(p, y^{s_h}, e_h, ...,\right) \equiv \min_k \left\{u_h\left(\underline{x}_{fh}^k\right)\right\}$. This condition will be used to discard constraint (2) in problem 3.5, and later in proving Lemma 4.4 (indeed, it amounts to a useful device to compactify problem 3.5).

Given our assumptions, if a solution to the programming problem 3.4 exists, it is unique, and for each k, the same is true for problem 3.5. Given Lemma 3.1, the only hurdle to using the Kuhn-Tucker conditions to characterize these solutions is constraint (2), which we now eliminate.

Lemma 3.2. *For any admissible p, y, e_h, (x, z) is a solution to problem 3.5 only if $0 < \phi\left(z_h^{s_h}, p, y^{s_h}, e_h\right) < 1$, for any s_h.*

Proof. See the Appendix.

Problem 3.5 satisfies both necessary and sufficient conditions to apply Kuhn-Tucker Theorem (see Mangasarian (1969), e.g.): all constraints and the objective function are pseudoconcave and the constraint set has nonempty interior from Lemmas 3.1 and 3.2.

We conclude this section establishing regularity of the solution to problem 3.5, all k.

Lemma 3.3. *The solution to problem 3.5 is a smooth function of the parameters, outside borderline cases.*

Proof. See the Appendix.

4. Existence of an equilibrium

We want to show existence of an equilibrium by using the associated system of Kuhn-Tucker and market clearing equations. For ease of notation, we set

$$f_{1h}^{s_h k}(.) = -p x_{fh}^{s_h k} + \left(1 - \phi_h\left(z_h^{s_h k}; .\right)\right) p(y^{s_h} + e_h)$$

$$f_{2h}^{k}(.) = \sum_{s_h} (\pi_h^{s_h k} - \pi_h^{s_h k'}) z^{s_h k} - a_h^k + a_h^{k'}$$

$$f_{3h}^{k}(.) = \sum_{s_h} \pi_h^{s_h k} z^{s_h k} - a_h^k - \underline{V}_h$$

$$g_h^{s_h k}(.) = -p x_{wh}^{s_h k} + \phi_h\left(z_h^{s_h k}; p, y^{s_h}, e_h\right) p(y^{s_h} + e_h)$$

and call $\alpha_h^{s_h k}$ the multiplier associated with $f_{1h}^{s_h k}$, β_h^k the multiplier associated with f_{3h}^k, $\delta_h^k \geq 0$ the multiplier associated with f_{2h}^k, and finally $\underline{\eta}_h$ and $\overline{\eta}^h$ the multipliers associated with the constraints $\theta_h^2 \geq 0$ and $1-\theta_h^2 \geq 0$, respectively. As we observed in the proof of Lemma 3.3, both $\alpha_h^{s_h k} > 0$ and $f_{1h}^{s_h k} = 0$, and $\beta_h^k > 0$ and $f_{3h}^k = 0$, all s_h, k, h. Also, let $\rho_h^{s_h k}$ be the multiplier associated with the constraint $g_h^{s_h k}$ in 3.3, again with $\rho_h^{s_h k} > 0$ and $g_h^{s_h k} = 0$. Using the first order conditions for the two problems, an equilibrium must be a solution to the following system of equations:

$$
\begin{aligned}
&\pi_h^{s_h k} Du_h\left(x_{fh}^{s_h k}\right) - \alpha_h^{s_h k} p = 0, \text{ all } s_h, k, h && (1)\\
&-\alpha_h^{s_h k} D\phi_h\left(z^{s_h k};.\right) p(y^{s_h} + e_h) + \beta_h^k \pi_h^{s_h k} + \delta_h^k(\pi_h^{s_h k} - \pi_h^{s_h k'}) = 0, \text{ all } s_h, k, h && (2)\\
&U_h^2 - U_h^1 + \underline{\eta}_h - \overline{\eta}^h = 0, \text{ all } h && (3)\\
&\min\left\{\underline{\eta}_h, \theta_h^2\right\} = 0, \text{ all } h && (4)\\
&\min\left\{\overline{\eta}^h, 1 - \theta_h^2\right\} = 0, \text{ all } h && (5)\\
&f_{1h}^{s_h k}(.) = 0, \text{ all } s_h, k, h && (6)\\
&\min\left\{\delta_h^k, f_{2h}^k(.)\right\} = 0, \text{ all } k, h && (7)\\
&f_{3h}^k(.) = 0, \text{ all } k, h && (8)\\
&\pi_h^{s_h k} Dv_h\left(x_{wh}^{s_h k}\right) - \rho_h^{s_h k} p = 0, \text{ all } s_h, k, h && (9)\\
&g_h^{s_h k}(.) = 0, \text{ all } s_h, k, h && (10)\\
&\textstyle\sum_h [(1-\theta_h^2) \sum_{s_h} \pi_h^{s_h 1}\left(x_h^{s_h 1\backslash} - y^{s_h\backslash} - e_h^{\backslash}\right) + \theta_h^2 \sum_{s_h} \pi_h^{s_h 2}\left(x_h^{s_h 2\backslash} - y^{s_h\backslash} - e_h^{\backslash}\right)] = 0 && (11)
\end{aligned}
\tag{4.1}
$$

Observe that in (4.1) we have deleted the last market clearing equation, corresponding to the numeraire commodity C, using Walras law. In order to find a solution to system (4.1) we apply a degree theorem (see Lloyd (1978)). Let $F_{\gamma,u} : \Xi \longrightarrow \mathbb{R}^{\nu}$, with $\nu = \dim \Xi$, be the function defining the equations in system (4.1). Here Ξ is the space of "endogenous" variables, which is a product of open spaces, or manifolds without boundaries. Note that we have actually parametrized a family of such functions, and by Corollary 2.3 we can build a continuous homotopy $H : \Xi \times [0,1] \longrightarrow \mathbb{R}^{\nu}$ between any two such functions, that is between any two economies $(\gamma, u), (\gamma', u') \in$. In order for $\deg_2 F_{\gamma,u}$ to be well defined at the value zero for these functions, we prove that the set $H^{-1}(0)$ is compact. We then can construct a test economy (γ^*, u^*) for the case of $S_h = S = 2$ for all h, with a unique equilibrium (x, z, θ, p), and show that the test economy is regular, that is, F_{γ^*,u^*} is continuously differentiable at least locally around the unique point ξ such that $F_{\gamma^*,u^*}(\xi) = 0$, and with full-ranked derivative $D_\xi F_{\gamma^*,u^*}$. The application of the degree theorem is completed after constructing another homotopy between the function defining the equilibrium for an economy with an arbitrary (finite) number of states and a related economy with two states.

These facts will lead to the key statement of the paper.

Theorem 4.1. *For any economy $(\gamma, u) \in$, an equilibrium exists.*

The proof follows from the four subsequent lemmas, which establish the properties we just summarized.

We build the test economy. Although the test economy is simplified and is based on the assumption that $S_h = S = 2$, it captures the fundamental elements of the model and will be used to extend the existence result to any finite S_h (Lemma 4.5). Assuming $S = 2$ allows us to easily show regularity, especially regarding the incentive and participation constraint equations (see Lemma 4.3).

Lemma 4.2. *There exists an economy* (γ^*, u^*) *with a unique associated equilibrium* $\xi^* = (x, z, \theta, p)$.

Proof. Recall that $\gamma_h = (e_h, y, a_h, \pi_h, v_h, \underline{V}_h)$. Assume without loss of generality that $S_h = S$, all h. Moreover, take $S = 2$. For all h, take $u_h(x_{fh}) = v_h(x_{wh}) = \frac{1}{C}\sum_{c=1}^{C} \log x^c$. Also, take $e_h = \underline{e}_h \cdot \mathbf{1}$, with $\underline{e}_h > 0$, as the endowments**,** and for all h and all s take $y_h^s = \underline{\widetilde{y}}^s \cdot \mathbf{1}$, with $\underline{\widetilde{y}}^s > 0$. Define $\underline{y}^s = \underline{\widetilde{y}}^s + \underline{e}_h$. Choose $\underline{e}_h$ and $\underline{\widetilde{y}}^s$ such that $\underline{y}^{\bar{s}} = 100$ and $\underline{y}^{\tilde{s}} = 1000$, for the states $\bar{s}$ and $\tilde{s}$. Take

$$a_h^1 = \log 1 = 0, \text{ and } a_h^2 = \log e = 1.$$

to be the effort levels for all h. As probabilities, take

$$\pi_h^{\bar{s}1} = \frac{2}{3}, \pi_h^{\bar{s}2} = \frac{1}{3}$$

Finally, set $\underline{V}_h = \log 1 = 0$, for all h.

