Quantitative analysis of multi-periodic supply chain contracts with options via stochastic programming

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Abstract

We propose a stochastic programming approach for quantitative analysis of supply contracts, involving flexibility, between a buyer and a supplier, in a supply chain framework. Specifically, we consider the case of multi-periodic contracts in the face of correlated demands. To design such contracts, one has to estimate the savings or costs induced for both parties, as well as the optimal orders and commitments. We show how to model the stochastic process of the demand and the decision problem for both parties using the algebraic modeling language AMPL. The resulting linear programs are solved with a commercial linear programming solver. We compute the economic performance of these contracts, giving evidence that this methodology allows to gain insight into realistic problems.

1 Introduction

The purpose of supply chain management (SCM) is to improve the overall efficiency of a network of producers, retailers and customers, while preserving a decentralized approach to the decision-making process. Coordination between independent units can be achieved through appropriate exchanges of information. In that respect, contracts offer a large variety of possibilities to the mutual benefit of the contractors.

To evaluate the impact of a given contract on the behavior of the agents and on the performance of the system, it is necessary to perform a quantitative analysis. This requires the building of a mathematical model that describes the environment (demand, capacities, etc.) and the use of mathematical tools to treat these models. To illustrate our point, we focus on a model for supply contracts with options. This model was proposed in [4], and the authors have provided an extensive theoretical analysis. In the present paper, we concentrate on the quantitative analysis of such supply contracts: given profit or costs reallocation rules, what are the potential savings and the future extra-costs, for the buyer and for the supplier? Such estimates constitute a difficult issue, since profit and costs depend not only on the demand pattern and the values of some base parameters, e.g., finished product selling price, raw material purchasing price, etc., but also on the procurement and production policies of the contractors. The traditional computing approaches are based on the news-vendor model and/or on dynamic programming

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[2, 3, 4, 5, 6, 25, 26]. They become intractable on general multi-periodic problems and/or on problems with constraints on the state variables. In this paper, we propose an approach based on stochastic programming, which does not suffer from the same limitations.

A main issue in a practical implementation of stochastic programming is the building and solving of the stochastic programming model from the deterministic and stochastic description of the problem. The thrust of our paper is that, in the case of supply chain with option contracts on a multi-period basis, the associate models can be compactly written in an algebraic modeling language such as AMPL [15], and solve by commercial linear programming solvers, such as CPLEX [12]. Those tools powerful enough to build a model and produce in a solution in a reasonable amount of time, allowing the decision-makers to evaluate the potential merit of a given contract on the fly and perform a sensitivity analysis on the parameters.

As far as modeling is concerned, models with several periods are also easily constructed. In order to illustrate our program, we mainly focus on the 2-period model of [4]. We show how one can reformulate it in the stochastic programming framework. We compute the optimal strategy for the buyer and for the supplier separately. We then compare the individual performance with the global optimum of a centralized policy in a vertical integrated framework, exploited as a benchmark.

The paper is organized as follows. In Section 2, we review the supply chain contracts that are studied in the literature. Section 3 discusses the formulation of stochastic models in production/inventory supply chain management as stochastic linear programming problems. Section 4 focuses on a specific multiperiodic supply chain contract with options. We provide the corresponding models for the buyer and the supplier. Section 5 deals with auxiliary models, such as vertically integrated system, or a Stackelberg game associated with the considered contract with options. Section 6 reports on numerical studies of the models presented in the preceding two sections. In the conclusion, we review some interesting extensions and new avenues of research.

2 Supply chain contracts

Supply chain management (SCM) deals with the management of information and material flow in a network of producers, retailers and customers. Basically the flow control issue in SCM can be related to the multi-echelon inventory theory, involving stochastic periodic demands, developed by Clark and Scarf [11], which considers that the whole system is under a centralized decision process. Using a dynamic programming formulation, these authors showed that, under strict assumptions, the optimal policy is of the "order-up-to" type for each installation in the system.

The assumption of shared information made in centralized multi-echelon inventory theory is not realistic in the supply chain context. Moreover, the multiple decision makers in a supply chain system run different firms or divisions, making it difficult to implement centralized control. As analyzed in [10, 22], locally rational behavior can be inefficient from a global perspective. Thus, efficient management requires the coordination of independently managed entities, seeking to maximize their own profit.

Classical markets mechanisms appear to be inefficient to achieve such a coordination, because decision makers have private information which they don't share with others. A way to circumvent this difficulty consists of resorting to contracts. Such contracts include the reallocation of decision rights, rules for sharing the costs of inventory and stockouts as well as rules for sharing information. In the SCM framework, the contract analysis is focused on operational details and requires an explicit modeling of materials flows, forecasting and planning process and the stochastic aspects which perturb the whole system (see for example [7, 8]).

To be efficient, the contract between the supplier and the buyer has to be rewarding to both parties, although each party faces different tradeoff. A focal point in supply contracting is flexibility and risk sharing when confronted to the dynamic and uncertain nature of the demand process. Full flexibility would allow the buyer to purchase and receive any amount and thus meet the uncertain demand with greater probability, at a lower cost, but at the expense of the supplier. Contracts aim at improving flexibility, while preserving the interests of both parties. Typically, they allow the buyer to adjust current orders and future commitments in a limited way, possibly at a given extra-cost paid to the supplier. In exchange, the supplier is bound to meet the buyer's demand within the range of the commitments. This entails additional costs in capacity and raw material to quickly meet the uncertain demand from the buyer. A good contract should compensate for those costs to create a win-win situation for both the buyer and the supplier.

Let us briefly review the main types of contracts that have been considered in the literature.

1. *The total minimum quantity commitment [6].* The buyer guarantees that his cumulative orders for all periods in the contract horizon will exceed a specified minimum quantity. In return, the supplier offers price discounts. In practice, the supplier provides a menu of (per unit price, total minimum commitment) pairs from which the buyer chooses a commitment at the corresponding price. As reported in [13], this kind of contract has been implemented in the textile industry.
2. *The total minimum quantity commitment with flexibility [6].* The supplier imposes restrictions on the total purchases at the discounted price (for example due to limited production capacity). Any quantity ordered above the restriction is available at a higher price.
3. *The periodical stationary commitment [2, 23, 24].* The buyer is required to purchase a fixed amount in each period. Discounts are given based on the level of minimum commitment. Additional units can be purchased at an extra cost. Often, the supplier imposes restrictions on the total purchases at the discounted price (for example due to limited production capacity). Any quantity ordered above the restriction is available at a higher price.
4. *The periodical commitment with order bands [25].* The buyer is required to restrict the order quantities to be within constant specified lower and upper limits. The unit price depends on the band-width (the difference between the upper and lower limits) and increases with this band-width.
5. *The periodical commitment with rolling horizon flexibility [3, 5].* At the beginning of the horizon, the buyer commits to purchase given quantities every period. The buyer has a limited flexibility to purchase quantities different from the original commitments. The buyer is also allowed to update the previously made commitments, within a given limitation. The unit price decreases with the allowed flexibility. This kind of contract is exploited in the electronics industry [5, 26].
6. *The periodical commitment with options [4].* At the beginning of the horizon, the buyer commits to purchase given quantities every period. The buyer has a limited flexibility to purchase options (at unit option price) from the supplier that allows him to buy additional units, by paying an exercise price. So, options permit the buyer to adjust orders quantities to the observed demands. It is shown in [4], that under some assumptions, these contracts with options encompass backup agreements contracts, periodical commitment contracts with flexibility and pay-to-delay arrangements.

To choose among the various alternatives and to design a good contract one needs to provide reliable estimates of the potential savings and risks for the parties. To meet this goal, the operating modes and the relationship among the actors must be apprehended in carefully designed quantitative models, and those models must be analyzed with appropriate quantitative tools. The quantitative methodology in the quoted literature dealing with the contracts reviewed above, is almost exclusively based on the news-vendor model and dynamic programming. To produce tractable computational schemes, this methodology imposes on the models special assumptions and approximations. For example, constraints on the state variables raise serious difficulties in dynamic programming. Correlated demands in multiperiodic problems also introduce enormous complications.

We believe there is a need for efficient and easily implementable formulation and computational schemes to analyze contracts, either in general or in specific instances. The methodology should be flexible enough to incorporate local specifications, and fast enough to provide on-the-fly evaluations. Our contention is that stochastic programming is an attractive solution method, that does not suffer from the same limitations as the other approaches. In particular, it is possible to account for the dynamics of the underlying demand and to handle general constraints on the decision and state variables, such as a minimum service level or a guaranteed minimum profit.

3 Stochastic programming models in production/inventory management

From a general perspective, there are two main components in a stochastic programming problem: a description of the underlying stochastic process and a deterministic discrete time dynamic model. It is relatively easy to describe those two components separately, but rather tedious and difficult to build the integrated stochastic programming model (see for example [20].) One of the active research stream in stochastic programming is the development of automatic procedures to build the stochastic programming model from its two basic components [16]. In the lack of widely available tools, we develop our own procedure based on the algebraic modeling language AMPL [15]. Our methodology is inspired by the principles exposed in [18]. It extends some of the ideas that appeared in [17].

In this section, we present our approach and we illustrate it on a simple stochastic production/inventory model. We hope that the simplicity of the deterministic production/inventory model makes it easier to follow the steps of the implementation; in sections 4 and 5, we shall describe explicitly supply chain contracts.

The main task in implementing stochastic programming is to build a so-called deterministic equivalent and to solve it by means of some mathematical programming technique. The basic assumption underlying the concept of deterministic equivalent is that the decision-maker does not influence the chance moves: i.e., for our problems of interest, the successive stochastic demands are not affected by the states and/or decision variables. We introduce the further assumption that the stochastic demand process takes discrete values only. This makes it possible to formulate the deterministic equivalent as a finite dimensional mathematical programming problem.

Let us briefly introduce our three step approach to model building based on an algebraic modeling language.

1. The first step consists in writing the deterministic model.
2. The second step models the demand process as a discrete event stochastic process. This process is represented by an event tree, i.e., a set of nodes that are linked by transition arcs.
3. Finally, the deterministic decision model and the stochastic demand process representation are merged to create a full-fledged stochastic programming model.

The thrust of our approach is that each step can be efficiently performed by means of short AMPL instructions. We detail each step separately.

