Ordering Pareto-Optima Through Majority Voting*

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Abstract

A commodity is shared between some individuals: There is an initial allocation; some selection procedures are used to choose an alternative allocation and; individuals decide between keeping the initial allocation or shifting to the alternative allocation. The selection procedures are supposed to involve an element of randomness in order to reflect uncertainty about economic, social and political processes. It is shown that for every allocation, λ, there exists a number, $\gamma(\lambda) \in [0;1]$, such that, if the number of individuals tends to infinity, then the probability that a proportion of the population smaller (resp. larger) than $\gamma(\lambda)$ prefers an allocation chosen by the selection procedure converges to 1 (resp. 0). The index $\gamma(\lambda)$ yields a complete order in the set of Pareto optimal allocations. Illustrations and interpretations of the selection procedures are provided.

Keywords: Pareto-optimal allocations, Infra-majority voting.

JEL-classification: D31, D72.

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1 Introduction

The present paper investigates the general question of whether social choice rules, especially voting rules, can be used to distinguish and even better rank Pareto-optimal allocations. The present framework is too simple to yield a general answer to this question. Nevertheless it provides a partial positive answer. For the sake of clarity, and in order to fix ideas it might be helpful to give a short introduction to the framework as well as the main results of the present paper before entering into its motivation.

A family of simple models is considered: The set of divisions of one unit of a commodity between m individuals. A division of the commodity is thus a vector λ with m nonnegative coordinates that sum up to one, i.e. it is a point in the $(m\text{-}1)$–simplex and it is called an allocation throughout the paper. Individuals are assumed to care only about their own share of the commodity so obviously all allocations are Pareto-optimal.

Fix the number of individuals, m, an allocation, λ, and an integer, n, $n \in \{1, \dots, m\}$. The main aim of the paper is to compute the "number" of other allocations, λ', that are such that at least n individuals are better off with λ' than with λ. Since we have a continuum of allocations, the natural way to count this "number" of allocations is to compute their volume or Lebesgue measure. Clearly this volume is 1 for $n = 1$ and 0 for $n = m$ because all allocations are Pareto optimal. The main result is the following: There exists a number, $\gamma(\lambda) \in [0, 1]$, such that these volumes converge to 1 for $n/m < \gamma(\lambda)$ and to 0 for $n/m > \gamma(\lambda)$ as the number of individuals tends to infinity. As an example, for the egalitarian allocation, $\lambda = (1/m, \dots, 1/m)$, the threshold value converges to $e^{-1} \approx 0.37$ as the number of individuals tends to infinity[1]. Of course the number, $\gamma(\lambda)$, depends on how volumes of allocations are measured as well as the allocation therefore a family of measures, all related to the Lebesgue measure, are considered. For every measure in this family a simple model is obtained and the same threshold phenomenon as for the Lebesgue measure is observed.

The occurence of this threshold effect as the number of individuals tends to infinity constitutes the main result of the paper. It is derived quite "mechanically" from computations in the sense that it is based on a purely parametric approach, where the Lebesgue measure in accordance with Laplace's advice is taken as the most neutral and natural measure. It should be noted that the case of a finite (but large) number of individuals

[1] Volume of the set of allocations that are better than the egalitarian allocation for a proportion larger than ρ of the population: For $\rho = 0.5$, it is $2 \cdot 10^{-5}$ for $m = 100$ and $2 \cdot 10^{-9}$ for $m = 200$ and; for $\rho = 0.4$, it is 0.08 for $m = 100$ and 0.03 for $m = 200$.

is of course the most interesting case; actually, going to the limit with a continuum of individuals, where the threshold is clear-cut, is a mean to extract information for the ¯nite case. Therefore the paper also contains results about the strength of the threshold e®ect for the ¯nite case.

The motivation for this parametric approach and the computation of those volumes is clear: Starting from an allocation, ¸, it is obviously very easy to ¯nd another allocation, ¸^{0}, that makes all individuals but one better o®: We just have to take the share of one individual and split it between the others. Thus, if most social choice rules behave as Maxwell's devil and select alternative allocations that are exactly in this almost zero-measure set where all individuals but one are better o® then the parametric approach followed in the present paper would be gratuitous and the main result would be a mere curiosity. However there are many upstream economic, social and political reasons why this is not the case. Indeed, uncertainty about characteristics, outcomes of social as well as political processes may be major reasons why Maxwell's devil is less relevant.

Even though it is beyond the primary concern of the present paper some attempts to justify the parametric approach by yielding microeconomic illustrations of why Maxwell's devil is less relevant and macroeconomic interpretations of the parametric approach's consequences are made. Of course our aim is to argue against Maxwell's devil-like arguments and, more boldly, to advocate for the interest of interpreting the Lebesgue measure as a probability distribution over the set of allocations as representing how alternative allocations are selected. The ¯rst two illustrations are very simple non-coorperative games where m individuals have to share one unit of a commodity. The ¯rst game being a one-shot game and the second game being bargaining a la Rubinstein. The last illustration is a pure exchange economy with consumption externalities involving two consumers and one good.

It is an implicit conjecture in the present paper that numerous upstream mechanisms to divide a commodity generate (and can be identi¯ed with) probability distributions over the (m-1){simplex of allocations[2] as soon as they involve an element of randomness which is a fair assumption in case of uncertainty about characteristics, outcomes of social as well as political processes. We conjecture that the nature of the result (the threshold e®ect) will appear very often in large populations. The paper focuses on a family of special distributions but ongoing research seeks to qualify \often".

[2] The induced distribution might be degenerate: A way to share a commodity could be to pull out knifes and give it to the surviving guy; the distribution is concentrated on the vertices of the (m-1){simplex, 1=m chance for everybody provided that there are no strong (wo)men.

Indeed, natural and easy interpretations of these distributions are given in terms of wealth distributions by use of a simple urn model that dates back to Polya and Eggenberger[3]. So the interest of the Lebesgue measure goes beyond the brutal parametric approach with the belief that Maxwell's devil does not strike too often.

The paper is organized as follows. Section 2 introduces the framework. Section 3 illustrates the choice of the Lebesgue measure as a selection mechanism. In Section 4 the main results are stated; it mainly contains the definition of the index γ and the study of its asymptotic properties. The family of selection mechanisms at scope in the paper are given an interpretation in Section 5. Finally section 6 offers interpretations of the results together with some concluding comments. All proofs are gathered in the appendix, which starts with a section on urn models, with emphasize on the notion of occupancy. Indeed, urn models turn out to be very helpful for the computation of the various probabilities, on which the proposed ranking of allocations is based.

2 The framework

Some commodity is allocated between m individuals and the preferences of each individual depends on his share only. Allocations are represented by points in the (m-1)–simplex

$$S^{m-1} = \left\{ \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} \lambda_i = 1 \right\};$$

where the i'th coordinate is the share of the i'th individual. Clearly, indifference sets for the i'th individual are linear manifolds indexed by the i'th coordinate.[4]

Generally, as argued in the introduction, uncertainty about characteristics, outcomes of social as well as economic processes introduces an element of randomness in the selection of allocations - some illustrations are provided in the next section. Therefore selection procedures are identified with probability measures over the set of allocations, S^{m-1}. In the present paper, the Lebesgue measure – called indifferently the uniform distribution – is studied so that computing probabilities merely reduces to computing volumes. But in order to study different selection procedures, the uniform distribution is considered over

[3] The literature on preference formation in a voting population facing a set of candidates has made extensive use of these Polya-Eggenberger urn models, see Berg (1985) and Berg & Gehrlein (1992).

[4] In Karni & Safra (1995) a similar framework is considered, but the commodity is supposed to be indivisible and points in the (m-1)–simplex are probability distributions, where the i'th coordinate is the probability that the i'th individual gets the commodity. However Karni & Safra is concerned with the existence of social welfare functions, while the present paper is concerned with allocative stability.

the set of all possible divisions of the commodity into cm pieces (with $c \in \mathbb{N}$), so that all individuals get c pieces. For different values of c, the uniform distribution over the set of divisions of the commodity into cm pieces induces different probability distributions over the set of allocations. Indeed, divisions induce allocations as described hereafter.

For a fixed c, divisions are represented by points in the (cm-1)–simplex

$$S^{cm-1} = \left\{ \psi = (\psi_1, \ldots, \psi_{cm}) \in \mathbb{R}_+^{cm} \mid \sum_{j=1}^{cm} \psi_j = 1 \right\};$$

where the j'th coordinate is the size of the j'th piece, and the (cm-1)–simplex is endowed with the Lebesgue measure. Given a division of the commodity, the share of an individual consists of c pieces with subsequent index, thus the share of the i'th individual is

$$\lambda_i = \sum_{j=c(i-1)+1}^{ci} \psi_j .$$

Clearly, indifference sets for the i'th individual in the (cm-1)–simplex are linear manifolds which are indexed by the sum of the individual's pieces. A probability measure over the set of allocations is induced by the Lebesgue measure over the set of divisions and the “projection” of the (cm-1)–simplex on the (m-1)–simplex, $\pi : S^{cm-1} \to S^{m-1}$, defined by

$$\pi(\psi_1, \ldots, \psi_{cm}) = \left(\sum_{j=1}^{c} \psi_j ; \ldots ; \sum_{j=(c-1)m+1}^{cm} \psi_j \right).$$

The density of the induced probability measure on S^{m-1} can easily be computed.

