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Kevin J. Lansing
Federal Reserve Bank of San Francisco

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Some New Variance Bounds for Asset Prices: A Comment*

Kevin J. Lansing†
Federal Reserve Bank of San Francisco

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Abstract

Engel (2005) derives a theoretical variance inequality involving the change in equilibrium stock prices $Var(\Delta p_t)$. Assuming that stock prices are "cum-dividend"and that investors are risk neutral, he shows that $Var(\Delta p_t)$ must be greater than or equal to the variance of the "perfect foresight"(or "ex post rational") price change $Var(\Delta p_t^*)$, where p_t^* is computed from the discounted stream of subsequent realized dividends. This paper expands the analysis to consider "ex-dividend" prices and risk aversion in a standard Lucas-type asset pricing model. I show that the direction of the price-change variance inequality can be reversed, depending on the values assigned to some key parameters of the model, namely the dividend AR(1) parameter, the investor's subjective time discount factor, and the coefficient of relative risk aversion. Overall, the results demonstrate that the present-value model of stock prices does not impose theoretical bounds on price-change volatility except in some special cases.

Keywords: *volatility bounds, variance bounds, asset price variability.*

JEL Classification: E44, G12, G14.

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† Research Department, Federal Reserve Bank of San Francisco, P.O. Box 7702, San Francisco, CA 94120-7702, (415) 974-2393, FAX: (415) 977-4031, email: kevin.j.lansing@sf.frb.org, homepage: www.frbsf.org/economics/economists/klansing.html

1 Introduction

In theory, the price of a stock represents the market's consensus forecast of the discounted sum of future dividends that will accrue to the owner of the stock. One characteristic of a rational forecast is that it should be less variable than the object being forecasted. Numerous empirical studies starting with Shiller (1981) and LeRoy and Porter (1981) have argued that this rationality principle appears to be violated in the case of stock prices. In particular, observed stock prices (the forecast) appear to be much more variable than the discounted sum of subsequent realized dividends (the object being forecasted).[1]

As noted originally by Marsh and Merton (1986) and Kleidon (1986), empirical tests based on the observed volatility of stock prices are misspecified if prices are nonstationary. The unconditional variance of the fundamental equilibrium stock price does not exist when real dividends are growing over time, as in long-run U.S. data. To retain validity, empirical tests for excess volatility must be applied to stationary variables, such as the price-dividend ratio, the price change, or the equity return.[2]

Engel (2005) derives a theoretical variance inequality involving the *change* in stock prices—a variable that remains stationary even when dividends exhibit a unit root.[3] Assuming that stock prices are "cum-dividend" and that investors are risk neutral, he shows that the variance of the equilibrium price change must be greater than or equal to the variance of the "perfect foresight" (or "ex post rational") price change computed from the discounted stream of subsequent realized dividends. Also assuming risk neutrality, LeRoy (1984) previously demonstrated a similar result numerically using a calibrated model where stock prices are "ex-dividend."[4] The price-change variance is closely related to the conditional variance of the price level. Either statistic can be interpreted as a measure of the "smoothness" of the underlying price series. Indeed, LeRoy (1984, p. 186) provides numerical examples where the conditional variance of the equilibrium stock price is greater than the conditional variance of the perfect foresight price. The foregoing results would appear to help reconcile the high volatility of observed stock prices with the smooth behavior of discounted realized dividends.

This paper expands the foregoing analysis to consider both "ex-dividend" prices and risk aversion in a standard Lucas (1978) type asset pricing model. I show that the direction of the price-change variance inequality can be reversed, depending on the values assigned to

[1] For an overview of this large literature, see Gilles and LeRoy (1991), Shiller (2003), and LeRoy (2010).

[2] West (1988a, p. 641) summarizes the various assumptions made in the literature regarding the stochastic process for dividends and prices.

[3] Engel assumes that dividends evolve as an arithmetic random walk or remain stationary.