Now we show that there is a unique equilibrium for the above-chosen economy (γ, u). Consider the associated standard exchange economy with individuals having initial allocation $y_h^s + e_h$, all s. Under the assumed separability of the utility functions and the absence of financial markets, their state-by-state maximization problem is

$$\max_{x_h} \frac{1}{C}\sum_{c=1}^{C} \log x_h^{s,c} \quad s.t. \quad p\left(x_h^s - y_h^s - e_h\right) = 0$$

Let μ_h^s be the associated Lagrange multiplier. The first order conditions give

$$\mu_h^s = \frac{1}{p(y_h^s + e_h)} \text{ and } x_h^{s,c} = \frac{p(y_h^s + e_h)}{Cp^c}.$$

Observe that if $p = \mathbf{1}$, then

$$\mu_h^s = \frac{1}{C\underline{y}_h^s}, \quad x_h^{s,c} = \frac{C\underline{y}_h^s}{C} = \underline{y^s}_h, \quad u_h\left(x_h^s\right) = \log \underline{y}_h^s.$$

$[(x_h^s = \underline{y}_h^s \cdot \mathbf{1},\ \mu_h^s = \frac{1}{C\underline{y}_h^s})_h,\ p = \mathbf{1}]$ is a no-trade walrasian equilibrium, therefore it is the unique equilibrium associated to the (full information) Pareto optimal allocation $\left(y_h^s + e_h\right)_{s,h}$. In other words, any initial allocation which is equal across commodities for each household is a Pareto optimal allocation with associated unique equilibrium price equal to $\mathbf{1}$. Hence, in these economies any sharing of the initial allocation is also Pareto optimal. This is the key observation. Take

$$x_{wh}^{sk} = \omega_h^{sk}\underline{y}^s \cdot \mathbf{1} \text{ and } x_{fh}^{sk} = \left(1 - \omega_h^{sk}\right)\underline{y}^s \cdot \mathbf{1}.$$

and let $p = \mathbf{1}$. Once ω_h^{sk} is given and unique, the uniqueness of equilibrium prices and allocations follows from a standard argument: there cannot be an equilibrium vector of consumption other than the one just computed because, due to the strict concavity of the utility function, this other vector of consumption would violate Pareto optimality; but if that is the case, then prices and multipliers are unique, too.

Once ω_h^{sk} is given, it is immediately shown that equations (4.1.6) and (4.1.10) are satisfied. If also $\theta_h^2 = 1$, as we will show, equation (4.1.11) is automatically satisfied. Now letting $\rho_h^{sk} = \frac{\pi_h^{sk}}{C\omega_h^{sk}\underline{y}^s}$, even equation (4.1.9) is satisfied, mirroring the first order condition in the walrasian equilibrium for each state s, and for each k. Similarly, equation (4.1.1) is satisfied once we set $\alpha_h^{sk} = \pi_h^{sk} \cdot D_{s,k,C}u_h\left(x_{fh}^{sk}\right)$. Observe that we can uniquely determine $z_h^{sk} = \log \omega_h^{sk}\underline{y}^s$.

We are therefore left with checking existence and uniqueness of ω_h^{sk}, for all s, k and h, satisfying system (4.1), and in particular the remaining equations (4.1.2) through (4.1.5), and equations (4.1.7) and (4.1.8), and with showing that $\theta_h^2 = 1$, all h. Note that this entails determining uniquely also the remaining variables β, δ and η (all indexes are dropped). Using the envelope condition,

$$\frac{\partial \widetilde{v}_h}{\partial \omega_h^{sk}}\left(\omega_h^{sk}; .\right) = \frac{1}{\omega_h^{s_h k}} \tag{4.2}$$

Moreover, $u_h\left(x_{fh}^{sk}\right) = \log\left(1 - \omega_h^{sk}\right)\underline{y}^s$ and

$$D_{s,k,C}u_h\left(x_{fh}^{sk}\right) = \mu_{fh}^{sk} = \frac{1}{C\left(1 - \omega_h^{sk}\right)\underline{y}^s}. \tag{4.3}$$

Substituting for α_h^{sk} in equation (4.1.2) and using (4.2) and (4.3), we get

$$\pi_h^{sk}\beta_h^k + \delta_h^k(\pi_h^{sk} - \pi_h^{sk'}) = \pi_h^{sk} \cdot \frac{\omega_h^{sk}}{1 - \omega_h^{sk}} \tag{4.4}$$

for $k = 1, 2$ and with $k' \neq k$. Therefore, dropped the subscript h, the main goal now is to show that there is a unique solution $\left(\beta^k, \delta^k, \omega^{1k}, ..., \omega^{Sk}\right)_{k=1,2}$ with $\omega^{sk} \in (0, 1)$ all s, k, to the system of equations: (4.4), for all s, k, and (4.1.7), (4.1.8).

If, as we will show, $U_h^1 < U_h^2$, then, from (4.1.3) we get $\underline{\eta}_h - \overline{\eta}^h < 0$; from (4.1.4) it follows that $\overline{\eta}^h > \underline{\eta}_h \geq 0$, and, finally, from (4.1.5), we get that $\theta_h^2 = 1$, all h, as desired. Now from equation (4.1.4), the only solution will be $\underline{\eta}_h = 0$. Equation (4.1.3) is then an equation in one unknown, $\overline{\eta}_h$, clearly with a unique solution, different from zero. A sketch of the computations is provided in the Appendix. ■

Differentiability and full rank of the derivative with respect to the endogenous variables now follow from the observation that the derivative of our system is essentially equivalent, after a change of basis, to the derivative of a standard incomplete markets economy, with no financial markets.

Lemma 4.3. *(γ^*, u^*) is a regular economy.*

Proof. See the Appendix.

It is maintained that $S_h \geq 2$ for all h.

Lemma 4.4. *The set $H^{-1}(0)$ is compact.*

Proof. See the Appendix.

Finally, here is the result that links economies with $S_h = 2$ to any finite S_h, all h.

Lemma 4.5. *There is a continuous homotopy κ from economies with $S_h = 2$ and economies with $S_h > 2$ (and finite), all h. The set $\kappa^{-1}(0)$ is compact, and the economy (γ^*, u^*) is still a regular test economy.*

Proof. See the Appendix.

5. Constrained suboptimality

In this section we establish the generic constrained suboptimality of the equilibrium allocations. In particular, we consider a central authority who could choose $z_h^{s_h k}$ (which is tantamount to choosing $\omega_h^{s_h k}$) in lieu of the principals.[7] Let $\overline{z}_h^{s_h k}$ be the control variables, and $\overline{z} \in \mathbb{R}^{KSH}$ the corresponding vector. With a slight modification from Citanna, Kajii and Villanacci (1998), the possibility of Pareto improving over the equilibrium outcome occurs if the following system of equations has no solution

$$\begin{array}{l} F(\xi; \gamma, u) = 0 \\ c^T \begin{bmatrix} D\tilde{F} \\ DG \end{bmatrix} = 0 \\ c^T c - 1 = 0 \end{array} \tag{5.1}$$

where: a) $\tilde{F}(\tilde{\xi}, \overline{z}; z, \beta, \gamma, u) = 0$ represents system (4.1) without equations (2) and (8) , with $\tilde{\xi} = \xi \backslash (z, \beta)$, while z is kept at the equilibrium level, that is, z solves $F(\xi; \gamma, u) = 0$ (i.e., (4.1)), and everywhere in (4.1) we substitute for $z_h^{s_h k} + \overline{z}_h^{s_h k}$ whenever only $z_h^{s_h k}$ appeared; b) $G(\xi)$, the utility function vector, is given by

$$G(\xi) = [..., \sum_k \theta_h^k U_h^k, \pi_h^1 (z_h^1 + \overline{z}_h^1), \pi_h^2 (z_h^2 + \overline{z}_h^2), ...];$$

c) the derivatives are taken with respect to $\tilde{\xi}$ and $\overline{z}$, and; d) c is a vector of real coefficients. For $\overline{z} = 0$, $\tilde{F}(\tilde{\xi}, \overline{z}; z, \beta, \gamma, u) = 0$ and equation (4.1.2) and (4.1.8) are equivalent to $F(\xi; \gamma, u) = 0$. If we kept the participation constraint, overall the agent would not be affected by the change in $\overline{z}$. This way we could only care to show that the principals can be made better off. However, we drop the participation constraint and make sure that the agents, as well as the principals, can be made better off.

Evidently, system (5.1) is well defined if $\tilde{F}$ is differentiable, at least locally around an equilibrium, which is true if there are no corner solutions at the minimum functions in (4.1). We will first assume that to be the case, and show that this restriction is weakly generic in the space of parameters. As usual, a property is weakly generic if it holds in an open and dense subset of the parameter space.

[7] It is clear that the methodology applies to a wider range of interventions on the equilibrium outcome. For instance, it is easier to establish that endowment reallocations through taxes and transfers would also lead to a Pareto improvement over the equilibrium outcome.

Given Theorem 4.1, system (5.1) has no solution if and only if the last two groups of equations have no solution. We write them explicitly as

$$
\begin{array}{ll}
c^k_{1h} D^2 U^k_h + (-1)^k c_{3h} DU^k_h - c^k_{6h}\Psi + c_{11}\tilde{\theta}^k_h X^{k\backslash}_h + c_{12h}\tilde{\theta}^k_h DU^k_h = 0 & (1) \\
-c^k_{1h}\Psi^T = 0 & (2) \\
c^k_{7h}\chi_{[\delta^k_h=0]} = 0 & (3) \\
c_{4h}\chi_{[\theta^2_h=0]} - c_{5h}\chi_{[\theta^2_h=1]} + c_{11}\zeta^{\backslash}_{fh} + c_{12h}(U^2_h - U^1_h) = 0 & (4) \\
c_{3h} + c_{4h}(1 - \chi_{[\theta^2_h=0]}) = 0 & (5) \\
-c_{3h} + c_{5h}(1 - \chi_{[\theta^2_h=1]}) = 0 & (6) \\
c^k_{9h} D^2 V^k_h - c^k_{10h}\Psi + c_{11}\tilde{\theta}^k_h X^k_h = 0 & (7) \\
-c^k_{9h}\Psi^T = 0 & (8) \\
\sum_h \sum_k (c^k_{1h} P^{k*}_{fh} + c^k_{6h} Z^k_{fh} + c^k_{9h} P^{k*}_{wh} + c^k_{10h} Z^k_{wh}) = 0 & (9) \\
-c^k_{6h}\phi'^{kT} + c^k_{7h}(\pi^k - \pi^{k'})^T (1 - \chi_{[\delta^k_h=0]}) + c^k_{10h}\phi'^k + c^k_{13h}\pi^k_h = 0 & (10) \\
c^T c - 1 = 0 & (11)
\end{array}
\qquad (5.2)
$$

where: χ is the indicator function which is one on the set in brackets and zero on its complement; $\tilde{\theta}^2 = \theta^2$, $\tilde{\theta}^1 = 1 - \theta^2$.[8] Counting equations and unknowns, note that there are KSH equations (10) (corresponding to the derivatives with respect to $\overline{z}_h$), but that we have added the $3H$ extra variables (the rows DG) c_{12h}, and c^k_{13h}, for $k = 1, 2$, all h. Moreover, notice that at any equilibrium where $\delta^k_h \neq 0$, we also lose equations (3) in (5.2). Hence, throwing away equations (3), we have $KSH - KH - 3H + 1$ too many equations. This number is positive provided that $S \geq 3$.