Lemma 1 The Lebesgue measure on S^{cm-1} induces a probability measure on S^{m-1} with density

$$p_c(\lambda) = \frac{(cm-1)!}{[(c-1)!]^m} \prod_{i=1}^{m} \lambda_i^{c-1}.$$

Remark Consider the egalitarian allocation, $\lambda = (1/m, \ldots, 1/m)$ and another allocation $\lambda' \neq \lambda$. Then

$$\lim_{c \to \infty} \frac{p_c(\lambda')}{p_c(\lambda)} = 0.$$

Therefore, the induced probability measure on S^{m-1} converges to the Dirac measure with support on the egalitarian allocation as c tends to infinity – the notion of convergence left undefined. This can be caught intuitively by remembering that c is the number of pieces that every individual gets and that the more pieces individuals get the more are shares averaged.

3 Illustration of the Lebesgue measure

3.1 Example of a simple game

A population of m individuals want to share one unit of a commodity: A cake, represented by the uniform unit disk. The individuals might simply decide to share the cake evenly. There are a lot of more or less complicated mechanisms proposed in the literature. This section introduces one that is quite simple and moreover can result in any member of the family of distributions with densities $(p_c)_{c2N}$.

The basic game: The m individuals choose simultaneously a point on the unit circle; µ 2 [0; 2¼]. Thus m points are chosen: $(µ_i)_{i=1}^m$. There exists a permutation of the agents, ¾, such that

$$0 \cdot µ_{¾(1)} \cdot µ_{¾(2)} \cdot ::: \cdot µ_{¾(m)} \cdot 2¼:$$

The share of agent i is the slice of the cake contained between the radii de¯ned by his chosen point $µ_i$ and the ¯rst one encountered counterclock-wise: suppose i = ¾(k), it is the portion $(µ_{¾(k)}; µ_{¾(k+1)})$, with the convention that ¾(m + 1) = ¾(1). For this game it is a Nash equilibrium in mixed strategies that all individuals choose their points on the unit circle according to the uniform distribution. It is shown that the induced distribution over the (m-1){simplex of shares is the uniform distribution.

Proposition 1 If all individuals choose their points on the unit circle according to the uniform distribution over the interval [0; 2¼], then the game induces the uniform distribution over the (m-1){simplex of allocations.

Proof Follows from Tovey (1997). Q.E.D.

Corollary 1 Suppose that ¯rst individuals divide the commodity into c small pieces of equal size (each of them represented as a uniform unit disk) second they share every piece by playing the described game. Then for the Nash equilibrium in mixed strategies where all individuals choose their points on the unit circle according to the uniform distribution the induced distribution over the (m-1){simplex of shares has density p_c.

3.2 Example of a less simple game

As in the previous game a population of m individuals want to share one unit of a commodity. There is an initial allocation and individuals can decide by voting whether they keep their initial shares or they enter into bargaining a la Rubinstein in order to select another allocation. However as explained in Osborne & Rubinstein (1990) and van Damme (1991) all allocations can be supported as subgame perfect equilibria provided that there are more than two individuals, $m \geq 3$. Thus the outcome of bargaining is characterized by indeterminacy, so unless individuals coordinate their strategies it is far from obvious how individuals should play.

Individuals can coordinate their strategies through some extrinsic random variable such that all individuals observe this variable and coordinate their strategies on it. Indeed suppose that all individuals believe that the choices of strategies of all other individuals depend on a random variable on the (m-1)–simplex such that if the random variable takes the value λ then they play strategies that make the allocation λ a subgame perfect equilibrium. Then all individuals play strategies that make the allocation λ a subgame perfect allocation. All distributions of the random variable seem to be equally reasonable, in particular the family of distributions with densities $(p_c)_{c \in \mathbb{N}}$ is just as reasonable as all other distributions. Perhaps the uniform distribution is a natural prior, because if all allocations are equally reasonable then the uniform distribution over the set of allocations seems to be a neutral belief.

In order to complete the description of the game suppose that the random variable is drawn from a distribution with density p_c for $c \in \mathbb{N}$ and that first the value of the random variable is revealed and second individuals vote about keeping their initial shares or entering into bargaining. For this game the main result of the present paper is that for any initial allocation, λ, there exists a number, $\gamma_c(\lambda) \in [0; 1]$, such that the probability that there is a n/m proportion that prefers an allocation selected according to the extrinsic random variable converges to 1 for $n/m < \gamma_c(\lambda)$ and to 0 for $n/m > \gamma_c(\lambda)$ as m tends to infinity.

It should be remarked that the coordination of individuals' strategies by some extrinsic random variable implies that sunspots matter, see Cass & Shell (1983) and Shell (1987). However in games with indeterminacy it seems quite natural that players coordinate their strategies – at least to some degree – and they can only coordinate on an extrinsic random variable.

3.3 Example of a market mechanism

A general model of consumption externalities was introduced in Arrow (1969) and extensively studied in Crès (1996). Consider an economy with 1 commodity and 2 consumers where agents inflict negative consumption externalities to each other. Let x_1

(resp. x_2) denote the consumption of consumer 1 (resp. consumer 2). Consumer 1's, respectively consumer 2's, utility function is given by $U_1(x_1, x_2) = \log x_1 - x_2$, respectively $U_2(x_1, x_2) = \log x_2 - x_1$. Externalities are individualized as in Arrow (1969) through markets: There are, beside the usual market for the proper commodity, also markets for externalities. Consumers face individualized prices on all markets, behave as price-takers and express demands for both their proper and external consumptions. Prices clear markets.

Both consumers are endowed with 1 unit of the commodity. Moreover, beside these endowments in physical consumption good, legal entitlements are distributed to the consumers that represent, in a Coasian world, the initial juridical situation from where they can trade on external effects; this legal entitlement, denoted ω is the same for both consumers, to respect the symmetry between them. In this example, we consider the parametric family of economies where ω is taken, to fix idea, in $[-1, 1]$. No value of ω sounds more "reasonable" or probable than another. So that assuming the Lebesgue measure on this parametric family seems a fair assumption.

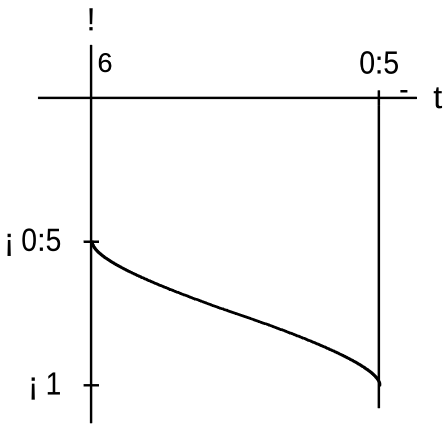
The system of equilibrium equations of this market economy is equivalent to the system of equilibrium equations derived from the usual planner's program, with the addition of consumer 1's budget constraint. Consumer 1's consumption x_1 can be expressed as a function of its welfare weight $t \in [0, 1]$:

$$x_1(t) = \frac{1}{2(1-2t)}\left(3 - 4t - \sqrt{32t^2 - 32t + 9}\right)$$

and of course $x_2(t) = x_1(1-t) = 2 - x_1(t)$. Both $x_1(t)$ and $U_1(x_1(t), x_2(t))$ are increasing function of t. For any value of the legal entitlement ω, there exists a symmetric equilibrium where the consumers are treated equally by the market; it is described by: $x_1 = x_2 = 1$, $t = 1/2$. But if $\omega \in [-1, -0.5]$ (this corresponds to the case where the legal authority wants to compensate for the negative externality through the legal entitlement), there exists also a pair of asymmetric equilibria. Thus we know that for $\omega \in [-1, -0.5]$, there exists a corresponding value $t(\omega) \neq 1/2$ and $x_1 \neq x_2$; the equilibrium allocations are $(x_1(t(\omega)), x_2(t(\omega)))$ and $(x_1(1-t(\omega)), x_2(1-t(\omega)))$. It is easy to compute, for an asymmetric welfare weight $t \neq 1/2$, the reciprocal function $\omega(t)$:

$$\omega(t) = \frac{1}{1-2t}\left(1 - 2t + t x_1(t) - \frac{t}{x_1(t)}\right).$$

Plotting this function yields the following curve: The curve being almosty linear, the uniform distribution over the set of parameters $\omega \in [-1, 1]$ generates a distribution over the



Welfare Weight at Equilibrium

1–simplex of welfare weights a distribution which is close to be uniform. Either there are no asymmetric equilibria $(\omega \in [-0.5; 1])$ or there are, and then they are evenly distributed in the 1–simplex of welfare weights. Suppose this example can be, to some extent, generalized: there are m consumers inflicting external effects on each other. Economies are parametrized by endowments and/or legal entitlements. Suppose that welfare weights yield reasonable ordinal comparisons of the utility level of the consumers[5]. Suppose the Lebesgue measure induces, on the (m-1)–simplex of welfare weights, a distribution whose density is close to p_c; since for any c, the value of γ_c at the center of the simplex is smaller than 0.5, it can be asserted that for most economies the symmetric equilibrium defeats asymmetric equilibria by pairwise comparison through majority voting: it is then a Condorcet winner.

This example is not introduced to lead to such an hypothetic statement. Its main virtue is to provide an example of a microeconomic model that generates a distribution over the simplex which is close to the uniform distribution. The starting point consists in considering a parametric model in which assuming the Lebesgue measure is the most neutral and natural assumption.

[5] This is true in the example: $U_1(x_1(t); x_2(t))$ is an increasing function of t. But it is difficult to admit it is still the case when there are more than 2 consumers: utility levels depend on all welfare weights; at most it can be conjectured that for nice enough classes of utility functions, even though “indifference surfaces” in the (m-1)–simplex of welfare weights are not hyperplanes, they can be straightened by application of a diffeomorphism.

