Proportional transaction costs on asset trades: a note on existence by homotopy methods.

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Abstract

We prove existence of equilibria with proportional transaction costs on asset trading, using homotopy methods.

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Introduction

We prove existence of equilibria with proportional transaction costs on asset trading, using homotopy methods. The issue of existence of such equilibria is also related to the existence of bid-ask spread equilibria, or equilibria with taxes and transfers.

Proportional transaction costs are among the most widely used fees in real world financial trading. Transaction costs have a significance at the individual level, as they are shown to reduce trading, and at the aggregate level, as they modify asset prices and welfare.

Macroeconomics has explored transaction costs as a reason to explain added consumption and asset price volatility. However, most of the macroeconomic analysis of the problem considers quadratic costs (see Heaton and Lucas (1996), e.g.). While technically more attractive, that formulation is considered economically less convincing. The partial equilibrium or no arbitrage analysis of transaction costs is copious, especially in continuous-time finance models of portfolio choice (see Magill and Costantinides (1976), Davis and Norman (1990), Jouini and Kallal (1995), Cvitanic and Karatzas (1996) are only few significant examples). The general equilibrium equivalent is scarce of sources. Foley (1970) studied the case of spot commodity markets; Hahn (1973) and Starrett (1973) extended the analysis to forward commodity markets, while Arrow and Hahn (1999) recently addressed asset markets in equilibrium. In these papers transaction costs are real, in the sense of requiring explicit or implicit use of commodities for purchases or sales. On the other hand, Préchac (1996) introduces nominal transaction costs on asset trading. However, all these contributions either do not deal with equilibrium existence issues, or avoid comparative statics exercises.[1] With homotopy methods, existence can be established through the use

[1] In Arrow and Hahn (1999), say, the constrained optimality analysis is done only for economies where asset markets are active.

of an extended system of equations which paves the way to comparative statics exercises.

A technical difficulty prevents the comparative statics analysis of the proportional or fixed transaction costs case. The main technical difficulty is twofold: possible nonconvexities in the budget set, and nondifferentiability of the budget line. We show how a degree proof can nevertheless be effectively applied in this context. We can then exploit the differential structure of the equilibrium system to study its generic properties, in particular the constrained optimality of equilibrium.

Nonconvexities arise when buying prices are lower than selling prices, or with fixed fees. The main idea is that nonconvexities can be studied through a traditional argument (see Starr (1969), say). Using a continuum of identical agents, we get rid of the effects of nonconvexities in the constraint set in the individual optimization problem. The insuing discontinuities in individual demand are integrated out by convexifying the aggregate demand function.

Nondifferentiabilities in the budget constraint arise when bid-ask spreads are positive, and also with fixed fees. They are treated using an argument which breaks down the optimization problem in several, but finite, convex and differentiable programming problems, and compares their solution at equilibrium. Again using the continuum of agents assumption, individuals indifferent across any of the finitely many solutions are arbitrarily assigned to anyone of these equivalent choices. In the aggregate, corresponding fractions of the population are determined, which sum up to the total size of the population.

In this sense, the concept of equilibrium with proportional transaction costs is reminiscent of a combined notion of Nash and competitive equilibrium for large economies (Minelli and Polemarchakis (1999)). It can be seen as an instance of more general setups where optimization problems mix discrete and continuum choice components.

It should be noted that the above-mentioned studies of transaction costs or taxes eliminate the possibility of transaction (or tax) arbitrage by assuming that costs (bid-ask spreads) and asset payoffs are both positive (see Jouini and Kallal (1995) and Préchac (1996), say). While many real assets have positive future payoffs (such as equity), many derivative products may well have negative payoffs and even negative expected (subjective) value in the absence of transaction costs. This fact is widely recognized in incomplete markets models, where no sign restriction is imposed on the asset payoff matrix. Therefore, a nominally positive bid-ask spread can translate into a real negative spread. Also in the case of equity trading, negative spreads are sometimes possible in real life situations.[2] Bid-ask spreads or transaction costs are adjusted more slowly than asset payoffs or changes in expectations relative to asset payoffs, due to information arrival. With multiple market makers, it may happen that

[2]For instance, Harris and Schultz (1997) report that Nasdaq's SOES for trading displays as a regular problem the presence of arbitrageurs called SOES bandits by market makers. These traders 'make money by spotting minor pricing discrepancies', and by executing trades across market makers. These kinds of problems are tackled exactly by imposing restricted participation.

traders see selling prices above buying prices. This is why the case of 'buying' prices lower than 'selling' prices should also be considered. From a normative viewpoint, it has been recently shown that negative bid-ask spreads can Pareto improve upon zero transaction costs equilibria when markets are incomplete (see Citanna, Polemarchakis and Tirelli (2000)). Hence, the case of negative spreads should also be considered based on its normative property.

We provide a unified approach to deal with all cases of zero, negative and positive spreads. To avoid transaction cost (or tax) arbitrage, a no arbitrage condition must be imposed, such as permitting only one side of trading at each time, either the purchase or the sale of the asset. Market participation is therefore restricted when spreads are nonpositive.

