Risk Aversion and Herd Behavior in Financial Markets*

Jean-Paul DECAMPS† and Stefano LOVO‡

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†GREMAQ-IDEI Université de Toulouse 1, 21 Allee de Brienne, 31000 Toulouse, France, e-mail: decamps@cict.fr

‡HEC, Finance and Economics Department, 1 Rue de la Liberation, 78351, Jouy en Josas, France, e-mail: lovo@hec.fr (corresponding author)

Abstract

We show that differences in investors risk aversion can generate herd behavior in stock markets where assets are traded sequentially. This in turn prevents markets from being efficient in the sense that financial market prices do not converge to the asset's fundamental value. The informational efficiency of the market depends on the distribution of the risky asset across risk averse agents. These results are obtained without introducing multidimensional uncertainty.

1 Introduction

In this paper we study some causes of herd behavior in stock markets, and its effects on price formation and market informational efficiency.

In the financial market microstructure theory, it is generally accepted that when trading occurs sequentially, agents gradually learn market fundamentals and, eventually, deviations between transaction prices and long-term market fundamentals vanish. [1] This happens because at each trading round, the investors' actions disclose, at least partially, their private information on fundamentals. This information is incorporated into the trading prices. Thus, by observing these prices, it is eventually possible to infer all the relevant private information that is dispersed among investors. In other words, in the long run, market is strong-form informational efficient.

The theoretical literature on "herd behavior" challenges this result. Loosely speaking, an investor engages in herd behavior when he imitates the action of other investors. The literature on herding proves that sequential interaction of rational investors can generate herd behavior and this prevents agents from learning the market fundamentals.[2]

More precisely, an investor "herds" when he takes an action he would not have taken had he not known that other investors have taken it. Thus, when there is herd behavior, investors' actions do not depend on their own private information, but only on other investors' past actions. Therefore, in the presence of herding, investors' actions do not disclose any private information on market fundamentals and consequently the social learning process stops.

However, most of the theoretical results on herding are based on the assumption that transaction prices are exogenously fixed and are not affected by the information provided by past trades. Therefore, the herding literature cannot be directly applied to stock markets, where prices potentially change at each trade, and it is clearly unfit to study the issue of price informational efficiency. On the other hand, there are numerous empirical studies [3] that detect imitative, or herd behavior in financial markets, suggesting that

[1] See Biais, Glosten and Spatt (2001) for a complete review of the literature in microstructure theory.

[2] See Chamley (2001) for an extensive and complete study on the causes of rational herding.

[3] See Bikhchandani and Sharma (2000) for a survey on the theoretical and empirical literature on herding.

this is a common widespread phenomenon in financial markets. Still, the standard financial market microstructure literature, where trading prices are endogenous, does not explain the phenomenon of herding.

Thus, in order to understand the actual causes of imitative behavior and its effect on market informational efficiency, it is crucial to understand whether the different results provided by the microstructure models and the herding models, follow from the assumption that prices are exogenous in the first case and endogenous in the latter.

In this paper we shall consider a financial market microstructure model where trades occur sequentially and trading prices are endogenously determined at each trade. We show that a simple difference in the degree of risk aversion between market makers and traders can generate history dependent behavior including herding behavior. Moreover, we prove that these behaviors drastically reduces the long run informational efficiency of financial markets. This suggests that standard microstructure theory over-estimates the markets informational efficiency and that the endogeneity of prices cannot, in general, prevent herd behavior.

The relations between rational herding, prices formation and informational efficiency, have been also analyzed by Avery and Zemsky (1998) and Lee (1998). Both these papers studied the occurrence of herding when trading is sequential and prices are endogenous.

In the sequential trade model from Avery and Zemsky (1998) (AZ henceforth), history dependent behavior follows from the presence of multidimensional uncertainty.

Indeed, in the AZ model, the observation of the trading history only assures that the agents in the economy learn about a single dimension of uncertainty at one time. So, multidimensional uncertainty can induce market makers and traders to interpret differently the histories of trades and, in the short run, this can generate history dependent behavior and short run mispricing. Nevertheless, in their model, the flow of information to market makers never stops[4] as market makers continue to learn about at least one of the dimensions of uncertainty. In brief, AZ show that multidimensional uncertainty is necessary to generate short run herding behavior but, in the long run, it cannot impede trading prices to eventually converge to the fun-

[4]Equivalently, using the terminology of AZ, an informational cascade cannot occur.

damental value of the asset[5].

Some of our findings are somewhat in contrast with AZ. Firstly, we show that multidimensional uncertainty is not necessary to obtain herd or contrarian behavior[6], unless market makers and traders are both risk neutral. Namely, we prove that when market makers and traders have different degrees of risk aversion, herd and contrarian behavior can occur even if there is only one dimension of uncertainty. Generally speaking, the occurrence of these behaviors in stock markets relies on the fact that the same history of trade affects differently market makers quotes and traders' valuations for the asset. In the AZ model, market makers and traders react differently to the same history because they interpret it in a different way. By contrast, in our model, traders and market makers interpret past histories in the same way. However, the same information affects market makers' quotes and traders' valuations differently because they differ in their risk aversion. This result, combined with the observation that investors differ in their aptitude toward risk, provides a reasonable explanation of the widespread observation of history dependent behaviors.

Secondly, we analyze the effect of herding and contrarian behavior on market informational efficiency. We demonstrate that when traders are risk averse and market makers are risk neutral, the informative content of trades vanishes as agents become more confident about the value of the security. This happens because a risk averse informed trader exchanges the asset on the one hand, to profit from his private information, and on the other hand, to reduce the risk of his portfolio. If the past trades induce strong beliefs about the value of the asset, then an additional private signal will slightly affect a trader's belief. Consequently, his action will mainly reflect the risk-sharing motivation without providing any additional information on the value of the asset. Thus, in general the long run information content of trades is smaller with respect to a situation where all agents are risk neutral.

We show that depending on the distribution of traders' portfolio composition, this phenomenon can generate informational cascade and cause long run mispricing in the sense that agents never learn the true value of the asset and that in the long run, prices are bounded away from fundamentals.[7] For

[5] This point is also emphasized by Hautière (2001) who extends the AZ's model to investigate the impact of short sales constraint on herd behavior.

[6] An investor engages in contrarian behavior when he takes an action he would not have taken had he not known that other investors have taken the opposite action.

[7] This results seems to hold even when risk averse market makers face risk neutral

example, when risk averse agents formed their portfolio in previous trading rounds and the size of trade per round is an integer multiple of a minimum size, then an informational cascade will eventually occur. This result is in contrast to the findings of AZ and of the classical microstructure theory where repeated interaction always leads to informational efficiency.

A similar inefficiency result is obtained by Lee (1998) by introducing transaction cost in a sequential trade model. He shows that information aggregation failure is due to the existence of transaction costs which may prevent traders from revealing their private information. In our model informational inefficiency is obtained in absence of transaction cost.[8]

Thirdly, we show how a market designer could improve long term informational efficiency by changing some parameters of the trading mechanism. Namely, an appropriate choice of the allowed size of trade could restore long term efficiency. We explicitly give the expression for this choice as a function of the distribution of risky asset among risk averse agents.

Finally, we will discuss the properties of quotes and bid ask spread when there is an informational cascade. The evolution of quotes in presence of an informational cascade depends on market makers' degree of risk aversion. Specifically, if market makers are risk neutral and an informational cascade occurs, then spread is zero, and public information, as well as prices, are constant. If market makers are risk averse, spread remains positive and quotes move even in the presence of informational cascade.

In section 2 the notations, the assumptions and the basic structure of the model are presented. In section 3 we provide the formal definitions of herd behavior, contrarian behavior, informational cascade and long run informational efficiency. In section 4 we present a simple specification of the model that shows that when market makers and traders attach different valuation to the asset, herd and contrarian behavior as well as informational cascade occur. Section 5 introduces risk averse investors in the model and studies the equilibrium of the economy where risk averse traders face risk neutral market makers. Section 6 discusses some extension of the model and namely the case of risk neutral traders versus risk averse market makers. Section 7 concludes. The appendix contains the proofs.

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traders.

[8] In Lee's model traders can trade the asset several times. The complexity following from this assumption requires the specification of traders utility function to solve the problem.

2 The model

We consider a sequential trade model similar to Glosten and Milgrom (1985): a risky asset is exchanged for money among market makers and traders. We denote with $\mathbf{v} = \mathbf{V} + \boldsymbol{\varepsilon}$ the liquidation value of the asset, where $\boldsymbol{\varepsilon}$ is symmetrically distributed with mean 0,[9] and $\mathbf{V}$ is equal to $\overline{V}$ with probability π_0 and to $\underline{V} < \overline{V}$ with probability $1 - \pi_0$. The random variables $\mathbf{V}$ and $\boldsymbol{\varepsilon}$ are independently distributed.

Trading mechanism. Trading occurs sequentially and each time interval is long enough to accommodate at most one trade. At the beginning of each trading period, market makers simultaneously set their bid and ask quotes. Then, traders are randomly selected. At any given period t, the selected trader can buy or sell a fixed quantity q of the asset, at the most attractive ask (A_t) or bid price (B_t) respectively.[10] Traders leave the market after they have had the opportunity to trade.

Traders. We denote by $U_T(\mathbf{v},x,m)$ the utility function for a given trader where x and m are respectively the amount of risky asset and money in his portfolio. We will refer to x as the *inventory* of the trader. For simplicity, we assume that all traders have the same utility function but they can differ for the compositions of their portfolios that are assumed to be independently and identically distributed. Specifically, considering a set $\Sigma \subset R^2$, we denote $F(\Sigma) = \Pr((x,m) \in \Sigma)$, i.e. the probability that in a generic period t, the portfolio composition of a trader is in Σ.