4 Main results

For $\lambda, \lambda' \in S^{m-1}$ let $N(\lambda, \lambda') \subset M = \{1, \ldots, m\}$ be the set of indices for which $\lambda_i < \lambda'_i$. Let $T_n(\lambda) \subset S^{m-1}$ be the set of points λ' for which $N(\lambda, \lambda')$ contains exactly n elements and; $U_n(\lambda) \subset S^{m-1}$ be the set of points for which $N(\lambda, \lambda')$ contains at least n elements. Hence $T_n(\lambda)$ is the set of allocations which are prefered by exactly n individuals to the allocation λ, and $U_n(\lambda)$ is the set of allocations which are prefered by at least n individuals to the allocation λ.

Proposition 2 The measures of $T_n(\lambda) \subset S^{m-1}$ and $U_n(\lambda) \subset S^{m-1}$ are

$$
\begin{aligned}
t_{c;n}(\lambda) &= \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m-j}{n} Q_{c;j}(\lambda) \\
u_{c;n}(\lambda) &= \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m-j-1}{n-1} Q_{c;j}(\lambda) \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{c;j}(\lambda) &= \sum_{J \in M_j} \sum_{k_1=0}^{c-1} \cdots \sum_{k_{m-j}=0}^{c-1} \binom{cm-1}{k_1, \ldots, k_{m-j}} \left(\sum_{i \in J} \lambda_i \right)^{cm-1-k} \left(\prod_{h=1}^{m-j} \lambda_{i_h}^{k_h} \right) \\
k &= \sum_{i=1}^{m-j} k_i \\
M \setminus J &= \{i_1, \ldots, i_{m-j}\}
\end{aligned}
$$

and M_j is the set of all subsets of M with j elements.

Remark As shown and developped in the appendix, the quantities $t_{c;n}(\lambda)$ and $u_{c;n}(\lambda)$ are known in discrete probability theory. Consider m urns and $(cm - 1)$ balls which are allocated into the m urns according to the probability distribution λ over the urns (i.e. every ball is allocated to urn i with probability λ_i). Then $t_{c;n}(\lambda)$ is the probability that exactly n urns contain less than c balls and $u_{c;n}(\lambda)$ is the probability that at least n urns contain less than c balls.

Clearly

$$
1 = u_{c;1}(\lambda) \geq \cdots \geq u_{c;n}(\lambda) \geq u_{c;n+1}(\lambda) \geq \cdots \geq u_{c;m}(\lambda) = 0;
$$

because Pareto optimal allocations are only considered. The quantity $u_{c,n}(\lambda)$ is the probability that an allocation chosen by the selection procedure is prefered to the initial allocation λ by at least n individuals. On the one hand if $u_{c,n}(\lambda)$ is small for some small ratio n/m, then λ is stable in the sense that it is quite unlikely that an alternative allocation that is chosen by the selection procedure is preferred by a r-majority for $r \geq n/m$. On the other hand if $u_{c,n}(\lambda)$ is large for some large ratio n/m, then λ is unstable in the sense that it is quite likely that an alternative allocation that is chosen by the selection procedure is preferred by a r-majority for $r \leq n/m$. Of course it is pretty subjective whether a ratio is small or large, but in order to fix ideas it is helpful to think of small ratios as being significantly smaller than $1/2$ and large ratios as being significantly larger than $1/2$.

Definition 1 Let $\lambda, \lambda' \in S^{m-1}$ then λ is at least as stable as λ' if and only if

$$u_{c,n}(\lambda) \leq u_{c,n}(\lambda')$$

for all $n \in M$.

Hence in order to compare the stability of the two allocations λ and λ', the two "curves", $(n/m, u_{c,n}(\lambda))_{n=1}^{m}$ and $(n/m, u_{c,n}(\lambda'))_{n=1}^{m}$, have to be compared, but these two curves may or may not cross. Indeed let $c = 1$, $m = 4$ and

$$\lambda = \left(\frac{32}{100}, \frac{32}{100}, \frac{32}{100}, \frac{4}{100}\right), \quad \lambda' = \left(\frac{37}{100}, \frac{37}{100}, \frac{13}{100}, \frac{13}{100}\right), \quad \lambda'' = (1, 0, 0, 0)$$

then

$$u_{1,2}(\lambda) \approx 0.730 \quad \text{and} \quad u_{1,3}(\lambda) \approx 0.098$$

$$u_{1,2}(\lambda') \approx 0.716 \quad \text{and} \quad u_{1,3}(\lambda') \approx 0.104$$

$$u_{1,2}(\lambda'') = 1 \qquad \text{and} \quad u_{1,3}(\lambda'') = 1.$$

Thus λ and λ' are both at least as stable as λ'' but they cannot be ranked. On the one hand, if the curves do not cross then the allocation with the curve to the left is at least as stable as the other one – see figure 1.a. On the other hand, if the curves do cross then the allocations cannot be ranked – see figure 1.b.

4.1 The index

In order to study asymptotic properties of the ranking of allocations relative to their stability, the relation between allocations for m individuals and allocations for $m + 1$ individuals has to be considered.

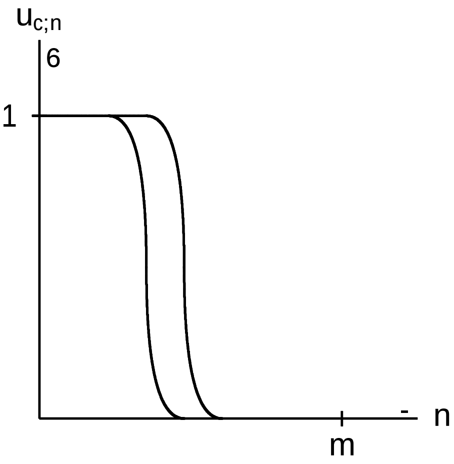


Figure 1.a

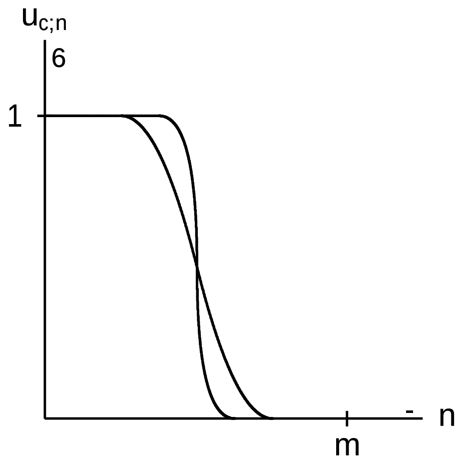


Figure 1.b

Hence let λ be a probability measure on the Borel sets of the unit interval then λ induces an allocation, $\lambda_m = (\lambda_{m;1}; \ldots; \lambda_{m;m}) \in S^{m-1}$, for all $m \in N$ by

$$\lambda_{m;i} = \lambda(I_{m;i})$$

$$I_{m;1} = \left[0; \frac{1}{m}\right] \text{ and } I_{m;i} = \left(\frac{i-1}{m}; \frac{i}{m}\right] \text{ for all other } i \in M$$

Let α be the Lebesgue measure then a probability measure, β, is absolutely continuous if and only if $\beta(A) = 0 \Leftrightarrow \alpha(A) = 0$ and it is singular if there exists B such that $\beta(B) = 1$ and $\alpha(B) = 0$. According to the Lebesgue decomposition theorem (see Itô (1984)) there exists a unique decomposition of λ into a convex combination of an absolutely continuous probability measure, γ, and a singular probability measure, δ, such that

$$\lambda = w\gamma + (1 - w)\delta$$

where $w \in [0; 1]$.

With the present relation between allocations and individuals the study of the asymptotic properties of the ranking of allocations relative to their stability reduces to the study of asymptotic properties of sequences of curves, $((n/m; u_{c;m;n}(\lambda_m))_{n=1}^{m})_{m \in N}$.

Theorem 1 For all λ, all c and all $(n_m)_{m \in N}$

$$\lim_{m \to \infty} u_{c;m;n_m}(\lambda) = \begin{cases} 1 & \text{for } \limsup_{m \to \infty} \frac{n_m}{m} < \eta_c(\lambda) \\ 0 & \text{for } \liminf_{m \to \infty} \frac{n_m}{m} > \eta_c(\lambda) \end{cases}$$

where

$$\zeta_c(\lambda) = \sum_{a=0}^{c-1} \int_{[0,1]} \frac{(cw\gamma(r))^a}{a!} e^{-cw\gamma(r)} dr.$$

For λ the associated sequence of curves, $((n/m, u_{c,m,n}(\lambda_m))_{n=1}^m)_{m \in N}$, converges pointwise to the following correspondence

$$d_c(\lambda, r) = \begin{cases} 1 & \text{for } r < \zeta_c(\lambda) \\ [0,1] & \text{for } r = \zeta_c(\lambda) \\ 0 & \text{for } r > \zeta_c(\lambda). \end{cases}$$

according to theorem 1. Since curves are decreasing the largest deviations between curves, $((n/m, u_{c,m,n}(\lambda_m))_{n=1}^m)_{m \in N}$, and correspondences, d_c, are for the smallest deviations between n/m and $\zeta_c(\lambda)$.

Observation 1 For all $m \in N$ and all $c \in N$

$$u_{c,m,n}(\lambda_m) \geq \frac{(n/m - \zeta_{c,m}(\lambda_m))^2}{(n/m - \zeta_{c,m}(\lambda_m))^2 + b_{c,m}} \quad \text{for} \quad n/m \leq \zeta_{c,m}(\lambda_m)$$

$$u_{c,m,n}(\lambda_m) \leq \frac{b_{c,m}}{(n/m - \zeta_{c,m}(\lambda_m))^2 + b_{c,m}} \quad \text{for} \quad n/m \geq \zeta_{c,m}(\lambda_m)$$

where

$$\zeta_{c,m}(\lambda_m) = \frac{1}{m} \sum_{a=0}^{c-1} \sum_{i=1}^{m} \binom{cm-1}{a} \lambda_{m,i}^a (1 - \lambda_{m,i})^{cm-1-a}$$

and

$$b_{c,m} = c^2 \left(\frac{1}{m} + \frac{(2(c-1))^{2c}}{cm-1} \right).$$

Remark Note that

$$\lim_{m \to \infty} \zeta_{c,m}(\lambda_m) = \zeta_c(\lambda)$$

$$\lim_{m \to \infty} b_{c,m} = 0.$$

Moreover the proof of observation 1 reveals that $b_{c,m}$ comes from a very conservative approximation, perhaps $b_{c,m}$ can be replace by a constant, which does not depend on c.