Notation

$2 < H < \infty$ (types of) traders, h
$1 < S < \infty$ states, s
$C > 1$ physical commodities, c
$G = C(S+1)$
$1 \leq I < \infty$ financial assets, i
$x_h^{s,c}$ = consumption of commodity c in state s by trader h
$e_h^{s,c}$ = corresponding endowment
b_h^i = quantity of asset i in trader h's portfolio, b_h
$b_h^{i+} = \max(0, b_h^i)$
$b_h^{i-} = \min(0, b_h^i)$
τ^i = transaction cost (tax or subsidy) on asset i
w_h = lump-sum transfer to (or profit share of) trader h
$p^{s,c}$ = c-th commodity price in state s, p^s the price vector in state s
q^i = price of asset i
Y = the payoff matrix (S rows, I columns)
y^s = the s-th row of Y

The Model

The notion of equilibrium is standard in two-period, finite exchange economies with incomplete financial assets. For each trader, we assume that the commodity space is $\mathbb{R}^C_{++}$ for each spot, and $\mathbb{R}^G_{++}$ overall. Preferences are representable by a utility function $u_h : \mathbb{R}^G_{++} \to \mathbb{R}$ which is smooth, differentially strictly increasing, differentially strictly quasi-concave and with indifference surfaces having closure contained in $\mathbb{R}^G_{++}$. Endowments are in the commodity space. Let $E = \mathbb{R}^{HG}_{++}$ be the endowment space, and U_h be the space of utility functions, endowed with the topology of C^2-uniform convergence. Let $U = \times_h U_h$. An economy will be a pair $(e, u) \in E \times U$.

Traders exchange commodities and financial assets. There are spot markets for physical commodities at each date and state. Financial asset trading occurs at $s = 0$. The I financial assets have state-contingent payoffs tomorrow, represented by the matrix Y and expressed in units of commodity $c = C$, the numéraire commodity, as in Geanakoplos and Polemarchakis (1986). We assume that rank $Y = I \leq S$. Asset markets can be incomplete, but no redundancies are allowed.

Asset trading occurs at a cost. This cost in general could take different forms. For instance, a trader may be asked to pay a fixed fee every time he enters a transaction, irrespective of the sign or size of the exchange. Or the cost could be proportional to the amount traded, whether an asset is purchased or sold.

In this paper, the cost is paid only if the asset is purchased, and it is proportional to the value of the amount bought. Buying asset i costs τ^i, with $\tau^i > -1$. A case of interest is when $\tau > 0$ (and $q > 0$). So, if trader h wants to buy asset i, i.e., $b_h^{i+} > 0$, he has to pay a higher price than if he sells it. This price differential, or bid-ask spread, is meant to represent a transaction cost. Of course, this could be motivated by some form of asymmetric information between the exchange house and the trader. We do not try to explain such spread in the model. Instead, we take it as given and explore its effects on equilibrium.

Transaction costs are collected by an exchange house. Each trader receives an amount of money w_h which is interpreted either as a lump-sum transfer or as the trader's share of the exchange house's profits from running the exchange operations. These profits come from the collected transaction costs, and are measured in the numéraire commodity.

Trader h's budget constraint is then:
$p^0(x_h^0 - e_h^0) + \sum_i [q^i(1+\tau^i)b_h^{i+} + q^i b_h^{i-}] - p^{0,C} w_h = 0$
$p^s(x_h^s - e_h^s) = p^{s,C} y^s b_h$, for all $s > 0$
also written more compactly as

$$-\Psi z_h + \Psi^C R b_h + \Psi^C W_h = 0 \tag{1}$$

where $z_h = x_h - e_h$, $W_h = (w_h, 0, .., 0)^T$ and

$$\Psi = \begin{bmatrix} p^0 & 0 & \\ 0 & \ddots & 0 \\ & 0 & p^S \end{bmatrix}$$

is $(S+1) \times G$-dimensional, and Ψ^C is the similar $(S+1)^2$-dimensional matrix containing only the commodity C prices on the diagonal; finally

$$R = \begin{bmatrix} -...[q^i(1+\tau^i)I(b_h^{i+}) + \ q^i I(b_h^{i-}) + m^i(1 - I(b_h^{i+}) - I(b_h^{i-}))] \\ Y \end{bmatrix}$$

$I()$ being the indicator function.[3]

The more general case, a cost for trading τ^i paid whether a trader buys or sells the asset, could also be studied. Then this cost would appear added to

[3] $I(x) = 1$ if $x > 0$, and $I(x) = 0$ otherwise. Here m^i is a real number in the interval $[q^i, q^i(1+\tau^i)]$ (assuming $q^i(1+\tau^i) > q^i$, or equal to zero otherwise) when $b_h^i = 0$.

both the buying and selling price. If the buying and selling prices are equal, the trader's financial trading balance is then $\sum_i [q^i(1+\tau^i)b_h^{i+} + q^i(1-\tau^i)b_h^{i-}]$. In fact, one could also easily consider a transaction cost different for purchases and sales. If the buying and selling prices are different, then having same transaction cost but different prices corresponds to the current formulation. Far from adding any conceptual or technical difficulty, the general case only adds notational burden to the model, and therefore it will not be considered here.

It should also be noted that we are implicitly imposing the restriction that individual trade occurs only on one side of the market, either buying or selling, and not both. This is without loss of generality when $q^i > 0$, and $\tau^i \geq 0$, so $q^i(1+\tau^i) \geq q^i$. As one can easily show, no trader would want to be on both sides of the market at the same time with these prices. When $q^i < 0$, then the bid-ask spread is reversed even if $\tau^i > 0$, and this constraint is not without loss of generality: transaction costs allow profits from buying and selling at the same time. To eliminate this possibility, we impose that no such trades are possible. We want to leave open the possibility that such negative bid-ask spreads exist, as previous work shows that they can be Pareto-improving (as mentioned in the Introduction).

From now on, $p^{s,C} = 1$ all s; this is the standard commodity-C normalization spot by spot.