Each trader receives a private signal $\mathbf{s} \in \{l, h\}$. Signals are i.i.d. across traders and independent from $\boldsymbol{\varepsilon}$ and from the compositions of agents' portfolios. We assume $\Pr(\mathbf{s} = l | \mathbf{V} = \underline{V}) = \Pr(\mathbf{s} = h | \mathbf{V} = \overline{V}) = p \in (1/2, 1)$. Signal l is more likely when $\mathbf{V} = \underline{V}$ and it can be interpreted as a "Bearish" signal. Similarly, $s = h$ can be interpreted as a "Bullish" signal. In other words, $E[\mathbf{v}|s = l] < E[\mathbf{v}] < E[\mathbf{v}|s = h]$. [11]

Market makers. We denote by $U_M(\mathbf{v},x,m)$ a market maker's utility function when x and m are respectively the amount of risky asset and money in his portfolio. Market makers know the ex-ante distribution of $\mathbf{v}$ and $\mathbf{s}$ but,

[9] In other words, $\Pr(\boldsymbol{\varepsilon} < x) = \Pr(-\boldsymbol{\varepsilon} < x)$

[10] The hypothesis of a fixed quantity of trade will be relaxed in section 5.1.

[11] The results of the paper do not rely on symmetric binary signals that are assumed for the tractability of the model.

unlike traders, they do not receive any private signals on the realization of $\mathbf{V}$.

Public belief. We denote H_t the history of trades up to time $t-1$. All the agents observe H_t but they do not know the identity of past traders. We denote $\pi_t = \Pr\left[\mathbf{V} = \overline{V}|H_t\right]$ the public belief at time t. For any given trader's action $\mathcal{A} \in \{buy, sell, no\ trade\}$, the dynamics of the public belief evolves according to the Bayes' rule: $\pi_{t+1} = \Pr\left[\mathbf{V} = \overline{V}|H_t, \mathcal{A}\right] \overset{def}{=} \Pi(\pi_t, \mathcal{A})$. We say that a trader's action $\mathcal{A}$ is informative if it affects the public belief: $\Pr\left[\mathbf{V} = \overline{V}|H_t, \mathcal{A}\right] \neq \pi_t$. We denote $v(\pi_t) = E[\mathbf{v}|H_t] = E[\mathbf{V}|H_t]$ the expectation of $\mathbf{v}$ when the public belief is π_t. Finally, we say that H_t is a *positive history* (resp. *negative history*) if $\pi_t > \pi_0$ (resp. $\pi_t < \pi_0$).

Private belief. An informed traders refines public information with that provided by his private signal. We denote $\pi_t^s = \Pr\left[\mathbf{V} = \overline{V}|H_t, s\right]$, $s \in \{h, l\}$, an informed traders' belief at time t. We then denote $v^s(\pi_t) = v(\pi_t^s)$.

Note that private signals provide information on the realization of $\mathbf{V}$ but not on the realization of $\boldsymbol{\varepsilon}$. For this reason the learning process in the economy only regards $\mathbf{V}$**,** whereas the presence of $\boldsymbol{\varepsilon}$ guarantees that the uncertainty on $\mathbf{v}$ remains even when the realization of $\mathbf{V}$ is known. The presence of $\boldsymbol{\varepsilon}$ is crucial for the existence of herding when market makers and traders differ in risk aversion.

3 The definitions of herding informational cascade and efficiency

We adopt exactly the same definition of herding, contrarian behavior and informational cascade as in Avery and Zemsky (1998). Herding occurs when a trader imitates others, whereas a trader engages in contrarian behavior when he behaves in the opposite way with respect to others.

A trader with private signal s engages in buy (sell) *herding behavior* if: (i) initially he strictly prefers not to buy the asset (resp. not to sell); (ii) after observing a positive history of trades H_t , i.e. $\pi_t > \pi_0$ (resp. negative history, i.e. $\pi_t < \pi_0$), he strictly prefers to buy (resp. sell).

A trader engages in buy (sell) *contrarian behavior* if: (i) initially he strictly prefers not to buy (resp. not to sell); (ii) after observing a negative (resp. positive) history of trades H_t, he strictly prefers to buy (resp. sell).

In other words, a trader engages in buy (resp. sell) herding or contrarian behavior when two conditions are met: first, if initially he has the option to trade, then he strictly prefers not to buy (resp. sell); second, if after observing an history of trade, he has the option to trade, then he strictly prefers to buy (resp. sell). If the history he has observed is positive, we say that he engages in buy herding (resp. sell contrarian behavior), whereas if the history is negative, he engages in buy contrarian behavior (resp. sell herding).

Following AZ, we distinguish between herding or contrarian behavior and informational cascade.

An *informational cascade* occurs when the actions of all informed traders are independent from their private information.

Note that at time t, an informational cascade occurs, if and only if $\pi_{t+1} = \pi_t$. Indeed, as the trade in time t is not affected by the trader's private information, it will provide no new information on the realization of $\mathbf{V}$, so that public beliefs will be unchanged. Note that herding, or contrarian, behavior and informational cascade are a relate but a distinct phenomena. For example, when all the traders engage in buy herding, then an informational cascade occurs. However, it is possible to observe an informational cascade even when different traders choose different actions provided that their choice is not affected by the signal received.

The occurrence of an informational cascade is related to the informational efficiency properties of the market. Indeed, if an informational cascade occurs, the learning process stops and prices cannot incorporate agents informations. More precisely, prices are strong-form efficient if they reflect all private information available in the economy:

The market is said to *be strong-form efficient in the long run*, if prices and beliefs ultimately converge to the true value of the asset.

$$\Pr\left(\lim_{t\to\infty} E[\mathbf{V}|H_t] = \mathbf{V}\right) = 1.$$

It is worth stressing that in the financial microstructure literature it is generally accepted that market is strong form efficient in the long run[12]. In the following sections we will show that even when prices are endogenous, herd, contrarian behavior, and informational cascades can occur and that market is not efficient in the long run.

4 A basic model

For expositional clarity we begin with a simple example that illustrates the occurrence of history dependent behavior.

We assume that $U_T(\mathbf{v},x,m) = \mathbf{v}x + m$ whereas $U_M(\mathbf{v},x,m) = (\theta\mathbf{v} + C)x + m$, with $\theta \geq 0$. An interesting aspect of this simple set up is that it includes as a particular case the simplest version of herding models, where price is exogenously fixed. Moreover, it also includes as particular case the simplest version of Glosten and Milgrom (1985) where price is endogenous and herding as well as informational cascade are impossible.

Competition among equally uninformed market makers leads to bid and ask price such that the expected profit from trading q risky asset is zero. This expectation is computed taking into account the informational content of the trade. Namely, at any time t bid and ask prices solve

$$\begin{aligned} E[(\theta\mathbf{v}+C)(x+q)+m-qB_t|\mathit{sell\ order}] - E[(\theta\mathbf{v}+C)x+m|\mathit{sell\ order}] &= 0 \\ E[(\theta\mathbf{v}+C)(x-q)+m+qA_t|\mathit{buy\ order}] - E[(\theta\mathbf{v}+C)x+m|\mathit{buy\ order}] &= 0 \end{aligned}$$

Therefore, market markers' quotes in period t are $A_t = \theta v(\Pi(\pi_t, buy)) + C$ and $B_t = \theta v(\Pi(\pi_t, sell)) + C$.

In this example, agents' valuation for the asset can be decomposed into a common value component, $v(\pi_t)$, and a private value distortion: $\theta \neq 1$ and $C \neq 0$. The difference between market makers quotes and traders valuations is responsible for trades. On one hand it depends on the difference in the common value component due to the asymmetry of information (i.e. $v(\pi_t) - v^s(\pi_t)$), and on the other hand on the private value distortion. At time t, a trader who received a signal s will buy the asset only if $v^s(\pi_t) \geq A_t$ and he will sell only if $v^s(\pi_t) \leq B_t$.

When $\theta = 0$ and $C \in (\underline{V}, \overline{V})$, the model corresponds to the simplest version of the herding model in Bikhchandani, Hirshleifer and Whelch (1992)

[12] See O'Hara (1995).

(BHW henceforth). In this case they show that herding can occur. This is illustrated in Figure 1 that plots the market makers quotes and traders' valuations, $v^s(\pi)$, as functions of the public belief π. No matter the public belief π, bid and ask prices are equal to C . Suppose that the initial prior $\pi_0 \in (\pi^*, \pi^{**})$, we have $v^l(\pi_0) < B_0 = C = A_0 < v^h(\pi_0)$. Thus, initially, an informed trader will sell the asset only if he has a Bearish signal, whereas he will buy the asset only if he has a Bullish signal. Consequently, traders orders are informative as long as $\pi_t \in (\pi^*, \pi^{**})$. However, an history of trade that is sufficiently unbalanced in one direction (buy or sell) will lead the public belief π_t to be larger than π^{**} or smaller than π^*. At this point buy or sell herding behavior will start respectively.

The parameters values $\theta = 1$ and $C = 0$, lead to the Glosten and Milgrom model. AZ have shown that in such a framework, herding is impossible since for any level of the public belief π_t, one gets $v^l(\pi_t) = B_t < A_t = v^h(\pi_t)$.[13] Figure 2 illustrates this case. The bid and ask prices are equal to $v^l(\pi_t)$ and $v^h(\pi_t)$ respectively so that informed traders with the Bullish signal (resp. Bearish signal) always buy (resp. sell) the risky asset no matter what is the evolution of the public beliefs π_t. In this case informed traders' actions always reflect the sign of their signal.

The following proposition shows that as soon as $\theta \neq 1$ or $C \neq 0$, herding or contrarian behavior can occur even in this basic one dimensional private signal setting.

Proposition 1 *There exist $\underline{C_h}$, $\overline{C_h}$, $\underline{C_c}$, $\overline{C_c}$ such that:*

i) If $C \in (\underline{C_h}, \overline{C_h})$ and $\theta \in (0,1)$, then there exists a level π_0 of initial belief such that buy herding as well as sell herding occur with positive probability.

ii) If $C \in (\underline{C_c}, \overline{C_c})$ and $\theta > 1$, then there exists a level π_0 of initial belief such that buy contrarian behavior as well as sell contrarian behavior occur with positive probability.