We are now ready to state the main result of this section.

Theorem 5.1. *For an open and dense subset of , if $S \geq 3$ and $C - 1 \geq KSH$, the equilibrium allocation is constrained suboptimal.*

---

[8] Also, c_{ih} is the coefficient for equation i in (4.1), and
$\zeta^{\backslash}_h = \sum_s \pi^{s2}_h (x^{s2\backslash} - y^{s\backslash} - e^{\backslash}_h) - \sum_s \pi^{s1}_h (x^{s1\backslash} - y^{s\backslash} - e^{\backslash}_h)$
while
$P^{k*}_{fh} = \begin{bmatrix} -\alpha^{1k} I^{\backslash} \\ -\alpha^{2k} I^{\backslash} \end{bmatrix}$ and $P^{k*}_{wh} = \begin{bmatrix} -\rho^{1k} I^{\backslash} \\ -\rho^{2k} I^{\backslash} \end{bmatrix}$
and
$Z^k_{fh} = \left[-x^{s_h kc}_{fh} + \left(1 - \Phi_h\left(z^{s_h k}_h; .\right)\right)(y^{s_h c} + e^c_h) \right]_{s=1,\ldots,S; c \neq C}$
$Z^k_{wh} = \left[-x^{s_h kc}_{wh} + \Phi_h\left(z^{s_h k}_h; .\right)(y^{s_h c} + e^c_h) \right]_{s=1,\ldots,S; c \neq C}$

Openness follows from the properness of the natural projection (Lemma 4.4). To prove density in Theorem 5.1, we will use a standard strategy to show that (5.1) has no solution, involving a transversality argument based on perturbations of system (5.1). This usually entails showing regularity of the equilibrium system first, and then applying quadratic perturbations of utility functions to show that the derivative of (5.2) with respect to c, and the parameters e, u, has full rank. Instead, mainly for compactness of exposition, we will establish the full rank property directly in one step, and then will make use of a transversality argument for infinite dimensional spaces to conclude that also the derivative of system (5.1) with respect to c and the endogenous variables would be full generically, but the resulting manifold with negative dimension is then empty. Hence the first step toward proving genericity of constrained suboptimality is to construct the perturbation technology for the infinite dimensional case.

5.1. The perturbation technology

We parametrize utilities in the following way. For any given economy (γ, u), we fix y and e, and the components of γ other than utilities, call them γ', with $\gamma'' = (u, v)$. We restrict our attention to a compact subset of $\mathbb{R}^C_{++}$ containing $y_h^{s_h} + e_h$ in its interior, all s_h and all h, and therefore bounded away from the axes of $\mathbb{R}^C_{++}$. Then the space $\mathcal{U}$ (and $\mathcal{V}$) of utilities previously defined and now restricted to such a compact set is a Banach space for a bounded metric induced by the C^2 compact-open topology.[9] Then $\ _{\gamma'}$ is also a Banach space. If a property is dense with respect to the set $\ _{\gamma'}$, then it is also dense with respect to . This is because density is essentially a local property, and we can imagine the functions in $\ _{\gamma'}$ to be extended to the original domain in the following way. If an economy does not satisfy the property, we wiggle the utilities defined over the compact set previously constructed, and keep them unchanged outside an open set containing this compact set. Clearly, if two functions are close in the topology on $\ _{\gamma'}$ then they are close in , given our construction and the metric on $\ _{\gamma'}$. Given this, from now on we drop the reference to the point γ' and simply write   for $\ _{\gamma'}$.

Given our economy (γ, u) and a corresponding function $u_h \in \mathcal{U}_h$, we consider for each equilibrium (in particular, for each $x_h^* \in F_{\gamma,u}^{-1}(0)$)

$$\overline{u}_h(x_{fh}^{s,k}) = u_h(x_{fh}^{s,k}) + \kappa_{fh}^{s,k} + a_{fh}^{s,k}(x_{fh}^{s,k} - x_{fh}^{s,k*}) + (x_{fh}^{s,k} - x_{fh}^{s,k*})^T A_{fh}^{s,k}(x_{fh}^{s,k} - x_{fh}^{s,k*})$$

[9] This is true because $\mathcal{U}$ (and $\mathcal{V}$) are G_δ subsets of the space of C^2 functions (for a proof, see Allen (1981), e.g.), and G_δ subsets are topologically complete (see Hocking and Young (1961)).

as a perturbed utility function, with $\kappa_{fh}^{s,k} \in \mathbb{R}$, $a_{fh}^{s,k} \in \mathbb{R}^C$ and $A_{fh}^{s,k}$ a $C \times C$ symmetric, negative definite matrix. All parameters $\kappa_{fh}^{s,k}$, $a_{fh}^{s,k}$ and $A_{fh}^{s,k}$ are assumed to be small in norm, so that $d(\overline{u}_h, u_h) < \varepsilon$, for a given $\varepsilon > 0$, and where $d(.)$ is the distance induced by the C^2 sup norm in $\mathcal{U}_h$. With this formulation, the set of all $\overline{u}_h$ so obtained is a linear subspace of $\mathcal{U}_h$. Similar conclusions hold for $\mathcal{V}_h$. That is so provided we are able to perturb utilities without transforming them from state-independent to state-dependent. This can be accomplished in steps. First, we use this parametrization on the subset of parameters where in equilibrium $x_{ih}^{s,k} \neq x_{ih}^{s',k'}$, when either $s \neq s'$ or $k \neq k'$, all $i = f, w$ and all h. This allows perturbations of utilities in the form expressed above without altering their state-independence. Then we show that the set of parameters for which in equilibrium $x_{ih}^{s,k} = x_{ih}^{s',k'}$, some s, s', k, k', i and h, with $s \neq s'$ or $k \neq k'$ is meager. An iterative procedure would then show that the condition $x_{ih}^{s,k} = x_{ih}^{s',k'}$, all s, s', k, k', i and h is also nongeneric. This is in essence the analogue of dealing with corner solutions.

Since the left hand side of (5.1) is a function Φ from a Banach space into $\mathbb{R}^\nu$ (with $\nu < \infty$), it suffices to show that $D_{\xi,c,\gamma''}\Phi(\xi, c, \gamma'')$, computed at all values such that $\Phi(\xi, c, \gamma'') = 0$, is onto when restricted to the particular linear subspace of $\mathcal{U}$ (and $\mathcal{V}$) identified by the above parametrization. Note that this subspace is finite for given x^*, but not necessarily so as we move x^*, since we do not know at this point that the number of equilibria is finite. The need to use the infinite dimensional version of the transversality theorem arises from the (uncountably) infinite dimensionality of this subspace, which in turn derives from the need to independently perturb the utility gradient and Hessian, hence to center the perturbation around x^*.

Once it is established that rank of the derivative is full, i.e. rank $D_{\xi,c,\gamma''}\Phi = \nu$, we can apply the Sard-Smale theorem (see Smale (1965) or Abraham et al. (1988), Theorem. 3.6.15) to establish transversality, that is, we conclude that there is a dense subset of such that rank $D_{\xi,c}\Phi_{\gamma,u}(\xi, c) = \nu$, for all γ, u in this subset.

We can apply the theorem since the projection $\pi : \Phi^{-1}(0) \to$ is a Fredholm map. It is a continuous linear map, in fact a smooth map, if we establish that rank $D_{\xi,c,\gamma''}\Phi = D\Phi = \nu$, so that $\Phi^{-1}(0)$ is a Banach smooth submanifold of $\Xi \times C \times$, a Lindelöf manifold, and:

a) π is (proper over a Hausdorff, locally compact and first countable topological space, hence) closed;

b) $\Xi \times C$ is finite dimensional;

c) $\Phi^{-1}(0)$ is a closed subspace of finite codimension (a consequence of *codim* $\Phi^{-1}(0) = \dim \mathbb{R}^\nu$)

Then the statement on π follows from Abraham et al (1988), Lemma 3.6.24, and:

d) π has finite kernel, since $\dim \ker \pi = \dim[(\Xi \times C) \cap \Phi^{-1}(0)] = \nu$; and,

e) π has finite co-kernel, a consequence of Abraham et al., Lemma 3.6.23.

Hence, by the Sard-Smale theorem, the set of regular values of π is open and dense in . The transversality proof is then concluded like in the standard finite dimensional case.