[4] In LeRoy's model, dividends are highly persistent but stationary. This result is also discussed by Gilles and LeRoy (1991, p. 771).

some key parameters of the model, namely the dividend AR(1) parameter ρ, the investor's subjective time discount factor β, and the coefficient of relative risk aversion α.

Following LeRoy (1984) and Engel (2005), I initially consider an economy where the representative investor is risk neutral ($\alpha = 0$). Dividends are assumed to follow an arithmetic AR(1) process that allows for a unit root as a special case. For this environment, it is possible to derive exact analytical expressions for both the variance of the equilibrium price change $Var\left(\Delta p_t\right)$ and the variance of the perfect foresight price change $Var\left(\Delta p_t^*\right)$. I show that $Var\left(\Delta p_t\right)$ can be greater or less than $Var\left(\Delta p_t^*\right)$, depending on the values assigned to ρ and β. The two variance statistics are exactly equal when the parameters satisfy the condition $\rho\left(1+\beta\right) = 1$. For a typical model calibration where dividends are a close to a random walk and the discount factor is close to unity, we have $\rho\left(1+\beta\right) > 1$ which in turn yields $Var\left(\Delta p_t\right) > Var\left(\Delta p_t^*\right)$, confirming the results obtained by LeRoy (1984) and Engel (2005). LeRoy's numerical computations employ values for ρ and β which satisfy the condition $\rho\left(1+\beta\right) > 1$. Engel employs a setup where p_t is defined as a cum-dividend price rather than an ex-dividend price. His model can be interpreted as imposing the parameter restriction $\rho\beta \simeq 1$ such that the condition $\rho\left(1+\beta\right) > 1$ is satisfied. If dividends are less persistent or the future is more heavily discounted such that $\rho\left(1+\beta\right) < 1$, then the variance inequality is reversed, yielding $Var\left(\Delta p_t\right) < Var\left(\Delta p_t^*\right)$.

The explanation for the variance inequality reversal is linked to the discounting mechanism. The parameters ρ and β both affect the degree to which future dividend innovations (which are known ex post) influence the perfect foresight price p_t^* via discounting from the future to the present. The future dividend innovations have no effect the equilibrium stock price p_t because the rational expected value of future innovations is zero. When dividends are highly persistent and the investor's discount factor is close to unity such that $\rho\left(1+\beta\right) > 1$, the discounting weights applied to successive future dividend innovations decay gradually. By taking the first-difference of the perfect foresight price series, the terms involving future innovations tend to cancel out. However, the *current* dividend innovation continues to have a impact—but one that acts differently on the current- versus prior-period values of the perfect foresight price. This differential impact of the current innovation serves to shrink the magnitude of Δp_t^*, resulting in a lower value for $Var\left(\Delta p_t^*\right)$ relative to $Var\left(\Delta p_t\right)$. In contrast, when $\rho\left(1+\beta\right) < 1$, the discounting weights applied to successive future innovations decay rapidly, so these terms do not tend to cancel out when taking the first difference of the perfect foresight price series. In this case, the positive impact of the future innovations on the variance dominates the negative impact of the current innovation, resulting in a higher value for $Var\left(\Delta p_t^*\right)$ relative to $Var\left(\Delta p_t\right)$.

In the model with risk aversion, I adopt a more-general specification for dividends. I specify the growth rate of dividends (and consumption) as an ARMA$(1, 1)$ process that allows for either a stochastic or a deterministic growth trend. Within this framework, I derive approximate analytical expressions for the variance of the equilibrium price change (in logarithms), i.e., $Var\left[\Delta \log\left(p_t\right)\right]$, and its perfect foresight counterpart $Var\left[\Delta \log\left(p_t^*\right)\right]$. I show that $Var\left[\Delta \log\left(p_t\right)\right]$ can be greater or less than $Var\left[\Delta \log\left(p_t^*\right)\right]$, depending on the value assigned to the coefficient of relative risk aversion α. Assuming plausible parameter values for the dividend growth process, I show that two variance statistics are exactly equal when $\alpha = 1$, representing logarithmic utility. When $\alpha < 1$, the model yields $Var\left[\Delta \log\left(p_t\right)\right] > Var\left[\Delta \log\left(p_t^*\right)\right]$, analogous to the risk-neutral result obtained by LeRoy (1984) and Engel (2005). When the $\alpha > 1$, the variance inequality is reversed such that $Var\left[\Delta \log\left(p_t\right)\right] < Var\left[\Delta \log\left(p_t^*\right)\right]$.