The main message of Proposition 1 is that even if uncertainty has one dimension, when market makers and traders attach even slightly different value to the traded asset, herd and contrarian behaviors can occur. Moreover, when herding or contrarian behavior start, the quotes do not change anymore and the flow of trade does not provide information on $\mathbf{V}$. Thus, in this simple setting, there is no difference between herding and informational cascade and

[13] In standard microstructure models the presence of liquidity traders implies $v^l(\pi_t) < B_t < A_t < v^h(\pi_t)$ that does not change the result in term of impossibility of herding.

whenever $\theta \neq 1$ or $C \neq 0$, agents' beliefs cannot converge to the truth (market is not strong- form informational efficient even in the long run).

As a general rule, the occurrence of herd or contrarian behavior in stock markets relies on the fact that market makers and traders can react differently to an history of trade. Namely, herding behavior is likely to occur when a trading history has unimportant effects on market makers' quotes, when compared to its effect on traders' valuation for the asset. Take for example, a sequence of buy orders that increases the public belief π_t. Suppose that this increases market makers quotes slightly in comparison to the increase in the traders' valuation. Eventually, traders's valuation for the asset will be so high that it becomes profitable to buy the asset even for those traders who received a Bearish signal, and would have initially sold it. Conversely, contrarian behavior occurs when market makers quotes over-react to the flow of trade. For example, a sequence of buy orders can increase market makers' quotes so dramatically, that it becomes profitable to sell the asset even for those traders that would have initially bought it. [14]

Proposition 1 illustrate exactly this phenomenon: the parameter θ represents the sensitivity of market makers' quotes to a change in public belief. When $\theta < 1$, this sensitivity is low and the information provided by the flow of trades will affect more the informed traders's valuations rather than market makers' quotes. Thus, herding behavior is likely to occur. When $\theta > 1$ prices over-react to the arrival of information leading to contrarian behavior. Figure 3 illustrates case i) in Proposition 1.

Parameter C introduces a bias in market makers' valuation that is independent on information. When C is negative (resp. positive), market makers quotes will be on average lower (resp. higher) than traders' valuation. Therefore traders will be prone to buy (resp. sell) no matter their information. Thus, a negative C is responsible for the existence of buy herding and buy contrarian behavior. This is illustrated by Figure 4. Similarly, a positive C will be responsible for sell herding or contrarian behavior.

[14] For example, in AZ multidimensional uncertainty induces market makers and traders to interpret differently the same trading history so that price sensitivity to information differs from that of informed traders' valuation.

5 Risk aversion and herding

Now, we study how differences in the level of risk aversion of informed traders and market makers can lead to history dependent behaviors and informational cascade. Firstly, we prove that, when market makers and traders have different degrees of risk aversion, their monetary valuations for the risky asset differ. Secondly, we show that this diversity can reduce market efficiency, and generate history dependent behaviors as well as informational cascade. To this purpose, we extend the method that was illustrated in the previous section to study the case of heterogenous traders.

We work thereafter under the assumption that market makers are risk neutral whereas informed traders are risk averse:[15]

Assumption 1: $U_M(\mathbf{v}, x, m) = \mathbf{v}x + m$ and $U_T(\mathbf{v}, x, m) = u(\mathbf{v}x + m)$ with $u : I\!R \to I\!R$, differentiable, strictly increasing and strictly concave.

We start with the analysis of agent's behavior.

5.1 Agents' behavior

Risk averse traders' behavior. Consider a trader whose utility function u satisfies Assumption 1. We denote with β and α the trader's buy and sell *reservation prices* that are implicitly defined by equations (1) and (2) respectively.

$$E\left[u\left(\mathbf{v}x + m\right) - u\left(\mathbf{v}(x+q) + m - q\beta\right)\right] = 0 \quad (1)$$

$$E\left[u\left(\mathbf{v}x + m\right) - u\left(\mathbf{v}(x-q) + m + q\alpha\right)\right] = 0 \quad (2)$$

Consider a trader whose portfolio is (x, m) and let π be his belief that $\mathbf{V} = \overline{V}$. The buy reservation price $\beta(\pi, x, m)$ represents the maximum asset's price that this trader is willing to pay for q additional assets. Similarly, the sell reservation price $\alpha(\pi, x, m)$ is the minimum asset's price at which he accepts to sell q assets.

If at time t market makers' ask and bid quotes are (A_t, B_t) and a trader with signal s and portfolio (x, m) is allowed to trade, then he will buy if $\beta(\pi_t^s, x, m) \geq A_t$, he will sell if $\alpha(\pi_t^s, x, m) \leq B_t$ and he will not trade elsewhere.

[15] In section 6 we briefly discuss the case of risk averse market makers versus risk neutral traders.

Thus, a trader's behavior is determined by the composition of his portfolio (x, m) on one hand, and by his belief π on the other hand. That is to say, a trader buys q risky assets at price A if his portfolio's composition is $(x, m) \in \Phi(\pi, A)$, where

$$\Phi(\pi, A) = \{(x, m) | \beta(\pi, x, m) \geq A\}.$$

The set $\Phi(\pi, A)$ represents all portfolios compositions (x, m) such that a risk averse trader with belief π is willing to buy at least q assets at price B.

Similarly, an agent with belief π will sell at price B, if his portfolio composition $(x, m) \in \Psi(\pi, B)$, where

$$\Psi(\pi, B) = \{(x, m) | \alpha(\pi, x, m) \leq B\}.$$

Here, $\Psi(\pi, B)$ is the set of portfolios compositions such that a risk averse agent with belief π is willing to sell at least q risky assets at price A.

The following lemma describes some relevant properties of reservation prices.

Lemma 1 *The mappings* $(\pi, x, m) \longrightarrow \beta(\pi, x, m)$ *and* $(\pi, x, m) \longrightarrow \alpha(\pi, x, m)$ *are continuous on* $[0, 1] \times \mathbb{R} \times \mathbb{R}$. *Moreover,*

1) For any m, *it results*
if $x < -\frac{q}{2}$ *then,* $\beta(0, x, m) > \underline{V}$, $\beta(1, x, m) > \overline{V}$,
if $x = -\frac{q}{2}$ *then,* $\beta(0, x, m) = \underline{V}$, $\beta(1, x, m) = \overline{V}$,
if $x > -\frac{q}{2}$ *then,* $\beta(0, x, m) < \underline{V}$, $\beta(1, x, m) < \overline{V}$.
2) For any m, *it results*
if $x < \frac{q}{2}$ *then,* $\alpha(0, x, m) > \underline{V}$, $\alpha(1, x, m) > \overline{V}$,
if $x = \frac{q}{2}$ *then,* $\alpha(0, x, m) = \underline{V}$, $\alpha(1, x, m) = \overline{V}$,
if $x > \frac{q}{2}$ *then,* $\alpha(0, x, m) < \underline{V}$, $\alpha(1, x, m) < \overline{V}$.

Lemma 1 states that a risk averse investor's reservation prices for a risky asset will in general be different from the expected value of the asset given the investor's information. In particular, when there is no uncertainty on $\mathbf{V}$, but it remains the uncertainty on $\boldsymbol{\varepsilon}$, the bid reservation price is equal to its expected value only if $x = -q/2$. When the investor possesses $x > -\frac{q}{2}$ (resp. $x < -\frac{q}{2}$) units of the risky asset, because of risk aversion, he is reluctant to increase his position (resp. inclined to increase his position) and the maximum asset's price that this investor is willing to pay is lower (resp.

larger) than the expected value of the asset. Result 2 of the lemma develops the same idea for the ask price and symmetric interpretations hold.

It is worth stressing here the role that the component ε in the value of the asset has in Lemma 1. Recall that the liquidation value of the risky asset is $\mathbf{v} = \mathbf{V} + \boldsymbol{\varepsilon}$. As traders receive signals that only depend on the realization of $\mathbf{V}$, they can only learn about $\mathbf{V}$. Therefore, even for an agent who knows exactly the realization of $\mathbf{V}$ (i.e. $\pi = 0$ or $\pi = 1$), the uncertainty about $\mathbf{v}$ is unresolved because of the presence of the random component $\boldsymbol{\varepsilon}$. Lemma 1 states that if this agent is risk averse then in general his reservation prices will be different from $\mathbf{V}$.

Risk neutral market makers' behavior. Bertrand competition among equally uninformed risk neutral market makers leads to bid and ask quotes that are equal to the expected value of the asset given the available information. Namely, time t bid quote, B_t, is equal to the maximum [16] of the solutions of the following equation

$$B_t = E[\mathbf{v}|H_t, \text{trader sells at } B_t]. \tag{3}$$

The ask quote A_t is equal to the minimum [17] of the solutions of the equation

$$A_t = E[\mathbf{v}|H_t, \text{trader buys at } A_t]. \tag{4}$$

Lemma 2: *Solutions of equations (3) and (4) always exist. Moreover,* $A_t \in [v^l(\pi_t), v^h(\pi_t)]$ *and* $B_t \in [v^l(\pi_t), v^h(\pi_t)]$.

Lemma 2 states that bid and ask quotes have an upper and a lower bounds that correspond to the expected value of the asset given the past history and a Bullish or a Bearish signal respectively. In other words, the information that market makers extract from a given trading history cannot be more precise than the one privately informed traders extract from the same history.

We conclude this section observing that from Lemma 2, when π_t approaches 1 (resp. 0) the bid and ask price tend to $\overline{V}$ (resp. $\underline{V}$). However, from Lemma 1 it follows that traders' reservation price are in general different from $\overline{V}$ (resp. $\underline{V}$) even when π_t approaches 1 (resp. 0). Similarly to what

[16] Any other solution of (3) would not be an equilibrium as there would exist a larger bid that would provides positive profit to the market makers.

[17] Any other solution of (4) would not be an equilibrium as there would exist a lower ask that would provides positive profit to the market makers.

happens for the simple model in the previous section, this difference between market makers and traders's valuations for the asset is the cause of herding, contrarian behavior and market informational inefficiency.