By an application of observation 1 it is possible to discuss how complete the ranking by stability is, i.e. for which relative sizes of groups is it possible to rank two allocations and for which relative sizes is it not possible to rank them? Consider two allocations, λ and λ', then

$$u_{c;m;n}(\lambda_m) \gtrless u_{c;m;n}(\lambda'_m) \text{ if and only if } \gamma_{c;m}(\lambda_m) \gtrless \gamma_{c;m}(\lambda'_m)$$

provided that

$$\frac{n}{m} \geq \frac{\gamma_{c;m}(\lambda_m) + \gamma_{c;m}(\lambda'_m)}{2} - \sqrt{\left(\frac{\gamma_{c;m}(\lambda_m) - \gamma_{c;m}(\lambda'_m)}{2}\right)^2 - b_{c;m}}$$

or

$$\frac{n}{m} \leq \frac{\gamma_{c;m}(\lambda_m) + \gamma_{c;m}(\lambda'_m)}{2} + \sqrt{\left(\frac{\gamma_{c;m}(\lambda_m) - \gamma_{c;m}(\lambda'_m)}{2}\right)^2 - b_{c;m}}$$

Hence the ranking of allocations by their stability becomes "more and more" complete as the number of individuals tends to infinity. Indeed if $\gamma_c(\lambda) < \gamma_c(\lambda')$ then the associated curves can only cross for n/m closer and closer to $\gamma_{c;m}(\lambda_m)$, where the curve for λ'_m converges to one, or $\gamma_{c;m}(\lambda'_m)$, where the curve for λ_m converges to zero, as m tends to infinity because $b_{c;m}$ converges to zero and $\gamma_{c;m}$ converges to γ_c as m tends to infinity.

Definition 2 For $m \in N$ and all $c \in N$ the index of an allocation, $\lambda_m \in S^{m-1}$, is $\gamma_{c;m}(\lambda_m)$.

The index of an allocation is the expected relative size of the group of individuals who prefer an allocation chosen by the selection procedure to the allocation in question as shown in the appendix. To pursue the translation in terms of occupancy, as it is shown in the appendix, the index is the expected ratio of urns that are allocated less than c balls when allocating $(cm - 1)$ balls into m urns according to the probability distribution λ.

Observation 2 For all $m \in N$ and all $c \in N$

$$\gamma_{c;m}(\lambda_m) = \frac{1}{m}\sum_{n=1}^{m} u_{c;m;n}(\lambda_m) = \frac{1}{m} + \frac{1}{m}\sum_{n=1}^{m}(n-1)t_{c;m;n}(\lambda_m).$$

Hence, the index is a weighted sum of the probabilities that either exactly or at least n individuals prefer an allocation chosen by the selection procedure to the allocation in question. Observation 1 shows that the index contains relevant information with respect to ranking by stability, while observation 2 shows that the index has a reasonable interpretation. For an allocation, λ, in S^{m-1} consider $m-2$ hypersurfaces of S^{m-1} through λ defined by $u_{c;n}(\lambda') = u_{c;n}(\lambda)$ for $n \in \{2, \ldots, m\}$. If these $m-2$ level surfaces coincide then ranking by stability is complete, but as shown after definition 1 it is not. Observation 2 is a first step toward showing that the iso-index hypersurface (defined by $\zeta_{c,m}(\lambda) = \zeta_{c,m}(\lambda'))$ are “in between” the $m - 2$ level surfaces. As an illustration reconsider the example after definition 1, then figure 2.a and figure 2.b illustrate this construction. The level curves on figure 2.a, Σ_2 and Σ_3, are the sections of the two hypersurfaces defined respectively by $u_{1,2}(\lambda^*) = u_{1,2}(\lambda)$ and $u_{1,3}(\lambda^*) = u_{1,3}(\lambda)$ by the hyperplane $\lambda_4 = 1/5$ (the triangle – here a 2-simplex of size 4/5 – is of course the section of the 3-simplex S^3 by the same hyperplane). On figure 2.a, the six small stripes of space in between Σ_2 and Σ_3 are the allocations that cannot be compared with λ using the ranking by stability.

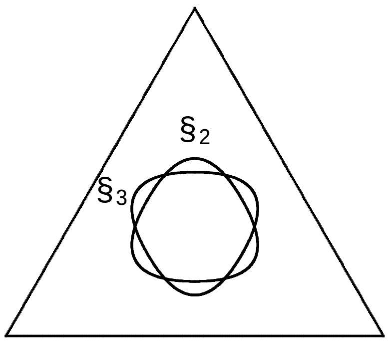


Figure 2.a

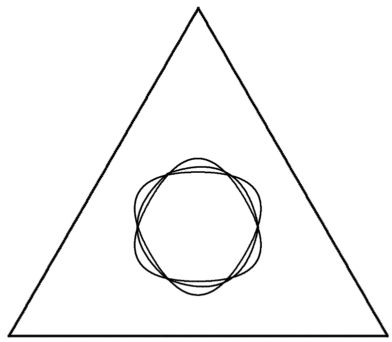


Figure 2.b

It is easy to check that λ' is in one of these stripes. On figure 2.b the iso-index curve is added and it is between Σ_2 and Σ_3, and to some extent sums up the information that they both give in terms of ranking by stability.

In case $c = 1$ the index takes the form

$$\zeta_{1,m}(\lambda_m) = \frac{1}{m}\sum_{i=1}^{m}(1 - \lambda_{m,i})^{m-1}$$

and

$$\lim_{m\to\infty} \zeta_{1,m}(\lambda_m) = \zeta_1(\lambda) = \int_{[0,1]} e^{-w^{\gamma}(r)}dr.$$

Consider the egalitarian allocation, where all individuals share the commodity, then

$$\zeta_1(\lambda) = e^{-1} \approx 0.37.$$

Therefore the expected number of individuals who prefer an allocation chosen by the selection procedure is approximately $0.37m$ for m large. For the egalitarian allocation:

$$u_{1,40} = 0.08 \text{ and } u_{1,50} = 2\cdot 10^{-5}$$

for $m = 100$ and

$$u_{1,80} = 0.03 \text{ and } u_{1,100} = 2 \cdot 10^{-9}$$

for $m = 200$.

Consider an allocation, where two thirds of the individuals share the commodity and one third of the individuals get nothing, then

$$\zeta_1(\lambda) = \int_{[0,2/3]} e^{-\frac{3}{2}} dr + \int_{]2/3,1]} 1 dr = \frac{2}{3} e^{-\frac{3}{2}} + \frac{1}{3} \approx 0.48.$$

Therefore the expected number of individuals who prefer an allocation chosen by the selection procedure is approximately $0.48m$ for m large. Hence the egalitarian allocation is more stable than the other allocation, where two thirds of the individuals share the commodity and one third of the individuals get nothing.

In case $c = 3$ the index takes the form

$$\zeta_{3,m}(\lambda_m) = \sum_{a=0}^{2} \sum_{i=1}^{m} \binom{3m-1}{a} \lambda_{m,i}^{a} (1 - \lambda_{i,m})^{3m-1-k}$$

and

$$\lim_{m \to \infty} \zeta_{3,m}(\lambda_m) = \zeta_3(\lambda) = \sum_{a=0}^{2} \frac{(3w\gamma(r))^a}{a!} \int_{[0,1]} e^{-3w\gamma(r)} dr.$$

For the egalitarian allocation, $\zeta_3(\lambda) \approx 0.423$ and for an allocation, where ten eleventh of the individuals share the commodity and one eleventh of the individuals get nothing, $\zeta_3(\lambda) \approx 0.418$. Hence the allocation, where ten eleventh of the individuals share the commodity and one eleventh of the individuals get nothing, is more stable than the egalitarian allocation for m large - even though it is not much. Recall that c is the number of pieces that individuals get and the more pieces that individuals get the more are shares averaged, therefore individuals, who have more than $1/m$ of the commodity, tend to prefer the allocation in question rather than an allocation chosen by the selection procedure.