When the investor's utility function is logarithmic, the income and substitution effects of dividend growth innovations exactly cancel, regardless of the stochastic process for dividend growth. As a result, both the equilibrium price-dividend ratio p_t/d_t and the perfect foresight price-dividend ratio p_t^*/d_t are constant under logarithmic utility. Given that the price-dividend ratios are constant, any variation in the two price series p_t and p_t^* must be driven solely by variations in the common stream of exogenous dividends. This explains why $Var\left[\Delta \log\left(p_t\right)\right] = Var\left[\Delta \log\left(p_t^*\right)\right]$ under log utility.

The explanation for the variance inequality reversal in the model with risk aversion is also linked to the discounting mechanism. When the risk aversion coefficient is below unity, the stochastic discount factors applied to successive future dividend growth innovations decay gradually. By taking the first-difference of the perfect foresight price series (in logarithms), the terms involving future innovations tend to cancel out. The differential impact of the current innovation serves to shrink the magnitude of $\Delta \log\left(p_t^*\right)$, thus resulting in a lower value for $Var\left[\Delta \log\left(p_t^*\right)\right]$ relative to $Var\left[\Delta \log\left(p_t\right)\right]$. When the risk aversion coefficient is above unity, the stochastic discount factors applied to successive future innovations decay rapidly, so these terms do not tend to cancel out when taking the first difference of the log price series. In this case, the positive impact of the future innovations on the variance dominates the negative impact of the current innovation, resulting in a higher value for $Var\left[\Delta \log\left(p_t^*\right)\right]$ relative to $Var\left[\Delta \log\left(p_t\right)\right]$.

The fact that a variance inequality involving price changes (or log price changes) can reverse direction depending on parameters should perhaps not be surprising. Unlike the situation with prices themselves, the equilibrium price-change cannot be represented as a conditional forecast of the price-change that would prevail under perfect foresight. In other words, while the present-value model implies $p_t = E_t p_t^*$, this relationship between prices does not imply $\Delta p_t = E_t \Delta p_t^*$ nor does it imply $\Delta \log(p_t) = E_t \Delta \log(p_t^*)$. It turns out that the present-value model does not impose theoretical bounds on price-change volatility except in some special cases.

2 Asset Pricing Model

Equity shares are priced using the frictionless pure exchange model of Lucas (1978). There is a representative investor who can purchase shares to transfer wealth from one period to another. Each share pays an exogenous stream of stochastic dividends in perpetuity. The investor's problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\alpha} - 1}{1-\alpha} \right], \tag{1}$$

subject to the budget constraint

$$c_t + p_t s_t = (p_t + d_t)\, s_{t-1}, \qquad c_t,\ s_t > 0 \tag{2}$$

where c_t is the investor's consumption in period t and α is the coefficient of relative risk aversion (the inverse of the intertemporal elasticity of substitution), d_t is the dividend, and s_t is the number of shares held in period t. The symbol E_t is the mathematical expectation operator, conditional on information available at time t. For simplicity, throughout the paper, I assume that the representative investor's information set consists only of current and past dividends. The symbol p_t denotes the equilibrium ex-dividend price conditional on the investor's information.[5]

The first-order condition that governs the investor's share holdings is given by

$$p_t = E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} (p_{t+1} + d_{t+1}) \right]. \tag{3}$$