3.1 The base deterministic model

The basic structure[1] of the models involved in the supply chain contracts problematic corresponds to the classical multi-periodic production/inventory problem:

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{T}(h_t I_t^+ + s_t I_t^- + \sum_{k=0}^{t-1} c_{t-k} X_{k,t}) && \text{(1a)}\\
\text{s. t.} \quad & I_t = I_t^+ - I_t^-,\ t = 1, \ldots, T && \text{(1b)}\\
& I_t = I_{t-1} + \sum_{k=0}^{t-1} X_{k,t} - D_t,\ t = 2, \ldots, T && \text{(1c)}\\
& I_1 = X_{0,1} - D_1, && \text{(1d)}\\
& I_t^+,\ I_t^-,\ X_{k,t} \geq 0,\ t = 1, \ldots, T,\ t = k, \ldots, t. && \text{(1e)}
\end{aligned}
$$

The decision variables, representing orders and their delivery, are denoted as X. The remaining variables, I, I^+ and I^-, are state variables. They can be interpreted as the state of the inventory; they are split into a positive and a negative part, respectively corresponding to physical inventory and backlog. The objective involves three types of costs: the ordering/production costs (which depend on the delivery lead-time), the holding costs and the shortage costs. Finally, the parameter D represents the successive demand values. In our models, the demands are exogenously defined.

Note that our decision variables $X_{k,t}$ are double indexed. The first index denotes the period at which the variable is fixed, while the second refers to the period at which the decision takes effect. It is not necessary to make such a distinction in a purely deterministic context, but this will prove to be a considerable help in transforming the deterministic model to match its stochastic environment.

This linear programming problem is most easily formulated in an algebraic modeling language, as shown in Figure B.1 in the appendix.

3.2 The stochastic demand process

The demand process for the successive time periods is modeled as a discrete stochastic process $\{D_t\}_{t=1,\ldots,T}$. The distinguishing features for such a process are the relationships among the successive random variables $D_1, \ldots, D_T$. Since we consider discrete state space processes, it suffices to list all possible realizations $\{d_1, \ldots, d_T\}$, and give their associate probabilities to get a probabilistic description of the entire process. Theoretically, the above information can be given as an exhaustive list, but in many practical situations it may be generated from a more restricted set of information. Without considerable loss of generality, we shall consider in this paper that the process is described via one-step transitions from D_t to D_{t+1}, $t = 1, \ldots, T-1$. Namely, given the value of an observed demand d_t, we assume that we know all possible realizations for D_{t+1} with their associate conditional probabilities.

We make now the following specific assumptions:

1. The number of possible discrete values for the demand D_t is denoted n_t and depends only on the time period t. This number n_t is independent from the previous demand value d_{t-1}.
2. The conditional transition probabilities from D_t to D_{t+1}, with $t = 1, \ldots, T-1$, depend on the period t at which the transition takes place, but not on the demand value d_t.

[1] For sake of clarity, we consider a particularly simple model in this section, which is mainly devoted to conceptual considerations. Supply chain related models are developed in sections 4 and 5.

3. The possible discrete values of the stochastic process D_{t+1} may be functions of the observed value at the preceding period d_t.

Clearly, those assumption are somewhat restrictive, but they allow an efficient formulation for the problem studied in Section 4. They could be relaxed to build more general models, but this is not the purpose of the present paper. It is then an easy matter to reconstruct the probability distribution for any random sequence $\{D_0, D_1, \dots, D_t\}$, as described further in the section.

3.2.1 The demand stochastic process as an event tree

Such a discrete stochastic process can be displayed on a so-called *event tree*. The nodes of the event tree represent the state of the process at a given period, while the (oriented) arcs correspond to the probabilistic transitions from one node at given period to another node at the next period. The main point is that there exists exactly one arc leading to a node, while there may be many arcs emanating from a node.

With our assumptions for the demand process, the associate event tree is symmetric. Let[2] $f[t]$ be the number of branches emanating from a node; it depends only on t. Let also denote $N[t]$ the number of nodes at period t.

As any tree, the event tree is unequivocally described by a node numbering and a predecessor function that gives the number of the node that immediately precedes the current node. We choose a rather natural node numbering by periods and define the predecessor mapping as a function that can be computed by the modeling language.

The tree is rooted at period $t = 0$, but the first realization takes place at $t = 1$. The number of nodes at period t is then recursively computed by

$$N[t] = f[t-1]N[t-1],$$

and $N[0] = 1$. Nodes on the event tree are indexed with a pair (t, n), with $t \in \{0, \dots T\}$ and $n \in \{1, \dots N[t]\}$. At period t, the nodes are numbered from 1 to $N[t]$ going from top to bottom. Thus, node (t, n) is the n-th node from the top in period t. Similarly, the transitions from a node to its immediate successors are numbered from top to bottom.

To illustrate our point, we considers the case of a three-period model (with $T = 2$) and we represent it on Figure 3.2.1 page 7. In this representation, the uncertainty unfolds in time from left to right. Nodes appearing in the same vertical slice belong to the same time period. At $t = 0$, there are three branches giving rise to three nodes in $t = 1$ ($f[0] = 3$ and N[1]=3). At $t = 1$, each node has two branches ($f[1] = 2$) and there are $N[2] = 6$ nodes in $t = 2$. Time $t = 2$ is the horizon: no branch emanates from those nodes.

It is possible now to list the data that are required to represent the discrete demand stochastic process as such an event tree:

- $f[t]$ is the number of transitions from a node in period t. These transitions are thus indexed from 1 to $f[t]$.
- The stochastic process value at node (t, n) is $D[t, n]$. The stochastic process value at $t + 1$ is given as a function of the origin node (t, n), of the value at this node, $D[t, n]$, and of the transition index $k \in \{1, \dots f[t]\}$.
- The conditional transition probabilities $p[t, k]$ are given as function of the period t and the transition index $k \in \{1, \dots f[t]\}$.

It is worth pointing out that the size of the tree is a direct multiplication of the transition numbers. With more than three periods, the tree size may grow enormous if

[2]From then on, we choose to insert the argument of a function within square brackets '[' and ']'. This notation matches the one in force in the algebraic modeling language AMPL, which we use to model and solve the examples.

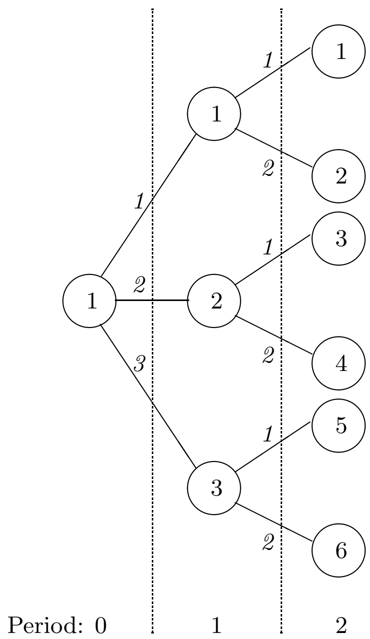

Figure 1: Event tree representation

one allows more than a very few transitions per period. This must be kept in mind in building models that are solvable by commercial solvers. Furthermore, the above quite general assumptions allow to model correlated demands.

3.2.2 Navigating through the tree

Since some constraints link two successive periods, we need to introduce an auxiliary function that help backtracking from a node to its predecessors. The predecessor function $a[t, n, k]$ maps the current node (t, n) to the index of its predecessor node in period $t - k$ along the unique path that goes from the root $(0, 1)$ to the node (t, n). This function is recursively defined by

$$a[t, n, k] = \begin{cases} \left\lceil \frac{n}{f[t-1]} \right\rceil & \text{if } k = 1, \\ a[t-1, a[t, n, 1], k-1] & \text{otherwise,} \end{cases} \tag{2}$$

for all $1 \leq k \leq t$, with $\lceil x \rceil$ defined as the smallest integer n with $n \geq x$.

We find it convenient, though not strictly necessary in the present case, to define a scenario as the path from the root $(0, 1)$ to any terminal node (leaf of the tree) (T, n).

Let's now introduce the function $b[t, n]$ which gives the index of the node in slice t that is traversed by the scenario leading to node (T, n). The mapping b is thus

$$b[t, n] = a[T, n, T - t]. \tag{3}$$

Via this function, the scenario associated with the terminal node (T, n) can be represented as the sequence of nodes $\{(0, 1), (1, b[1, n]), (2, b[2, n]), \ldots (T, b[T, n]), (T, n)\}$.

Finally, we need the auxiliary function $\ell[t, n]$ defined as

$$\ell[t, n] = n - (a[t, n, 1] - 1) f[t - 1], \tag{4}$$

which gives the transition index that caused the transition from $(t-1, a[t,n,1])$ to (t,n).

3.2.3 Computing nodes probabilities

In sake of clarity in this introductory section, we consider the case that the demand values $D[t,n]$ are exogenously given[3]. Consider then $p[t,j]$ the conditional transition probabilities from a node (t,n) to a successor $(t+1,m)$, where $n = a[t+1,m,1]$ and $\ell[t+1,m] = j$. From our assumptions, those probabilities depend on t and the transition index j, but not on the node index n; they are exogenously given as parameters.

The functions $a[\cdot,\cdot,\cdot]$ and $\ell[\cdot,\cdot]$ make it possible to define as computable functions the probabilistic parameters of the problem, e.g., the node probabilities, associated with the different demand values. As a matter of fact, the unconditional occurrence probability $P[t,n]$ of node (t,n) can be recursively computed by

$$P[t,n] = p[t-1, \ell[t,n]]\ P[t-1, a[t,n,1]], \tag{5}$$

with $P[0,1] = 1$. Note that $P[T,n]$ can be viewed as the probability of the scenario leading to node (T,n).

3.3 Building the full stochastic programming model

The full stochastic programing model superposes decisions and chance moves. It is important to specify in which order decisions and chance moves occur. To illustrate our convention, we pictured the sequence of decisions and chance moves on Figure 3.3, page 8.

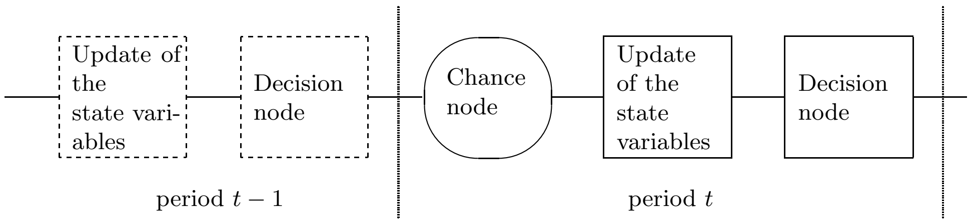

Figure 2: Sequence in the decision process

In this picture, the terminal period does not include a decision node; on the other hand, a unique chance node has to be introduced at the beginning of the process. Clearly, this representation calls for one more period than the horizon T. As previously stated, our choice is to start with $t = 0$ and terminate with $t = T$.