Suppose that $\lambda = w\gamma + (1 - w)\delta$ where γ is an absolutely continuous probability measure and δ is a singular probability measure then $D_w \zeta_c(\lambda) < 0$ according to some straight forward calculations. So, if λ minimizes $\zeta_c(\lambda)$ then λ is absolutely continuous. Let

$$f_c(x) = \sum_{a=0}^{c-1} \frac{(cx)^a}{a!} e^{-cx} \text{ and } g_c(x) = 1 - \frac{1}{x}(1 - f_c(x)).$$

Then $f_c(x)$ is the marginal contribution to the index of an individual who gets x and $g_c(x)$ is the index of an allocation where some group of size $1 - 1/x$ gets nothing and some group of size $1/x$ shares the commodity. $f_c(x)$ is strictly decreasing on $[0;\infty[$, strictly concave on $[0;(c-1)/c]$ and strictly convex on $[(c-1)/c;\infty[$ according to some straight forward calculations. Therefore, in order to minimize $\gamma_c(\lambda)$, individuals should get either 0 or $(c-1)/c$ provided that they get something in $[0;(c-1)/c]$ and individuals should all get the same provided that they get something in $[(c-1)/c;\infty[$. So, if λ minimizes $\gamma_c(\lambda)$ then there exists $x \in [(c-1)/c;\infty[$ – actually $x \geq 1$ – such that some group of size $1/x$ shares the commodity and the rest gets nothing. Therefore, the solution to

$$\min \; g_c(x)$$

$$\text{s.t.} \quad x \geq 1$$

characterizes the allocations that minimize $\gamma_c(\lambda)$ in the sense that if x solves the problem then some group of size $1/x$ should share the commodity and some group of size $1 - 1/x$ should get nothing. In order to study whether the egalitarian allocation minimizes $\gamma_c(\lambda)$, the derivative of $g_c(x)$ could be evaluated at $x = 1$ where

$$\begin{aligned} D_x g_c(x) &= \frac{1}{x^2}\left(1 - \left(\sum_{a=0}^{c} \frac{(cx)^a}{a!} + (1-c)\frac{(cx)^c}{c!}\right) e^{-cx}\right) \\ &= \frac{1}{x^2} e^{-cx}\left(\sum_{a=c+1}^{\infty} \frac{(cx)^a}{a!} + (1-c)\frac{(cx)^c}{c!}\right) \end{aligned}$$

according to some straight forward calculations. We conjecture that $D_x g_c(1) < 0$ for all $c \geq 3$ which implies that the egalitarian allocations does not minimize $\gamma_c(\lambda)$ for $c \geq 3$ but we have not been able to prove this conjecture.

4.2 Interpretation of the Lebesgue measure

The selection procedures studied in the present paper are taken to be the uniform distribution over the set of divisions, i.e. the (cm-1)–simplex, or equivalently the induced family of probability distributions on the set of allocations, i.e. the (m-1)–simplex. These probability distributions can be generated by the Pólya-Eggenberger urn models that give an alternative interpretation of the selection procedures. Indeed in this alternative interpretation “equalitarianism”, i.e. the extent to which it is easier for a wealthy individual than for a poor individual to become wealthier, becomes important.

The structure of Pólya-Eggenberger urn models[6] is the following: An urn contains cm balls of m different colours, c balls of each colour; balls are drawn at random and after

[6] In Berg (1985) and Gehrlein and Berg (1992) Pólya-Eggenberger distributions are used to model

each draw it is put back into the urn with s more balls of the same color; k draws are made. Hence Pólya-Eggenberger urn models are parametrized by c and s.

Let k_i be the number of times balls of color i have been drawn then the probability of $(k_i)_{i=1}^m$ with $\sum_{i=1}^m k_i = k$ is (see Johnson and Kotz (1978))

$$Pr[k_1;\ldots;k_m] = \frac{k!}{k_1!\ldots k_m!}\prod_{i=1}\frac{(c+s(k_i-1))\ldots c}{(cm+s(k-1))\ldots cm} \tag{2}$$

For s = c = 1 the uniform probability measure on distributions of balls is generated

$$Pr[k_1;\ldots;k_m] = \frac{k!}{(m+k-1)\ldots m} = \begin{pmatrix} m+k-1 \\ m-1 \end{pmatrix}^{-1}$$

Let $\lambda_i = k_i/k$ then $(\lambda_i)_{i=1}^m$ is a point in the (m-1)–simplex $(k_i)_{i=1}^m$. Clearly the probability measures on allocations of balls induces a probability measure on the (m-1)–simplex, but only

$$\begin{pmatrix} m+k-1 \\ m-1 \end{pmatrix}$$

points are in the support of the probability measure. Indeed the selection procedures considered in the present paper can be obtained as limits of Pólay-Eggenberger urn models.

Proposition 3 If s = 1 and k tends to infinity then the induced probability measure on the (m-1)–simplex converges to a probability measure with density

$$p_c(\lambda) = \frac{(cm-1)!}{[(c-1)!]^m}\prod_{i=1}\lambda_i^{c-1}$$

in the weak topology.

Proof For s = 1, equation (2) is

$$P[k_1;\ldots;k_m] = \begin{pmatrix} cm+k-1 \\ cm-1 \end{pmatrix}^{-1}\prod_{i=1}\begin{pmatrix} c+k_i-1 \\ c-1 \end{pmatrix}$$

for $\sum_{i=1}^m k_i = k$. Let $\lambda = (\lambda_i)_{i=1}^m \in S^{m-1}$ and suppose that

$$\lim_{k\to\infty}\frac{k_{i;k}}{k} = \lambda_i.$$

homogeneity of a voting population, i.e. the degree of “similarity" of preferences of voters in a fixed population.

Then

$$p_c[\lambda] = \lim_{k \to \infty} (m-1)! \begin{pmatrix} m+k-1 \\ m-1 \end{pmatrix} Pr[k_{1;k}; \ldots ; k_{m;k}]$$

$$= \frac{(cm-1)!}{[(c-1)!]^m} \prod_{i=1}^{m} \lambda_i^{c-1}$$

because the Lebesgue measure of S^{m-1} is $1/(m-1)!$ if it is projected on $m-1$ coordinates. Q.E.D.

Remark According to lemma 1 the induced probability measure on the S^{m-1} simplex is identical to the probability measure on allocations induced by the uniform probability distribution on divisions into cm pieces.

Pólya-Eggenberger urn models lead to an alternative interpretation of probability distributions on allocations: The commodity is divided into k pieces of equal size and; every time a ball of color i is drawn individual i receives a piece of the commodity. Recall that if a ball of color i is drawn then it is put back into the urn with s more balls of the same color. Therefore if s/c is small then the initial distribution of balls is important compared with the number of balls that are put into the urn after the draws and if s/c is large then the number of balls that are put into the urn after the draws are important compared with the the initial distribution of balls. Hence s/c seems to be a natural measure of the degree of equalitarianism, i.e. the extent to which it is easier for a wealthy individual than for a poor individual to become wealthier.

On the one hand, if $s = 0$ then the probability, that a ball of color i is drawn, does not depend on the history of draws. In this case the multinomial distribution is obtained

$$Pr[k_1; \ldots ; k_m] = \frac{1}{m^k} \frac{k!}{k_1! \ldots k_m!},$$

that converges to the Dirac measure concentrated on the egalitarian allocation as k tends to infinity. On the other hand, if s is very large then the first draw almost completely determines the allocation. In this case the probability distribution obtained is concentrated on the m vertices of the S^{m-1} simplex hence it corresponds to a kind of "winner takes it all" selection procedure. Neither case seems to be relevant from an empirical point of view because they result in trivial distributions where one individual gets everything while the others get nothing. This is the main reason to exclude these cases from the present paper.

5 Concluding Comments

In the present paper the stability of allocations has been studied from a combinatorial point of view in a quite simple model where power of groups is related to their size as in voting. Allocations were ranked according to their stability unfortunately this ranking turned out not to be complete. However as the number of individual tends to in¯nity the ranking becomes more and more complete. Indeed it was shown that every allocation can be associated with an index such that the ranking of allocations by this index converge to the ranking of allocations by stability in the sense that these two rankings deviate only for groups of smaller and smaller or larger and larger relative size as the number of individuals tends to in¯nity.

All allocations are unstable provided that groups of small relative size are allowed to in°uence as in infra-majority voting, i.e. there is always somebody who wants another allocation. Similarly all allocations are stable provided that groups of large relative size are allowed to in°uence as in supra-majority voting, i.e. there is never unanimity to want another allocation. Therefore if respect for minorities is interpreted as allowing groups of small relative size to in°uence then there is a trade-o® between stability and respect for minorities. The index for an allocation is more or less the in¯mum of the relative sizes of groups that can be allowed to in°uence while keeping the allocation stable. Indeed if the index for an allocation is close to zero then the allocation is stable even if groups of very small relative size are allowed to in°uence and if the index is close to one then the allocation is stable only if groups of very large relative size are allowed to in°uence. Hence the index of an allocation is a measure of the degree of the trade-o® between stability and respect for minorities.

Clearly the assumptions, i.e. the uniform distribution on the set of divisions, are very important for the results of the present paper. The robustness of the results with regard to other distributions, especially that the ranking of allocations by stability becomes more and more complete as the number of individuals tends to in¯nity, remains to be explored.

6 References

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7 Appendix

As noted in the remark to proposition 2 and section 4, urn models are very useful in relation to the present paper. First urn models are introduced and studied with emphasize on the notion of occupancy and asymptotic properties. Second results in the present paper are established.

7.1 Urn Models and Occupancy

This subsection of the appendix is mainly expository (see Kolchin, Sevast'yanov & Chistyakov (1978) and Johnson & Kotz (1978) for more details) and all results that are not established are stated and established in either Kolchin, Sevast'yanov & Chistyakov (1978) or Johnson & Kotz (1978). Let $m \geq 2$ be a natural number and let $M = \{1, \ldots, m\}$. Moreover let M_j be the set of all subsets of M with $j \in M$ elements then there are $\binom{m}{j}$ subsets in M_j.

For m distiguishable urns let $\mu = (p_i)_{i \in M}$ be a probability measure on the set of urns, i.e. $p_i \in R_+$ for all $i \in M$ and

$$\sum_{i \in M} p_i = 1;$$

and consider random distributions of r indistinguishable balls into the urns. The balls are supposed to be distributed independently according to μ, i.e. p_i is the probability that a ball is assigned to the i'th urn.