Equilibrium with transaction costs (τ, w) requires that:

(H) trader h maximizes $u_h(x_h)$ s.t. (1) taking prices, transaction costs and transfers as given (and of course payoffs and endowments);

(M) markets clear, i.e., $\sum_h z_h = 0$ and $\sum_h b_h = 0$; and

(T) zero profits for the exchange house, or $\sum_i q^i \tau^i (\sum_h b_h^{i+}) = \sum_h w_h$.

Condition (T) is consistent with a notion of competition among exchange houses. Nothing excludes that the left-hand side in this equation be negative, hence w_h can also be negative.

It should be noted that in (1) there is a discontinuity at $b_h^i = 0$ when $\tau^i \neq 0$. This discontinuity is the cause of nondifferentiability at $b_h^i = 0$ and of possible nonconvexity of the budget constraint. However, these problems will be bypassed by the construction below. First, we describe the equilibrium set in the absence of transaction costs, when $\tau = 0$ and $w = 0$. The results for this case essentially mimick those in Citanna, Polemarchakis and Tirelli (2000), and are presented here for the sake of completeness.

Zero and quasi-zero transaction costs equilibria

At $\tau = 0$ and $w = 0$, the equilibrium is nothing but a standard financial equilibrium; hence we have the following preliminary result.

Lemma 1 *At $\tau = 0$ and $w = 0$, an equilibrium exists for all economies $(e, u) \in E \times U$. (see Geanakoplos and Polemarchakis (1986).)*

The equilibrium *at* $\tau = 0$ *and* $w = 0$ can be represented as a system of equations (including the redistribution equation for (T)):

$$
\begin{array}{cr}
\vdots & \\
Du_h - \lambda_h \Psi = 0 & (1) \\
\lambda_h R = 0 & (2) \\
-\Psi z_h + R b_h + W_h = 0 & (3) \\
\vdots & \\
\sum_h z_h^{\setminus} = 0 & (4) \\
\sum_h b_h = 0 & (5) \\
-\sum_i q^i \tau^i (\sum_h b_h^{i+}) + \sum_h w_h = 0 & (6)
\end{array}
\tag{2}
$$

where $\lambda_h \in \mathbb{R}_{++}^{S+1}$ is a vector of Lagrange multipliers, for all h. Call this system $F(\xi, \tau; e, u) = 0$ where $\xi = ((x_h, b_h, \lambda_h), p, q)$.

We compute the derivative of the equilibrium system with respect to the endogenous variables and controls at $\tau = 0$ and $w = 0$, or $D_{\xi,\tau,w}F$. We notice that this matrix has full row rank when restricting attention to all the columns excluding the derivatives with respect to τ and w, and all rows except the last one (denote this matrix by $D_\xi F^{\setminus}$). This is standard regularity of zero transaction costs equilibrium.

When restricting attention to columns up to one corresponding to the derivative with respect to w_h, some h (or τ^i some i), $D_{\xi,\tau,w}F$ is square and has full rank at $\tau = 0$ and $w = 0$ for a generic set of economies, as it is easily shown. Formally, we state this result as a lemma (the proof is standard and omitted). Let $\tau = (\tau', \tau'')$ with $\tau' = \tau^i$ some i, and similarly $w = (w', w'')$ with $w' = w_h$ some h.

Lemma 2 *At $\tau = 0$ and $w = 0$, at any equilibrium:*

i) $D_\xi F^{\setminus}(\xi, 0; e, u)$ has full rank for an open, full-measure subset of endowments E^*.

ii) $b_h^i \neq 0$ all i, h and $q^i \neq 0$ in an open, full-measure subset of endowments $E^{**} \subset E^*$.

iii) At $\tau = 0$ and $w = 0$, $D_{\xi,\tau'[w']}F(\xi, 0; e, u)$ has full rank for an open, full-measure subset of endowments $E^{***} \subset E^{**}$.

As a corollary to this Lemma, we have existence of equilibria for small $(\tau, w) \neq 0$.

Lemma 3 *A solution to $F(\xi, \tau; e, u) = 0$ exists for all $(\tau, w) \in T_{e,u} \subset \mathbb{R}^{H+I}$, an open set, for all economies (e, u) with $e \in E^{***}$.*

Proof. See Citanna, Polemarchakis and Tirelli (2000).□

Note that Lemma 3 does not show existence for any (τ, w), no matter how big. For a general existence proof, i.e., for any τ, w, we need to bypass the previously noted lack of continuous differentiability of the equilibrium system when $b_h^i = 0$ and $\tau^i \neq 0$. That global existence is nevertheless possible is shown in the next section.

We conclude this section with an accessory result, which is useful in studying the properties of transaction cost equilibria. It states that, typically in incomplete markets economies with zero transaction costs, there is sufficient variation of traders' evaluation of commodity price effects.

Lemma 4 *At $\tau = 0$ and $w = 0$, at any equilibrium when $S + 1 - I \geq H$, the matrix*

$$\begin{bmatrix} \lambda_1^0 z_1^{0,1} & \cdots & \lambda_1^{H-1} z_1^{H-1,1} \\ \vdots & & \vdots \\ \lambda_H^0 z_H^{0,1} & \cdots & \lambda_H^{H-1} z_H^{H-1,1} \end{bmatrix}$$

has full rank H in a generic subset $E^{****} \subset E^{***}$.

Proof. Straightforward: see Citanna, Polemarchakis and Tirelli (2000), say.□

Lemma 4 can be used to establish nice properties of transaction cost equilibrium. It means that there is sufficient heterogeneity across traders in equilibrium.