3.3.1 The deterministic equivalent in the event tree representation

Now that we have in hands the tools to handle the stochastic process, we can formulate the stochastic counterpart of Problem (1).

The general stochastic linear program. In the stochastic version, the demands are stochastic, but also the decision and state variables. Once the criterion in (1) has been

[3] More generally, the demand values $D[t,n]$ are not exogenously given. Then these quantities can be computed by

$$D[t,n] = h[D[t-1, a[t,n,1]], l[t,n]],$$

where h is some function computable through the algebraic language replaced by an expected value, the problem becomes

$$
\begin{aligned}
\min \quad & E\left[\sum_{t=1}^{T}(h_t I_t^{+} + s_t I_t^{-} + \sum_{k=0}^{t-1} c_{t-k} X_{k,t})\right] && \text{(6a)}\\
\text{s. t.} \quad & I_t = I_t^{+} - I_t^{-},\ t = 1, \ldots, T && \text{(6b)}\\
& I_t = I_{t-1} + \sum_{k=0}^{t-1} X_{k,t} - D_t,\ t = 2, \ldots, T && \text{(6c)}\\
& I_1 = X_{0,1} - D_1, && \text{(6d)}\\
& I_t^{+},\ I_t^{-},\ X_{k,t} \geq 0,\ t = 1, \ldots, T,\ k = 1, \ldots, t. && \text{(6e)}
\end{aligned}
$$

Problem (6) is a stochastic linear programming problem, but we need to make a few notational changes to make this property transparent w.r.t. the event tree description of the demand stochastic process.

The event tree related formalism. Two main operations are in order. First, we must index variables, parameters and constraints with respect to the tree nodes. Second, we must introduce the probabilistic elements that were defined in the previous sections; namely the parameters a, b and ℓ defined in (2), (3) and (4), and the probabilities P given in (5).

All constraints indexed by t, namely constraints (6b)-(6e), should now be indexed by t and by n. As an illustration, consider the case of a variable, say I, with a single time index $\tau \leq t$ appearing in a constraint associated with the node (t, n). The index τ specifies the date at which the variable has been fixed. Then I_τ should be replaced by $I[\tau, a[t, n, t-\tau]]$. A similar treatment is to be applied to all parameters, such as D_t, that are time and node dependent. This notational changes are not necessary for the parameters which are not node dependent, as for example c_t, h_t or s_t.

Variables that are endowed with two time indices such as $X_{\tau,t}$, with $\tau \leq t$, appearing in a constraint associated with the node (t, n), should be treated with care. The first index τ should be expanded to include a node reference, since it refers to the date at which the value of the variable is fixed; this choice is contingent on node $(\tau, a[t, n, t-\tau])$. The second index t refers to the date at which the decision takes effects: the value of the variable is thus the same on all nodes in slice t that have common predecessor $(\tau, a[t, n, t-\tau])$ in period τ. Consequently, the variable $X_{\tau,t}$ that appears in constraint (t, n), becomes $X[\tau, a[t, n, t-\tau], t]$.

The event tree related stochastic linear program. By introducing the event tree related formalism in the general stochastic linear program (6a)-(6e), we find the following explicit model. At first, let the total costs incurred up to time T, when the process ends in node (T, n), be reformulated as

$$
\mathcal{E}[T, n] = \sum_{t=1}^{T}(h_t I^{+}[t, b[t, n]] + s_t I^{-}[t, b[t, n)]] + \sum_{k=0}^{t-1} c_{t-k} X[k, b[k, n], t]).
$$

Using this expression, we can rewrite the deterministic equivalent linear programming problem associated with the event tree representation of the stochastic process $(D_t)_{t=1,\ldots,T}$:

$$
\begin{aligned}
\min \quad & \sum_{n=1}^{N[T]} P[T, n]\mathcal{E}[T, n] && \text{(7a)}\\
\text{s.t.} \quad & I[1, n] = X[0, 1, 1] - D[1, n], \qquad n = 1, \ldots N[1], && \text{(7b)}\\
& I[t, n] = I^{+}[t, n] - I^{-}[t, n], \\
& \qquad\qquad t = 1, \ldots T, \quad n = 1, \ldots N[t], && \text{(7c)}
\end{aligned}
$$

$$I[t,n] = I[t-1, a[t,n,1]] + \sum_{k=0}^{t-1} X[k, a[t,n,t-k], t] - D[t,n], \qquad t = 2, \ldots T,\ n = 1, \ldots N[t], \tag{7d}$$

$$I^{+}[t,n] \geq 0,\ I^{-}[t,n] \geq 0,\ X[t-1,n,t+k] \geq 0, \qquad t = 1, \ldots T, \quad n = 1, \ldots N[t],\ k = 1, \ldots T-t \tag{7e}$$

A complete AMPL formulation of this problem is given in the appendix B.3.

4 The basic model of multi-periodic supply contract with options

In this paper we focus on a general case of single buyer-single supplier contract with periodical commitment with options, in presence of periodical correlated demands, as studied in [4]. It is shown in [4], that under some assumptions, this kind of contract with options encompasses most of the contracts developed in the literature.

The supply chain instance considered in [4] involves two actors: a supplier and a buyer. The supplier transforms raw materials into a finished good that is sold and delivered to the buyer. The buyer is an intermediary between the market and the supplier, selling the finished product to clients on the end market. The buyer position is that of a wholesale dealer or a factory performing final assembly. In the latter case, the assembly time is considered to be negligible.

The management decisions for the actors are as follows. The supplier acquires raw material at the beginning of the campaign and stores it for future use. He produces, stores and delivers the finished product to the buyer. Production is limited by the availability of raw material. The buyer receives, stores and sells to end markets the finished product. The buyer faces a periodic uncertain demand for the finished product. Leftover products after the campaign are sold at salvage price, while demand on excess of availability is lost.

The buyer and the supplier agree on a contract which specifies the amount to be delivered in each period, with possible adjustment from the buyer. The supplier is committed to deliver the amount of finished product that has been agreed upon in the contract. The model in [4] has only two periods, but can be extended to a larger number of periods. Each period offers the possibility to adjust production and ordering decisions to match the revealed demand. The paper [4] considers this relationship between the supplier and the buyer as a two-person game with a leader (the supplier) and a follower (the buyer), which induces that the appropriate solution concept is a Stackelberg optimum. This game interpretation will be also considered here.

As the objective of the paper is to analyze how apply stochastic programming to a given supply contract, we choose, without loss of generality, to keep the notations of the flow models for the buyer and the supplier as simple as possible. Characteristics specific to a given practical situation could be added.

4.1 The discrete market demand process

The frequent assumption that demands are uncorrelated (except in [4, 23]) does not reflect reality. For example, the widespread exponential smoothing techniques exploits such correlations in the computation of the forecasts. Charnes et al. in [9] describe the importance of considering serial correlation in the optimization of inventory control policies, even when the magnitude of the autocorrelation is low. Furthermore, Fisher et al. [14] show that when managing short life-cycle products, simple extrapolation of a small amount of early sales provides much better forecasts that traditional experts approaches.

In this case, autocorrelation proves to be the necessary condition for an efficient forecasting and planning alternative, called Quick Response.

In the present paper, we consider the case of correlated periodic demands, as in [4], but, as we exploit an event tree formulation, we have to implement a discretization procedure in order to get a discrete demand process. Let's first resume the assumptions underlying the demand process in [4].

Structure of the correlated demand process. This market demand process is described by the conditional distribution of D_{t+1}, conditionally to $D_t = d_t$. This conditional distribution is assumed to be Gaussian[4], i.e.,

$$D_{t+1} \mid d_t = E(D_{t+1} \mid d_t) + \epsilon_{t+1}\sqrt{\mathrm{Var}(D_{t+1} \mid d_t)},$$

where $\{\epsilon_t\}_{t=1,\dots,T}$ is a set of independent identically distributed random variables, with normal distribution with mean 0 and unit standard deviation.

Let μ_t and σ_t be the unconditional mean and variance of D_t. Then, the considered conditional scheme is defined by the equations

$$E(D_{t+1} \mid d_t) = \mu_{t+1} + \rho_{t,t+1}\frac{\sigma_{t+1}}{\sigma_t}(d_t - \mu_t) \tag{8}$$

and

$$\mathrm{Var}(D_{t+1} \mid d_t) = \sigma^2_{t+1}(1 - \rho^2_{t,t+1}), \tag{9}$$

where $\rho_{t,t+1}$ is the correlation coefficient between the successive demands D_t and D_{t+1}.

The discretization procedure. The discretization of the continuous random variables D_t is obtained by discretization of the standard normal distribution ϵ_t. The basic discretization scheme we use is described in Appendix A. The scheme generates $2n_t + 1$ points on the grid, where n_t is a parameter that can take different values at different time period t.

Deciding on the level of discretization is a delicate issue. Clearly, finer grids lead to better approximation, provided the assumed normal distribution is itself a good approximation of the true demand process. Unfortunately, the event tree, and the problem, size grow exponentially with the grid sizes. The numerical studies reported in Section 6 show that a fine discretization in the first period coupled with cruder grids on later periods still yields satisfactory results.

4.2 The buyer's problem

Consider a classical periodic review inventory problem with a finite horizon stochastic demand. Let T be the horizon, i.e., the number of periods involved in the contract. The sequence of events and actions taken by the buyer are as follows. At the beginning of the horizon, the buyer orders $Q_{0,t}$ units to be delivered in period $t \in \{1, .., T\}$, at a unit wholesale price of p_t. These $Q_{0,t}$ are referred as *firm orders*.

In addition to this classical rigid purchasing process, the buyer has a limited flexibility to purchase *options* (at unit option price) from the supplier that allow him to buy additional units, by paying an exercise price. We assume that one option gives the buyer a right to purchase one unit. Formally, at the initial period the buyer purchases $M_{0,t}$ options, with $t \in \{1, .., T-1\}$, at a unit option price of o_t. We assume that each decision variable $M_{0,t}$ is bounded above by a given constant $\overline{M}_t$. Then, in period t, after observing

[4] In fact, for practical applications it is necessary to consider truncated Gaussian distributions, as any demand has to be positive. For sake of clarity in the formulation, we don't introduce this feature in the models, even if the numerical examples in section 6, we consider truncated distribution in the numerical procedures.

demand value d_t, the buyer chooses to exercise $m_{t,t+1} \leq M_t$ options to be delivered in period $t+1$, at a unit price of e_t.[5]

In each period, excess demand is assumed to be backlogged and excess inventory is carried to the next period. In time period t, the unit holding cost is h_t^{bf} and the unit shortage cost is equal to s_t^{bf}. At the end of the horizon T, the left-over inventory can be either sold to an external specific market for a unit salvage price v_e^{bf}, either returned to the supplier for a unit salvage price v_s^{bf}. Clearly, if $v_e^{bf} < v_s^{bf}$, the return option is preferred, otherwise the left-over inventory are salvaged on the external market. So, we denote $v^{bf} := \max(v_e^{bf}, v_s^{bf})$. Let us summarize below the variables and parameters involved in the buyer's problem.