7.1.1 Empty Urns

For distributions of balls into urns, some urns are empty and some urns are occupied. Let X_0 be the number of empty urns, then the probability that n urns are empty is

$$\Pr[X_0 = n] = \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m-j}{n} P_j^r(\mu)$$

where

$$P_j^r(\mu) = \sum_{J \in M_j} \left(\sum_{i \in J} p_i \right)^r;$$

as shown by an application of the inclusion-exclusion principle[7], and the probability that at least n urns are empty is

$$\Pr[X_0 \geq n] = \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m-j-1}{n-1} P_j^r(\mu):$$

[7] See the proof of proposition 2

The expected value and variance of the occupancy distribution are

$$
\begin{aligned}
E[X_0] &= \sum_{i \in M} (1 - p_i)^r \\
Var[X_0] &= E[X_0] - \sum_{i \in M} (1 - 2p_i)^r \\
&+ \sum_{i \in M} \sum_{j \in M} ((1 - p_i - p_j)^r - (1 - p_i)^r (1 - p_j)^r)
\end{aligned}
$$

Clearly the expected value attains its minimum in the symmetric case where $p_i = 1/m$ for all $i \in M$, this case is called the classical occupancy distribution $P_n(r; m)$ and

$$
Pr[X_0 = n] = P_n(r; m) = \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m}{n; j} \left(\frac{j}{m} \right)^r ;
$$

where

$$
\binom{m}{n; j} = \frac{m!}{n!\, j!\, (m - n - j)!}
$$

is a trinomial number. Obviously for the classical occupancy distribution

$$
\begin{aligned}
E[X_0] &= m \left(1 - \frac{1}{m} \right)^r \\
Var[X_0] &= m \left(1 - \frac{1}{m} \right)^r + m(m - 1) \left(1 - \frac{2}{m} \right)^r - m^2 \left(1 - \frac{1}{m} \right)^{2r} .
\end{aligned}
$$

7.1.2 Occupied Urns

Let X_a be the number of urns that contain exactly a balls after the distribution of r balls. In particular, the expected value and the variance of the random variable X_a are

$$
\begin{aligned}
E[X_a] &= \binom{r}{a} \sum_{i \in M} p_i^a (1 - p_i)^{r-a} \\
Var[X_a] &= \binom{r}{a} \sum_{i \in M} p_i^a (1 - p_i)^{r-a} \left(1 - \binom{r}{a} \sum_{i \in M} p_i^a (1 - p_i)^{r-a} \right) \\
&+ \binom{r}{a; b} \sum_{i \in M} \sum_{j \neq i} p_i^a p_j^a (1 - p_i - p_j)^{n-2a} \\
&- \binom{r}{a}^2 \sum_{i \in M} \sum_{j \neq i} p_i^a p_j^a (1 - p_i)^{n-a} (1 - p_j)^{n-a} .
\end{aligned}
$$

Let Y_c be the sum of c random variables, X_0, ..., X_{c-1}, $Y_c = X_0 + \cdots + X_{c-1}$.

Lemma 2 The expected number of urns that contain less than c balls each is

$$E[Y_c] = \sum_{a=0}^{c-1} \sum_{i=1}^{m} \binom{r}{a} p_i^a (1-p_i)^{r-a}$$

where $c \le r$.

Lemma 3 The probabilitiy that there are exactly n urns that contain less than c balls each is

$$P[Y_c = n] = \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m-j}{n} Q^r_{c;j}(\mu); \quad (3)$$

and the probabilitiy that there are at least n urns with less than c balls is

$$P[Y_c \ge n] = \sum_{j=0}^{m-n} (-1)^{m-n-j} \binom{m-j-1}{n-1} Q^r_{c;j}(\mu); \quad (4)$$

where

$$Q^r_{c;j}(\mu) = \sum_{J \in M_j} \sum_{k_1=0}^{c-1} \cdots \sum_{k_{m-j}=0}^{c-1} \binom{r}{k_1;\ldots;k_{m-j}} \left(\sum_{i \in J} p_i \right)^{r-k} \prod_{h=1}^{m-j} p_{i_h}^{k_h}$$

$$k = \sum_{i=1}^{m-j} k_i$$

$$M \setminus J = \{i_1;\ldots;i_{m-j}\}$$

Proof Only the probability that all urns in a specified subset of $m-j$ urns contain less than c balls has to be computed because the probability that exactly $m-j$ urns contain less than c balls each follows from an application of the inclusion-exclusion principle.

Consider $m-j$ urns then the probability that they contain less than c balls each is computed as the sum over $(k_{h_i})_{i=1}^{m-j} \in \{0;\ldots;c-1\}^{m-j}$ of the probabilities that there are k_{h_i} balls in the h_i'th urn for all $i \in \{1;\ldots;m-j\}$. For $(k_{h_i})_{i=1}^{m-j}$ the probability that there are k_{h_i} balls in the h_i'th urn for all $i \in \{1;\ldots;m-j\}$ is

$$\sum_{k_1=0}^{c-1} \cdots \sum_{k_{m-j}=0}^{c-1} \binom{r}{k_1;\ldots;k_{m-j}} \left(\sum_{i \in J} p_i \right)^{r-k} \prod_{h=1}^{m-j} (p_{i_h})^{k_h}$$

where

$$k = \sum_{i=1}^{m-j} k_i$$

$$M \setminus J = \{i_1;\ldots;i_{m-j}\}$$

$$\binom{r}{k_1;k_2;\ldots;k_{m-j}} = \frac{r}{k_1!\ldots k_{m-j}!\,(r-(k_1+\ldots k_{m-j}))!}$$

where the multinomial coefficient is the number of possible distributions such that the h_i'th urn contain k_{h_i} urns for all $i \in \{1;\ldots;m-j\}$. Q.E.D.

Lemma 4 Suppose that $r \le cm - 1$ then

$$\sum_{n=2}^{m-1} \Pr[Y_c \ge n] = -1 + Q^r_{c;m-1}(\mu).$$

Proof On the one hand for $r \le cm - 1$ and $n = 0$ the occupancy formula (3) implies that

$$1 + \sum_{j=1}^{m-1} (-1)^{m-j} Q^r_{c;j}(\mu) = 0; \qquad (5)$$

provided that $Q^r_{c;m}(\mu) = 1$. Hence if less than cm balls are distributed, then the probability that no urn contain less than c balls is zero.

On the other hand

$$\begin{aligned}
\sum_{n=2}^{m-1} \Pr[Y_c \ge n] &= \sum_{n=2}^{m-1} \sum_{j=1}^{m-n} (-1)^{m-n-j} \binom{m-j-1}{n-1} Q^r_{c;j}(\mu) \\
&= \sum_{j=1}^{m-2} (-1)^{m-j} Q^r_{c;j}(\mu) \sum_{n=2}^{m-j} (-1)^n \binom{m-j-1}{n-1} \\
&= \sum_{j=1}^{m-2} (-1)^{m-j} Q^r_{c;j}(\mu) \sum_{k=1} \binom{p}{k} \\
&= \sum_{j=1}^{m-2} (-1)^{m-j} Q^r_{c;j}(\mu) \\
&= -1 + Q^r_{c;m-1}(\mu)
\end{aligned}$$

for $p = m - j - 1$ because

$$\sum_{k=0} (-1)^{k+1} \binom{p}{k} = 0.$$

according to Johnson & Kotz (1978). Q.E.D.

7.1.3 Asymptotic Properties

Let $Z_a(m)$ be the normalized random variables of $X_a(m)$, i.e. $Z_a(m) = X_a(m)/m$ and let $V_c(m)$ be the sum of c normalized random variables, i.e. $V_c(m) = Y_c(m)/m = \sum_{a=0}^{c-1} Z_a(m)$ then $E[Z_a] = m^{-1}E[X_a]$ and $Var[Z_a] = m^{-2}Var[X_a]$. In order to study the asymptotic properties of the $Z_a(m)$'s and the $V_c(m)$'s as m converge to infinity the relation between the number of urns, the number of balls and the distributions of balls has to be considered. Hence, let μ be a probability measure on the Borel sets of the unit interval and $p_i(m) = \mu(I_i(m))$ where

$$I_1(m) = \left[0, \frac{1}{m}\right] \quad \text{and} \quad I_i(m) = \left(\frac{i-1}{m}, \frac{i}{m}\right]$$

for all other $i \in M$ and all $M \in N$. Let λ be the Lebesgue measure then a probability measure, ν, is absolutely continuous if and only if $\nu(A) = 0 \Leftrightarrow \lambda(A) = 0$ and it is singular if there exists B such that

$\beta(B) = 1$ and $\alpha(B) = 0$. According to the Lebesgue decomposition theorem (see Itô (1984)) there exists a unique decomposition of μ into a convex combination of an absolutely continuous probability measure, γ, and a singular probability measure, δ, i.e.

$$\mu = w\gamma + (1 - w)\delta$$

where $w \in [0;1]$.

Lemma 5 Suppose that

$$\lim_{m \to \infty} \frac{r(m)}{m} = s \in \mathbb{R}_+ \cup \{\infty\}$$

then

$$\lim_{m \to \infty} E[Z_a(m)] = \int_{[0;1]} \frac{(sw\gamma(t))^a}{a!} e^{-sw\gamma(t)} dt$$

$$\limsup_{m \to \infty} mVar[Z_a(m)] \leq 1 + \frac{(2a)^{2(a+1)}}{s}.$$

Proof The mean values and the variances are treated in two separate parts.