Global existence

In the previous section we noted that existence of equilibria with transaction costs and transfers could be obtained locally around zero, by using an implicit function theorem argument. While that technique (Lemma 3) is sufficient for the constrained suboptimality analysis of standard incomplete markets equilibrium, it leaves open the question of general existence of equilibria for arbitrary τ, w. We close the gap in this section. Somewhat surprisingly, we will show existence by means of a degree proof. As we mentioned earlier, the difficulty arises as the budget constraint is either nondifferentiable or nonconvex, precisely at the individual no trade point, when buying and selling prices are different. At first sight, this seems to prevent the use of the Kuhn-Tucker conditions and of the extended system of equations to represent an equilibrium. Hence it looks as if degree theory could not be applied to the extended system, which is so useful to study the constrained optimality of equilibrium (see Citanna, Polemarchakis and Tirelli (2000) for the case $\tau = 0$).

However, it turns out that the extended system can be effectively used, by patching together several, but finitely many differential problems which together represent the individual optimum.[4]

We consider solving $L = 2^I$ differential problems for each trader, each for one combination of constraints on purchases and sales of assets, everything else equal. That is, all these problems are solved at the same commodity and asset prices, and at the same transaction costs and transfers. For each l, let $b_h^{i,l}$ be

[4] This mirrors Theorem 2.2 in Jouini and Kallal (1995), which establishes that the nonlinear no arbitrage equations can be written as a collection of linear equations, one for each admissible no arbitrage positive linear functional.

the holding of asset i for trader h. There is going to be a subset of assets $\mathbf{I}^+(l)$ defined as

$$\mathbf{I}^+(l) = \{i \in I \mid b_h^{i,l} \geq 0\}$$

and a subset $\mathbf{I}^-(l)$ defined as

$$\mathbf{I}^-(l) = \{i \in I \mid b_h^{i,l} \leq 0\}$$

Let

$$\widetilde{b}_h^{i,l} = \begin{cases} b_h^{i,l} & \text{if } i \in \mathbf{I}^+(l) \\ -b_h^{i,l} & \text{otherwise} \end{cases}$$

Hence, we solve the utility maximization problem at prices p, q, transaction costs τ and transfers w_h with the additional nonnegativity constraints $\widetilde{b}_h^{i,l} \geq 0$, all i. Notice that when $q^i(1+\tau^i) > q^i$ these restrictions are without loss of generality because traders only consider being on one side of the market for each asset at the optimum. When the bid-ask spread is negative, this entails the additional no arbitrage restriction that trades can only be made at most on one side of the market, for each asset. In both cases, we proceed as follows. Once we have solved the L differential problems, we compute the indirect utility for each case, utilities are compared and problem l is selected if it yields the highest possible utility. Consumption and asset portfolios are chosen correspondingly. Note that there may be multiple l satisfying the utility maximizing condition. Then we let θ_h^l be the weight assigned to problem l, with $\theta_h^l \in [0,1]$ and with $\sum_l \theta_h^l = 1$. Then $\theta_h^l > 0$ only if problem l is a utility maximizer, and $\theta_h^l = 1$ if problem l is the only utility-maximizing choice.

It is obvious that this procedure equivalently solves the original utility maximization problem (H). The advantage of this method of solution is that it leads to an equilibrium representation through a system of equations.

To illustrate what happens, consider an economy with $H = 2$, $C = 1$, $I = 2 = S$, two Arrow securities, and no consumption at time zero. This is a standard walrasian economy, where purchases and sales have different prices, $q^i(1+\tau^i)$ and q^i, respectively. The budget line is:

$$\sum_i [q^i(1+\tau^i)I(b_h^{i+}) + q^i I(b_h^{i-})] b_h^i = 0$$

and $b_h^i = x_h^i - e_h^i$, for $i = 1, 2$. If $\tau^2 = 0$ and $\tau^1 > 0$, the budget line has a kink at $x_h = e_h$, but it is convex. If Arrow securities have negative payoffs, their prices q will be negative, and the situation with $\tau^1 > 0$ corresponds to one with standard Arrow securities and $\tau^1 < 0$. In this case, the budget constraint has a kink at the no trade point, and it is nonconvex. In both cases, direct use of the first order conditions seems impossible. However, it is easily observed that each trader solves the equivalent (discrete) choice problem of looking for the optimum assuming $b_h^1 \geq 0$, and then assuming $b_h^1 \leq 0$, and finally comparing the indirect utilities and choosing the highest. When $\tau^1 > 0$, the utility will be the same only if we are at the no trade. In this case, for an open set of preferences the solution will not be a tangency condition, and both α_h^1 and α_h^2 will be positive. Note that θ_h^l could be strictly less than one and greater than zero. However, since $x_h^1 = x_h^2 = e_h$, this is immaterial at the market clearing level. When $\tau^1 < 0$, the no trade case cannot occur. On the other hand, it may occur that $u_h(x_h^1) = u_h(x_h^2)$ even though $x_h^1 \neq x_h^2$. In this situation, a correct choice of θ_h^l is necessary for the interpretation. Hence we shall think of θ_h^l as the proportion of traders of type h choosing trading strategy l, with a continuum of traders of type h. So equilibria for $\tau^i < 0$ some i (and $Y > 0$) will be approximate equilibria in large but finite economies. Note also that the system will typically (in utilities) not be in this situation.

A second issue in definition of equilibrium is the value of w_h, which cannot be too large when negative, otherwise it forces traders to negative consumption.[5] In order to deal with this problem, a lower bound on w_h must simply be imposed, such that

$$w_h + e_h^{0,C} > 0.$$

In fact, we will specialize to the uniform redistribution case, where $w_h = (1/H)\pi$, so that $\sum_h w_h = \pi$, the total profit (or loss) from collecting transaction fees τ. The general case of nonuniform w_h can also be dealt with, and more easily as we would have extra variables, to be treated as parameters.