The first-order condition can be iterated forward to substitute out p_{t+1+j} for $j = 0, 1, 2, \ldots$ Applying the law of iterated expectations and imposing a transversality condition yields the following expression for the fundamental equilibrium stock price

$$p_t = E_t \sum_{j=1}^{\infty} M_{t+j}\, d_{t+j}, \tag{4}$$

[5] Specifically, the price p_t corresponds to the information set $H_t = \{d_t,\ d_{t-1},\ d_{t-2}, \ldots\}$. Engel (2005) and West (1988b) make a distinction between H_t and an unspecified larger information set I_t that contains at least H_t. They use the symbol $\widehat{p}_t$ for the price conditional on H_t and the symbol p_t for the price conditional on I_t. For simplicity, I assume that H_t and I_t are identical here such that $\widehat{p}_t = p_t$.

where $M_{t+j} \equiv \beta^j \left(c_{t+j}/c_t\right)^{-\alpha}$ is the stochastic discount factor. The perfect foresight price is given by

$$p_t^* = \sum_{j=1}^{\infty} M_{t+j}\, d_{t+j.} \tag{5}$$

Equity shares are assumed to exist in unit net supply. Market clearing therefore implies $s_t = 1$ for all t. Substituting this equilibrium condition into the budget constraint (2) yields, $c_t = d_t$ for all t.

3 Risk-Neutral Investor

When $\alpha = 0$, the representative investor is risk neutral and the pricing equations can be written as follows

$$\begin{aligned} p_t &= E_t\, \beta \left(p_{t+1} + d_{t+1}\right) \\ &= E_t \left\{\beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ...\right\}, \end{aligned} \tag{6}$$

$$\begin{aligned} p_t^* &= \beta \left(p_{t+1}^* + d_{t+1}\right) \\ &= \beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + ..., \end{aligned} \tag{7}$$

To facilitate an analytical solution for both p_t and p_t^*, I assume that dividends are governed by the following AR(1) process

$$d_{t+1} = \rho\, d_t + (1-\rho)\, \overline{d} + \varepsilon_{t+1}, \qquad \begin{array}{l} \varepsilon_{t+j} \sim N\left(0, \sigma_\varepsilon^2\right), \\ |\rho| \leq 1, \end{array} , \tag{8}$$

which allows for a unit root when $\rho = 1$.

Repeated substitution of equation (8) into equation (6) and then imposing $E_t \varepsilon_{t+j} = 0$ for $j = 1, 2, ...$ yields the following expression for the fundamental equilibrium stock price

$$\begin{aligned} p_t &= d_t \left\{\beta\rho + (\beta\rho)^2 + (\beta\rho)^3 + ...\right\} + \overline{d} \left\{\beta(1-\rho) + \beta^2\left(1-\rho^2\right) + \beta^3\left(1-\rho^3\right) + ...\right\}, \\ &= d_t \left[\frac{\beta\rho}{1-\beta\rho}\right] + \overline{d} \left[\frac{\beta(1-\rho)}{(1-\beta)(1-\beta\rho)}\right], \end{aligned} \tag{9}$$

which shows that the equilibrium price-dividend ratio p_t/d_t is constant when $\rho = 1$ or $\overline{d} = 0$.

Repeated substitution of equation (8) into equation (7) yields the following expression for the perfect foresight price

$$p_t^* = \underbrace{d_t \left[\frac{\beta\rho}{1-\beta\rho}\right] + \overline{d} \left[\frac{\beta(1-\rho)}{(1-\beta)(1-\beta\rho)}\right]}_{p_t} + \frac{\beta}{1-\beta\rho} \sum_{j=1}^{\infty} \beta^{j-1} \varepsilon_{t+j}, \tag{10}$$

which satisfies the relationship $p_t = E_t p_t^*$. Since p_t is the rational forecast of p_t^*, Shiller (1981) argued that market efficiency requires $Var\left(p_t\right) \leq Var\left(p_t^*\right)$. Marsh and Merton (1986) and Kleidon (1986) later pointed out that neither variance will exist if dividends (and hence prices) are nonstationary. Shiller's derivation assumed that prices and dividends were rendered stationary by removing a common deterministic time trend. However, in the present model with $\rho = 1$, the trend in prices and dividends is stochastic, so Shiller's detrending procedure would not eliminate the unit root. To analyze the $\rho = 1$ case, we can detrend prices by taking the first difference of the respective price series. Taking the first difference of equation (10) yields

$$\Delta p_t^* = \Delta p_t - \left[\frac{\beta}{1-\beta\rho}\right]\varepsilon_t + \left[\frac{\beta\left(1-\beta\right)}{1-\beta\rho}\right]\left[\varepsilon_{t+1} + \beta\varepsilon_{t+2} + \beta^2\varepsilon_{t+3} + \ldots\right]. \qquad (11)$$