"Mean Values" Suppose that $t \in I_{i(m)}(m)$ for all $m \in \mathbb{N}$ then

$$\limsup_{m \to \infty} m\mu(I_{i(m)}(m)) = \liminf_{m \to \infty} m\mu(I_{i(m)}(m)) = w\gamma(t) \in \mathbb{R}_+$$

for almost all $t \in [0;1]$ (see Itô (1984)). Therefore

$$\lim_{m \to \infty} E[Z_a(m)] = \int_{[0;1]} \frac{(sw\gamma(t))^a}{a!} e^{-sw\gamma(t)} dt$$

because

$$\begin{aligned} E[Z_a(m)] &= \frac{1}{m} \binom{r(m)}{a} \sum_{i \in M} p_i(m)^a (1 - p_i(m))^{r(m)-a} \\ &= \frac{1}{m} \binom{r(m)}{a} \frac{1}{m^a} \sum_{i \in M} (mp_i(m))^a (1 - p_i(m))^{r(m)-a} \end{aligned}$$

and

$$\begin{aligned} \lim_{m \to \infty} \binom{r(m)}{a} \frac{1}{m^a} &= \frac{s^a}{a!} \\ \lim_{m \to \infty} (mp_{i(m)}(m)) &= w\gamma(t) \\ \lim_{m \to \infty} (1 - p_{i(m)}(m))^{r(m)-a} &= e^{-sw\gamma(t)}. \end{aligned}$$

"Variances" The variance can be bounded

$$
\begin{aligned}
Var[X_a(m)] \leq\; & \binom{r}{a} \sum_{i\in M} p_i^a (1-p_i)^{r(m)-a} \\
&+ \binom{r(m)}{a,b} \sum_{i\in M}\sum_{j\neq i} p_i^a p_j^a (1-p_i-p_j)^{r(m)-2a} \\
&- \binom{r(m)}{a,b} \sum_{i\in M}\sum_{j\neq i} p_i^a p_j^a (1-p_i)^{r(m)-2a}(1-p_j)^{r(m)-2a} \\
&+ \binom{r(m)}{a}^2 \sum_{i\in M}\sum_{j\neq i} p_i^a p_j^a (1-p_i)^{r(m)-2a}(1-p_j)^{r(m)-2a} \\
&- \binom{r(m)}{a}^2 \sum_{i\in M}\sum_{j\neq i} p_i^a p_j^a (1-p_i)^{r(m)-a}(1-p_j)^{r(m)-a}.
\end{aligned}
$$

The first term is less than $E[X_a(m)]$, the sum of the second term and the third term is negative, because

$$(1-p_i-p_j) \leq (1-p_i-p_j+p_ip_j) = (1-p_i)(1-p_j);$$

and the sum of the fourth term and the fifth term is less than

$$m^2 \frac{(2a)^{2(a+1)}}{r(m)} \left(1-\frac{a}{r(m)}\right)^{2(r(m)-2a)}.$$

Therefore

$$Var[Z_a(m)] \leq \frac{1}{m} + \frac{(2a)^{2(a+1)}}{r(m)} \left(1-\frac{a}{r(m)}\right)^{2(r(m)-2a)};$$

thus

$$\limsup_{m\to\infty} m Var[Z_a(m)] \leq 1 + \frac{(2a)^{2(a+1)}}{s}.$$

Hence $Var[Z_a(m)] = O(m)$ for $s \in R_{++}$.

In order to show that the sum of the fourth and the fifth term is less than

$$m^2 \frac{(2a)^{2(a+1)}}{r(m)} \left(1-\frac{a}{r(m)}\right)^{2(r(m)-2a)};$$

first note that

$$1-(1-p_i)^a(1-p_j)^a \leq a(p_i+p_j)$$

secondly note that if p_i and p_j solve

$$
\begin{aligned}
\max\quad & p_i^a p_j^a (p_i+p_j)(1-p_i)^{r(m)-2a}(1-p_j)^{r(m)-2a} \\
\text{s.t}\quad & p_i, p_j \in [0,1]
\end{aligned}
$$

then

$$\frac{a}{r(m)} \leq p_i = p_j = \frac{2a+1}{2(r(m)-a)+1} \leq \frac{2a}{r(m)}$$

for $r(m) \geq 2a$ and thirdly use these bounds on the p_i's to find an upper bound the sum of the fourth and the fifth term. Q.E.D.

Corollary 2 Suppose that

$$\lim_{m\to\infty}\frac{r(m)}{m} = s \in R_+ \cup \{\infty\}$$

then

$$\lim_{m\to\infty} E[V_c(m)] = \sum_{a=0}^{c-1}\int_{[0;1]}\frac{(sw\gamma(t))^a}{a!}e^{-sw\gamma(t)}dt$$

$$\limsup_{m\to\infty} mVar[V_c(m)] \le c^2\left(1+\frac{(2(c-1))^{2c}}{s}\right):$$

Proof First

$$\lim_{m\to\infty} E[\sum_{a=0}^{c-1} Z_a(m)] = \sum_{a=0}^{c-1}\int_{[0;1]}\frac{(sw\gamma(t))^a}{a!}e^{-sw\gamma(t)}dt$$

because $E[V+W] = E[V]+E[W]$. Secondly

$$Var[\sum_{a=0}^{c-1} Z_a(m)] \le \sum_{a=0}^{c-1}\sum_{b=0}^{c-1}\sqrt{Var[Z_a(m)]Var[Y_b(m)]}$$

$$\le c^2\left(\frac{1}{m}+\frac{(2(c-1))^{2c}}{r(m)}\right);$$

therefore

$$\limsup_{m\to\infty} mVar[\sum_{a=0}^{c-1} Z_a(m)] \le c^2\left(1+\frac{(2(c-1))^{2c}}{s}\right):$$

Hence $Var[\sum_{a=0}^{c-1} Z_a(m)] = O(m)$ for $s \in R_{++}$. Q.E.D.

If $s > 0$ then the corollary implies that for a fixed probability distribution on the unit interval the distribution of $V_c(m)$ converges to a degenerate distribution because the variance converges to zero. Hence $(Pr[V_c \le z])_{z\in[0;1]}$ converges to the following "function"

$$d_c(\mu; z) = \begin{cases} 0 & \text{for } z < E[V_c] \\ [0;1] & \text{for } z = E[V_c] \\ 1 & \text{for } z > E[V_c] \end{cases}$$

Hence for fixed $c \in N$ different probability measures on the unit interval can be ranked by stochastic dominance by comparing the mean values of the induced, normalized random variables. However for $m \in N$ the induced probability measures on the unit interval cannot be ranked only by comparing the mean values of the induced, normalized random variables because their distribution need not be degenerate.

7.1.4 Comparisions of Distributions

Consider two probability measures on the unit interval, μ and ν, by use of the construction in the previous subsection these probability measures induces two distributions for every $m \in N$. Clearly if $m \in N$ is large compared to $c \in N$ then the distributions can be ranked by stochastic dominance – almost.

Lemma 6 Suppose that a random variable, $T \in \mathbb{R}$, has mean value $E \in \mathbb{R}$ and variance $V \in \mathbb{R}_{++}$ then the distribution, $\Pr : \mathbb{R} \to [0;1]$, satisfies the following inequalities

$$\Pr[T \leq t] \leq \frac{V}{V + (E - t)^2} \quad \text{for} \quad t \leq E$$

$$\Pr[T \leq t] \geq \frac{(E - t)^2}{V + (E - t)^2} \quad \text{for} \quad t \geq E.$$

Proof Suppose that $u \leq 0$ and $\Pr[T - E \leq u]u + (1 - \Pr[T - E \leq u])v = 0$ then $V \geq \Pr[T - E \leq u]u^2 + (1 - \Pr[T - E \leq u])v^2$ therefore

$$\Pr[T - E \leq u] \leq \frac{V}{V + u^2}.$$

Suppose that $u \geq 0$ and $(1 - \Pr[T - E \leq u])u + \Pr[T - E \leq u]v = 0$ then $V \geq \Pr[T - E \leq u]u^2 + (1 - \Pr[T - E \leq u])v^2$ therefore

$$\Pr[T - E \leq u] \geq \frac{u^2}{V + u^2}.$$

Q.E.D.

Corollary 3 Suppose that two random variables, $S; T \in \mathbb{R}$, have mean values $E_S; E_T \in \mathbb{R}$ with $E_S \leq E_T$ and variances $V_S; V_T \in \mathbb{R}_{++}$. If

$$(E_S - t)^2 (E_T - t)^2 \geq V_S V_T$$

then

$$\Pr[T \leq t] \geq \Pr[S \leq t]$$

for $t \in]E_S; E_T[$.

Proof Follows from

$$\Pr[S \leq u] \geq \frac{(u - E_S)^2}{V_S + (u - E_S)^2} \geq \frac{V_T}{V_T + (u - E_T)^2} \geq \Pr[T \leq u]$$

by simple manipulations. Q.E.D.

Remark Consider two probability measures on the unit interval, μ and ν, and $c \in \mathbb{N}$ and suppose that $E[Z_{\mu;c}(m)] \leq E[Z_{\nu;c}(m)]$. If

$$(t - E[Z_{\mu;c}(m)])(E[Z_{\nu;c}(m)] - t) \geq \frac{1}{m} + \frac{(2a)^{2(a+1)}}{r(m)}$$

then

$$\Pr[Z_{\mu;c}(m) \leq t] \geq \Pr[Z_{\nu;c}(m) \leq t]$$

for $t \in [E[Z_{\mu;c}(m)]; E[Z_{\nu;c}(m)]]$. Hence for two probability measures on the unit interval if the mean values of the two induced normalized random variables and m as well as $r(m)$ are large then the distributions can be ranked by stochastic dominance – almost.