Let $\omega = (\tau, e)$, and let $\omega^* = (\tau^*, e^*)$ be a 'test' point. The lower bound condition here must hold for π. Also letting $t \in [0, 1]$ be a homotopy parameter, we can represent an equilibrium for arbitrary ω as the solution to the following system of equations at $t = 0$:

[5] Since traders take w_h as given, equilibria with $w_h < 0$ may not satisfy individual rationality. However, the requirement $w_h \geq 0$ can be added without any substantial change, as we show below.

$$
\begin{array}{cr}
\vdots & \\
Du_h(x_h^l) - \lambda_h^l \Psi = 0 & (1) \\
-\lambda_h^{0,l} q^i (1 + \tau_h^{i,l}(t)) + \lambda_h^{\mathbf{1},l} y^i + \widetilde{\alpha}_h^{i,l} = 0, \text{ all } i & (2a) \\
\min\{\alpha_h^{i,l}, \widetilde{b}_h^{i,l} + t\} = 0, \text{ all } i & (2b) \\
-\Psi(x_h^l - e_h(t)) + R b_h^l + W_h(t) = 0; & (3) \\
-t + u_h(x_h^l) - u_h(x_h^L) + \underline{\nu}_h^l - \nu_h = 0 & (4) \\
\min\{\underline{\nu}_h^l, \theta_h^l\} = 0 & (5) \\
\min\{\nu_h, 1 - \sum_{l \neq L} \theta_h^l\} = 0 & (6) \\
\vdots & \\
\sum_h [\sum_l \theta_h^l (x_h^{l\backslash} - e_h^{\backslash}(t))] = 0 & (7) \\
\sum_h (\sum_l \theta_h^l b_h^l) = 0 & (8) \\
- \sum_i q^i \tau^i(t) (\sum_h b_h^{i+}) + \sum_h w_h(t) = 0 & (9) \\
\min(\beta, \pi + a(t)) = 0 & (10)
\end{array}
\qquad (\mathrm{E})
$$

Equations $(1)-(3)$ hold for all $l = 1, ..., L$, all h, and represent the Kuhn-Tucker conditions for problem l; equations (4) and (5) hold for $l \neq L$, all h, equation (6) holds for all h; together, they represent how we patch together all the L problems to solve for (H) using a differential approach; equations $(7) - (9)$ are nothing but the market clearing and zero net profit conditions; finally, equation (10) guarantees that each trader always has a positive endowment of the numéraire commodity.

As for the notation, $b_h^{i+} = \max\{0, \sum_l \theta_h^l b_h^{i,l} + t\}$; $\alpha_h^{i,l}$ is the Lagrange multiplier associated to the corresponding nonnegativity constraint on $\widetilde{b}_h^{i,l}$, with

$$
\widetilde{\alpha}_h^{i,l} = \begin{cases} \alpha_h^{i,l} & \text{if } i \in \mathbf{I}^+(l) \\ -\alpha_h^{i,l} & \text{otherwise} \end{cases}
$$

The homotopy links the arbitrary ω to the test point ω^*, through

$$
\tau_h^{i,l}(t) = \begin{cases} \tau^i(t) & \text{if } i \in \mathbf{I}^+(l) \\ 0 & \text{otherwise} \end{cases}
$$

and $\tau^i(t) = (1-t)\tau^i + t\tau^{i*}$, $e_h(t) = (1-t)e_h + te_h^*$, $W_h(t) = (w_h(t), 0, .., 0)^T$, where $w_h(t) = w_h + (1-t)\beta/H$, all h. Here β is a slack variable, as well as $\underline{\nu}_h^l$ and ν_h. Finally, $a(t) = \min_h e_h^{0,C}(t)/2$.[6]

Note that an equilibrium is expressed as the zeros of the *continuous* function $F(\xi, t; \omega) = 0$ computed at $t = 0$. Here F represents the left-hand side of (E), and

[6] If one wishes to have $w_h \geq 0$, we can simply impose this condition on π and substitute (10) with $\min(\beta, \pi + ta(t)) = 0$. Everything else goes through unchanged.

$$\xi = ((x_h, b_h, \lambda_h, \alpha_h, \underline{\nu}_h, \nu_h, \theta_h)_h, p^{\backslash}, q, \pi, \beta) \in \Xi$$

Ξ is an open subset of a Euclidean space, and $\dim \Xi$ is equal to the number of equations in (E). Moving t away from zero deforms the equilibrium system into a system of equations which is better suited for analysis and takes the arbitrary ω to the test point ω^*, when $t = 1$. We will show that this homotopy satisfies all the assumptions to claim that the degree modulo 2 of the function F at the test point, i.e., $\deg_2(F_{t=1}, \{0\})$, is the same as the degree computed at $t = 0$, therefore at the function representing a transaction cost equilibrium and for arbitrary parameters ω.