Buyer's decision variables

for $t = 1, ..., T-1$
- $M_{0,t} \geq 0$: number of options, which can be exercised in period t
- $m_{t,t+1} \geq 0$: number of options exercised in period t to be delivered in period $t+1$

for $t = 1, ..., T$
- $Q_{0,t} \geq 0$: firm order to be delivered to the buyer in period t

Buyer's state variables

for $t = 0, ..., T$
- I_t : finished good inventory at the end of period t
- $I_t^+ \geq 0$: physical finished good inventory at the end of period t
- $I_t^- \geq 0$: backorder of finished good inventory at the end of period t

Buyer's parameters
- v^{bf}: unit salvage value of finished goods

for $t = 1, ..., T-1$
- $\overline{M}_t$: bound on the number of options to be exercised in period t
- o_t : unit price for an option which can be exercised in period t
- e_t : unit price for an option exercised in period t to be delivered in period $t+1$

for $t = 1, ..., T$
- D_t : stochastic demand in period t
- r_t : selling price of finished product on the end market in period t
- p_t : unit purchasing cost of finished product from the supplier (for units of the firm order $Q_{0,t}$ to be delivered in period t)
- s_t^{bf}: unit shortage cost for finished goods in period t
- h_t^{bf}: unit period holding cost for finished goods in period t

To formulate the buyer's problem, we define revenues and expenses separately. The buyer's revenues $\mathcal{R}^b$ have three terms,

$$\mathcal{R}^b(I^-, I^+) = r_1(D_1 - I_1^-) + \sum_{t=2}^{T} r_t(D_t + I_{t-1}^- - I_t^-) + v^{bf} I_T^+. \tag{10}$$

The middle term is the cumulated revenues from the effective sales in period $t = 2, ..., T$; the first term pertains to sales in the first period (for which there is no backlog from a previous period) and the last term is the salvage value of the left-over inventory at the end of the horizon.

[5] We assume here a single period delay between the decision to exercise the option and the delivery of the order. This assumption could be easily relaxed.

The $\mathcal{E}^b$ expenses are written with two sums, because there are no available options for period 1,

$$\mathcal{E}^b(I^-, I^+, m, Q, M) = \sum_{t=1}^{T-1}(e_t m_{t,t+1} + o_t M_{0,t}) + \sum_{t=1}^{T}\left(h_t^{bf} I_t^+ + s_t^{bf} I_t^- + p_t Q_{0,t}\right). \quad (11)$$

Note that the decision variables M and Q are fixed at the beginning of the horizon (i.e. in $t = 0$) and, in a sense, are deterministic. In contrast, I^+, I^- and m are random variables depending on the demand pattern. Thus, the expenses and the revenue are also random variables. Consequently, along the lines of section 3, the buyer's problem can be reformulated as the stochastic programming problem

$$\begin{aligned}
\min \quad & E\left[\mathcal{R}^b(I^-, I^+) - \mathcal{E}^b(I^-, I^+, m, Q, M)\right] && (12a)\\
& I_t = I_t^+ - I_t^-, \quad t = 1, \ldots T, && (12b)\\
& I_1 = Q_{0,1} - D_1, && (12c)\\
& I_t = I_{t-1} + Q_{0,t} + m_{t-1,t} - D_t, \quad t = 2, \ldots T, && (12d)\\
& 0 \leq m_{t,t+1} \leq M_{0,t}, \quad t = 1, \ldots T-1, && (12e)\\
& 0 \leq M_{0,t} \leq \overline{M}_t, \quad t = 1, \ldots T-1, && (12f)\\
& Q_{0,t}, I_t^+, I_t^- \geq 0, \quad t = 1, \ldots T. && (12g)
\end{aligned}$$

The objective function could be modified to account for the buyer's attitude towards risk. This can be done using a utility function (see [19]). A common practice is to take a piecewise linear concave function with break point at a given target level of profit.

A mathematical programming formulation of this problem, using an-AMPL like language, is given in the appendix C.1.

4.3 The supplier's problem

In order to produce the finished goods for the buyer, the supplier purchases raw material from an upstream supplier. Due to long lead-times, purchased orders must be all issued in the first period. Thus, we assume that the supplier has to order and stock all the necessary raw material to meet the maximal demand of the buyer as defined by the contract: no shortage of raw material or finished product delivery is permitted. Thus, the supplier must purchase enough raw material and produce enough finish products to meet all planned demands $Q_{0,1} + \ldots + Q_{0,T}$ and all purchased option rights $M_{0,1} + \ldots + M_{0,T-1}$.

The relationship between the buyer and the supplier are best described as a Stackelberg game, in which the supplier is the leader and the buyer is the follower. In this situation, the supplier announces wholesale, option and exercise prices and, for given bounds on the option exercises, the buyer places orders and options which maximizes his expected revenues. These orders act as demand functions for the supplier, who determines his optimal parameters choice (i.e., optimal wholesale, option and exercise prices and optimal bounds) and his optimal production and inventory variables[6].

We assume that the supplier is able to reconstruct the optimal policy of the buyer and thus determine the issuing buyer's demands along the same event tree as the one used in the buyer's case. In particular, the supplier is able to give numerical values to

[6] As a matter of fact in a complete Stackelberg game, the supplier, in response to buyer decisions, determines also his optimal parameters choice (i.e., optimal wholesale, option and exercise prices and optimal bounds) in the same time as his optimal production and inventory variables. In section 6, we present such a game, but restricted to a single parameter choice, due to the excessive computation time required to solve a game including simultaneously all the supplier's parameters

the buyer's decision variable at each node of the event tree. Thus the buyer's decision variables appear as parameters to the supplier, and should be single indexed with respect to the time period at which they take effect. For this reason, we shall systematically replace the quantities $Q_{0,t}$, $M_{0,t}$, $m_{t-1,t}$ and I_T^+ by $\widetilde{Q}_t$, $\widetilde{M}_t$, $\widetilde{m}_t$ and $\widetilde{I}_T^+$.

The supplier has multiple production opportunities depending on the incurred production lead-time, with corresponding production costs. So, let w_k be the unit labor cost of production for an order with a delay $k \geq 1$. At the beginning of each period $t \geq 1$, after observing $m_{t,t+1}$ (the total number of exercised options for this period), the supplier decides on production orders for future periods $X_{t,t+k}$ with $k = 1, ..., T-t$ and $t = 0, \ldots, T-1$. The initial finished goods inventory hold by the supplier is assumed to be zero. At the end of the process, the left-over raw material and finished product inventories can be sold to an external specific market for a unit salvage prices v^{sr} and v^{sf}, respectively. We recall that the left-over finished inventory of the buyer can be returned to the supplier. The supplier costs are v_s^{bf} per unit, with a unit-extra transport cost of t_{bs}.

Prior to formulating the supplier's problem, let us summarize the variables and parameters involved in this model.

Supplier's decision variables

for $t = 0, ..., T-1$

$Y_{0,t} \geq 0$: raw material firm order issued in period $t = 0$ and to be delivered in period t

$X_{t,t+k} \geq 0$: finished good firm production order issued in period t and to be delivered in period t+k (with $k = 1, ..., T-t$)

Supplier's state variables

for $t = 0, ..., T$

$J_t \geq 0$: finished good inventory at the end of period t

$R_t \geq 0$: raw material inventory at the end of period t

Parameters for the supplier

v^{sf}: unit salvage value of finished goods for the supplier

v_s^{bf}: unit return cost of left-over finished product from the buyer

t_{bs} : unit-extra transport cost of left-over returned finished product from the buyer

for $t = 2, ..., T$

$\widetilde{m}_t$: the demand resulting from the option exercised by the buyer in period $t-1$ (to be delivered in period t)

$\widetilde{M}_t$: number of options, which can be exercised in period t

o_t : unit price for the buyer's right to exercise an option in period t

e_t : unit price for an option exercised in period t

for $t = 1, ..., T$

$\widetilde{Q}_t$: firm order for period t to be delivered to the buyer

p_t : unit selling cost of finished product

c_t : unit purchasing cost of raw material to be delivered at the beginning of period t from firm order $\widetilde{Q}_t$

w_k : unit labor cost of production for an order with a k-period lead-time

h_t^{sf}: unit holding cost for finished product in period t

h_t^{sr}: unit holding cost of raw material in period t

for $t = T$

$\widetilde{I}_T^+ \geq 0$: physical finished good inventory to the buyer at the end of period T

To formulate the supplier's problem we define the revenues $\mathcal{R}^s$

$$\mathcal{R}^s(J,R,\widetilde{Q},\widetilde{m},\widetilde{M},\widetilde{I}^+) = (\sum_{t=1}^{T-1} p_t\widetilde{Q}_t + e_t\widetilde{m}_{t+1} + o_t\widetilde{M}_t) \\ +p_T\widetilde{Q}_T + v^{sr}R_{T-1} + v^{sf}(J_T + \delta_{e<s}\widetilde{I}^+_T), \tag{13}$$

The first term is the cumulated revenues from the firm orders, the option rights and the exercise price of the options exercised by the buyer. The remaining terms correspond to the last firm order and the salvage value of the left-over raw material and finished product inventories at the end of the horizon.