5.2. Density

We are now ready to prove the second part of Theorem 5.1, the density statement.
Proof. Assume conditions A) through D), that is:

A) $c_{1h}^{sk} \neq 0$, and $c_{9h}^{sk} \neq 0$, all s, k, h.

B) Writing equations

$$\sum_h \sum_k c_{6h}^k Z_{fh}^k = 0 \tag{5.3}$$

in matrix form as $Ab = 0$, where

$$A = \begin{bmatrix} \cdots & x_{fh}^{sk1} - [(1-\phi(z_h^{sk})]\tilde{y}^{s1} + \frac{\partial \Phi_h}{\partial p^1} p^1 \tilde{y}^{s1} & \cdots \\ & \vdots & \\ \cdots & x_{fh}^{skc} - [(1-\phi(z_h^{sk})]\tilde{y}^{sc} + \frac{\partial \Phi_h}{\partial p^1} p^c \tilde{y}^{sc} & \cdots \\ & \vdots & \\ \cdots & x_{fh}^{sk(C-1)} - [(1-\phi(z_h^{sk})]\tilde{y}^{s(C-1)} + \frac{\partial \Phi_h}{\partial p^1} p^{C-1} \tilde{y}^{sC-1} & \cdots \end{bmatrix}$$

is a $(C-1) \times KSH$-dimensional matrix of individual excess demands, $\tilde{y}^s = y^s + e_h$, and for ease of notation $s = s_h$, while $b = (\cdots, c_{6h}^{sk}, \cdots)$ is a KSH-dimensional vector. There exists a square submatrix of A with KSH rows, and full rank, denoted by A'. This explains why we assume that $C - 1 \geq KSH$.

C) There exists at least a household h such that
$\zeta_h^{\backslash} \equiv \sum_s \pi_h^{s2}(x^{s2\backslash} - y^{s\backslash} - e_h^{\backslash}) - \sum_s \pi_h^{s1}(x^{s1\backslash} - y^{s\backslash} - e_h^{\backslash}) \neq 0.$

D) In equilibrium, there are no corner solutions, and $x_{ih}^{s,k} \neq x_{ih}^{s',k'}$, when either $s \neq s'$ or $k \neq k'$, all $i = f, w$ and all h;

First observe that if A), and after throwing out $(H-1)$ of equations (4), and equation (11), which under our assumptions we do without compromising the excess number of equations in the system, the following perturbation of system (5.2) works:

$$\begin{array}{lll} A_{fh}, \text{ all } k,h & (1) \\ c_{1h}^{skC}, \text{ all } s,k,h & (2) \\ c_{11}^{c}, \text{ some } c & (4) \\ c_{3h} \text{ if } \theta_h^2 = 0, \ c_{4h} \text{ otherwise, all } h & (5) \\ c_{3h} \text{ if } \theta_h^2 = 1, \ c_{5h} \text{ otherwise, all } h & (6) \\ A_{wh,} \text{ all } k,h & (7) \\ c_{9h}^{skC}, \text{ all } s,k,h & (8) \\ c_{9h}^{sk\backslash C}, \text{ some } h,s,k & (9) \\ c_{6h}^{sk}, \text{ all } s,k,h & (10) \end{array} \tag{5.4}$$

Note that (4) is perturbed using condition C). The next step is to show that $D_{\xi,\gamma''}F$ has full rank (without using $\alpha_{h'}^{s'k'}$), which would imply that rank $D_{\xi,c,\gamma''}\Phi = \nu$. We can apply the transversality theorem to state that system (5.1) has no solution as desired, provided the stated conditions hold. This is immediately verified using the following sketched perturbations:

$$\begin{array}{lll} \Delta a_{fh}^{s,k,c} & (1) \\ \Delta \alpha_h^{s,k} & (2) \\ \Delta \kappa_{fh}^{s,k}, \text{ some } s,k & (3) \\ \Delta \underline{\eta}_h \text{ or } \Delta\theta_h^2 \text{ [if } \theta_h^2 > 0 \text{ or } = 0, \text{ resp.]} & (4) \\ \Delta \overline{\eta}_h \text{ or } \Delta\theta_h^2 \text{ [if } \theta_h^2 < 1 \text{ or } = 1, \text{ resp.]} & (5) \\ \Delta x_{fh}^{s,k,C} & (6) \\ \Delta \delta_h^k \text{ or } \Delta z_h^k \text{ [if } f_{2h} > 0 \text{ or } = 0, \text{ resp.]} & (7) \\ \Delta z_h^k & (8) \\ \left(\Delta x_{wh}^{s,k}, \ \Delta \rho_h^{s,k}\right) & (9), (10) \\ \Delta x_{fh}^{s,k,c}, \text{ some } s,k,h \text{ all } c \neq C & (11) \end{array} \tag{5.5}$$

Note that equations (11) do not contain the market clearing condition for $c = C$.

As for condition A), observe that if $c_{1h}^k = 0$ and $c_{9h}^k = 0$, all h,k, then also $c_{10h}^k = 0$, and $c_{11} = 0$, using the numéraire commodity normalization. Using equations (4.1.1) and (5.2.1), we have

$$c_{6h}^{sk} = \alpha_h^{sk}(c_{12h}\tilde{\theta}^k + (-1)^k c_{3h}) \tag{5.6}$$

Concentrating on equations (5.2.9), one can see that these form the subsystem (5.3). Under condition B), if $C - 1 \geq KSH$, (5.3) has a unique solution, that is,

$$c^k_{6h} = 0 \tag{5.7}$$

all h, k. Indeed, if such a submatrix A' exists, A has maximal rank. From

$$0 = \alpha^{sk}_h (c_{12h} \tilde{\theta}^k + (-1)^k c_{3h})$$

we have $c^k_{12h} \tilde{\theta}^k + (-1)^k c_{3h} = 0$, all k, h. But this expression implies $c_{3h} = c_{12h} = 0$, all h, hence from equations (5.2.4) we have $c_{4h} = c_{5h} = 0$, while from equations (10) in (5.2) and using Assumption 1, we get $c^k_{7h} = c^k_{13h} = 0$. Note that this we do even in the case when $\delta^k_h = 0$, by using equation (3). This means $c = 0$, a contradiction to the equation $c^T c - 1 = 0$, implying that system (5.1) cannot have a solution in this case.[10]

If there is an s', k', h' such that $c^{s'k'}_{1h'} = 0$, then equation (5.2.2) drops for s', and we can drop $C-1$ equations (5.2.1), leaving the numéraire equation, and obtaining (5.6) for s', k', h'. We drop $(H-1)$ equations of (5.2.4) and equation (5.2.11) as before. Taking into account condition C), we can perturb the remaining equation in (5.2.4) using c_{11}, we can use $c^{s'k'}_{6h'}$ to perturb (5.6), and use $c^{s'k'}_{10h'}$ to perturb (5.2.10). Then everything else goes through as in the first case. If there is an s', k', h' such that $c^{s'k'}_{9h'} = 0$, but $c^{s'k'}_{1h'} \neq 0$, then from (5.2.7) we obtain $c^{s'k'}_{10h'} = 0$, and we can drop the C equations (5.2.7) and proceed to perturb this subsystem as before. If $c^{s'k'}_{1h'} = 0$, we cannot use the procedure used above, because $c^{s'k'}_{10h'} = 0$. But

$$-c^k_{10h[1\times S]} \begin{bmatrix} p & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p \end{bmatrix}_{[S\times(CS)]} + \widetilde{\theta}^k_h c_{11[1\times(C-1)]} \begin{bmatrix} \dots & \pi^{sk}_h \, [I_{C-1}|0] & \dots \end{bmatrix}_{[(C-1)\times(CS)]} = 0,$$

[10] If $C = 1$, then system (5.2) reduces to

$$\begin{aligned}
&c^k_{6h} = [(-1)^k c_{3h} + c_{12h}\theta^k]\alpha^k_h \\
&c^k_{7h}\chi_{[\delta^k_h=0]} = 0 \\
&c_{4h}\chi_{[\theta^2_h=0]} - c_{5h}\chi_{[\theta^2_h=1]} = 0 \\
&c_{3h} + c_{4h}(1-\chi_{[\theta^2_h=0]}) = 0 \\
&-c_{3h} + c_{5h}(1-\chi_{[\theta^2_h=1]}) = 0 \\
&c^k_{10h} = 0 \\
&-c^k_{6h}\Phi'^{kT} + c^k_{7h}(\pi^k - \pi^{k'})^T(1-\chi_{[\delta^k_h=0]}) + c^k_{13h}\pi^{kT} = 0 \\
&c^T c - 1 = 0
\end{aligned}$$

which always has a nonzero solution (compare it with (4.1)).

and

$$-c_{10h'}^{k's'}p^{s'} + \pi_{h'}^{s'k'}\widetilde{\theta}_{h'}^{k'} \cdot c_{11[1\times(C-1)]}\left[[I_{C-1}|0]\right]_{[(C-1)\times C]} = 0,$$

and also

$$\pi_{h'}^{s'k'}\widetilde{\theta}_{h'}^{k'} \cdot c_{11[1\times(C-1)]}\left[[I_{C-1}|0]\right]_{[(C-1)\times C]} = 0,$$

which implies that $c_{11} = 0$, if $\widetilde{\theta}_{h'}^{k'} \neq 0$, so we gain an extra degree of freedom, and throw away the one equation among (5.2.4), (5.2.1) and (5.2.7) which we cannot perturb. If $\widetilde{\theta}_{h'}^{k'} = 0$, we lose the equation we are not able to perturb.