7.2 Proofs

Proof of proposition 2 The proof uses two intermediate results

Lemma 7 For $N \subset M$ let $S_N(\lambda) \subset S^{m-1}$ be the set of allocations, λ', such that $N \subset N(\lambda;\lambda')$. Then for the distribution with density p_c the measure of $S_N(\lambda)$ relative to the measure of S^{m-1} is

$$s_{c;N}(\lambda) = \sum_{k_1=0}^{c-1} \cdots \sum_{k_{m-j}=0}^{c-1} \binom{cm-1}{k_1;\ldots;k_{m-j}} \left(\sum_{i \in N} \lambda_i\right)^{cm-1-k} \prod_{h=1}^{m-j} (\lambda_{i_h})^{k_h}$$

$$k = \sum_{i=1}^{m-j} k_i$$

$$M \setminus N = \{i_1;\ldots;i_{m-j}\}$$

Proof Consider λ and let ϕ be defined by

$$\phi_i = \begin{cases} \lambda'_i - \lambda_i & \text{for } i \in N \\ \lambda'_i & \text{for } i \in M \setminus N \end{cases}$$

for λ' such that $N \subset N(\lambda;\lambda')$. Then $\phi_i \geq 0$ for all i and

$$\sum_{i=1}^{m} \phi_i = 1 - \sum_{i \in N} \lambda_i = \sum_{i \in M \setminus N} \lambda_i$$

thus ϕ is in a (m-1)–simplex of size

$$\sum_{i \in M \setminus N} \lambda_i.$$

Clearly for $c = 1$ (the uniform distribution) the measure of a (m-1)–simplex of size ρ relative to S^{m-1} is ρ^{m-1}.

For $c \geq 2$ the integral of the density over the relevant simplex has to be computed. Let $N = \{1;\ldots;n\}$ and suppose that the integral of

$$\prod_{i=2} \frac{\lambda_i^{c-1}}{(c-1)!}$$

over allocations in the $(m-2)$–simplex of size α, i.e. $\sum_{i=2}^{m} \lambda'_i = \alpha$, that make the individuals $\{2;\ldots;n\}$ better off than λ is

$$\sum_{k_2=0}^{c-1} \cdots \sum_{k_n=0}^{c-1} \frac{\lambda_2^{k_2}}{k_2!} \cdots \frac{\lambda_n^{k_n}}{k_n!} \frac{\left(\alpha - \sum_{i=2}^{n} \lambda_i\right)^{c(m-1)-1-\sum_{i=2}^{n} k_i}}{\left(c(m-1)-1-\sum_{i=2}^{n} k_i\right)!}:$$

Then the measure of the allocations in the (m-1)–simplex of size β, which is defined by $\sum_{i=1}^{m}\lambda'_i=\beta$, that make the individuals $\{1,\ldots,n\}$ better off is ($\alpha=\beta-\lambda_1$)

$$\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\lambda_2^{k_2}}{k_2!}\cdots\frac{\lambda_n^{k_n}}{k_n!}$$

$$\int_{\lambda_1}^{\beta-\sum_{i=2}^{n}k_i}\sum_{k_1=0}^{\infty}\frac{\lambda'_1}{(c-1)!}\frac{(\beta-\lambda'_1-\sum_{i=2}^{n}\lambda_i)^{c(m-1)-1-\sum_{i=2}^{n}k_i}}{(c(m-1)-1-\sum_{i=2}^{n}k_i)!}d\lambda'_1$$

$$=$$

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\lambda_1^{k_1}}{k_1!}\cdots\frac{\lambda_n^{k_n}}{k_n!}\frac{(\beta-\sum_{i=1}^{n}\lambda_i)^{cm-1-\sum_{i=1}^{n}k_i}}{(cm-1-\sum_{i=1}^{n}k_n)!}.$$

It remains to be shown that the integral of

$$\prod_{i=2}^{m}\frac{\lambda_i^{c-1}}{(c-1)!}$$

over allocations in the $(m-2)$–simplex of size α, i.e. $\sum_{i=2}^{m}\lambda'_i=\alpha$, that make the individuals $\{2,\ldots,n\}$ better off than λ is

$$\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\lambda_2^{k_2}}{k_2!}\cdots\frac{\lambda_n^{k_n}}{k_n!}\frac{(\alpha-\sum_{i=2}^{n}\lambda_i)^{c(m-1)-1-\sum_{i=2}^{n}k_i}}{(c(m-1)-1-\sum_{i=2}^{n}k_i)!}.$$

First note that the integral of

$$\prod_{i=n+1}^{i=m}\frac{\lambda_i'^{c-1}}{(c-1)!}$$

over the $(m-n-1)$–simplex defined by $\sum_{i=n+1}^{i=m}\lambda'_i=\alpha-\lambda_n$ is

$$\frac{(\alpha-\lambda_n)^{c(m-n-1)-1}}{(c(m-n-1)-1)!}.$$

Second a computation of the integral of

$$\frac{\lambda_n'^{c-1}}{(c-1)!}\frac{(\alpha-\lambda'_n)^{c(m-n-1)-1}}{(c(m-n-1)-1)!}$$

over the set $\lambda'_n\in[\lambda_n;\alpha]$ gives the result. Q.E.D.

Lemma 8

$$\sum_{k=0}^{i}(-1)^k\binom{n+i}{n+k}\binom{n+k-1}{k}=1.$$

Proof First the preceding formula may be rewritten as

$$\sum_{k=0}^{i}(-1)^k\binom{n+i}{i-k}\binom{n+k-1}{n-1}$$

second exchange k with $k' = i - k$

$$\sum_{k'=0} (-1)^{i-k'} \binom{n+i}{k'} \binom{n+i-1-k'}{n-1} :$$

If the standard binomial formula,

$$\sum_{k=0} (-1)^{k} \binom{b-k}{b-i} \binom{a}{k} = \binom{b-a}{i} ;$$

is applied with $a = n+i$, $b = n+i-1$ then

$$\sum_{k'=0} (-1)^{i-k'} \binom{n+i}{k'} \binom{n+i-1-k'}{n-1} = (-1)^{i} \binom{-1}{i} :$$

The basic identity,

$$\binom{-r}{i} = (-1)^{i} \binom{r+i-1}{i} ;$$

gives the result for $r = 1$. Q.E.D.

Proposition 2 can be established by use of lemma 7 and lemma 8. First the formula for $u_{c;n}(\lambda)$ has to be established. The proof follows from the principle of inclusion and exclusion as in the original problem of occupancy[8] and it depends on induction on the index j.

The set $T_n(\lambda)$ is the union of all sets $S_N(\lambda)$ for all $N \in M_n$, i.e. $T_n(\lambda) = \cup_{N \in M_n} S_N(\lambda)$. If the relative p_c-measures of these sets are added then the first element of expression (1) is obtained,

$$\sum_{J \in M_{m-n}} s_{c;M \setminus J}(\lambda); \tag{6}$$

but these sets intersect, so some parts of are counted more than once. As an example, let N' be a set of $(n+1)$ integers then it contains

$$\binom{n+1}{n}$$

sets of n integers. Therefore the volume of the set $S_{N'}(\lambda)$ has then to be discounted $\binom{n+1}{n} - 1 = \binom{n}{1}$ times then second element of expression (1) is obtained,

$$-\binom{n}{1} \sum_{J \in M_{m-n-1}} \left(\sum_{i \in J} \lambda_i \right)^{m-1} : \tag{7}$$

[8] Any standard proof of this result would hold here by noticing that, for any fixed subset $N \in M$, the probability that all urn in N contains less than c balls is equal to the probability that all families in N end up being better off when choosing an alternative allocation according to the density p_c, given the identification $p_i = \lambda_i$. However since the principle of inclusion-exclusion is crucial in the present paper, it seems adequate to give the argument.

The induction hypothesis is that the volume corresponding to a set of $(n+i-1)$ elements has to be discounted

$$(-1)^{i-1}\binom{n+i-2}{i-1}$$

times from the initial quantity (expresion 6). Then the volume corresponding to a set of $(n+i)$ integers was counted

$$\underbrace{\binom{n+i}{n}}_{\text{in (6)}} - \underbrace{\binom{n+i}{n+1}\binom{n}{1}}_{\text{in (7)}} + \dots + (-1)^{i}\binom{n+i}{n+i-1}\binom{n+i-2}{i-1}$$

times that is equal to

$$1-(-1)^{i}\binom{n+i-1}{i}$$

times according to lemma (8). Hence a volume corresponding to a set of $(n+i)$ integers has to be counted $(-1)^{i}\binom{n+i-1}{i}$ times more in order to be counted exactly one time.

This implies that

$$u_{c;n}(\lambda) = \sum_{i=0}^{m-n}(-1)^{i}\binom{n+i-1}{i}\left(\sum_{I\in M_{m-n-i}} s_{c;M\setminus I}(\lambda)\right)$$

hence if $j = m-n-i$ the expression of proposition 2 is obtained. The volume of $t_{c;n}(\lambda)$ follows directly because $t_{c;n}(\lambda) = u_{c;n}(\lambda) - u_{c;n+1}(\lambda)$, thus[9]

$$\sum_{j=0}^{m-n}(-1)^{m-n-j}\underbrace{\left(\binom{m-j-1}{n-1}+\binom{m-j-1}{n}\right)}_{=\binom{m-j}{n}}\left(\sum_{J\in M_j} s_{c;M\setminus J}(\lambda)\right).$$

Hence the expression of proposition 2 is obtained. Q.E.D.

Proposition 2 implies that the results on urn models can be applied in order to establish theorem 1, observation 1 and observation 2. Thus if μ is replaced by λ and $Pr[V_c(m) \geq n]$ is replaced with $u_{c;m;n}(\lambda_m)$ then the proofs are applications of results on urn models.

Proof of theorem 1 Apply corollary 2. Q.E.D.

Proof of observation 1 Apply lemma 6. Q.E.D.

Proof of observation 2 Apply lemma 4. Q.E.D.

[9] Note that by convention $\binom{a}{b} = 0$ for $a \leq b-1$.