The expenses $\mathcal{E}^s$ are written in three sums, corresponding to the costs related to decisions taken before the first period, to the cumulated cost during all the horizon (except the last period) and to the last period T,

$$\mathcal{E}^s(J,R,Y,X,\widetilde{I}^+) = h^{sr}_0 R_0 + c_0 Y_{0,0} + \sum_{k=1}^{T} w_k X_{0,k} \\ + \sum_{t=1}^{T-1}(h^{sr}_t R_t + c_t Y_{0,t} + \sum_{k=1}^{T-t} w_k X_{t,t+k} + h^{sf}_t J_t) \\ +h^{sf}_T J_T + (v^{bf}_s + t_{bs})\delta_{e<s}\widetilde{I}^+_T. \tag{14}$$

From the view point of the supplier, the parameters $\widetilde{m}_t$ and $\widetilde{I}^+_T$ are random variables depending on the buyer's decisions to exercise the options. In contrast, M and Q are deterministic. Thus the expenses and the revenues are also random variables. Consequently, the supplier's optimization problem can be formulated as the stochastic programming problem

$$\min \quad E\left[\mathcal{R}^s(J,R,\widetilde{Q},\widetilde{m},\widetilde{M},\widetilde{I}^+) - \mathcal{E}^s(J,R,Y,X,\widetilde{I}^+)\right]$$

$$\text{s.t.} \quad R_0 = Y_{0,0} - \sum_{k=1}^{T} X_{0,k}, \tag{15a}$$

$$R_t = R_{t-1} + Y_{0,t} - \sum_{k=1}^{T-t} X_{t,t+k}, \quad t = 1,\dots T-1, \tag{15b}$$

$$J_1 = X_{0,1} - \widetilde{Q}_1, \tag{15c}$$

$$J_t = J_{t-1} + \sum_{k=0}^{t-1} X_{k,t} - \widetilde{m}_t - \widetilde{Q}_t, \quad t = 2,\dots T, \tag{15d}$$

$$R_t,\ J_t,\ X_{k,t} \geq 0, \quad t = 1,\dots T,\ k = 0,\dots t-1. \tag{15e}$$

A complete formulation of this optimization problem, using an-AMPL like language, is given in the appendix C.2.

5 Auxiliary models

We present in this section the pair of models we exploit as benchmark for estimating the performances of the multi-periodic contract with options : a basic contract without options and a fully integrated system.

5.1 The basic contract without options

The assumptions of this model are similar to those of the basic multi-periodic contract with options, except that no option can be exercised. So, the buyer places firm orders at the begin of the process for all the periods. No additional amounts can be ordered after observing demands. In this situation, the whole risks are endowed by the buyer and the supplier faces a deterministic situation once the buyer has placed the orders.

5.2 Vertical integration

Option contracts help improving the joint performance of the buyer and the supplier acting independently within the contract framework. A bound on this improvement is given by the performance of a fully integrated supply chain. To this end, we provide a model of a fully integrated system. In this new model, the structure of the logistic system is preserved: purchased raw materials are transformed into a finished goods that are stored in a retail inventory to serve clients on the end market. In contrast with the buyer-supplier model, the decisions are taken by a single decision-maker. The raw materials are still assumed to be purchased at the beginning of the campaign and periodically delivered or stored for future use. Production is limited by the availability of raw material. The retail level receives, stores and sells to end markets the finished product. Each period offers the possibility to adjust production and ordering decisions to match the revealed demand. There is no necessity of a contract between the production level and the retail level as the different decision variables are controlled by a single decision maker.

Prior to formulating the integrated model, let us summarize the variables and parameters involved.

Decision variables in the integrated model
for $t = 0, ..., T-1$

- $Y_{0,t} \geq 0$: raw material firm order, issued in the first period, to be delivered in period t
- $X_{t,t+k} \geq 0$: finished good firm production order issued in period t and to be delivered in period t+k (with $k = 1, ..., T-t$)

State variables in the integrated model
for $t = 0, ..., T$

- I_t : retail finished good inventory at the end of period t
- $I_t^+ \geq 0$: retail physical finished good inventory at the end of period t
- $I_t^- \geq 0$: retail backorder of finished good inventory at the end of period t
- $R_t \geq 0$: raw material inventory at the end of period t

Parameters for the integrated model

- w_k: unit labor cost of production for an order with a k-period lead-time
- v^r : unit salvage value of raw material
- v^f : unit salvage value of finished goods

for $t = 1, ..., T$

- D_t: stochastic demand in period t
- r_t : selling price of finished product on the end market in period t
- c_t : unit purchasing cost of raw material to be delivered at the beginning of period t
- h_t^f : unit holding cost for finished product in period t
- h_t^r : unit holding cost of raw material in period t
- s_t^f : unit shortage cost for finished product in period t

The global revenue $\mathcal{R}^i$ has three terms

$$\mathcal{R}^i(I^-, I^+, R) = r_1(D_1 - I_1^-) + \sum_{t=2}^{T} r_t(D_t + I_{t-1}^- - I_t^-) + v^r R_T + v^f I_T^+. \tag{16}$$

The expenses $\mathcal{E}^i$ are written as

$$\mathcal{E}^i(I^+, I^-, R, X) = \sum_{t=1}^{T} \left(h_t^f I_t^+ + s_t^f I_t^- + h_t^r R_t + c_t Y_{0,t} + \sum_{k=0}^{T-t} w_k X_{t,t+k} \right) \tag{17}$$

Consequently, the present optimization problem can be formulated as the stochastic programming problem

$$\min \quad E\left[\mathcal{R}^i(I^-, I^+, R) - \mathcal{E}^i(I^+, I^-, R, X)\right] \tag{18a}$$

$$I_t = I_t^+ - I_t^-, \quad t = 1, \dots T, \tag{18b}$$

$$I_1 = X_{0,1} - D_1, \tag{18c}$$

$$I_t = I_{t-1} + \sum_{k=1}^{t} X_{t-k,t} - D_t, \quad t = 2, \dots T, \tag{18d}$$

$$R_0 = Y_{0,0} - \sum_{k=1}^{T-1} X_{1,1+k}, \tag{18e}$$

$$R_t = R_{t-1} + Y_{0,t} - \sum_{k=1}^{T} X_{0,k}, \quad t = 1, \dots T. \tag{18f}$$

6 Example of analysis of a base contract

The main purpose of this section is to show that stochastic programming allows in-depth analysis of contract performance. The work has been carried out on a laptop Toshiba Satellite Pro 4200, 500 Mhz with 196 Mb of core memory. The modeling language is AMPL [15] and the commercial solver is CPLEX 6.1 [12]. The models are those of Sections 4 and 5.

Our base model has two periods. The demand parameters are $\mu_t = 1000$, $\sigma_t = 330$ and $\rho_t = 0.5$. The cost and price parameters are (for $t = 1, 2$): $c_t = 3$, $h_t^{bf} = 0.5$, $h_t^{sf} = 0.25$, $h_t^{sr} = 0.125$, $r_t = 12$, $p_t = 8$, $o_t = 1.5$, $e_t = 8$, $s_t^{bf} = 6$, $(w_1, w_2) = (4, 3)$, $v^{bf} = 4$, $v^{sf} = 5$, $v^{sr} = 2$, $v_s^{bf} = 2$ and $t_{bs} = 4$. The constraint parameters on the option right level is $\overline{M}_t = 10000$.

The continuous demands are approximated by discrete values according to the scheme described in Section 4 and Appendix A. The base choice is to use grids with the same number of points in each period : this base number is 81, yielding a tree with 6642 nodes. The deterministic equivalent linear programs have respectively 19929 variables and 13285 constraints for the buyer's model and only 6809 variables and 6724 constraints in the supplier's problem. Solving these two problems is a matter of a few seconds only.

6.1 Impact of the discretization grid

The choice of an appropriate approximation of the stochastic process, is an important issue in stochastic programming. A finer grid is liable to yield more reliable results, but the size of the deterministic equivalent program increases dramatically with the size of the grid. We have carried a few experiments to test the impact of the grid sizes on the objective function and on the decision variables values for the two problems, the buyer's and the supplier's. The buyer's problem is the largest of the two models: it constitutes the bottleneck in the grid refinement.

The finer grid we considered has 321 points per period. This lead to a tree with 103362 nodes and deterministic equivalent linear program for the buyer with approximatively 200000 constraints and 300000 variables. This is the maximal size we could handle on our somewhat limited hardware. The solution required about an hour, but much of the time was spent in swap. Clearly, larger size can be handled on faster computers endowed with large core memory.

We tested different grid sizes in order to investigate the robustness of the model. The optimal values of the large buyer's model – and the corresponding supplier's model – constitute our benchmarks for other discretization schemes. Table 1 displays the decision variables and the profits for the different grid sizes. An 81×81 grid reasonably approximates the 321×321 reference grid.

Grid	First stage decisions			Expected profit		
	M_1	$Q_{0,1}$	$Q_{0,2}$	Buyer	Supplier	Joint
5 x 5	845,00	1396,00	353,00	4220,87	3234,24	7455,10
11 x 11	499,00	1540,00	346,00	4291,80	2791,63	7083,43
21 x 21	484,00	1472,00	410,00	4326,83	2928,51	7255,34
41 x 41	470,00	1483,00	410,00	4328,71	2904,89	7233,60
81 x 81	469,00	1465,00	434,00	4329,08	2924,88	7253,95
161 x 161	471,00	1468,00	429,00	4329,68	2913,82	7243,50
321 x 321	470,00	1469,00	428,00	4329,75	2921,19	7250,94

Table 1: Simultaneous refinement of the 1st and 2nd stage grids.

In the 2-stage stochastic programming framework, the first stage decisions are the most important, since they are committing for the second stage. Second stage decisions are recourse to adjust to the chance outcome. This suggests that a looser approximation of the optimal recourse may be sufficient to gear good first stage decisions. Thus, a coarser grid for the second stage may be enough. We tested this idea; the results displayed on Table 2 confirm the analysis: a 41 point grid for the second stage seems to be accurate enough. In particular, the first stage decision quickly stabilize as the grid for the second period demand becomes moderately fine.

Grid	First stage decisions			Expected profit		
	M_1	$Q_{0,1}$	$Q_{0,2}$	Buyer	Supplier	Joint
321 x 5	500,00	1469,00	411,00	4313,88	2860,47	7174,347
321 x 11	456,00	1469,00	437,00	4307,53	2830,25	7137,781
321 x 21	473,00	1469,00	428,00	4330,43	2921,47	7251,906
311 x 41	471,00	1469,00	427,00	4329,92	2921,08	7251,001
321 x 81	470,00	1469,00	428,00	4329,46	2921,36	7250,822
321 x 161	471,00	1469,00	428,00	4329,73	2921,85	7251,58
321 x 321	470,00	1469,00	428,00	4329,75	2921,19	7250,942

Table 2: Refinement of the 2nd stage grid alone.

6.2 Probability distribution of profit, orders and options

The stochastic programming solution provides a much richer information than the sole optimal expected profit. Indeed, the program computes the profit at each leaf of the tree: thus, one can easily reconstruct the probability distribution of the profit under the optimal policy. Similarly, one can provide a full probabilistic description of the optimal decisions, in particular those pertaining to the exercised options by the buyer.

In Figure 3, we depict the probability distributions of the buyer and the supplier. We also provide a picture of the profit density function of the buyer. This information helps visualize the risk.