Finally, we show that conditions B), C) and D) hold generically. The argument is standard and is sketched in the Appendix (see Regularity results). To conclude, we obtain a dense subset of parameters where system (5.1) has no solution, as we were to show. ■

A. Proof of Lemma 3.1

Drop the subscript h. The set $B(p, e, y, \phi, a^k)$ could be empty because of constraints (3) and (4). Indeed, for any s, once ω^s is given, if $x_f^s = (1-\omega^s)(y^s+e)/2$ then (1) is satisfied as a strict inequality. We distinguish the cases when $k=1$ or $k=2$. Without loss of generality assume $a^2 \geq a^1$.

a) First, we examine the constraint set for $k=1$, and with $a^2 > a^1$. We note that

$$z = \min_{s'} \widetilde{v}\left(\omega^{s'} = 1, p, y^{s'}, e\right) > a^1 + \underline{V}$$

by Assumption 6 and by definition of indirect utility. By continuity and monotonicity of $\tilde{v}$, for all s there exists an $\omega^s \in (0,1)$ such that $z > \tilde{v}(\omega^s, p, y^s, e) > a^1 + \underline{V}$. Let $\overline{z} = \min_{s'} \widetilde{v}\left(\omega^{s'}, p, y^{s'}, e\right)$. Since $\tilde{v}$ is a diffeomorphism, for all s there exists an $\overline{\omega}^s$ such that $\overline{z} = \tilde{v}(\overline{\omega}^s, p, y^s, e)$, and $\overline{\omega}^s < 1$ by construction and $\overline{\omega}^s > 0$ by Assumption 3. So (2) is also satisfied with strict inequalities. Let $z^s = \overline{z}$ and $\omega^s = \overline{\omega}^s$. Then (4) is immediately satisfied and nonbinding, since

$$\sum_s \pi^{s1}\overline{z} - a^1 - \underline{V} = \overline{z} - a^1 - \underline{V} > 0$$

As for (3), we observe that

$$\sum_s [\pi^{s1} - \pi^{s_h 2})]\overline{z} - a^1 + a^2 = a^2 - a^1 > 0$$

by assumption.

b) If $k = 1$ but $a^2 = a^1$, let $\overline{s}$ be a state such that $\pi^{\overline{s}2} - \pi^{\overline{s}1} > 0$ and $\tilde{s}$ a state where $\pi^{\tilde{s}2} - \pi^{\tilde{s}1} < 0$. Such states exist by Assumption 1. Since $\lim_{\omega \to 0} \widetilde{v}(\omega, p, y^s, e) = -\infty$ and, by Assumption 6, for any s,

$$v(y^s + e) > \left|a^2 - a^1\right| = 0$$

the continuity of $\widetilde{v}$ and the connectedness of $[0,1]$ imply that for all s there exists an $\overline{\omega}^s \in (0,1)$ such that $\widetilde{v}(\overline{\omega}^s, p, y^s, e) = 0$. Let $\omega^s = \overline{\omega}^s$ and $z^s = \widetilde{v}(\overline{\omega}^s, p, y^s, e) = 0$ for all $s \neq \tilde{s}$. Then (3) and (4) are satisfied with inequalities. By Assumption 6, there exists a $\omega^{\tilde{s}}$ and a corresponding $z^{\tilde{s}} = \widetilde{v}\left(\omega^{\tilde{s}}, p, y^s, e\right)$ such that the above inequalities are satisfied in a strict sense and such that $\omega^{\tilde{s}} = \phi(z^{\tilde{s}}; p, y^{\tilde{s}}, e) \in (0,1)$, so that (2) is satisfied too.

c) Finally, for $k = 2$, again for all s there exists an $\overline{\omega}^s \in (0,1)$ such that $\widetilde{v}(\overline{\omega}^s, p, y^s, e) = 0$. Let $z^s = 0$ and $\omega^s = \overline{\omega}^s$ for all $s \neq \overline{s}$, where $\overline{s}$ is as above. Using Assumption 6, again we immediately can choose $z^{\overline{s}}$ such that both constraints (3) and (4) are satisfied with strict inequalities (this is where the full power of Assumption 6 is used). Then, letting $\omega^{\overline{s}} = \phi(z^{\overline{s}}; p, y^{\overline{s}}, e)$, we have $0 < \omega^{\overline{s}} < 1$, and constraint (2) is satisfied and nonbinding. ■

B. Proof of Lemma 3.2

$>$From the participation constraint (4) it must be that $\omega_h^{s_h} > 0$, because

$$\lim_{\omega_h^{s_h} \to 0} \widetilde{v}_h(\omega_h^{s_h}, p, y^{s_h}, e_h) = -\infty.$$

Hence at an optimum for problem 3.5, the constraint $\phi_h(z_h^{s_h}; p, y^{s_h}, e_h) \geq 0$ is never binding. Similarly, it must be that $\omega_h^{s_h} < 1$. We know that the firm can lock in the utility $\underline{u}_h\left(\underline{x}_{fh}^{s_h k}\right)$ in each state, as a consequence of Lemma 3.1. Recall that $\underline{x}_{fh}^{s_h k}$ is obtained with $\underline{\omega}_h^{s_h k} < 1$. Moreover, for any s_h, $u_h\left(x_{fh}^{s_h}\right) < u_h(\overline{x}_{fh}) < +\infty$, where $\overline{x}_{fh} = \left(\overline{x}_{fh}^c\right)_{c=1}^C$ and $\overline{x}_{fh}^c = \frac{p(y^{s_h}+e_h)}{\min_c p^c}$. (This last statement shows that the very low utility in state s_h can be compensated by a very high utility in the other states only up to a given extent.). Now assume that there is k and s_h such that $\omega_h^{s_h k} \to 1$. Then $x_{fh}^{s_h k} \to 0$, and by Assumption 2,

$$\lim_{x_{fh}^{s_h} \to 0} u_h(x_{fh}^{s_h k}) = -\infty.$$

which ends our proof. ■

C. Proof of Lemma 3.3.

The Kuhn-Tucker conditions for problem 3.5 are

$$
\begin{array}{lll}
x_h^{s_hk} & \pi_h^{s_hk} D_{x^{s_hk}} u_h \left(x_{fh}^{s_hk} \right) - \alpha_h^{s_hk} p = 0 & (1) \\
z_h^{s_hk} & -\alpha_h^{s_hk} \phi_h' \left(z_h^{s_hk}; . \right) p(y^{s_h} + e_h) + \beta_h^k \pi_h^{s_hk} + \delta_h^k (\pi_h^{s_hk} - \pi_h^{s_hk'}) = 0 & (2) \\
\alpha_h^{s_hk} & \min\{\alpha_h^{s_hk}, -p x_{fh}^{s_hk} + \left(1 - \phi_h \left(z_h^{s_hk}; . \right)\right) p(y^{s_h} + e_h)\} = 0 & (3) \\
\delta_h^k & \min\left\{\delta_h^k, \sum_{s_h} (\pi_h^{s_hk} - \pi_h^{s_hk'}) \cdot z_h^{s_hk} - a_h^k + a_h^{k'} \right\} = 0 & (4) \\
\beta_h^k & \min\{\beta_h^k, \sum_{s_h} \pi_h^{s_hk} z_h^{s_hk} - a_h^1 - \underline{V}_h\} = 0 & (5)
\end{array} \quad \text{(C.1)}
$$

One immediately sees that $\alpha_h^{s_hk} > 0$, all s_h, k, h, that $\beta_h^k > 0$ all k, h. To show $\beta_h^k > 0$, solve for $\alpha_h^{s_hk}$ in $(C.1.1)$ and substitute it into $(C.1.2)$ to get

$$
\begin{aligned}
& \pi_h^{s_hk} \beta_h^k + \delta_h^k (\pi_h^{s_hk} - \pi_h^{s_hk'}) \\
= & \left(\pi_h^{s_hk} D_{s_hk} u_h \left(x_{fh}^{s_hk} \right) (y^{s_h} + e_h) \right) \phi_h' \left(z^{s_hk}; . \right)
\end{aligned}
$$

Suppose $\beta_h^1 = 0$. If $\delta_h^k > 0$, there is a state $\overline{s}$ such that $\pi_h^{\overline{s}k} - \pi_h^{\overline{s}k'} < 0$, while the right-hand side is always positive, clearly an absurd. Similarly, with $\delta_h^k = 0$, we would have the absurd that the right-hand side is both positive and zero. Hence $\beta_h^k > 0$.

We limit ourselves to showing that if it is not the case that constraint (3) holds with equality *and* $\delta_h^k = 0$, then the derivative of system (C.1) with respect to the choice variables and the multipliers (listed to the left-hand side of the system) has full row rank. Hereafter, we drop the subscript h and fix a k.

If $\delta^k > 0$ and $\sum_s (\pi^{sk} - \pi^{sk'}) \cdot z^{sk} - a^k + a^{k'} = 0$, taking the derivative, we get the $CS + 2S + 2$- dimensional square matrix

$$
M^k = \begin{bmatrix}
D_k^2 & 0 & -\Psi^T & 0 & 0 \\
0 & -\phi''^k & -\phi'^k & \pi^k - \pi^{k'} & \pi^k \\
-\Psi & -\phi'^{kT} & 0 & 0 & 0 \\
0 & (\pi^k - \pi^{k'})^T & 0 & 0 & 0 \\
0 & \pi^{kT} & 0 & 0 & 0
\end{bmatrix}
$$

where

$$
\Psi = \begin{bmatrix}
p & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & p
\end{bmatrix},
$$

an $S \times CS$ matrix of prices, and D_k^2, ϕ'^k and ϕ''^k are matrices of dimension $SC \times SC$ (the first) and $S \times S$ (the last two) with zeros off the diagonal, and with generic diagonal elements $D_{sk}^2 = \pi^{sk} D^2 u \left(x_f^{sk}\right)$, $\phi'^{sk} = \phi'(z^{sk}; p, y^s, e) p(y^s + e)$ and $\phi''^{sk} = \alpha^{sk} \cdot D_{sk}^2 \phi \left(z^{sk}; p, y^s, e\right) \cdot p(y^s + e)$, respectively. Note that $\phi''^{sk} > 0$, all s, k.