5.2 Individual herding

In order to understand the role of risk aversion in the occurrence of herding and contrarian behavior, it is useful to distinguish two components in the trading motivations of a risk averse investor: the *information component* and the *inventory component*. The information component reflects the changes in an investor's reservation prices that follow a good or bad news regarding $\mathbf{V}$. If the news about the realization of $\mathbf{V}$ is not perfectly informative, then, *ceteris paribus*, its effect on trading motivation will decrease when the prior belief π approaches 0 or 1. For example, the impact of a Bullish signal on an trader's belief, and thus on his reservation price, is negligible when his prior belief π is close to 0 or to 1. That is to say, $\lim_{\pi \to 0}(v^h(\pi) - v^l(\pi)) = 0$ and $\lim_{\pi \to 1}(v^h(\pi) - v^l(\pi)) = 0$.[18]

The inventory component reflects the agent's preference for low-risk-portfolio. For a given level of π and a given trading price, a risk averse investor is prone to sell risky assets if his inventory is sufficiently large, and prone to buy if his inventory is sufficiently small. The inventory component increases with the investor's degree of risk aversion, the portfolio's exposure to risk $|x|$, and the unresolved uncertainty about the asset fundamentals.

In the following proposition we show that when the public belief is sufficiently close to 0 or to 1, an informed trader's signal affects his action only if his inventory x is sufficiently close to $q/2$ or $-q/2$. Therefore, all informed traders whose inventory x is not equal to $q/2$, or to $-q/2$, will eventually ignore the information provided by their private signals.

Denote with Θ^s_{buy} the set $\{(x,m)|x < -q/2\} \cap (\Phi(\pi^s_0, A_0))^c$. Similarly, denote Θ^s_{sell} the set $\{(x,m)|x > q/2\} \cap (\Psi(\pi^s_0, B_0))^c$. The set Θ^s_{buy} (resp. Θ^s_{sell}) represents the portfolio compositions of those traders with signal s that at time 0 prefer not to buy (resp. sell) the asset, despite at time 0 they have the option to trade and their inventory x is smaller than $-q/2$ (resp. greater than $q/2$).

[18] This property is usual in models with a binary not perfectly informative signals: strong prior overwhelm a bounded private signal. (See Chamley 2001)

Proposition 2: *Under assumption 1, a trader who receives a signal s and whose portfolio $(x, m) \in \Theta^s_{buy}$ (resp. $(x, m) \in \Theta^s_{sell}$) engages in buy (resp. sell) herding or contrarian behavior with positive probability.*

A trader engages in herding or contrarian behavior when two requirements are met: (i) initially he strictly prefers not to buy (or to sell); (ii) after observing an history of trade he strictly prefers to buy (resp. sell). Condition $(x, m) \in \Theta^s_{buy}$ obviously guarantees requirement (i). To understand why when $x < -q/2$, he will eventually decide to buy, take for example the case where the public belief π_t converges to 1. Then, the bid and ask quotes will converge to $\overline{V}$ from Lemma 2, whereas $\beta(\pi^s_t, x, m) > \overline{V}$ from Lemma 1. When π_t is sufficiently close to 1 we have $\beta(\pi^s_t, x, m) > \overline{V} \simeq A_t$, and if the trader has the option to trade, he will buy the asset. In other words, in the long run the information component of a traders' action vanishes and his trade only reflects the inventory component. Consequently, traders whose portfolio $(x, m) \in \Theta^s_{buy}$, may engage in buy herding or buy contrarian behavior depending on the realization of a positive or negative history respectively. Similarly, if a trader's portfolio composition belongs to Θ^s_{sell}, a positive or negative history may induce him to engage in sell contrarian behavior or sell herding respectively.

5.3 Informational cascade and efficiency

What are the consequences of agents herding or contrarian behavior on market informational efficiency?

In this section, we study under which conditions a sufficiently long history of trade guarantees that agents ultimately learn the true value of the asset. Moreover, we analyze the efficiency properties of the economy as it is measured by the speed of convergence of prices (and beliefs) to the true value of the asset. Following Diamond and Verrecchia (1987), this speed of convergence can be measured by the rate at which the likelihood ratio $L_t = \frac{\pi_t}{1-\pi_t}$ tends to infinity, when $\mathbf{V} = \overline{V}$, or to zero when $\mathbf{V} = \underline{V}$.

Proposition 3 states that the long term informational efficiency of the market, as well as the speed of convergence of beliefs to the truth, depends on the distribution of traders' portfolio composition $F(.)$.

Proposition 3: *Under assumption 1, the following holds.*

1) *The presence of risk averse traders reduces the informational efficiency of the market when compared to a situation where agents are risk neutral.*

2) *If F is such that there is a zero probability that a trader's inventory is close to either $q/2$ or $-q/2$, then long run informational inefficiency occurs almost surely.*

3) *If the distribution F is a continuous distribution function, then the rate of convergence of the asset price to its fundamental value tends to zero.*

Result 1 in Proposition 3 establishes that, in an economy with risk averse traders, trading prices converge to market fundamental at a lower rate [19] with respect to an economy where market makers and traders are risk neutral. This is not surprising as, in presence of risk aversion there might be a proportion of informed traders whose inventory is so unbalanced that their trade is actually not informative. This result is in tune with the standard rational expectations equilibrium models. Indeed, the presence of an inventory component in informed traders' motivation can be reinterpreted as the existence of an additional noise (for example coming from liquidity traders) that reduces market informational efficiency.

Result 2 is more striking as it contradicts the common wisdom that repeated interaction in financial market always leads to informational efficiency and that in the long run prices always converge to fundamentals. Result 2 is also somewhat in contrast with AZ that prove that multidimensional uncertainty is not sufficient to produce long run mispricing. To understand Result 2, note that, for Proposition 2, when the public belief π_t is sufficiently close to 1 or to 0 a trader's order will only reflect his inventory unbalance unless his inventory x is sufficiently close to $q/2$ or to $-q/2$. However, if the inventory of all traders is bounded away from $q/2$ and from $-q/2$, when π_t is sufficiently close either to 0 or 1, trades stop providing information on $\mathbf{V}$ and an informational cascade starts. It is worth stressing that, as any finite history of trade is in general compatible with any realization of $\mathbf{V}$, there is a positive probability that the public belief will be entrapped around 0 (or around 1) even if $\mathbf{V} = \overline{V}$ (resp. $\mathbf{V} = \underline{V}$). In other words, not only the public belief does not converge to the truth, but agents' beliefs can be completely

[19] For example a sequence of buy orders (resp. sell orders) changes the likelihood ratio L_t at a lower rate.

incorrect even in the long run. Thus, an informational cascade in the wrong direction can occur with positive probability.

Note that when an informational cascade starts, the bid-ask spread is zero and quotes stop moving because orders stop providing information. Thus, history dependent behavior can cause long run mispricing. Note also that whenever traders formed their portfolio with previous transactions in the same market, then the condition on F required for Result 2 is met. Indeed, considering that the size of each trade is q, if a trader bought his portfolio in previous trading rounds, then his inventory x will be an integer multiple of $\pm q$ and the hypothesis of Result 2 will be satisfied.

Result 3 extends result 2 emphasizing that, even when F is continuous, and in the long run the market is efficient, the process of convergence of the public belief to the truth is so slow that it might be indistinguishable from an informational cascade. By contrast, if traders and market makers were risk neutral, the ratio at which π_t converges to the truth would be strictly positive and independent from π_t. This is for example what happens in Diamond and Verrecchia (1987).

5.4 Market mechanism and informational efficiency

In this section we study how the market designer should choose the parameters of the trading mechanism in order to improve the long term informational efficiency of the market. To begin with, we assume that the market regulator can choose the amount of risky assets q that can be traded at each period. We denote with q_s and q_b the amount of asset that can be sold or bought respectively. We show how the choice of appropriate q_s and q_b can prevent the occurrence of informational cascade. From the previous discussion we know that for a given allowed size of trade q, in the long run the only informative buy orders (sell orders) come from investors whose inventory is sufficiently close to $-q/2$ (resp. $q/2$) and that informational cascade arises when there is no such investor in the economy (see result 2 in Proposition 3). Therefore, for a given distribution of traders' portfolio composition, F, conditions that guarantee long run informational efficiency are that for any $\delta > 0$,

$$\begin{aligned}
\Pr\left(x \in \left(\frac{q_s}{2} - \delta, \frac{q_s}{2} + \delta\right)\right) &> 0, \\
\Pr\left(x \in \left(-\frac{q_b}{2} - \delta, -\frac{q_b}{2} + \delta\right)\right) &> 0.
\end{aligned}$$

Indeed by choosing q_s and q_b such that these two conditions are satisfied, the market designer ensures that there is a positive probability of observing informative trades that come from trades whose inventory is arbitrarily close to $\frac{q_s}{2}$ or $-\frac{q_b}{2}$.

When trader's inventories are discretely distributed, long run efficiency can be improved by choosing trading sizes for the buy and sell orders which maximize the speed of convergence of beliefs to the truth. We have the following result.

Lemma 3: *When trader's inventories are discretely distributed, long run informational efficiency can be improved by choosing the trading sizes for the buy and sell orders, q_b and q_s, such that*

$$q_s = \max_q \Pr(x = q/2), \tag{5}$$

$$q_b = \max_q \Pr(x = -q/2). \tag{6}$$

Now, we prove that the inefficiency results of the previous section do not rely on the fact that the size of trade per period is fixed. We show that inefficiency follows from the fact that the space of traders' action is discrete. This is in tune with the findings of the literature on herding (See Chamley 2001).

Consider a financial market whose trading mechanism differs from the one described in section 2 only for the fact that traders can decide to buy (sell), at the ask price (resp. bid price), any quantity $Q \in nq$, with $n \in \mathbf{N}$, with $n < N$, where N is a natural number. In other words, in every period each market maker quotes one bid and one ask price at which he is willing to sell or buy respectively any quantity Q of the risky asset up to a maximum quantity of Nq. This is what happens for example in Nasdaq SOES market. Traders can buy or sell round lots of the risky asset, the size of the lot is q and no more than N lots can be traded at each round. The following proposition shows that the result of Proposition 3 can be extended to the case of round lots trading.