Proposition 1. When the representative investor is risk neutral and dividends are governed by the AR(1) process (8), then:

$$Var\left(\Delta p_t\right) \geq Var\left(\Delta p_t^*\right) \quad \text{if} \quad \rho\left(1+\beta\right) \geq 1,$$

$$Var\left(\Delta p_t\right) < Var\left(\Delta p_t^*\right) \quad \text{if} \quad \rho\left(1+\beta\right) < 1.$$

Proof: Taking the variance of both sides of equation (11) yields

$$\begin{aligned} Var\left(\Delta p_t^*\right) &= Var\left(\Delta p_t\right) + \frac{\beta^2\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2} - \frac{2\beta\, Cov\left(\Delta p_t, \varepsilon_t\right)}{\left(1-\beta\rho\right)} \\ &+ \frac{\beta^2\left(1-\beta\right)^2\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2}\left[1+\beta^2+\beta^4+\beta^6+\ldots\right]. \end{aligned}$$

From equations (8) and (9), we have $Cov\left(\Delta p_t, \varepsilon_t\right) = \beta\rho\,\sigma_\varepsilon^2/\left(1-\beta\rho\right)$. The infinite sum inside the square brackets of the above expression is equal to $1/\left(1-\beta^2\right)$. Inserting these results into the variance expression and then simplifying yields the following result

$$Var\left(\Delta p_t^*\right) = Var\left(\Delta p_t\right) + \frac{2\beta^2\sigma_\varepsilon^2}{\left(1-\beta\rho\right)^2\left(1+\beta\right)}\left[1-\rho\left(1+\beta\right)\right],$$

which shows that the direction of the variance inequality is governed by the sign of the term $\left[1-\rho\left(1+\beta\right)\right]$. ■

Proposition 1 shows that when $\rho\left(1+\beta\right) > 1$, we have $Var\left(\Delta p_t\right) > Var\left(\Delta p_t^*\right)$, confirming the results obtained by LeRoy (1984) and Engel (2005). But when $\rho\left(1+\beta\right) < 1$, the variance inequality is reversed such that $Var(\Delta p_t) < Var(\Delta p_t^*)$. The numerical examples in LeRoy (1984, p. 186) employ the values $\rho \in (0.8,\ 0.99)$ and $\beta = 0.9$, which satisfy the condition $\rho\,(1+\beta) > 1$.

Engel (2005) employs a cum-dividend pricing equation where the stock price at time t includes a guaranteed dividend, unlike the Lucas-model setup where prices are ex-dividend. Specifically, Engel employs the following asset pricing equation:

$$p_t = d_t + E_t\,\beta p_{t+1}. \tag{12}$$

The above pricing equation can be obtained from the ex-dividend pricing equation (6) by substituting in for d_{t+1} from (8) and then imposing $\rho\beta \simeq 1$. By effectively imposing $\rho\beta \simeq 1$, the cum-dividend pricing equation ensures that the condition $\rho\,(1+\beta) > 1$ is satisfied.