If $\delta^k = 0$ and $\sum_s (\pi^{sk} - \pi^{sk'}) \cdot z^{sk} - a_h^k + a_h^{k'} > 0$, the matrix of derivatives has the same rank of

$$M^{k\prime} = \begin{bmatrix} D_k^2 & 0 & -\Psi^T & 0 \\ 0 & -\phi''^k & -\phi'^k & \pi^k \\ -\Psi & -\phi'^{kT} & 0 & 0 \\ 0 & \pi^{kT} & 0 & 0 \end{bmatrix}$$

We want to show both that $0 = M^k \Delta$, where $\Delta = (\Delta x, \Delta z, \Delta \alpha, \Delta \delta, \Delta \beta)$ implies that $\Delta = 0$, and that $0 = M^{k\prime} \Delta'$ where $\Delta' = (\Delta x, \Delta z, \Delta \alpha, \Delta \beta)$ implies $\Delta' = 0$. The argument is standard, based on the convexity of preferences, and the details are left to the reader. ■

D. Proof of Lemma 4.2.

Drop the subscript h. The relevant subsystem composed of equations (4.4) and (4.1.7) and (4.1.8) is written more compactly as

$$\begin{array}{ll} (1) & \pi^{s1}\beta^1 + \left[\pi^{s1} - \pi^{s2}\right]\delta^1 = \pi^{s1}k^{s1} \\ (2) & \sum_s (\pi^{s1} - \pi^{s2}) \log \omega^{s1} \underline{y}^s - a^{12} \geq 0 \\ (3) & \sum_s \pi^{s1} \log \omega^{s1} \underline{y}^s = a^1 \\ (4) & \pi^{s2}\beta^2 + \left[\pi^{s2} - \pi^{s1}\right]\delta^2 = \pi^{s2}k^{s2} \\ (5) & \sum_s (\pi^{s1} - \pi^{s2}) \log \omega^{s2} \underline{y}^s - a^{12} \leq 0 \\ (6) & \sum_s \pi^{s2} \log \omega^{s2} \underline{y}^s = a^2 \end{array} \tag{D.1}$$

where (1) and (4) correspond to equations (4.1.2), (2) and (5) correspond to equations (4.1.7) and (3) and (6) correspond to equations (4.1.8), once it is understood that $\delta^k \geq 0$, for $k = 1, 2$, and that if (2) is a strict inequality, $\delta^1 = 0$, and if (8) is a strict inequality, $\delta^2 = 0$. Also, we have defined

$$k^{sk} = \frac{\omega^{sk}}{1 - \omega^{sk}} > 0,$$

$$a^{12} = a^1 - a^2,$$

and $y = \frac{y^{\bar{s}}}{y^{\tilde{s}}} = 1/10$. Once the parameters are fixed, system (D.1) is formed of two independent blocks, expressions (1) through (3), and expressions (4) through (6). Moreover, each block can be solved recursively. The strategy to solve the above system is decomposed in steps (all equation numbers refer to system (D.1), unless otherwise stated):

(i) Since (2) cannot hold as equality, and therefore $\delta^1 = 0$, solve for $\beta^1, \delta^1, \omega^{s1}$, all s, from equations (1) and (3) and verify that (2) is satisfied.

(ii) Since (5) cannot hold as inequality, solve first for $\left(\log \omega^{\bar{s}2} \underline{y}^{\bar{s}}, \log \omega^{\tilde{s}2} \underline{y}^{\tilde{s}}\right)$ and hence for $\omega^{\bar{s}2}$ and $\omega^{\tilde{s}2}$ from equations (5) and (6).

(iii) Solve for β^2, δ^2 as functions of ω^{s2}, for $s = \bar{s}, \tilde{s}$, from equation (4), checking that $\delta^2 > 0$.

(iv) Check that the restrictions on ω^{s2} are satisfied.

(v) Check that $U_{fh}\left(x^1_{fh}\right) < U_{fh}\left(x^2_{fh}\right)$, so that $\theta^2_h = 1$, for all h.

The details of the computation are available from the authors upon request. ■

E. Proof of Lemma 4.3.

The derivative of the extended system $F_{\gamma^*,u^*}(\xi^*)$ with respect to ξ^* is summarized below as

$$\begin{bmatrix} M^{1\prime}_h & 0 & 0 & 0 & 0 & P^1_{fh} \\ B^1_h & D_h & B^2_h & 0 & 0 & 0 \\ 0 & 0 & M^2_h & 0 & 0 & P^2_{fh} \\ C^1_h & 0 & 0 & W^1_h & 0 & P^1_{wh} \\ 0 & 0 & C^2_h & 0 & W^2_h & P^2_{wh} \\ 0 & 0 & X^2_h & 0 & X^2_h & 0 \end{bmatrix} \tag{E.1}$$

where:

a) $M^k_h = M^k$, the previously defined matrix (see Lemma 3.3) of derivatives of the principal's first order conditions, and similarly, $M^{k\prime}_h$ is $M^{k\prime}$ with the added column of the derivative with respect to δ^k and the row corresponding to equation (4.1.7) (calculations in Lemma 4.2 show that $\delta^1_h = 0$ while $f^1_{2h}(.) = 0$, all h, and $\delta^2_h > 0$).

b) B_h^k is a $3\times(CS_h+2S_h+2)$ matrix of zeros, except for the first CS_h elements of the first row, which are equal to the vector $(...,\pi^{s1}Du(x^{s1}),...)$, if $k=1$, and to $(...,-\pi^{s2}Du(x^{s2}),...)$, if $k=2$.

c) D_h is the 3×3 matrix of derivatives of equations[11] (3), (4) and (5) with respect to θ_h^2, $\underline{\eta}_h$ and $\overline{\eta}_h$, that is

$$D_h=\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

d) C_h^k is a $(CS_h+S_h)\times(CS_h+2S_h+2)$ matrix of derivatives of the agent's first order conditions (equations (9) and (10)) with respect to the principal's choice variables, for $k=1,2$. In particular, it has all zeros except for the derivative of equation (10) with respect to z_h^{sk}.

e) Finally, $W_h^k=\begin{bmatrix} D^2v_h^k & -\Psi^T \\ -\Psi & 0 \end{bmatrix}$, the matrix of derivatives of the agent's first order conditions with respect to x_{wh}^{sk} and ρ_h^{sk}, for $k=1,2$. $X_h^2=\begin{bmatrix} ... & \pi_h^{s2}I^{\backslash T} & ... \end{bmatrix}$ (recall that $\theta_h^2=1$, all h), while

$$P_{fh}^k=\begin{bmatrix} -\alpha^{1k}I^{\backslash} \\ -\alpha^{2k}I^{\backslash} \\ *_h^k \\ 0 \end{bmatrix} \text{ and } P_{wh}^k=\begin{bmatrix} -\rho^{1k}I^{\backslash} \\ -\rho^{2k}I^{\backslash} \\ 0 \end{bmatrix}$$

are a $(CS_h+2S_h+2)\times(C-1)$ and a $(CS_h+S_h)\times(C-1)$, respectively, matrices of price derivatives. Here $I^{\backslash T}=\begin{bmatrix} I & 0 \end{bmatrix}$, a $(C-1)\times C$ matrix, with a last column of zeros, and I is the usual $(C-1)$ square identity matrix.

The simple idea of the proof is to first perform some elementary operations on rows and columns of (E.1) (that is, without affecting its rank) and then to observe that the obtained matrix has full row rank, as a consequence of a known result. Namely, and for each h:

1. Using the full rank of D_h, clean[12] the matrices B_h^k of their nonzero elements, for $k=1,2$.

2. Using the super-columns of (x_{wh}^1,ρ_h^1), and the fact that W_h^1 has full row rank, clean C_h^1 of its nonzero elements. Keep in mind that $\theta_h^2=1$.

[11] All equation numbers refer to system (4.1), unless otherwise stated.

[12] By "clean" we mean: use an appropriate linear combination of columns (or rows) of the given matrix and add it to the targeted matrix (or row(s), or column(s)), so that the result is zero.

3. For $k = 2$, the matrix of derivatives of (7) and (8) with respect to z_h^{s2} has full rank (recall that $S = 2$). Hence we use it to clean $-\phi''^2$ and $-\phi'^{2T}$ in M_h^2, and the nonzero elements of C_h^2. Similarly, the matrix of derivatives of (2) with respect to β_h^2 and δ_h^2 has full rank. Hence we use it to clean the rows of $-\phi'^2$ in M_h^2 (i.e. the derivatives of equation (2) with respect to α_h^{s2}), and $*_h^2$, a matrix of nonzero derivatives in P_{fh}^k.