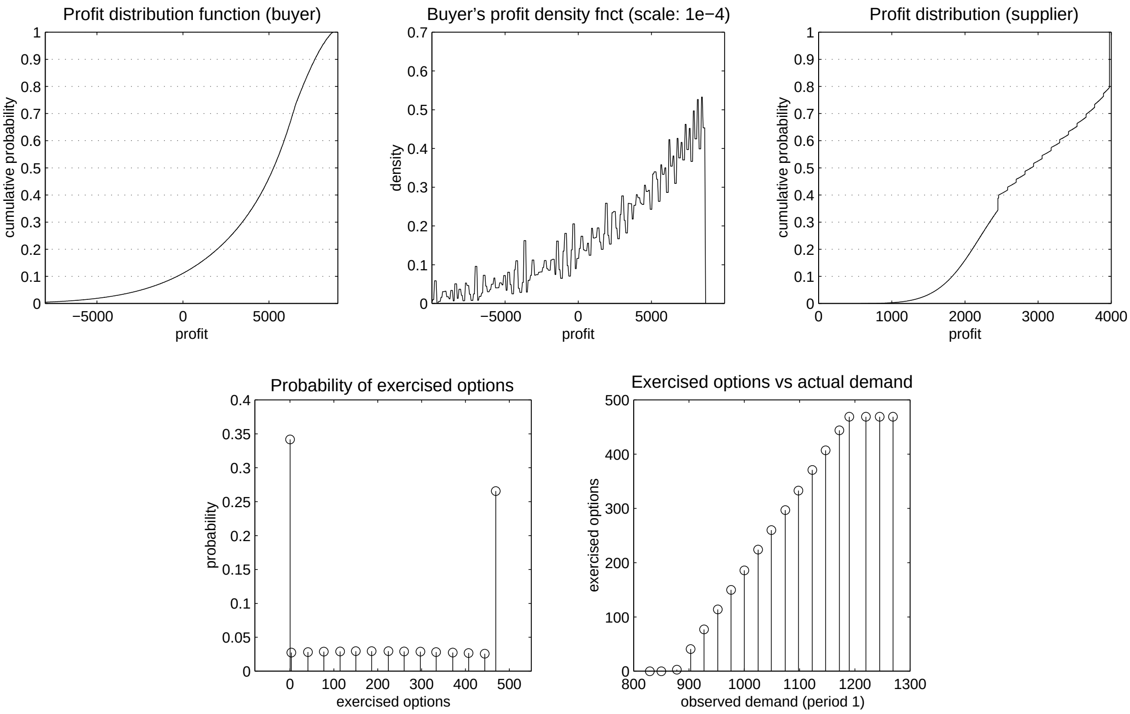

Figure 3: Profit distribution and exercised options.

We also represent the probability distribution of the exercised options and the exercised option as a function of the observed demand in the first period.

6.3 Downside risk constraint

The expected profit criterion is valid for risk-neutral decision-makers. For risk-averse decision-makers, one would probably want to use a criterion based on a utility function [19]. To avoid complication, it is possible to look for a policies focusing on downside risk. The analysis of the optimal policy in the base model shows that the buyer's profit can be as low as $-14,000$ and that it achieves a negative value with probability 0.15. It is a minor change in the model to add a constraint on the minimum profit, or on the conditional expectation of negative profits (we shall later discuss a similar constraint on the service level). As an illustration, we run a model with a constraint limiting the buyer's deficit to $-10,000$. The corresponding numerical results are given on 4.

One notices that the buyer's policy shifts towards an increasing use of exercised options. The number of option rights is now 993 instead of 469. The distribution function of the buyer is steeper, with less dispersion. High and low profits are both limited.

6.4 Sensitivity on option and buy-back prices

The objective of this subsection is to give some insight on how the profits and the ordering policy change when some of the key parameter in the option contract vary. In particular,

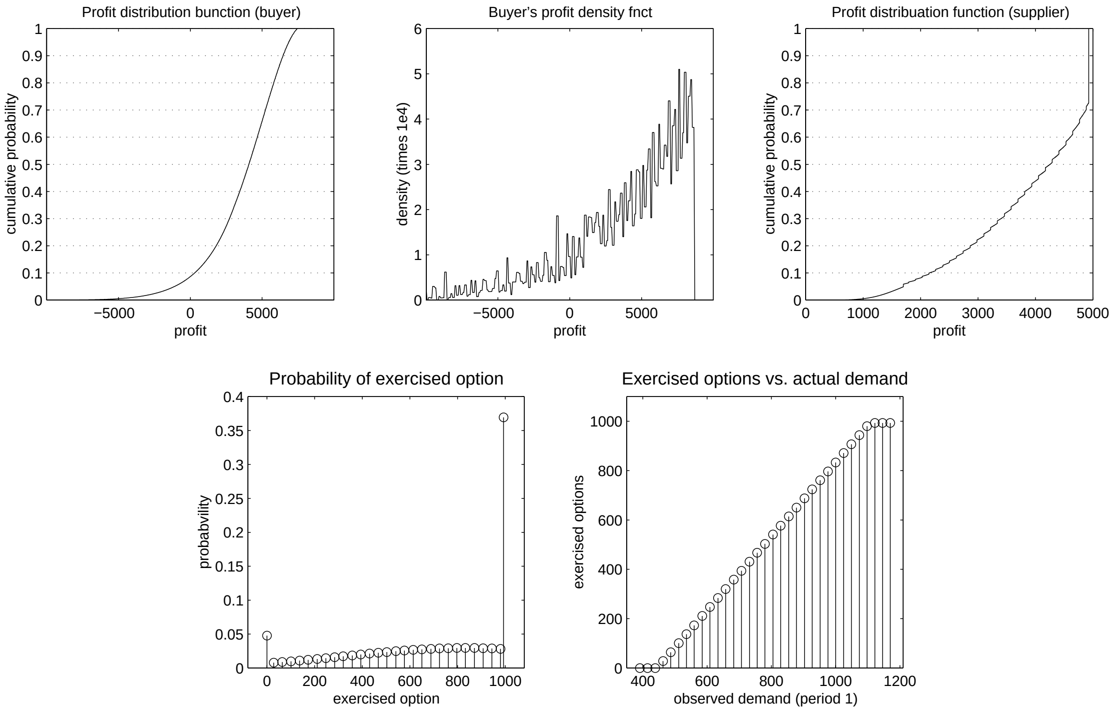

Figure 4: Profit distribution and exercised options with a downside risk constraint.

we concentrate on the impact of the option right purchasing price c_o, the option exercise price c_e and the buy-back (or salvage) price v_s^{bf}.

We let each parameter vary independently, while letting the others at their base value. The results are reported in Tables 3, 4 and 5.

Option price	Expected profit			M_1	$Q_{0,1}$	$Q_{0,2}$
o_t	Buyer	Supplier	Joint			
0	5817.437	907.771	6725.208	10,000	1392	0
0.25	5412.789	1992.316	7405.105	1407	1440	0
0.5	5093.326	2372.814	7466.140	1145	1465	76
0.75	4837.169	2629.032	7466.201	922	1465	192
1	4629.987	2791.614	7421.601	744	1465	287
1.25	4462.506	2885.741	7348.247	597	1465	364
1.5	4329.076	2924.875	7253.951	469	1465	434
1.75	4226.059	2916.696	7142.755	355	1465	496
2	4150.795	2858.695	7009.490	247	1465	553
2.25	4101.365	2765.389	6866.754	149	1465	607
2.5	4076.385	2625.958	6702.343	53	1465	659
2.75	4072.616	2532.186	6604.802	0	1465	690

Table 3: Impact of a variation of the option right price.

As expected, Table 3 shows that the buyer's profit decreases when the price of option rights increases, while the supplier's profit first increases and then decreases. If the price of the option right is too high, the buyer becomes more conservative, which turns out to be harmful to the supplier. A similar comment holds for the impact of the price of option exercise, see Table 4

Finally, Table 5 shows the impact of the buy-back price. It appears that the buyer

Exercise price	Expected Profit			M_1	$Q_{0,1}$	$Q_{0,2}$
e_t	Buyer	Supplier	Joint			
5.75	5503.163	2222.650	7725.813	1231	1269	0
6.00	5299.949	2396.527	7696.476	1168	1318	0
6.25	5109.878	2549.730	7659.608	1106	1367	0
6.50	4935.500	2661.190	7596.690	1040	1416	0
6.75	4775.342	2742.848	7518.190	976	1465	0
7.00	4643.808	2831.888	7475.696	806	1465	157
7.25	4541.091	2867.934	7409.025	688	1465	261
7.50	4457.475	2907.761	7365.236	604	1465	330
7.75	4387.994	2912.204	7300.198	533	1465	385
8.00	4329.080	2924.880	7253.960	469	1465	434
8.25	4279.487	2912.624	7192.111	417	1465	470
8.50	4236.919	2912.044	7148.963	369	1465	502
8.75	4201.249	2882.237	7083.486	321	1465	532
9.00	4170.617	2869.973	7040.590	279	1465	559
9.25	4145.424	2835.369	6980.793	240	1465	581
9.50	4124.188	2807.507	6931.695	200	1465	602

Table 4: Impact of the price of the option exercise.

takes advantage of higher buy-back prices, but in the present model this trend is not strong enough to induce higher profits for the supplier.

6.5 Stackelberg equilibrium

In the game theoretic framework the relation between the buyer and the supplier can be depicted as a game with a leader and a follower. The leader (supplier) chooses the option and buy-back prices (possibly the selling price also) and the follower (buyer) decides on orders and purchase rights. The Stackelberg equilibrium is defined as the optimal choice of the leader, under the assumption that the buyer reacts optimally to the leader's choice. The problem of computing a Stackelberg equilibrium is known to be very difficult. However, by performing sensitivity analysis it is possible to estimate the Stackelberg equilibrium when only one component is allowed to vary.

Here, we have considered the case of the optimal choice by the supplier of the price of the option right, when the price of of the option exercise is fixed ($e_t = 8$). The optimal choice $o_t = 1.5$ can can be read from Table 3. In Table 3 we compare the Sackelberg equilibrium with two other extreme cases: a contract with no option rights ($o_t = \infty$), and a vertically integrated supply chain. We note that at the Stackelberg equilibrium, both actors are better off, but their joint profit still falls short from the results achieved in the vertically integrated supply chain.