Starting instead with cum-dividend pricing equation (12) and its perfect foresight counterpart and then following the same methodology described above, it is straightforward to show that the result is

$$Var(\Delta p_t^*) \;=\; Var(\Delta p_t) \;-\; \frac{2\beta\sigma_\varepsilon^2}{(1-\beta\rho)^2\,(1+\beta)} \tag{13}$$

which implies $Var(\Delta p_t) > Var(\Delta p_t^*)$, in agreement with Proposition 2 in Engel (2005).[6]

The reason for the variance inequality reversal can be understood from equations (10) and (11). When the parameters ρ and β are both close to unity, the discounting weights applied to future dividend innovations in the solution for p_t^* decay gradually, as shown by equation (10). By taking the first-difference of the p_t^* series to obtain Δp_t^*, the terms involving future innovations tend to cancel each other out, as can be seen from equation (11), where these terms are multiplied by the coefficient $\beta(1-\beta)/(1-\beta\rho)$. However, equation (11) shows that the *current* dividend innovation ε_t continues to have a strong impact on Δp_t^*. A positive value of ε_t serves to shrink Δp_t^* relative to Δp_t, whereas positive *future* innovations ε_{t+j}, $j = 1, 2, ...$serve to magnify Δp_t^* relative to Δp_t. The negative influence of ε_t on the variance of Δp_t^* dominates the positive influence of ε_{t+j}, $j = 1, 2, ...$when the discounting weights in the solution for p_t^* decay sufficiently gradually, as measured by the condition $\rho\,(1+\beta) > 1$.

Given that a typical calibration satisfies $\rho\,(1+\beta) > 1$, the model would predict $Var(\Delta p_t) > Var(\Delta p_t^*)$, in agreement with the results obtained by LeRoy (1984) and Engel (2005).

[6] Proposition 1 in West (1988b) derives a related variance inequality involving price changes that also makes use of the cum-dividend pricing equation (12).

4 Risk Averse Investor

In this section, I allow for risk aversion and consider a more-general specification for dividends I assume that the growth rate of dividends $x_t \equiv \log(d_t/d_{t-1})$ is governed by the following ARMA (1,1) process

$$x_{t+1} = \overline{x} + \rho(x_t - \overline{x}) + \varepsilon_{t+1} - \phi\varepsilon_t, \qquad \begin{array}{l} \varepsilon_{t+j} \sim N(0, \sigma_\varepsilon^2), \\ |\rho| < 1, \\ \phi \in \{0, 1\}, \end{array} \tag{14}$$

where $\phi = 0$ implies a stochastic trend in log dividends, while $\phi = 1$ implies a deterministic trend.[7] The unconditional moments of dividend growth are given by

$$E(x_t) = \overline{x}, \tag{15}$$

$$Var(x_t) = \frac{\left(1 + \phi^2 - 2\rho\phi\right) \sigma_\varepsilon^2}{1 - \rho^2}, \tag{16}$$

$$Corr(x_t,\, x_{t-1}) = \frac{(\rho - \phi)(1 - \rho\phi)}{1 + \phi^2 - 2\rho\phi}. \tag{17}$$

The fundamental equilibrium price-dividend ratio is denoted by $y_t \equiv p_t/d_t$. The perfect foresight counterpart is denoted by $y_t^* \equiv p_t^*/d_t$. By substituting the equilibrium condition $c_t = d_t$ into the first-order condition (3), the first-order condition and its perfect foresight counterpart can now be written as

$$y_t = E_t\left[\beta \exp(\theta x_{t+1})(y_{t+1} + 1)\right], \tag{18}$$

$$y_t^* = \beta \exp(\theta x_{t+1})\left(y_{t+1}^* + 1\right), \tag{19}$$

where $\theta \equiv 1 - \alpha$.

The corresponding expressions for the log price change can be written as follows:

$$\Delta \log(p_t) = \Delta \log(y_t) + x_t, \tag{20}$$

$$\Delta \log(p_t^*) = \Delta \log(y_t^*) + x_t, \tag{21}$$

[7] The $\phi = 1$ case corresponds to the following specification for dividends: $\log(d_t) = \rho \log(d_{t-1}) + \mu t + \varepsilon_t$, where μt is the the deterministic time trend. Lagging this equation by one period and then subtracting one equation from the other yields equation (14) with $\phi = 1$, where $(1 - \rho)\,\overline{x} = \mu$.