Proposition 4: *Under assumption 1, if at each stage traders can trade round lots up to a maximum of N lots, and if each market makers can quote only one ask and one bid at which he satisfies any buy or sell order respectively up to N lots , then the following holds.*

1) *The presence of risk averse traders reduces the informational efficiency of the market when compared to a situation where agents are risk neutral.*

2) *If for any* $n \in \{-N, -N+1, ..., N\}$ *the probability that a trader inventory is arbitrarily close to* $q(n+\frac{1}{2})$ *is zero, then long run informational inefficiency occurs.*

3) *If the distribution* F *is a continuous distribution function, then the rate of convergence of the asset price to its fundamental value tends to zero.*

Proposition 4 suggests that with round-lots trading mechanisms informational cascade and long run mispricing are possible. For example, this is particularly true when traders bought their portfolio using the same trading mechanism. Indeed, if agents traded discrete quantity they will have discrete portfolios compositions and the hypothesis for result 2 in proposition 4 will be satisfied. Notice that, passing from a round lots mechanism to an odd lots[20] mechanism could improve market efficiency. However, strong-form informational efficiency cannot be completely reached as an odd lots mechanism can be seen as a round lots mechanism where q is one share of the risky asset.

6 Extension

In this section, we briefly discuss the occurrence of herding and informational cascade when risk neutral traders face risk averse market makers. The study of the multi-period price competition among risk averse market maker is more involved than in the risk neutral case. We believe that a complete study of the risk averse market maker case would over complicate the analysis without affecting the main implications on the existence of herding and informational cascade. To be thorough, we discuss how the results of the previous sections would change with risk averse market makers versus risk neutral informed trader. Following Ho and Stoll (1983) and Biais (1993) one could say that at some time t the market ask (bid) price is equal (or close) to the second lowest (resp. highest) sell (resp. buy) reservation price among

[20] i.e. a situation where in each trading round, it is possible to exchange any amount of the risky asset up to a maximum amount.

market makers.[21] Still, as in our set up traders are privately informed, market makers take into account the informational content of a trade when choosing their quotes. Therefore, traders' orders act on market makers' quotes in two ways: on the one hand an order can provide some information on the value of the asset and change the public belief. We call this effect the *information effect*[22]. On the other hand, traders' orders change the composition of market makers' portfolio, and consequently affect their reservation prices and quotes even when they do not change the public belief. We denote this phenomenon the *inventory effect*. Note that if a market maker's reservation prices are decreasing functions of his inventory[23], then the information and the inventory effects move quotes in the same direction magnifying the sensitivity of prices to the history of trade.

There are two relevant differences between the case discussed here and the one studied in section 5.1. The first difference is that with risk averse traders, prices become gradually less informative as the fraction of traders whose trade is not informative increases. By contrast, when traders are risk neutral, herding (or contrarian) behavior starts simultaneously for all traders. Indeed, when traders are risk neutral, they actually differ only for the sign of their information. Thus, when their actions do not reflect their private signal, they all behave exactly in the same way. For this reason, with risk neutral traders, history dependent behavior always implies an informational cascade.

The second difference is that an informational cascade never ends when market makers are risk neutral, whereas it might end if market makers are risk averse. In particular the bid-ask spread is equal to zero in presence of informational cascades when trader are risk averse whereas spread continues to evolve in presence of informational cascade when market maker are risk averse. Indeed, when market makers are risk averse, quotes move because of the inventory effect even when orders do not conceal any new information. This can end an informational cascade. For example, consider a situation where market makers' inventories are so low, that quotes will be sufficiently high to induce informed traders to sell the asset no matter their signal. In this

[21]Where, a market maker' buy and sell reservation prices could be defined in the same whay traders reservation prices are defined in the previous section.

[22]This is the only force that moves quotes in the risk neutral market makers case (including AZ model).

[23]For example this is the case when market makers have a CARA utility functions.

situation, sell herding behavior occurs, an informational cascade starts and the public belief is steady. As on average, there will be more sell orders than buy orders, market makers' inventory will gradually increase and this might decrease the bid and ask quotes until the point that only informed traders that received a "Bearish" signal still want to sell. This would temporarily end the informational cascade. This example also illustrates that market makers risk aversion can easily produce contrarian behavior because when the information and the inventory effects act in the same direction, prices over react to the trading history.

7 Conclusion

We studied how the difference in risk aversion between traders and market makers can originate herd and contrarian behavior in a sequential trade framework. Furthermore, we examined how these phenomena affect the market long run informational efficiency. In general, herd and contrarian behavior may occur when trading histories affect differently market makers' and traders' reservation prices. Herd behavior may arise when quotes sensitivity to the flow of trade is lower than traders' valuation sensitivity, whereas contrarian behavior follows from over-reaction of quotes to the flow of trade. We proved that both these phenomena are compatible with a situation where (i) market participants differ in their degrees of risk aversion (ii) the minimum and the maximum size of trade per period are fixed. This suggests that herd and contrarian behavior in stock market should be seen as the norm rather than the exception.

The difference between market makers and traders risk aversion can also generate informational cascade and thus, long run mispricing and market informational inefficiency.

If traders are risk averse and market makers are risk neutral, when an informational cascade starts, the information flow stops, prices are constant, spread is zero. The probability of observing an informational cascade depends on the distribution of the risky asset across informed traders and it is one whenever traders built their portfolio in previous trading stages. Still, a market designer can improve market informational efficiency by appropriately choosing the allowed size of trade per period.

If risk averse market makers face risk neutral traders, when informational cascade starts, the spread shrinks but remains positive. Moreover in presence of informational cascade quotes move because market makers' portfolio change. This can temporarily restore the informativeness of trades, but in general it is not sufficient to guarantee long run strong form efficiency of the market.

8 Appendix

Proof of proposition 1: In order to simplify the notation we assume $\overline{V} = \underline{V} + 1$. Here π_t corresponds to $\Pr(\mathbf{V} = \underline{V} + 1 | H_t)$. Let $\underline{C_h} = \underline{V}(1-\theta)$, $\overline{C_h} = (\underline{V}+1)(1-\theta)$, $\underline{C_c} = (\underline{V}+1)(1-\theta)$, $\overline{C_c} = \underline{V}(1-\theta)$. Consider case (i): $C \in (\underline{V}(1-\theta), (\underline{V}+1)(1-\theta))$ and $\theta \in (0,1)$. A direct application of the intermediate value theorem shows there exists an interval I in $(0,1)$such that:

$$\forall \pi \in I \quad v^l(\pi) < \theta(\underline{V} + \pi) + C < v^h(\pi),$$

which induces that the informed traders with low signal will sell whereas the informed traders with high signal will buy[24]. In other words informed agents trading strategies change with their private signal. We now consider π_0 in I. Taking advantage of the relation $v^h(1) = v^l(1) = \underline{V} + 1 < \theta(\underline{V}+1) + C$, we apply again the intermediate value theorem and deduce there exist $\mathcal{V}$, a left neighborhood of 1 such that:

$$\forall \pi \in \mathcal{V} \quad v^h(\pi) \geq v^l(\pi) > \theta(\underline{V} + \pi) + C.$$

This implies that in such beliefs' region any informed agents buy no matter his signal and ask (and bid) prices are equal to $\theta(\underline{V} + \pi) + C$. We deduce that for a number of buy orders sufficiently large the public belief π_t lies in a left neighborhood of 1. To conclude the proof, simply remark that as signals are not perfectly correlated to the realization of $\mathbf{V}$, a finite history that is sufficiently unbalanced in one direction occurs with positive probability. Hence assertion (i) for buy herding. Part (ii) is proved using analogous arguments.■

Proof of lemma 1: The continuity of $\alpha(.)$ and $\beta(.)$ rely on the the differentiability and monotonicity of u. We prove result 1) for $\pi = 1$. The proof

[24]Notice that the strict inequality $v^l(\pi) < v^h(\pi)$ comes from the informativeness of the private signal (that is $p > \frac{1}{2}$).

for $\pi = 0$ and the proof of result 2) are symmetric. First, it follows from equation (1) that $\beta(1,x,m)$ is implicitly defined by the equality:

$$E\left[u\left(\left(\overline{V}+\boldsymbol{\varepsilon}\right)x+m\right)\right] - E\left[u\left(\left(\overline{V}+\boldsymbol{\varepsilon}\right)(x+q)+m-q\beta\right)\right]=0 \quad (7)$$

From the symmetric distribution of $\boldsymbol{\varepsilon}$, we deduce that $\mathbf{W}_0 = \left(\overline{V}+\boldsymbol{\varepsilon}\right)x+m$ (resp. $\mathbf{W}_1 = \left(\overline{V}+\boldsymbol{\varepsilon}\right)(x+q)+m-q\overline{V})$) has the same probability distribution than $\mathbf{W}_0' = \overline{V}x+m+\boldsymbol{\varepsilon}|x|$ (resp. $\mathbf{W}_1' = \overline{V}x+m+\boldsymbol{\varepsilon}|x+q|$). Note that if $x=-q/2$, then $|x|=|x+q|$. Thus, if $x=-\frac{q}{2}$, equality (7) is satisfied if and only if $\beta(1,\frac{q}{2},m)=\overline{V}$. Note also that if $x<-q/2$, then $|x|>|x+q|$. Moreover, as the mapping $b \longrightarrow E\left[u\left(\overline{V}x+m+b\boldsymbol{\varepsilon}\right)\right]$ is decreasing[25], we have that

$$E[u(\mathbf{W}_0)] = E\left[u\left(\mathbf{W}_0'\right)\right] < E\left[u\left(\mathbf{W}_1'\right)\right] = E\left[u\left(\mathbf{W}_1\right)\right]. \quad (8)$$

Finally, remarking the equality

$$E\left[u\left(\mathbf{v}x+m\right)-u\left(\mathbf{v}(x+q)+m-q\beta\right)\right] = E\left[u\left(\mathbf{W}_0\right)-u\left(\mathbf{W}_1+q(\overline{V}-\beta)\right)\right],$$

we deduce from (7) and (8) that $\beta(1,x,m) > \overline{V}$ for $x<\frac{q}{2}$. Similarly, if $x>-q/2$, then $|x|<|x+q|$ and the same reasoning gives $\beta(1,x,m)<\overline{V}$ for $x>\frac{q}{2}$. ■