6.6 Service level

Stochastic programming can easily handle extra constraints. As an illustration, we extend the base model to include a service level constraint. The service level is defined as the ratio of served demand on total demands. This constraint can be written as follows

$$\sum_{n \in N[T]} P[T,n] I^-[T,n] \leq \alpha \sum_{n \in N[T]} P[T,n] \sum_{t=1,\ldots T} D[t, b[t,n]].$$

The service level parameter is α. Note that this constraint is linear. Table 7 displays the evolution of profits and decision variables versus the service level

Salvage price v_s^{bf}	expected profit			M_1	$Q_{0,1}$	$Q_{0,2}$
	Buyer	Supplier	Joint			
0.000	3925.717	3274.076	7199.793	574	1465.000	297.000
0.125	3947.968	3251.465	7199.433	566	1465.000	307.000
0.250	3971.067	3262.300	7233.367	563	1465.000	311.000
0.375	3994.394	3236.999	7231.393	556	1465.000	319.000
0.500	4017.962	3213.690	7231.652	550	1465.000	328.000
0.750	4065.919	3164.556	7230.475	539	1465.000	343.000
1.000	4114.932	3111.368	7226.300	525	1465.000	360.000
1.250	4166.097	3087.816	7253.913	516	1465.000	375.000
1.500	4218.842	3028.597	7247.439	501	1465.000	392.000
1.750	4272.958	2967.371	7240.329	485	1465.000	412.000
2.000	4329.079	2924.875	7253.954	469	1465.000	434.000
2.250	4387.475	2858.699	7246.174	455	1465.000	452.000
2.500	4447.660	2787.785	7235.445	438	1465.000	475.000
2.750	4510.316	2724.863	7235.179	418	1465.000	497.000
3.000	4575.636	2645.999	7221.635	401	1465.000	522.000
3.250	4643.337	2556.945	7200.282	378	1465.000	549.000
3.500	4714.475	2476.991	7191.466	359	1465.000	574.000
3.750	4788.731	2369.843	7158.574	330	1465.000	605.000
4.000	4866.659	2265.101	7131.760	304	1465.000	638.000

Table 5: Effect of the salvage price.

Scenario	**NO Model**	**Stackelberg:** $o_t = 1.5$	**VI Model**
Buyer expected profit	4072.616	4329.076	-
Buyer's profit standard deviation	3702.604	3317.175	-
Supplier expected profit	2532.186	2924.875	-
Supplier's profit standard deviation	1587.475	1423.715	-
Joint expected profit	6604.802	7253.951	9125.992
Joint profit standard deviation	4776.265	5090.167	6426.418
M_1	-	469	-
$Q_{0,1}$	1465	1465	-
$Q_{0,2}$	690	434	-

Table 6: No Options, Stackelberg and Vertically integrated models.

service level α	Expected profit			M_1	$Q_{0,1}$	$Q_{0,2}$
	Buyer	Supplier	Joint			
0,960	4866,66	2265,10	7131,76	304,00	1465,00	638,00
0,960	4866,66	2265,10	7131,76	304,00	1465,00	638,00
0,969	4856,72	2258,85	7115,57	341,00	1502,92	617,08
0,970	4852,73	2307,62	7160,35	358,45	1514,00	610,00
0,971	4847,55	2255,32	7102,87	356,00	1514,00	617,00
0,973	4834,48	2250,14	7084,62	365,00	1527,72	612,28
0,974	4826,65	2246,91	7073,56	375,97	1538,00	605,00
0,975	4817,28	2401,94	7219,22	385,45	1538,00	611,00
0,976	4806,73	2258,96	7065,69	392,62	1538,00	615,00
0,980	4750,29	2237,10	6987,39	425,00	1576,57	598,43
0,990	4430,80	2185,10	6615,89	544,80	1660,00	597,00
0,000	628,13	1978,77	2606,90	1686,00	1978,00	650,00

Table 7: Impact of the service level.

7 Conclusion

Stochastic programming appears to be a versatile and powerful tool to model the relationship between a buyer and a supplier in a supply chain context. It permits a fine analysis the impact of contracts on the performance of the chain. In this paper, we focused on contracts with purchase option rights and a buy-back policy. The contract we studied is quite general and encompasses many other contracts (see [4]).

The formulation via algebraic modeling languages appears to be simple enough to be implementable for on-the-fly analyses. The main limitation of stochastic programming stems from the natural tendency of event tree to become enormous when the demand discretization and the number of periods increase. However, we have shown that the current modeling languages and linear programming solvers are powerful enough to handle satisfactory approximations. Anyhow, the quality of the approximation is not a severe requirement: the purpose of the models is to give reliable hints on the effects of supply chain contracts models, knowing that decision-makers need guidelines much more than precise instructions on how to choose their policies.

Two extensions seem to be in order. One would consist in refining the risk analysis, for instance by considering complex utility functions. There is no difficulty in dealing with piece-wise linear concave utility function, contrary to other approaches based on dynamic programming or similar approaches. The other extension would deal with multiperiodic problems. While there is no difficulty in formulating such problems, it is clear that increasing the number of periods quickly leads to enormous and intractable event trees. In that respect, it is interesting to note that satisfactory results in the 2 stage model were obtained when the grid size in the second period was kept small, if not very small. This perhaps suggests that using grids of decreasing size for later periods may lead to accurate enough solutions, while controlling the event tree explosion and achieving tractable models.

Implementation of the stochastic programming approach for rolling horizon contracts involving several time-periods as in [3, 5] and explicit consideration of the decision-maker attitude towards risk constitutes an ongoing research.

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A Discretization of the standard normal variable

We propose a scheme that generates a discrete random variable with $2m+1$ values that approximates the normal random variable with 0 mean and unit standard deviation. The number m may take arbitrary integer values. Clearly, the parameter m may take different values at different time period. In this appendix, we do not subscript it with t for the sake of clarity.

We now explain how we choose the discrete values $\{\delta_{t,1}, \dots \delta_{t,2m+1}\}$ and compute the associate probabilities $\{p_{t,1}, \dots p_{t,2m+1}\}$. The values δs are chosen to be equally spaced, symmetric around the origin, with a total spread of 6 units. We introduce the step

$$\tau \;=\; \tfrac{6}{2m+1},$$

and we define

$$\begin{aligned} \delta_1 &= \delta_{2m+1} = 3, \\ \delta_{m+1} &= 0, \\ \delta_{k+1} &= \delta_k + \tau, \quad k = 1, ..., 2m-1. \end{aligned}$$

For any intermediary point δ_k, with $k = 2, ..., 2m$, the associate probability p_k is given by

$$p_k = \int_{\delta_k - \frac{\tau}{2}}^{\delta_k + \frac{\tau}{2}} f(x)dx,$$

where $f(\cdot)$ is the density of the standard normal distribution. For the end points the probabilities are

$$p_1 = \int_{-\infty}^{\delta_1 + \frac{\tau}{2}} f(x)dx,$$

and

$$p_{2m+1} = \int_{\delta_{2m+1} - \frac{\tau}{2}}^{+\infty} f(x)dx,$$

The computation of these integral are programmed in many software packages.

B The simple production/inventory model formulated in AMPL

B.1 The base deterministic model

```
param T > 0; # number of periods
param demand {1..T} >= 0; # periodic demands
param holdingcost >= 0; # inventory holding cost
param shortagecost >= 0; # inventory shortage cost
param procurementcost {k in 1..T} >= 0; # procurement cost (via production
      or purchasing) associated with a k period lead-time

var procurement {k in 0..T-1, l in k+1..T} >= 0; # units ordered in period
      k and delivered at the beginning of period l (with l-k>=1)
var Inventory {1..T} >= 0; # inventory surplus at the end
      of period t
var Inventoryplus {t in 1..T} >= 0; # physical inventory at the end
      of period t
var Inventoryminus {t in 1..T}>= 0; # backorder at the end of
      period t

minimize total_expenses:
   sum {t in 1..T} (sum {k in 0..t-1}procurement[k,t]*procurementcost[t-k]
      + holdingcost[t]*Inventoryplus[t] + shortagecost[t]*Inventoryminus[t]);
   # Total cumulated costs for all periods

subject to {t in 1..T}:
    Inventory[t] = Inventoryplus[t] - Inventoryminus[t];
     # Definition of physical inventories and shortages
subject to {t in 2..T}:
    Inventory[t] = Inventory[t-1] + sum {k in 0..t-1}procurement[k,t]
    - demand[t];
     # Definition of procurements/inventories dynamics
subject to:
    Inventory[1] = procurement[0,1] - demand[1];
     # Definition of the first period inventory
```

B.2 The stochastic process formulation

In this example, we consider independent demands. In case of correlation schemes between successive demands, as considered in the example of this paper, the demand values are computed parameters, instead of simple parameters.

```
param T > 0; # number of periods

# Parameters whose values are to be read in the user data file
param f {0..T-1};  # number of branches arising from each node in period t
param p {t in 0..T-1, k in 1..f[t]} >= 0;   # transition probabilities from
   # any node in period t to its successors in period t+1

# Computed parameters
param N {t in 0..T}:= if t=0 then 1 else N[t-1]*f[t-1];
  # number of nodes in period t

# Parameters whose values are to be read in the user data file
param demand {t in 1..T, n in 1..N[t]};   # demand values associated with
  # the nodes of the event tree description

# Computed parameters
param a {t in 1..T, n in 1..N[t], k in 1..t}:=
   if k=1 then
               if t=1 then 1 else ceil(n/f[t-1])
          else a[t-1,a[t,n,1],k-1]; # k-periods predecessor

param l {t in 1..T, n in 1..N[t]}:= n-f[t-1]*(a[t,n,1]-1); # transition index of the
last transition
param P {t in 0..T, n in 1..N[t]}:=
   if t=0 then 1 else p[t-1,l[t,n]]*P[t-1,a[t,n,1]]; # node probability
param b {t in 0..T, n in 1..N[T]}:= if t < T then a[t+1,b[t+1,n],1] else n;
   # index of the node traversed at time t by scenario ending at (T,n)
```