4.1 Fundamental Equilibrium Solution

The fundamental equilibrium is obtained by solving the first-order condition (18), subject to the dividend growth process (14). To facilitate an approximate analytical solution, it is convenient to transform the first-order condition using a nonlinear change of variables to obtain

$$z_t = \beta \exp(\theta x_t)\left[E_t z_{t+1} + 1\right], \tag{22}$$

where $z_t \equiv \beta \exp(\theta x_t)(y_t + 1)$. Under this formulation, z_t represents a composite variable that depends on both the growth rate of dividends and the price-dividend ratio. Equation (22) shows that the value of z_t in period t depends on the investor's conditional forecast of that same variable. By making use of the definition of z_t, the first-order condition (18) can be written as $y_t = E_t z_{t+1}$. Hence, the fundamental equilibrium price-dividend ratio is simply the rational forecast of the composite variable z_{t+1}.

The following proposition presents an approximate analytical solution for z_t.

Proposition 2. *An approximate analytical solution for the fundamental equilibrium value of the composite variable $z_t \equiv \beta \exp(\theta x_t)(y_t + 1)$ is given by*

$$z_t = \exp\left[a_0 + a_1(x_t - \overline{x}) + a_2\,\varepsilon_t\right],$$

where a_1 and a_2 solve the following system of nonlinear equations

$$a_1 = \frac{\theta}{1 - \rho\beta \exp\left[\theta\,\overline{x} + \frac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]},$$

$$a_2 = -\,a_1 \phi\beta \exp\left[\theta\,\overline{x} + \frac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]$$

and $a_0 \equiv E[\log(z_t)]$ is given by

$$a_0 = \log\left\{\frac{\beta \exp(\theta\,\overline{x})}{1 - \beta \exp\left[\theta\,\overline{x} + \frac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]}\right\},$$

provided that $\beta \exp\left[\theta\,\overline{x} + \frac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] < 1$.

Proof: See appendix.

Two values of a_1 satisfy the nonlinear system in Proposition 2. The inequality restriction selects the value of a_1 with lower magnitude to ensure that the point of approximation

$\exp(E[\log(z_t)])$ is positive. Given the solution for the composite variable z_t, we can recover the fundamental equilibrium price-dividend ratio as follows

$$y_t \;=\; E_t z_{t+1} \;=\; \exp\left[a_0 + a_1\rho\,(x_t - \overline{x}) - a_1\phi\,\varepsilon_t + \frac{1}{2}\,(a_1 + a_2)^2\,\sigma_\varepsilon^2\right]. \tag{23}$$

Lansing (2010) compares the approximate solution from Proposition 2 to the exact theoretical solution derived by Burnside (1998) for the case of $\phi = 0$, which implies a stochastic trend in log dividends. The approximate solution is extremely accurate for low and moderate levels of risk aversion $(\alpha \simeq 2)$. But even for high levels of risk aversion $(\alpha \simeq 10)$, the approximation error for y_t remains below 5 percent.

As shown in the appendix, the approximate solution can be used to derive the following unconditional moments of the equilibrium asset pricing variables:

$$E[\log(y_t)] \;=\; \log\left\{\frac{\beta\exp\left[\theta\,\overline{x} + \frac{1}{2}\,(a_1+a_2)^2\,\sigma_\varepsilon^2\right]}{1 - \beta\exp\left[\theta\,\overline{x} + \frac{1}{2}\,(a_1+a_2)^2\,\sigma_\varepsilon^2\right]}\right\}, \tag{24}$$

$$Var[\log(y_t)] \;=\; \frac{[a_1(\rho-\phi)]^2\,\sigma_\varepsilon^2}{1-\rho^2}, \tag{25}$$

$$E[\Delta\log(p_t)] \;=\; \overline{x} \tag{26}$$

$$Var[\Delta\log(p_t)] \;=\; \left[(a_1\rho)^2 + (1+a_1\rho)^2 - 2a_1\rho\,(1+a_1\rho)\,Corr(x_t,\,x_{t-1})\right] Var(x_t)$$

$$-2a_1\phi\left[a_1(\rho-\phi) + (1-\rho+\phi)(1+a_1\rho)\right]\,\sigma_\varepsilon^2 \tag{27}$$

where $Var(x_t)$ and $Corr(x_t,\,x_{t-1})$ are given by equations (16) and (17).