Proof of lemma 2: To prove the existence of the solution of equation (4), notice first that

$$\begin{aligned}
E[\mathbf{v}|H_t, \text{trader buys at } A_t] &= E[\mathbf{V}+\boldsymbol{\varepsilon}|H_t, \text{trader buys at } A_t] = \\
&= E[\mathbf{V}|H_t, \text{trader buys at } A_t] = \\
&= \overline{V}\Pr(\mathbf{V}=\overline{V}|H_t, \text{trader buys at } A_t) + \\
+\underline{V}(1-\Pr(\mathbf{V} &= \overline{V}|H_t, \text{trader buys at } A_t)) \in [\underline{V},\overline{V}],
\end{aligned}$$

Suppose that $\Pr(\text{trader buys at } A_t|H_t)>0$ for any $A_t \in [\underline{V},\overline{V}]$, in this case the probabilities in the previous expression are well defined, and so we have

$$\Pr(\mathbf{V}=\overline{V}|H_t, \text{ trader buys at } A_t) = \quad (9)$$
$$\frac{\pi_t\Big(pF(\Phi(\pi_t^h,A_t))+(1-p)F(\Phi(\pi_t^l,A_t))\Big)}{\pi_t\Big(pF(\Phi(\pi_t^h,A_t))+(1-p)F(\Phi(\pi_t^l,A_t))\Big)+(1-\pi_t)\Big((1-p)F(\Phi(\pi_t^h,A_t))+pF(\Phi(\pi_t^l,A_t))\Big)}.$$

[25] Simply remark that as u' is decreasing, for any real a and any positive real b, we have $cov(\boldsymbol{\varepsilon},u'(a+b\boldsymbol{\varepsilon})) = E[\boldsymbol{\varepsilon}u'(a+b\boldsymbol{\varepsilon})]<0$, thus $E[u(a+b\boldsymbol{\varepsilon})]$ is decreasing in b.

Now, according to (4), a market maker ask price is a fixed point of the mapping $g : \ \xi \longrightarrow E[\mathbf{v}|H_t$, trader buys at $\xi]$ that satisfies $g(\underline{V}) \geq \underline{V} > 0$ and $g(\overline{V}) \leq \overline{V}$. Assuming $F(.)$ is continuous, the existence of a solution of (4) comes from the mean value theorem. If traders' portfolios composition are discretely distributed (i.e. $F(.)$ is not continuous), then the existence of the solution of (4) can be obtained by allowing traders to use any mixed strategies when they are indifferent between trading or not. In this case $E[\mathbf{v}|H_t$, trader buys at $\xi]$ depends on the traders' mixed strategy, and so the mapping g becomes a upper hemicontinous convex valued correspondence that maps the set $[\underline{V}, \overline{V}]$ in itself. Such correspondence has a fixed point for the Kakutani's fixed point theorem. If Pr(trader buys at $A_t|H_t) = 0$ for some $A_t \in [\underline{V}, \overline{V}]$, then the probability (9) is not well defined. In this case it is possible to prove existence by completing the mapping g with an out-of-equilibrium-path belief that satisfies $E[\mathbf{v}|H_t$,trader buys at $A_t] = E[\mathbf{v}|H_t]$ when A_t is such that Pr(trader buys at $A_t|H_t) = 0$. This is equivalent to assume that in the economy there is a positive measure of liquidity traders that buy at any price and whose action provides no information on $\mathbf{V}$.

The relation $v^l(\pi_t) \leq A_t \leq v^h(\pi_t)$ is equivalent to $\pi_t^l \leq \Pr(\mathbf{V} = \overline{V}|H_t$, trader buys at $A_t) \leq \pi_t^h$. In order to prove $A_t \leq v^h(\pi_t)$, notice that (9) is maximum when $F(\Phi(\pi_t^l, A_t)) = 0$ and $F(\Phi(\pi_t^h, A_t)) = 1$, and in this case $\Pr(\mathbf{V} = \overline{V}|H_t$, trader buys at $A_t) = \pi_t^h$. Similarly, (9) is minimum when $F(\Phi(\pi_t^l, A_t)) = 1$ and $F(\Phi(\pi_t^h, A_t)) = 0$, and in this case $\Pr(\mathbf{V} = \overline{V}|H_t$, trader buys at $A_t) = \pi_t^l$, that proves $A_t \geq v^l(\pi_t)$. The proof for the bid quote can be obtained with a symmetric argument.■

Proof of proposition 2: We provide the proof for buy herding and contrarian behaviors. A similar argument applies to sell herding and contrarian behaviors. Consider an informed trader who received signal s and whose portfolio $(x, m) \in \Theta_{buy}^s$. First, notice that as $x \notin \Phi(\pi_0^s, A_0)$, initially the trader does not want to buy. Now, assume a positive history is realized and no information cascade occurs; then public belief π_t as well as π_t^s will be close to 1. Thanks to the presence of partially informative signals, this happens with positive probability. Notice that $\overline{V} = v(1) = \beta(1, -q/2, m) < \beta(1, x, m)$, where the second equality and the last inequality comes from result 1) in Lemma 1. Moreover, from Lemma 2 we know that $A_t \leq v(\pi_t^h)$. From the continuity of $\beta(.)$ and $v(.)$ with respect to π_t, we have that $\lim_{\pi_t \to 1} v(\pi_t^h) < \lim_{\pi_t \to 1} \beta(\pi_t^s, x, m)$ and therefore it results $A_t \leq v(\pi_t^h) < \beta(\pi_t^s, x, m)$ for π_t sufficiently close to 1. Thus, the trader eventually strictly prefers to buy and he engages in buy herd behavior. To prove that contrarian behavior occurs with positive probability, assume that a negative history is realized and no information cascade occurs. From lemma 1 we have $\underline{V} = v(0) = \beta(0, -q/2, m) < \beta(0, x, m)$. Thus, $\lim_{\pi_t \to 0} v(\pi_t^h) < \lim_{\pi_t \to 0} \beta(\pi_t^s, x, m)$, and therefore $A_t \leq v(\pi_t^h) < \beta(\pi_t^s, x, m)$ for π_t sufficiently close to 1. Thus, the trader eventually strictly prefers to buy and he engages in buy contrarian behavior. ■

Proof of proposition 3: Before proceeding to the proof some methodological remarks have to be done. Our analysis is based on the approach developed in Diamond and Verrecchia (1987). Following these authors and defining the ratio $L_t = \frac{\pi_t}{1-\pi_t}$, the informational efficiency properties of the market can be identified with the speed of convergence of L_t to infinity, if $\mathbf{V} = \overline{V}$, and to zero if $\mathbf{V} = \underline{V}$. Notice that $L_t = \Pi_{i=1}^{i=t} r(\mathcal{A}_i, \pi_i)$ where $r(\mathcal{A}_i, \pi_i) = \frac{\Pr(\mathcal{A}_i \,|\, V=\overline{V}, H_i)}{\Pr(\mathcal{A}_i \,|\, V=\underline{V}, H_i)}$. The ratio $r(\mathcal{A}_i, \pi_i)$ is interpreted as the informative content of action $\mathcal{A}_i$. If $r(\mathcal{A}_i, \pi_i) > 1$, then action $\mathcal{A}_i$ is a good news on the value of $\mathbf{V}$; $\mathcal{A}_i$ is a bad news if $r(\mathcal{A}_i, \pi_i) < 1$, and it provides no information if $r(\mathcal{A}_i, \pi_i) = 1$. In other words, the farther $r(\mathcal{A}_i, \pi_i)$ from 1, the larger the information content of $\mathcal{A}_i$ and so the larger the rate at which L_t changes. In Diamond and Verrecchia, the ratio $r(\mathcal{A}_t, \pi_t)$ does not depend on the current public belief π_t and the beliefs' speed of convergence to the truth can be computed explicitly[26]. In our setting, because of risk aversion, $r(\mathcal{A}_t, \pi_t)$ does depend on π_t and the study of information efficiency is more involved.

Proposition 3 is a direct consequence of the lemma:

Lemma 4: *Let denote $\mathcal{A}_i \in \{buy,\ sell,\ no\ trade\}$ the action at date $t = i$ and consider the ratio $r(\mathcal{A}_i, \pi_i) = \frac{\Pr(\mathcal{A}_i \,|\, V=\overline{V}, H_i)}{\Pr(\mathcal{A}_i \,|\, V=\underline{V}, H_i)}$. Then, under assumption 1, the following holds.*

$$1) \qquad \frac{(1-p)}{p} < r(\mathcal{A}_t, \pi_t) < \frac{p}{(1-p)},$$

2) *If there exist w and w' neighborhoods of $q/2$ and $-q/2$ respectively, such that $F((x,m)|x \in w) = F((x,m)|x \in w') = 0$, then there exist $\lambda \in (0,1)$ such that for any $\pi_t \notin (\lambda, 1-\lambda)$, $r(\mathcal{A}_t, \pi_t) = 1$.*

[26] Diamond and Verrechia define the speed of convergence to the true value of the asset by the expected length of time needed for the process L_t to cross a given threshold.

3) *If F is a continuous distribution function, then*

$$\lim_{\pi_t \to 1} r(\mathcal{A}_t, \pi_t) = \lim_{\pi_t \to 0} r(\mathcal{A}_t, \pi_t) = 1.$$

Proof: We prove Lemma 4 considering buy orders. Similar argument applies for sell orders and no trade.

Proof of result 1: Take $\mathcal{A}_t = buy$, then

$$\Pr(\mathcal{A}_t \,|\, \mathbf{V} = \overline{V}, H_t) = pF(\Phi(\pi_t^h, A_t)) + (1-p)F\left(\Phi(\pi_t^l, A_t)\right)$$

and therefore

$$r(\mathcal{A}_t, \pi_t) = \frac{pF(\Phi(\pi_t^h, A_t)) + (1-p)F(\Phi(\pi_t^l, A_t))}{(1-p)F(\Phi(\pi_t^h, A_t)) + pF(\Phi(\pi_t^l, A_t))}. \tag{10}$$

Now, remark that $r(\mathcal{A}_t, \pi_t)$ reaches its minimum for $F(\Phi(\pi_t^h, A_t)) = 0$ and $F(\Phi(\pi_t^l, A_t)) = 1$ and reaches its maximum for $F(\Phi(\pi_t^h, A_t)) = 1$ and $F(\Phi(\pi_t^l, A_t)) = 0$. This implies $r(\mathcal{A}_t, \pi_t)$ lies between $\frac{1-p}{p}$ and $\frac{p}{1-p}$. Finally, note that if traders were risk neutral, then $r(\mathcal{A}_t, \pi_t)$ would be equal to $\frac{p}{1-p}$. This proves that the presence of risk averse informed traders reduces the informative content of buy orders.