To prove existence we compute $\deg_2(F_t, \{0\})$ for $t = 1$ (see Lloyd, (1978)). We proceed by finding first the test point ω^*. Let $\tau^* = 0$. Note that this is now a standard incomplete markets economy. Let u^* be any utility satisfying the maintained assumptions, and let $e^* = e^{PO} \in \mathbb{R}^{GH}_{++}$, a corresponding Pareto optimal allocation for total resources $r \in \mathbb{R}^G_{++}$. That is, e^{PO} solves

$$\begin{array}{cc} \max u_1^*(x_1) & s.t. \\ u_h^*(x_h) \geq \overline{u}_h & h > 1 \\ \sum_h x_h \leq r & \end{array} \qquad (PO)$$

where $\overline{u}_h \in u_h^{*-1}(\mathbb{R}_{++})$, $h > 1$. At this Pareto optimum, which exists and is unique for all $r, \overline{u}_2, ..\overline{u}_H$, there is a unique set of multipliers p^{PO} and μ_h^{PO} such that

$$\begin{array}{cc} Du_1^*(e_1^{PO}) - p^{PO} = 0 & \\ \mu_h^{PO} Du_h^*(e_h^{PO}) - p^{PO} = 0 & h > 1 \\ u_h^*(e_h^{PO}) = \overline{u}_h & h > 1 \\ \sum_h e_h^{PO} = r & \end{array} \qquad (3)$$

We then show that there is only one zero of the function F_1, that is, at ω^*.

Lemma 5 *There is a unique array ξ^* such that $F(\xi^*, 1; \omega) = 0$.*

Proof. We compare equations (3) with equations (1) and (7) in (E). Given the price normalization $p^{s,C} = 1$, we uniquely identify from (3) $\lambda_1^{s,l_1*} = (p^{s,C})^{PO}$ and $p^{s,c*} = (p^{s,c})^{PO}/(p^{s,C})^{PO}$ in equations (1), for $l_1 = 1, ..., L$, $h = 1$. We also uniquely identify $\lambda_h^{l*} = (1/\mu_h^{PO})\lambda_1^{1*}$, for all l and all $h > 1$. Then (1) and (7) have a unique solution with $x_h^{l*} = e_h^*$, all l, all h, and prices and multipliers as above. Note that in (7) the solution is independent of θ_h^l. Equations (3), $s > 0$, are satisfied with $b_h^{l*} = 0$, all l, h, due to rank $Y = I$. Now from equation $(2b)$, $\alpha_h^{l*} = 0$ and from equation $(2a)$, $q^{i*} = (\lambda_h^{1*}/\lambda_h^{0*})y^i$, all i. Therefore, equations (3) for $s = 0$ imply $w_h = 0$ all h, or $\pi^* = 0$. From (11), $\beta^* = 0$.

Finally, from equation (4) and since $u_h(x_h^{l*}) = u_h(x_h^{L*})$, all l, we have $\underline{\nu}_h^{l*} - \nu_h^* > 0$, so that $\underline{\nu}_h^{l*} > 0$, and $\theta_h^{l*} = 0$, all $l \neq L$, and from (6), $\nu_h^* = 0$, implying $\underline{\nu}_h^{l*} = 1$, all $l \neq L$ and all h. Finally, $\theta_h^{L*} = 1$, all h. ■

Notice that the function F_1 is actually continuously differentiable in ξ. Let $A = D_\xi F(\xi^*, 1; \omega)$. We now want to show that the only solution ξ^* has nice regularity properties.

Lemma 6 *A has full rank.*

Proof. To see this, compute the derivative A and premultiply by the vector $\Delta\xi$, to get:

$$\begin{array}{cr} D^2 u_h(x_h^{l*})\Delta x_h^l - \Psi^{*T}\Delta\lambda_h^l - \Lambda_h^{l*}\Delta p^{\backslash} = 0 & (1) \\ -\lambda_h^{0*}\Delta q^i + r^{iT}\Delta\lambda_h^l = 0 & (2a) \\ -\Psi^*\Delta x_h^l + R\Delta b_h^l = 0 & (3) \\ Du_h(e_h^*)(\Delta x_h^l - \Delta x_h^L) - \Delta\underline{\nu}_h^l = 0 & (4) \\ \sum_h \Delta x_h^{L\backslash} = 0 & (7) \\ \sum_h \Delta b_h^L = 0 & (8) \end{array}$$

where $r^i = (-q^{i*}, y^i)^T$. We will show that the only solution to this system of equations is $\Delta\xi = 0$. Note that $\Delta\alpha_h^{i,l} = 0$ from $(2b)$, $\Delta\theta_h^l = 0$ all $l \neq L$, from (5) (hence $\Delta\theta_h^L = 0$), $\Delta\nu_h = 0$ from (6). Also, $\Delta\pi = 0$ from (9), $\Delta\beta = 0$ from (10) (so $\Delta w_h = 0$ all h). We premultiply equation (1) by Δx_h^{lT}, and divide by λ_h^0 to get

$$\Delta x_h^{lT} D^2 u_h(x_h^{l*})/\lambda_h^{0*}\Delta x_h^l - \Delta x_h^{lT}\Psi^*\Delta\lambda_h^l/\lambda_h^{0*} - \Delta x_h^{lT}\Lambda^*\Delta p^{\backslash} = 0 \qquad ((1'))$$

where we note that $\Lambda_h^{l*}/\lambda_h^{0*} = \Lambda^*$ independent of h, as λ_h^{l*} and λ_1^{1*} are colinear at the test economy, all l, h. From $(2a)$, we have $-\Delta b_h^{lT}\Delta q + \Delta b_h^{lT} R^T \Delta\lambda_h^l/\lambda_h^{0*} = 0$, and where $R^T = (r^i)_i$. Combining with (3), we have $-(\Delta\lambda_h^{lT}/\lambda_h^{0*})\Psi^*\Delta x_h^l + \Delta q^T \Delta b_h^l = 0$. Substituting in $(1')$, we have

$$\Delta x_h^{lT} Du_h(x_h^{l*})/\lambda_h^{0*}\Delta x_h^l - \Delta b_h^{lT}\Delta q - \Delta x_h^{lT}\Lambda^*\Delta p^{\backslash} = 0 \qquad (4)$$