4. Because of Lemma 3.3, we can use $M_h^{1\prime}$ to clean the matrix P_{fh}^1.

Rearranging rows and columns, the transformed matrix (E.1) is shown to have full rank if and only if the following matrix has full row rank (zeros are omitted):

$$\begin{array}{|c|c|c|c|c|}
\hline
D^2u_h^2 & -\Psi^T & & & \begin{bmatrix} -\alpha_h^{12} I^{\backslash} \\ -\alpha_h^{22} I^{\backslash} \end{bmatrix} \\
\hline
-\Psi & & & & \\
\hline
 & & D^2u_{wh}^2 & -\Psi^T & \begin{matrix} -\rho_h^{12} I^{\backslash} \\ -\rho_h^{22} I^{\backslash} \end{matrix} \\
\hline
 & & -\Psi & & \\
\hline
X_h^2 & & X_h^2 & & \\
\hline
\end{array}$$

where $D^2u_h^2$ $(D^2u_{wh}^2)$ is the Hessian of the utility function of firm (worker) if effort $k = 2$ is chosen. The above matrix has the same structure of the analogous derivative for a standard exchange economy with completely incomplete financial markets computed at a no trade equilibrium and therefore it has full row rank. ■

F. Proof of Lemma 4.4

In what follows, "Converges" means "converges (or so does a subsequence) to an element which belongs to the set containing the sequence". Consider a pair of economies (γ, u) and (γ^*, u^*) and any sequence

$$\{x^n, z^n, \theta^n, \alpha^n, \beta^n, \delta^n, \eta^n, \rho^n, p^n, t^n\}_{n=1}^{\infty} \subset H^{-1}(0).$$

Then:

$\left\{ \left(\theta_h^2\right)^n \right\}_{n=1}^{\infty}$ Converges, say to $\overline{\theta}_h$, because it is contained in the compact set $[0, 1]$. For a similar reason, $\{t^n\}_{n=1}^{\infty}$ Converges to t. For at least one k, $\left\{x_{wh}^{kn}\right\}_{n=1}^{\infty}$ is bounded above by total resources and below by zero, all h. Since for any s we have $z_h^{skn} = v(x_{wh}^{skn})$, using the participation constraint and the boundary condition on preferences it is easily seen that x_{wh}^{kn} Converges to x_{wh}^k. $\left\{ (\rho_h^{s_h k})^n \right\}_{n=1}^{\infty}$ Converges because of (4.1.9) and the fact that $p^C = 1$. $\{p^n\}_{n=1}^{\infty}$ Converges because $\rho^{s_h k}$ does and because of equations (4.1.9). Now equation (4.1.10) shows that $\{x_{wh}^{k'n}\}$ also Converges for $k' \neq k$ (a necessary step if $\overline{\theta}_h \in \{0, 1\}$). Since $z_h^{s_h kn} = v_h\left(x_{wh}^{skn}\right)$, then $\{z_h^{s_h n}\}_{n=1}^{\infty}$ Converges. On the other hand, $\{x_{fh}^n\}$ is bounded above by total resources and below away from the axes as a consequence of Lemmas 3.1 and 3.2, and the convergence of p^n; then x_{fh}^n Converges, using the boundary condition on preferences. $\left\{(\alpha_h^{s_h k})^n\right\}_{n=1}^{\infty}$ Converges because of the price normalization and of (4.1.1).

$\{\beta_h^n\}_{n=1}^{\infty}$ and $\{\delta_h^n\}_{n=1}^{\infty}$ Converge because of equations (4.1.2) $= 0$, i.e., $\begin{bmatrix} \beta^n \\ \delta^n \end{bmatrix} = [A(\pi^n)]^{-1} \chi^n$ - with obvious notation - and $A(\pi^n)$ is a square matrix (recall that $S_h \geq 2$, all h) which has full rank along the sequence and in the limit and χ^n Converges.

According to the value of $\overline{\theta}_h^2$, either $\underline{\eta}_h^n \to 0$ or $\overline{\eta}_h^n \to 0$ (or both), while the other Converges from (4.1.4) and (4.1.5). More precisely, if $\overline{\theta}_h = 0$, since $\min\{\overline{\eta}_h^n, 1 - (\theta_h^2)^n\} = 0$, all n, then $0 = \lim_{n\to+\infty} \min\{\overline{\eta}_h^n, 1 - (\theta_h^2)^n\} = \lim_{n\to+\infty} \overline{\eta}^n$. Then, from (4.1.3), $\lim_{n\to+\infty} \underline{\eta}^n = \lim_{n\to+\infty} U(1, n) - U(2, n)$, a converging limit, with obvious notation. The same argument works in the remaining cases, and completes the proof. ■

G. Proof of Lemma 4.5

The proof of existence by homotopy is carried through inductively. It is enough to show that a homotopy exists between economies with $S = 2$ and $S = 3$. Define $\pi_h \in \Pi_h \subseteq \mathbb{R}_{++}^2$ and $\pi_h^* \in \Pi_h^* \subseteq \mathbb{R}_{++}^3$, where the two probability spaces satisfy the assumptions of the paper in the case in which the number of the states is 2 and 3, respectively. Moreover, let

$$\pi_h^k(t) = (1 - t)\left(\pi_h^k, 0\right) + t\pi_h^{k*}.$$

Define the space of endogenous variables of an economy with S states as Ξ^S and let $\dim \Xi^S = n(S)$. Let ^S be the space of exogenous variables of an economy with S states (which for $S = 3$ includes both π_h and π_h^*). Fix $\varepsilon \in \mathbb{R}_{++}$. Consider the function

$$\kappa : \Xi^3 \times [0, 1] \to \mathbb{R}^{n(3)}$$

defined:

a) by the left-hand side of equations (4.1.4) and (4.1.5), and of equations (4.1.1), (4.1.2), (4.1.6), (4.1.9), (4.1.10), but only for $s_h = 1, 2$, replacing everywhere $\pi_h^{s_h k}$ by $\pi_h^{s_h k}(t)$, while the corresponding equations for $s_h = 3$ are given by the left-hand side of the following system:

$$
\begin{array}{lr}
(1-t)(x_{fh}^{3k} - \varepsilon \mathbf{1}) + t[\pi_h^{3k}(t) D u_h \left(x_{fh}^{3k}\right) - \alpha_h^{3k} p] = 0, \text{ all } k, h & (1) \\
(1-t) z_h^{3k} + t[-\alpha_h^{s3k} D\phi_h \left(z^{3k}; .\right) p(y^3 + e_h) + \beta_h^k \pi_h^{3k}(t) + \delta_h^k (\Delta \pi_h^3(t)] = 0, {}^{13} \text{ all } k, h & (2) \\
(1-t)\alpha_h^{3k} + t\left[-p x_{fh}^{3k} + \left(1 - \phi_h(v_h(x_{wh}^{3k}))\right) p(y^3 + e_h)\right] = 0, \text{ all } k, h & (6) \\
(1-t)\left(x_{wh}^{3k} - \varepsilon \mathbf{1}\right) + t[\pi_h^{3k}(t) D v_h \left(x_{wh}^{3k}\right) - \rho_h^{3k} p] = 0, \text{ all } k, h & (9) \\
(1-t)\rho_h^{3k} + t\left[-p x_{wh}^{3k,} + \phi_h\left(z_h^{3k}; p, y^3, e_h\right) p(y^3 + e_h)\right] = 0, \text{ all } k, h & (10)
\end{array}
\tag{G.1}
$$

b) by transforming equations (4.1.3), (4.1.7), (4.1.8) and (4.1.11) into

$$
\begin{array}{lr}
t\sum_k (-1)^k \pi_h^{3k*} u(x_{fh}^{3k}) + \sum_{s_h} \pi_h^{s_h 2}(t) u_h\left(x_{fh}^{s_h 2}\right) - \sum_{sh} \pi_h^{s_h 1}(t) u_h \left(x_{fh}^{s_h 1}\right) + \underline{\eta}_h - \overline{\eta}^h = 0, \text{ all } h & (3) \\
\min\left\{\delta_h^k, \sum_{s=1,2} \Delta\pi_h^s(t) z_h^{sk} + t\Delta\pi_h^{3*} z_h^{sk} - a^k + a^{k'}\right\} = 0, \text{ all } k, h & (7) \\
t\pi_h^{3k*} v_h(x_{wh}^{3k}) + \sum_{s=1,2} \pi_h^{sk}(t) z_h^{sk} - a^k - \underline{V}_h = 0, \text{ all } k, h & (8) \\
\sum_h [(1-\theta_h^2) \sum\limits_{s=1,2} \pi_h^{s1}(t)\left(x_h^{s_h 1 \backslash} - y^{s_h \backslash} - e_h^{\backslash}\right) + \theta_h^2 \sum\limits_{s=1,2} \pi_h^{s2}(t)\left(x_h^{s_h 2\backslash} - y^{s_h\backslash} - e_h^{\backslash}\right)] & \\
+t\sum_k \theta_h^k \sum_h \pi_h^{3k*}\left(x_h^{3k\backslash} - y^{3\backslash} - e_h^{\backslash}\right) = 0 & (11
\end{array}
\tag{G.2}
$$

Observe that for $t = 0$, we get the system defining equilibrium for the case of two states of the world, plus some values related to a "fake" third state; as a consequence, uniqueness and regularity of the test economy are not affected by the extension of the original function defining the equilibrium (the left-hand side of (4.1)). For $t = 1$, we get the system defining the equilibrium for the case of three states.