Proof of result 2: The sketch of the proof is as follows. If the distribution of the traders' portfolios is bounded away from $\frac{q}{2}$ and from $-\frac{q}{2}$ then, for any belief π_t sufficiently close to 1 or 0, any trader acts no matter the private signal. This implies $A_t = B_t = v(\pi_t)$ which in turn implies $r(\mathcal{A}_t, \pi_t) = 1$. We now give the formal proof. By assumption there exists $\delta > 0$ such that $\Pr\left((x, m) \,/\, x \in (-\delta - \frac{q}{2}, \delta - \frac{q}{2})\right) = 0$. As $\beta(1, -\frac{q}{2}, m) = \overline{V}$, we deduce that it exists $\eta_\delta > 0$ such that almost surely for any x, $\beta(1, x, m) \notin (\overline{V} - \eta_\delta, \overline{V} + \eta_\delta)$. Taking advantage from the continuity of β with respect to π, we deduce that there exist $\lambda_\beta > 0$ such that for any π in $(1 - \lambda_\beta, 1)$, almost surely for any x, $\beta(\pi, x, m) \notin (\overline{V} - \eta_\delta, \overline{V} + \eta_\delta)$. This implies either $\beta(\pi^h, x, m)$ and $\beta(\pi^l, x, m)$ are lower than $\overline{V} - \eta_\delta$ either $\beta(\pi^h, x, m)$ and $\beta(\pi^l, x, m)$ are larger than $\overline{V} + \eta_\delta$. Now, recall that $v(1) = \overline{V}$ and $v(\pi_t^l) < A_t < v(\pi_t^h)$. Thus, η_δ can be chosen such that $A_t \in (\overline{V} - \eta_\delta, \overline{V} + \eta_\delta)$. This implies, for almost surely any x, no matter their signal, traders either decide not to buy, either decide to buy. Thus traders' actions are not informative and so $A_t = v(\pi_t)$.

In a similar way it is possible to find $\lambda'_\beta \in (0, 1)$ and $\lambda_\alpha \in (0, 1)$, such that buy orders are not informative when $\pi_t < \lambda'_\beta$ and sell orders are not informative when $\pi_t \notin (\lambda_\alpha, 1-\lambda_\alpha)$. Then it is sufficient to choose $\lambda = \min\{\lambda_\beta, \lambda'_\beta, \lambda_\alpha\}$.

Proof of result 3: Take $\mathcal{A}_t = buy$. It can be easily checked that $F(\Phi(\pi, A))$ is not increasing in A. Indeed, all the agents that are willing to buy the asset at price A will also buy at price $A' < A$. And so for any given level of prior π, the probability of observing a buy order does not increase with the price at which market makers sell. Moreover, from lemma 2 we know that $v^l(\pi_t) \leq A_t \leq v^h(\pi_t)$. Thus, considering expression (10), the quantity $r(\mathcal{A}_t, \pi_t)$ has an upper and a lower bound:

$$\begin{aligned}\frac{pF(\Phi(\pi_t^h, v^h(\pi_t))) + (1-p)F(\Phi(\pi_t^l, v^h(\pi_t)))}{(1-p)F(\Phi(\pi_t^h, v^l(\pi_t))) + pF(\Phi(\pi_t^l, v^l(\pi_t)))} &\leq r(\mathcal{A}_t, \pi_t) \\ &\leq \frac{pF(\Phi(\pi_t^h, v^l(\pi_t))) + (1-p)F(\Phi(\pi_t^l, v^l(\pi_t)))}{(1-p)F(\Phi(\pi_t^h, v^h(\pi_t))) + pF(\Phi(\pi_t^l, v^h(\pi_t)))}.\end{aligned}$$

Moreover, from Lemma 1 and the continuity of F, we deduce that when π_t goes to 0 or 1, the upper and the lower bound of $r(\mathcal{A}_t, \pi_t)$ continuously converge to

$$\frac{F(\{(x,m)|x < -q/2\})}{F(\{(x,m)|x < -q/2\})} = 1.$$

This proves result 3).■

Proof of lemma 3: When trader's inventories are discretely distributed, conditions (5) and (6) state that q_b and q_s should be chosen such that the probability of observing trades from agents whose inventory is equal to $q_s/2$ or $-q_b/2$ is maximized. Indeed, for π_t sufficiently close to 1 or to 0, it results that $r(\mathcal{A}, \pi_t)$, the informative content of sell (resp. buy) orders is close to

$$\begin{aligned} r(sell, \pi_t) &= \frac{F(\Psi(\pi_t)) + \Pr(x = q_s/2))(1-p)}{F(\Psi(\pi_t)) + \Pr(x = q_s/2))p}, \\ \text{resp. } r(buy, \pi_t) &= \frac{F(\Phi(\pi_t)) + \Pr(x = -q_s/2))p}{F(\Phi(\pi_t)) + \Pr(x = -q_s/2))(1-p)} \end{aligned}$$

where $\Psi(\pi_t) = \{(x,m)|$ trader sells no matter his signal, given $H_t\}$, $\Phi(\pi_t) = \{(x,m)|$ trader buys no matter his signal, given $H_t\}$. Therefore the speed of convergence of beliefs to the truth will be maximized by choosing

$$\begin{aligned} q_s &= \max_q \Pr(x = q/2)), \\ q_b &= \max_q \Pr(-q/2). \end{aligned}$$

$\blacksquare$

Proof of proposition 4: Remark that similarly to the case where the size of trade is fixed, the informativeness of a trade $\mathcal{A}_i \in \{-N\delta, ..., -\delta, 0, \delta, ..., N\delta\}$ can be measured by the ratio $r(\mathcal{A}_i, \pi_i) = \frac{\Pr(\mathcal{A}_i \mid V=\overline{V}, H_i)}{\Pr(\mathcal{A}_i \mid V=\underline{V}, H_i)}$. We follow the same method used for the proof of proposition 3: Proposition 4 is a direct consequence of the following lemma.

Lemma 5: *Under assumption 1, if traders can buy and sell round lots of minimum size q, then*

1)

$$\frac{1-p}{p} < r(\mathcal{A}_t, \pi_t) < \frac{p}{1-p}$$

2) *If for any $n \in \{-N, -N+1, ...N\}$ there exist w^n neighborhoods of $q(n+\frac{1}{2})$, such that $F((x,m)|x \in w^n) = 0$, then there exist $\lambda > 0$ such that for any $\pi_t \notin (\lambda, 1-\lambda)$, $r(\mathcal{A}_t, \pi_t) = 1$.*

3) *If the distribution F is a continuous distribution function, then*

$$\lim_{\pi_t \to 1} r(\mathcal{A}_t, \pi_t) = \lim_{\pi_t \to 0} r(\mathcal{A}_t, \pi_t) = 1$$

Proof of Lemma 5: The proof of result 1 is the same than the proof of result 1 in lemma 4.

In order to prove result 2 consider a trader with belief π and portfolio (x, m), we denote with $z(\pi, x, m, P) \in \mathbf{R}$ his excess demand for the risky asset when he can trade any quantity at price P:

$$z = \arg\max_{D \in \mathbf{R}} E[u(\mathbf{v}(x+D) + m - PD)].$$

Note that as u is strictly concave and twice continuously differentiable $z(.)$ is continuous in all its arguments. Result 2 relies on the following lemma:

Lemma 6: *If π is 0 or 1, then i) $z(\pi, x, m, v(\pi)) = -x$; ii) a trader is indifferent between buying (selling) nq or $(n+1)q$ risky assets at price $v(\pi)$ if and only if $x = -q(n+\frac{1}{2})$ (resp. $x = q(n+\frac{1}{2})$).*

Proof: We provide the proof for $\pi = 0$. The proof for $\pi = 1$ uses the same arguments. If $\pi = 0$, then $v(0) = \underline{V}$ and

$$z(\pi, x, m, v(\pi)) = \arg\max_{D \in \mathbf{R}} E[u((\underline{V} + \boldsymbol{\varepsilon})(x+D) + m - \underline{V}D]$$

and so, $z(0, x, m, \underline{V}) = -x$ follows from the concavity of u and the symmetry of the distribution of $\boldsymbol{\varepsilon}$. To prove assertion (ii), consider $\mathbf{W} = \underline{V}x + m + \boldsymbol{\varepsilon}(x + nq)$ and $\mathbf{W}^* = \underline{V}x + m + \boldsymbol{\varepsilon}(x + (n+1)q)$ and note that for $x = -q(n + 1/2)$ we have $\mathbf{W} = \underline{V}x + m - \boldsymbol{\varepsilon}q/2$ and $\mathbf{W}^* = \underline{V}x + m + \boldsymbol{\varepsilon}q/2$. Observing that $\boldsymbol{\varepsilon}$ is symmetrically distributed with mean 0 it follows $E[u(\mathbf{W}^*)] = E[u(\mathbf{W})]$. Thus, result ii).■

Now, in order to prove result 2 in lemma 5, it is sufficient to show that when π_t is close to 0 or to 1, informed trader's orders do not depend on their private signal. Notice first that as long as $p \in (1/2, 1)$, when π_t is sufficiently close to 1 or 0 an informative signal affects slightly the informed trader belief, i.e. $\lim_{\pi_t \to 1}(\pi_t^h - \pi_t^l) = \lim_{\pi_t \to 0}(\pi_t^h - \pi_t^l) = 0$. From the continuity of traders' demand in the belief π, we have that for π_t sufficiently close to 1 or 0, it results from lemma 6

$$|z(\pi_t^h, x, m, P) - z(\pi_t^l, x, m, P)| < q.$$

Thus, for π_t sufficiently close to 1 or 0, an order coming from a informed trader changes with his signal only if before receiving the signal the trader was almost indifferent between trading nq or $(n+1)q$ for some $n \in \mathbf{N}$. This is because the discreteness in the set of informed traders actions. However, from lemma 6, if π_t is sufficiently close to 1 or 0 then a trader is indifferent between trading nq or $(n+1)q$ only if his inventory x is sufficiently close to $q(n + \frac{1}{2})$ or $-q(n + \frac{1}{2})$. That never happens under the assumption we made on $F(.)$.