Now, from equations (7) and (8), premultiplying by $(\Lambda^*\Delta p^{\backslash})^T$ and Δq^T respectively, we get $\sum_h(\Lambda^*\Delta p^{\backslash})^T\Delta x_h^L = 0$ and $\sum_h \Delta q^T \Delta b_h^L = 0$. Summing over h in (4) for $l = L$, we obtain

$$\sum_h \Delta x_h^{LT} D^2 u_h(x_h^{L*})/\lambda_h^{0*}\Delta x_h^L - \sum_h \Delta b_h^{LT}\Delta q - \sum_h \Delta x_h^{LT} I^{\backslash}\Delta p = 0$$

or

$$\sum_h \Delta x_h^{LT} D^2 u_h(x_h^{L*})/\lambda_h^{0*} \Delta x_h^L = 0$$

while $\Delta x_h^{LT} D^2 u_h(x_h^{L*})/\lambda_h^{0*} \Delta x_h^L < 0$ if $\Delta x_h^L \neq 0$, all h, by differential strict concavity of u_h. Then $\Delta x_h^L = 0$ all h, and from equation (1) and the price normalization $\Delta \lambda_h^L = 0$, so that $\Delta p^{\backslash} = 0$, $\Delta q = 0$ and $\Delta b_h^L = 0$, using $(2a)$ and (3). It is now immediate to show, using $(1) - (3)$ and differential strict concavity that $\Delta x_h^l = 0$ and that $\Delta \lambda_h^l = 0$ and $\Delta b_h^l = 0$, all $l \neq L$, all h. From (4), $\Delta \underline{\nu}_h^l = 0$, all l, h. To conclude, $\Delta \xi = 0$, as we wanted to show. ■

Hence we know that $\deg_2(F_1, \{0\}) = 1$. We are left to show that we can correctly homotope this function to all functions F_t at all parameter values ω, without changing the degree. For this, we have the following lemma.

Lemma 7 *The set $F^{-1}(0)$ is compact.*

Proof. We show that the set is sequentially compact: starting from an arbitrary sequence $\{\xi^n, t^n\}_{n=1}^{\infty} \subset F^{-1}(0)$, we show that it has a subsequence (to simplify notation, the sequence itself) converging to (ξ, t), with $(\xi, t) \in F^{-1}(0)$.

[Equation numbers refer to system (E).] First, since $t^n \in [0,1]$, $t^n \to t$. Similarly, $\{\theta_h^{ln}\}$ converges to $\theta_h^l \in [0,1]$, all l, h.

Suppose that $\theta_h^1 = 1$, all h. Then from equation (7), $\{x_h^{1n}\}$ converges since it is bounded above by total resources and below by zero. From equation (10), $e_h^{0,C}(t^n) + w_h(t^n) > 0$ all n, h, and it is bounded away from zero. Using the boundary condition on utilities, it must be that $x_h^1 \gg 0$, all h. From equation (1) and the price normalizations, $\{\lambda_h^{1n}\}$ converges to $\lambda_h^1 \gg 0$. Then from the same equation, for $c \neq C$, $\{p^n\}$ converges to $p \gg 0$. Using the assumption that rank $Y = I$, from equation (3) for $s > 0$ we get convergence of $\{b_h^{1n}\}$, all h.

From $(2a)$, for no $i \in \mathbf{I}^+(1)$, $q^{in} \to -\infty$, and for no $i \in \mathbf{I}^-(1)$, $q^{in} \to \infty$.

Suppose there is $i \in \mathbf{I}^+(1)$ such that $q^{in} \to \infty$. Then, $b_h^{i,1} = -t$, since otherwise from $(2b)$ we get $\alpha_h^{i,1n} \to 0$, a contradiction. If $t = 0$, there must be $l' \neq 1$ such that $x_h^{l'} = x_h^1$ all h, and $i \in I^-(l')$ (this is because $b_h^{i,1} = 0$), so that $q^{in} \to \infty$ contradicts $(2a)$. So assume $t > 0$.

We have that $q^{in}(1+\tau^i) b_h^{i,1n} \to -\infty$. Since at $s = 0$, $|\sum_i q^{in}(1+\tau^{i,1}) b_h^{i,1n}| < M$ for some finite M and sufficiently large n, all $h > 2$, there is i' such that $q^{i'n}(1+\tau^{i',1}) b_h^{i',1n} \to \infty$, for $h > 1$. Suppose that $q^{i'n} \to \infty$. But then $i' \in \mathbf{I}^+(1)$, and $b_h^{i',1n} \to b_h^{i',1} = -t$, which contradicts $q^{i'n}(1+\tau^{i',1}) b_h^{i',1n} \to \infty$. [Here we use that $\tau^i > -1$, all i.] Suppose that $q^{i'n} \to -\infty$; then $i' \in \mathbf{I}^-(1)$, and $b_h^{i',1n} \to b_h^{i',1} = t$, again same contradiction. Therefore, if $i \in \mathbf{I}^+(1)$, $q^{in} \to q^i$. Similar reasoning shows that $q^{in} \to q^i$ even when $i \in \mathbf{I}^-(1)$, hence $\{q^n\}$ converges.

From (3) and convergence of $\{x_h^{1n}\}$, $\{p^n\}$, $\{q^n\}$ and $\{b_h^{1n}\}$ we get convergence of $\{w_h^n + (1-t^n)\beta^n\}$.