We claim that the set $\kappa^{-1}(0)$ is compact. As in Lemma 4.4, in what follows "Converges" means "converges (or so does a subsequence) to an element which belongs to the set containing the sequence". Consider a pair $(\gamma, u) \in {}^{3}$ and $(\gamma^*, u^*) \in {}^{3}$, and any sequence

$$\{x^n, z^n, \theta^n, \alpha^n, \beta^n, \delta^n, \eta^n, \rho^n, p^n, t^n\}_{n=1}^{\infty} \subset \kappa^{-1}(0).$$

[13] Only in this proof, $\Delta\pi_h = \pi_h^k - \pi_h^{k'}$.

We denote the limit of the sequence of a given variable by that variable with an upper bar. Since $t \in [0,1]$, $\{t^n\}_{n=1}^{\infty}$ Converges. Although we would need to consider three distinct cases, when $\bar{t}=0, \bar{t}=1$ or $\bar{t} \in (0,1)$, the first two cases essentially amount to repeating the proof of Lemma 4.4. We consider the remaining case $\bar{t} \in (0,1)$. [Note that we are omitting the other homotopy parameter, which links (γ, u) to (γ^*, u^*): this part repeats the proof as in Lemma 4.4, and that is why it is omitted.]

First, since $t \in (0,1)$, $\pi_h^{kn}(t^n) \to \overline{\pi}_h^k(\bar{t}) \gg 0$. $\left\{\left(\theta_h^2\right)^n\right\}_{n=1}^{\infty}$ Converges, say to $\overline{\theta}_h$, because it is contained in the compact set $[0,1]$. For at least one k, $\left\{x_{wh}^{kn}\right\}_{n=1}^{\infty}$ is bounded above by equation $(G.2.11)$ and below by zero. Observe that for $s=1,2,$ we have $z_h^{skn} = v_h\left(x_{wh}^{skn}\right)$ and we repeat the argument in Lemma 4.4, to obtain Convergence of $\{x_{wh}^{skn}\}_{n=1}^{\infty}$. Similarly, for $s=1,2$, ρ_h^{skn} Converges because of (4.1.9) and the fact that $p^C=1$. From equation (4.1.9), $\{p^n\}_{n=1}^{\infty}$ Converges to $\pi_h^{s_h k}(\bar{t}) D v_h\left(\overline{x}_{wh}^{s_h k}\right)\left(\overline{\rho}_h^{s_h k}\right)^{-1} \gg 0$. As for the other k, using equation (4.1.10) and convergence of prices, we get Convergence of $\{x_{wh}^{skn}\}_{n=1}^{\infty}$for $s=1,2$.

For $s=1,2$, since $z_h^{skn} = v_h\left(x_{wh}^{sk}\right)$, we have that $\{z_h^{skn}\}_{n=1}^{\infty}$ Converges. Again for at least one k, $\left\{x_{fh}^{kn}\right\}$is bounded above by total resources and below by zero. Moreover, for any $s=1,2$, any h,n

$$u\left(x_{fh}^{skn}\right) \geq u\left(\left[1-\phi_h\left(z_h^{skn}\right)\right]\left(y^{sn}+e^n\right)\right)$$

and therefore

$$u\left(\overline{x}_{fh}^{sk}\right) \geq u\left(\left(1-\phi_h\left(\overline{z}^{sk}\right)\left(\overline{y}^s+\overline{e}\right)\right)\right)$$

>From the boundary condition $x_{fh}^{skn} \to \overline{x}_{fh}^{sk} \gg 0$, i.e., $\{x_{fh}^{skn}\}_{n=1}^{\infty}$ Converges, and $\{\alpha_{fh}^{skn}\}_{n=1}^{\infty}$ Converges, for $s=1,2$.

For $s=3$, observe that $\{x_{wh}^{3kn}\}_{n=1}^{\infty}$ Converges because of (G.2.8) and of the boundary condition on preferences. Also, $\rho_h^{3kn} \to \frac{1-\bar{t}}{\bar{t}}\left(\overline{x}_{wh}^{3k}-\varepsilon\mathbf{1}\right)+\overline{\pi}_h^{3k}(\bar{t}) D v_h\left(\overline{x}_{wh}^{3k}\right) \lesseqgtr 0$, and therefore Converges. Then we use (G.1.1) and (G.1.6) to get, for $c = 1,...,C$,

$$\begin{aligned} 0 \;=\; & (1-\bar{t})(\overline{x}_{fh}^{3kc}-\varepsilon)+ \\ & \bar{t}[\pi_h^{3k}(\bar{t}) D_c u_h\left(\overline{x}_{fh}^{3k}\right)-\frac{\bar{t}}{1-\bar{t}}\left[\overline{p x}_{fh}^{3k}-\left(1-\phi_h\left(v_h\left(\overline{x}_{wh}^{3k}\right)\right)\right)\overline{p}(\overline{y}^3+\overline{e}_h)\right]\overline{p}^c] \end{aligned} \tag{G.3}$$

Then if there exists c such that $x_{fh}^{3kcn} \to 0$, by Assumption 2 we have

$$\lim_{x_{fh}^{3kcn}\to 0} D_c u_h\left(x_{fh}^{sk}\right) = +\infty$$

and from $(G.3)$ we get a contradiction. Therefore, $\{x_{fh}^{3kn}\}_{n=1}^{\infty}$ Converges. Then $\{\alpha_h^{3kn}\}_{n=1}^{\infty}$ Converges because of the price normalization and of (4.1.1) and (G.1.1).

As for z_h^{3kn}, observe the following. If $\delta_h^{kn} \to +\infty$ or $\delta_h^{kn} \to \overline{\delta}_h^k > 0$, from equation (G.2.7), z_h^{3kn} Converges. If $\delta_h^{kn} \to \overline{\delta}_h^k = 0$, then from equation (4.1.2), β_h^{kn} converges. Now if $z_h^{3kn} \to +\infty$, then, by Assumption 3, $\phi_h\left(z_h^{3kn}\right) \to +\infty$, which contradicts equation $(G.1.10)$. If $z_h^{3kn} \to -\infty$, then again by Assumption 3 $\phi_h'\left(z_h^{3kn}\right) \to 0$, which contradicts equation $(G.1.2)$. $\{\beta_h^n\}_{n=1}^{\infty}$ and $\{\delta_h^n\}_{n=1}^{\infty}$ Converge because of equations $(G.1.2)$, i.e.,

$$\begin{bmatrix} \beta^n \\ \delta^n \end{bmatrix} = \left[A\left(\pi^n, \pi^{*n}\right)\right]^{-1} \chi^n$$

with obvious notation - and $A\left(\pi^n, \pi^{*n}\right)$ has full rank along the sequence and in the limit and χ^n Converges. The remaining of the argument is identical to that of Lemma 4.4. ■

H. Regularity results

First, assume smoothness and that there is h, k and s, s' with $s \neq s'$such that, say, $x_{fh}^{s,k} = x_{fh}^{s',k}$. Then these equations and, from (4.1.1),

$$\begin{array}{l} \pi_h^{sk} D u_h\left(x_{fh}^{sk}\right) - \alpha_h^{sk} p = 0 \\ \pi_h^{s'k} D u_h\left(x_{fh}^{s'k}\right) - \alpha_h^{s'k} p = 0 \end{array} \tag{H.1}$$

are equivalent to

$$\begin{array}{l} \pi_h^{sk} D u_h\left(x_{fh}^{sk}\right) - \alpha_h^{sk} p = 0 \\ \pi_h^{sk}/\alpha_h^{sk} - \pi_h^{s'k}/\alpha_h^{s'k} = 0 \\ x_{fh}^{s,k} = x_{fh}^{s',k} \end{array} \tag{H.2}$$

After substituting equations (H.2) for (H.1) in (4.1), we count one too many equations. It is immediate now that we can essentially perturb this modified equation system as in (5.5), with the additional use of π_h^{sk}, and therefore conclude that there is an open and dense subset of the parameters where the function F is smooth and for which no solution to system (4.1) exists when $x_{fh}^{s,k} = x_{fh}^{s',k}$.

Second, we remove the smoothness assumption. At any solution to (4.1): (i) either $\underline{\eta}_h = 0$ or $\theta_h^2 = 0$, all h; (ii) either $\overline{\eta}_h = 0$ or $\theta_h^2 = 1$, all h; (iii) either $\delta_h^k = 0$ or $f_{2h}^k(.) = 0$, all k, h; and never both.

It is sufficient to consider one case, say (i), as all the others follow the same way, and they are at most finitely many combinations.

Suppose that $\underline{\eta}_h = 0$ and $\theta_h^2 = 0$, some h. Then we can write system (4.1) equivalently by substituting equation (4.1.4) and adding the equations $\theta_h^2 = 0$. Let $F' = 0$ represent this modified system. We know we can perturb this modified system (this is trivially shown) and apply transversality to obtain a dense subset of where $DF'_{\gamma,u}$ has full rank. Hence for γ, u in this dense subset, $F'^{-1}_{\gamma,u}(0)$ is a manifold of negative dimension, or the empty set. To see this, simply notice that in system (4.1) the number of equations and unknowns is equal, so that in the modified system $F'_{\gamma,u} = 0$ there is one too many equations. Of course, $\delta_h^1 = 0$ and $f_{2h}^1(.) > 0$ in equilibrium for all economies, a standard result for principal-agents models.

To show that conditions B) holds generically, we proceed as follows. We append the equations $\gamma A' = 0$, with $\gamma \in \mathbb{R}^{C-1}$ and $b^T b = 1$ to (4.1). Then, using a perturbation of the gradient of the agent utility function which leaves unchanged the direct and indirect utility function at equilibrium, we show that the augmented version of (4.1) generically has no solution. A standard argument applies to show the genericity of condition C).

■

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