Given result 2, the proof of result 3 is equivalent to the proof of result 3 in lemma 4: as long as π_t approaches 0 or 1 the proportion of informative trade shirks reducing the speed of convergence of the public belief to the truth. ■

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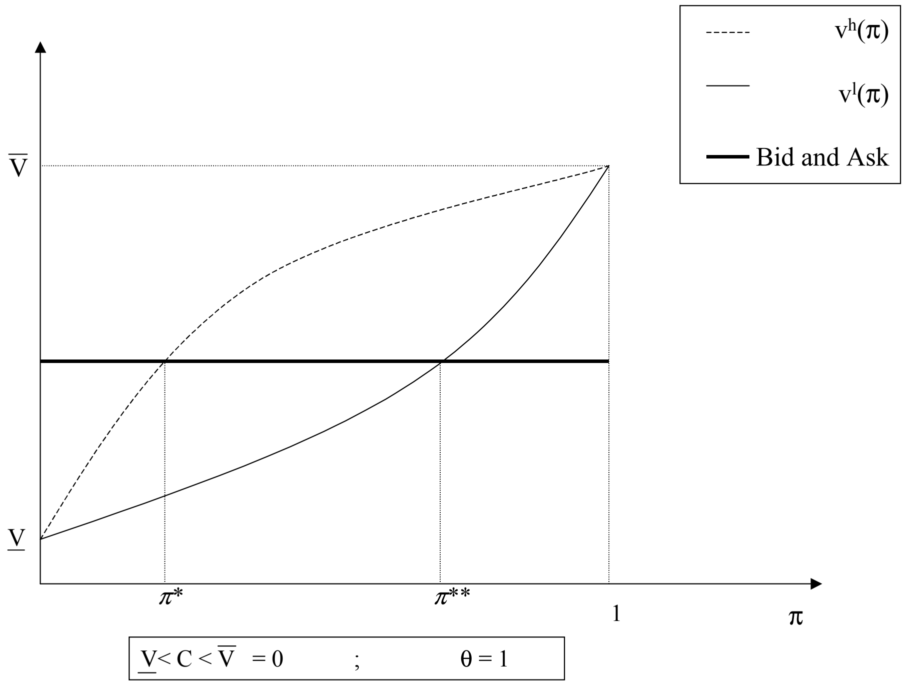

Figure 1: Traders' orders are informative as long as $\pi_t \in (\pi^*, \pi^{**})$. When $\pi_t < \pi^{**}$, all traders buy, whereas when $\pi_t > \pi^*$, all traders sell.

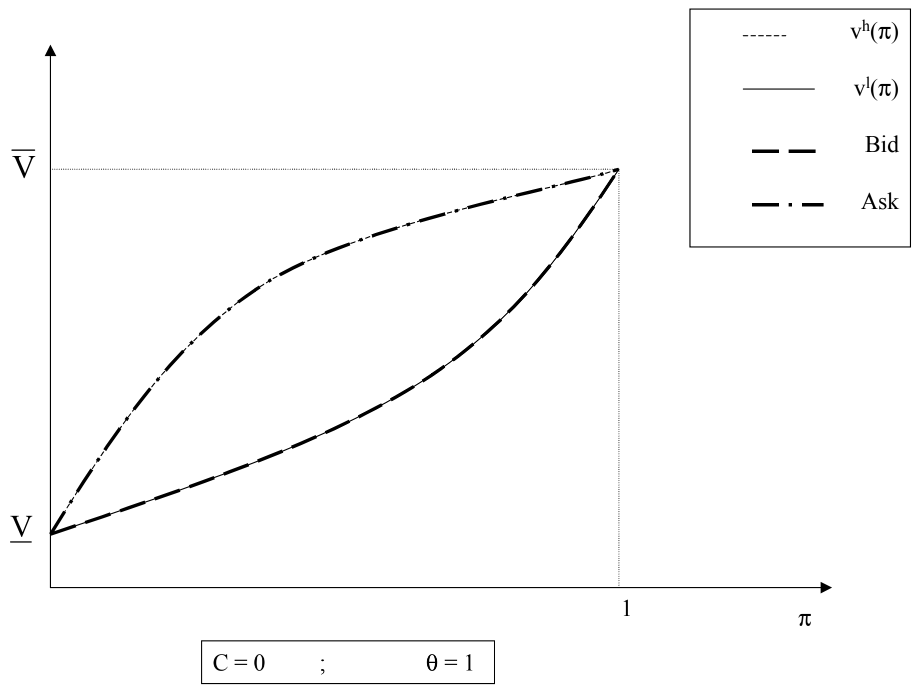

Figure 2: When $C = 0$ and $\theta = 1$, and a traders receive a signal $s = h$ ($s = l$), his valuation for the object is equal to the ask price (resp. bid price) no matter the past history of trade. Thus, traders' actions always reflect the sign of their signal and herding is impossible.

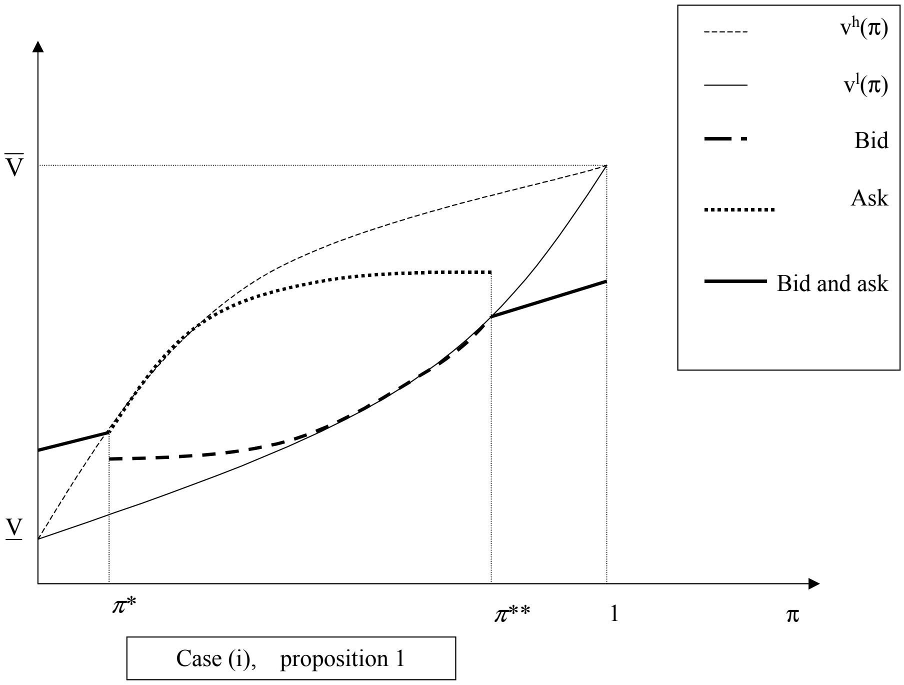

Figure 3: Traders' orders are informative as long as $\pi_t \in (\pi^*, \pi^{**})$. When a negative history is observed and $\pi_t < \pi^*$, all traders' valuation are lower than the bid price and sell herding starts. Note that public posterior beliefs stop to evolve, consequently ask and bid prices are identical and linear in π_t. Similarly, when $\pi_t > \pi^{**}$, buy herding occurs.

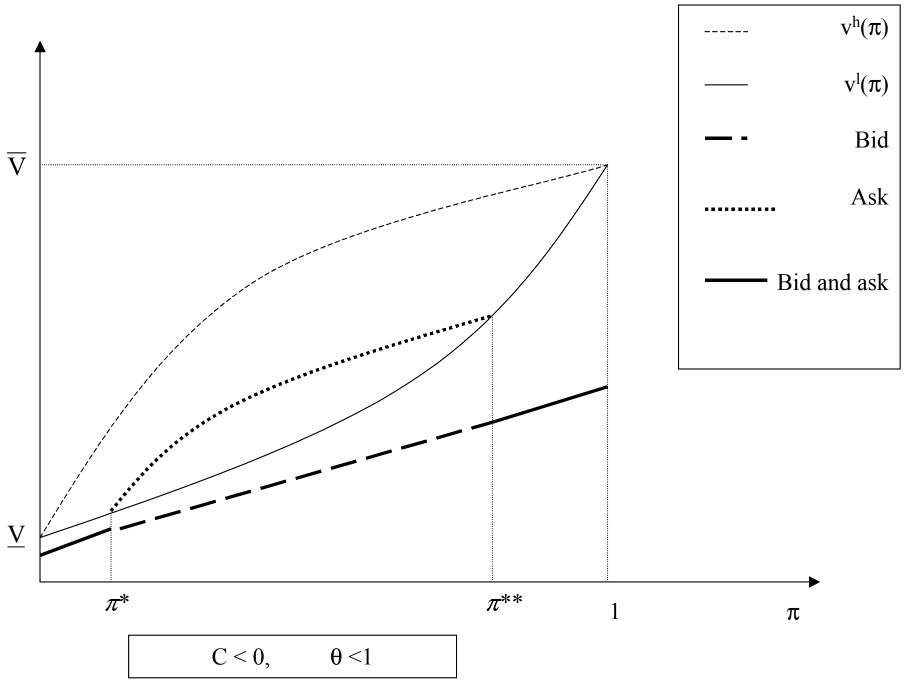

Figure 4: When the initial prior is between π^* and π^{**}, an informed trader buys if and only if he receives a bullish signal and he does not trade if his signal is bearish. However an informed trader always buys when $\pi_t < \pi^*$ or $\pi_t > \pi^{**}$. It follows that, if $\pi_0 \in (\pi^*, \pi^{**})$ a sequence of buy orders can prime buy herding behavior, while a sequence of no-trade can trigger buy contrarian behavior.