Now suppose that $t = 1$. Since at $t = 1$ there is only one solution ξ^* and the function F is continuous both in ξ and t, convergence is guaranteed, and in particular of $\{\pi^n\}$ (from (3)) and $\{\beta^n\}$ (from (10); observe that $\pi^* = 0$).

Suppose that $1 > t \geq 0$. Using (3) and (10), again we get convergence of $\{w_h^n\}$ and $\{\beta^n\}$.

Now it is standard to see that $\left(x_h^{ln}, b_h^{ln}\right)$ converges for $l \neq 1$, given convergence of prices and transfers. Moreover, from the boundary condition and equation (11), once more $x_h^l \gg 0$, all l, h. From (1) we get $\lambda_h^{ln} \to \lambda_h^l \gg 0$ all l, h, and from $(2a)$, $\alpha_h^{ln} \to \alpha_h^l \geq 0$.

From convergence of $\theta_h^1 \to 1$, $\underline{\nu}_h^{1n} \to \underline{\nu}_h^1 = 0$, and $\nu_h^n \to \nu_h$, all h, so that $\{\underline{\nu}_h^{ln}\}$ converges as well, all l, h and $\underline{\nu}_h^{ln} \to 0$, ending this case.

Second, suppose that $\theta_h^1 = 0$. Then we get as before convergence of x_h^{ln}, some $l \neq 1$, to a strictly positive vector, of λ_h^l, b_h^l all h and p. The rest of the argument follows as above.

Finally, the cases where $0 < \theta_h^1 < 1$, all h, as well as where θ_h^1 has a value different across traders, follow from the previous ones. ∎

To conclude, we have the following existence theorem.

Theorem 8 *For any economy (e, u), an equilibrium with transaction costs τ exists.*

Proof. Since Ξ is open, hence a boundaryless manifold, $\dim \Xi$ is equal to the number of equations in (E), $F(.;\omega)$ is continuous, and the space of ω satisfying our assumptions is path-connected, F_t is a continuous homotopy. Using Lemma 7, $\deg_2(F_t, \{0\})$ is well-defined and identical for all $t \in [0,1]$ and all ω. Lemmas 5 and 6 show that $\deg_2(F_{t=1}(.;\omega), \{0\}) = 1$, hence following Lloyd [14] $\deg_2(F_t(.;\omega), \{0\}) = 1$ for $t = 0$ and all ω. Concluding, an equilibrium with transaction costs τ exists for all economies e, u. ∎

This theorem provides the rigorous basis for studying comparative statics issues in asset markets with transaction costs, the subject of further research. One weakness of the theorem is that the payoff matrix has exogenous rank, fixed at I, the number of assets. To study the effects of transaction costs on trading decisions, we need to extend the framework to allow for the possibility that the rank depend on endogenous variables, such as future asset prices. Again, this can be the subject of further research.

References

[1] Arrow, K. and F. Hahn, 1999, Notes on sequence economies, transaction costs and uncertainty, J. Econ. Theory 86, 203-218.

[2] Citanna, A., Kajii, A., Villanacci, A., 1998. Constrained suboptimality in incomplete markets: A general approach and two applications. Economic Theory 11, 495-521.

[3] Citanna, A., Polemarchakis, H. and Tirelli, M., 2000, The taxation of asset trades. Mimeo, December 2000.

[4] Cvitanic, J. and I. Karatzas, 1996, Hedging and portfolio optimization under transaction costs: a martingale approach, Math. Fin. 6, 133-165.

[5] Davis, M. and A. Norman, 1990, Portfolio selection with transaction costs, Math. Oper. Res. 15, 676-713.

[6] Duffie, D. and T. Sun, 1990, Transaction costs and portfolio choice in a discrete-continuous time setting, J. Econ. Dyn. Control 14, 35-51.

[7] Foley, D., 1970, Economic equilibrium with costly marketing, J. Econ. Theory 2, 276-291.

[8] Geanakoplos, J., Polemarchakis, H., 1986. Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete. In: Heller, W., Starr, R., Starrett, D. eds., Uncertainty, Information and Communication: Essays in Honor of Kenneth J. Arrow, Vol. III. Cambridge University Press, Cambridge, MA.

[9] Hahn, F., 1971, Equilibrium with transaction costs, Econometrica 39, 417-439.

[10] Hahn, F., 1973, On transaction costs, inessential sequence economies and money, Rev. Econ. Studies 40, 449-462.

[11] Harris, J. and P. Schultz, 1997, The importance of firm quotes and rapid executions: evidence from the Jan 1994 SOES rules change, J. Fin. Economics 45, 135-166.

[12] Heaton, J. and D.J. Lucas, 1996, Evaluating the effects of incomplete markets on risk sharing and asset pricing, J. Pol. Economy 104, 443-487.

[13] Jouini, E. and H. Kallal, 1995, Martingales and arbitrage in securities markets with transaction costs, J. Econ. Theory 66, 178-197.

[14] Lloyd, N.: Degree Theory. Cambridge: Cambridge University Press 1978.

[15] Magill, M. and G. Costantinides, 1976, Portfolio selection with transaction costs, J. Econ. Theory 13, 245-263.

[16] Minelli, E. and H. Polemarchakis, 1999, Nash-Walras Equilibria of a Large Economy, mimeo, CORE, Louvain-la-Neuve.

[17] Préchac, C., 1996, Existence of equilibrium in incomplete markets with intermediation costs, J. Math. Econ. 25, 373-380.

[18] Starr, R., 1969, Quasi-equilibria in markets with nonconvex preferences, Econometrica 37, 25-38.

[19] Starrett, D., 1973, Inefficiency and the demand for money in a sequence economy, Rev. Econ. Studies 40, 437-448.