Figure 5: Model for the stochastic process with AMPL

B.3 The deterministic equivalent linear programming problem

```
# GLOBAL PARAMETER
param T;

# PROBABILISTIC SECTION


# Parameters whose values are to be read in the user data file
param f {0..T-1};   # number of branches arising from each node in period t
param p {t in 0..T-1, k in 1..f[t]} >= 0;    # transition probabilities from
   # any node in period t to its successors in period t+1

# Computed parameters
param N {t in 0..T}:= if t=0 then 1 else N[t-1]*f[t-1];
  # number of nodes in period t

# Parameters whose values are to be read in the user data file
param demand {t in 1..T, n in 1..N[t]};    # demand values associated with
  # the nodes of the event tree description

# Computed parameters
param a {t in 1..T, n in 1..N[t], k in 1..t}:=
   if k=1 then
              if t=1 then 1 else ceil(n/f[t-1])
          else a[t-1,a[t,n,1],k-1]; # k-periods predecessor

param l {t in 1..T, n in 1..N[t]}:= n-f[t-1]*(a[t,n,1]-1); # transition index for
the last transition
param P {t in 0..T, n in 1..N[t]}:=
   if t=0 then 1 else p[t-1,l[t,n]]*P[t-1,a[t,n,1]]; # node probability
param b {t in 0..T, n in 1..N[T]}:= if t < T then a[t+1,b[t+1,n],1] else n;
   # index of the node traversed at time t by scenario ending at (T,n)
```

```
# DETERMINISTIC EQUIVALENT

param T > 0; # number of periods
param holdingcost >= 0; # inventory holding cost
param shortagecost >= 0; # inventory shortage cost
param procurementcost {k in 1..T}>= 0; # procurement cost (via production
   or purchasing) associated with a k period lead-time

var procurement {t in 0..T-1, n in 1..N[t], k in t+1..T} >= 0; # units
   ordered in node (t,n) and delivered at the beginning of period k
var Inventory {t in 1..T, n in 1..N[t]} >= 0; # inventory surplus at the end of
period t
var Inventoryplus t in 1..T, n in 1..N[t]>= 0 ; # physical inventory at the end
  # of period t
var Inventoryminus t in 1..T, n in 1..N[t]>= 0 ; # backorder at the end of
   period t

subject to {t in 1..T, n in 1..N[t]}:
   Inventory[t,n] = Inventoryplus[t,n] - Inventoryminus[t,n];
     # Definition of physical inventories and shortages
subject to {n in 1..N[1]}:
   Inventory[1,n] = procurement [0,1,1] - demand[1,n];
     # Definition of initial inventory
subject to {t in 2..T, n in 1..N[t]}:
   Inventory[t,n] = Inventory[t-1,a[t,n,1]]
                  + sum {k in 0..t-1}procurement[k,a[t,n,k],t]
                  - demand[t,n];
     # Definition of procurements/inventories dynamics

minimize total_expenses:
  sum {n in 1..N[T]} P[T,n]*sum {t in 1..T-1}(holdingcost * Inventoryplus[t,b[t,n]]
       + shortagecost*Inventoryminus[t,b[t,n]]
       + sum {k in 1..T-t}procurement[t,b[t,n],t+k]*procurementcost[k]);
     # Total costs in all periods
```

C Linear programming models

We give here a mathematical programming formulation of the deterministic equivalent linear programming problem using AMPL-like notation. It is an easy matter to build the full-fledged AMPL model using the methodology discussed in Section 3

C.1 The buyer model in pre-AMPL formulation

The objective is conveniently expressed in terms of revenues and expenses along each scenario. For each scenario, indexed by $n \in \{1, \ldots, N[T]\}$, the revenues are given by

$$\begin{aligned}\mathcal{R}^b[n] = {} & r_1\Big(D[1, b[1, n]] - I^-[1, b[1, n]]\Big) + v^{bf} I^+[T, n] \\ & + \sum_{t=2}^{T} r_t\Big(D[t, b[t, n]] - I^-[t, b[t, n]] + I^-[t-1, b[t-1, n]]\Big)\end{aligned} \tag{19}$$

and the expenses by

$$\mathcal{E}^b[n] = \sum_{t=1}^{T-1}\Big(o_t M[0, 1, t] + e_t m[t, b[t, n], t+1]\Big)$$

$$+\sum_{t=1}^{T}\left(h_t^{bf}I^+[t,b[t,n]]+s_t^{bf}I^-[t,b[t,n]]+p_tQ[0,1,t]\right). \tag{20}$$

The extensive formulation of Problem (12) as a linear programming problem is given by

$$\begin{aligned}
\max \quad & \sum_{n=1}^{N[T]} P[T,n]\Big(\mathcal{R}^b[n]-\mathcal{E}^b[n]\Big) && \text{(21a)}\\
\text{s.t.} \quad & I[t,n]=I^+[t,n]-I^-[t,n], \quad t=1,\dots T,\ n=1,\dots N[t], && \text{(21b)}\\
& I[1,n]=Q[0,1,1]-D[1,n], \quad n=1,\dots N[1], && \text{(21c)}\\
& I[t,n]=I[t-1,a[t,n,1]]+Q[0,1,t]+m[t-1,a[t,n,1],t]-D[t,n], && \\
& \qquad t=2,\dots T,\ n=1,\dots N[t], && \text{(21d)}\\
& 0\le m[t,n,t+1]\le M[0,1,t], \quad t=1,\dots T-1,\ n=1,\dots N[t], && \text{(21e)}\\
& 0\le M[0,1,t]\le \overline{M}_t, \quad t=1,\dots T-1, && \text{(21f)}\\
& I^+[t,n]\ge 0,\ I^-[t,n]\ge 0,\ Q[0,1,t]\ge 0, && \\
& \qquad\qquad t=1,\dots T, \quad n=1,\dots N[t], && \text{(21g)}\\
& M[0,1,t]\ge 0, \quad t=1,\dots T-1. && \text{(21h)}
\end{aligned}$$

C.2 The supplier model in pre-AMPL formulation

The objective is conveniently expressed in terms of revenues and expenses along each scenario. For each scenario, indexed by $n\in\{1,\dots,N[T]\}$, the revenues are given by

$$\begin{aligned}
\mathcal{R}^s[n] = {} & \Big(\sum_{t=1}^{T-1} p_t\widetilde{Q}[t]+e_t\widetilde{m}[t+1,b[t+1,n]]+o_t\widetilde{M}[t]\Big)\\
& +p_T\widetilde{Q}[T]+v^{sr}R[T-1,b[T-1,n]]+v^{sf}(J[T,n]+\delta_{e<s}\widetilde{I}^+[T,n]),
\end{aligned} \tag{22}$$

and the expenses

$$\mathcal{E}^s[n]=h_0^{sr}R[0,1]+c_0Y[0,1,0]+\sum_{k=1}^{T}w_kX[0,1,k] \tag{23}$$

$$\begin{aligned}
& +\sum_{t=1}^{T-1}(h_t^{sf}J[t,b[t,n]]+h_t^{sr}R[t,b[t,n]]+\sum_{k=1}^{T-t}w_kX[t,b[t,n],t+k])\\
& +c_tY[0,1,t]+h_T^{sf}J[T,n]+(v_s^{bf}+t_{bs})\delta_{e<s}\widetilde{I}^+[T,n].
\end{aligned} \tag{24}$$

Consequently, the supplier's optimization problem can be formulated as the stochastic programming problem

$$\begin{aligned}
\max \quad & \sum_{n=1}^{N[T]} P[T,n]\Big(\mathcal{R}^s[n]-\mathcal{E}^s[n]\Big)\\
& R[0,1]=Y[0,1,0]-\sum_{k=1}^{T}X[0,1,k], && \text{(25a)}\\
& R[t,n]=R[t-1,a[t,n,1]]+Y[0,1,t]-\sum_{k=1}^{T-t}X[t,n,t+k], && \\
& \qquad\qquad t=1,\dots T-1,n=1,\dots N[t], && \text{(25b)}
\end{aligned}$$

$$J[1,n] = X[0,1,1] - \widetilde{Q}[1], n = 1, \dots N[1], \tag{25c}$$

$$J[t,n] = J[t-1, a[t,n,1]] + \sum_{k=1}^{t} X[t-k, a[t,n,k], t] - \widetilde{m}[t,n] - \widetilde{Q}[t],$$
$$n = 1, \dots N[t],\ t = 2, \dots T, \tag{25d}$$

$$R[t,n] \geq 0, \quad t = 0, \dots T-1, n = 1, \dots N[t], \tag{25e}$$

$$J[t,n] \geq 0, \quad t = 1, \dots T, n = 1, \dots N[t]. \tag{25f}$$

$$X[t,n,t+k] \geq 0, \quad 0 = 1, \dots T-1, n = 1, \dots N[t], k = 1, \dots T-t. \tag{25g}$$

C.3 The vertical integration model in pre-AMPL formulation

The objective is conveniently expressed in terms of revenues and expenses along each scenario. For each scenario, indexed by $n \in \{1, \dots, N[T]\}$, the revenues are given by

$$\begin{aligned}\mathcal{R}^b[n] = {} & r_1\Big(D[1, b[1,n]] - I^-[1, b[1,n]]\Big) + v^f I^+[T,n] + v^r R[T,n] \\ & + \sum_{t=2}^{T} r_t \Big(D[t, b[t,n]] - I^-[t, b[t,n]] + I^-[t-1, b[t-1,n]]\Big)\end{aligned} \tag{26}$$

and the expense by

$$\begin{aligned}\mathcal{E}^b[n] = \sum_{t=1}^{T} \Big(& h_t^f I^+[t, b[t,n]] + s_t^f I^-[t, b[t,n]] + h_t^r R[t, b[t,n]] \\ & + c_t Y[0,1,t] + \sum_{k=0}^{T-t} w_k X[t, b[t,n], t+k]\Big).\end{aligned} \tag{27}$$

The extensive formulation of Problem (18) as a linear programming problem is given by

$$\max \quad \sum_{n=1}^{N[T]} P[T,n]\Big(\mathcal{R}^b[n] - \mathcal{E}^b[n]\Big) \tag{28a}$$

$$\text{s.t.} \quad I[t,n] = I^+[t,n] - I^-[t,n], \quad t = 1, \dots T,\ n = 1, \dots N[t], \tag{28b}$$

$$I[1,n] = X[0,1,1] - D[1,n], \quad n = 1, \dots N[1], \tag{28c}$$

$$I[t,n] = I[t-1, a[t,n,1]] + \sum_{k=1}^{t} X[t-k, a[t,n,k], t] - D[t,n], \tag{28d}$$
$$t = 1, \dots T,\ n = 1, \dots N[t],$$

$$R[0,1] = Y[0,1,0] - \sum_{k=1}^{T} X[0,1,k], \tag{28e}$$

$$R[t,n] = R[t-1, a[t,n,1]] + Y[0,1,t] - \sum_{k=1}^{T-t} X[t,n,t+k],$$
$$t = 1, \dots T-1, n = 1, \dots N[t], \tag{28f}$$

$$I^+[t,n],\ I^-[t,n],\ Y[0,1,t], R[t,n] \geq 0$$
$$t = 1, \dots T, \quad n = 1, \dots N[t], \tag{28g}$$

$$X[t,n,t+k] \geq 0, \quad t = 0, \dots T, \quad k = 1, \dots T-t, \quad n = 1, \dots N[t]. \tag{28h}$$