From Proposition 2, the magnitude of the solution coefficient a_1 increases as the risk aversion coefficient α rises above unity. An increase in the magnitude of a_1 serves to magnify the volatility of the price-dividend ratio, as shown by equation (25).[8] For the case of log utility $(\alpha = 1)$, we have $\theta = 1 - \alpha = 0$, such that $a_1 = a_2 = 0$. In this case, the fundamental equilibrium price-dividend ratio is constant at the value $y_t = \beta/(1-\beta)$. This result obtains because the income and substitution effects of an innovation to dividend growth are exactly offsetting.

[8] LeRoy and LaCivita (1981) demonstrate that risk aversion magnifies the volatility of the price-dividend ratio in a Lucas-type model where the level of dividends is governed by a two-state Markov process.

4.2 Perfect Foresight Solution

The perfect foresight price-dividend ratio is governed by equation (19), which is a nonlinear law of motion. To derive analytical expressions for the perfect foresight variances, I approximate equation (19) using the following log-linear law of motion (details are contained in the appendix):

$$\log\left(y_t^*\right) - E\left[\log\left(y_t^*\right)\right] \;\simeq\; \theta\left(x_{t+1} - \overline{x}\right) + \beta\exp\left(\theta\,\overline{x}\right)\left\{\log\left(y_{t+1}^*\right) - E\left[\log\left(y_t^*\right)\right]\right\}. \tag{28}$$

As shown in the appendix, the approximate law of motion (28) and the dividend growth process (14) can be used to derive the following unconditional moments

$$E\left[\log\left(y_t^*\right)\right] \;=\; \log\left[\frac{\beta\exp\left(\theta\,\overline{x}\right)}{1-\beta\exp\left(\theta\,\overline{x}\right)}\right], \tag{29}$$

$$Var\left[\log\left(y_t^*\right)\right] \;=\; \frac{\theta^2\, Var\left(x_t\right)\left\{1+\beta\exp\left(\theta\,\overline{x}\right)\left[2Corr\left(x_t,\, x_{t-1}\right)-\rho\right]\right\}}{\left[1-\rho\beta\exp\left(\theta\,\overline{x}\right)\right]\left[1-\beta^2\exp\left(2\theta\,\overline{x}\right)\right]}, \tag{30}$$

$$E\left[\Delta\log\left(p_t^*\right)\right] \;=\; \overline{x}, \tag{31}$$

$$\begin{aligned} Var\left[\Delta\log\left(p_t^*\right)\right] \;=\; & \left[1-\beta\exp\left(\theta\,\overline{x}\right)\right]^2\, Var\left[\log\left(y_t^*\right)\right] \\ & +\left\{\alpha^2+\frac{2\alpha\theta\left[1-\beta\exp\left(\theta\,\overline{x}\right)\right]Corr\left(x_t,\, x_{t-1}\right)}{\left[1-\rho\beta\exp\left(\theta\,\overline{x}\right)\right]}\right\} Var\left(x_t\right), \end{aligned} \tag{32}$$

where $Var\left(x_t\right)$ and $Corr\left(x_t,\, x_{t-1}\right)$ are again given by equations (16) and (17).

4.3 Volatility Comparison

Given the complexity of the variance expressions for the log price change it is not immediately obvious whether the equilibrium price-change variance given by equation (27) is greater or less than the perfect foresight variance given by equation (32). To gain some insight, it is helpful to consider some special cases.

Proposition 3. For the special case of logarithmic utility ($\alpha = 1$), we have:

$$Var\left[\Delta\log\left(p_t\right)\right] \;=\; Var\left[\Delta\log\left(p_t^*\right)\right] \;=\; Var\left(x_t\right),$$

Proof: With log utility, we have $\theta = 1-\alpha = 0$. Proposition 1 then implies $a_1 = a_2 = 0$ for any values of ρ and ϕ. Plugging these values into the appropriate expressions yields the above result. ■

When the utility function is logarithmic, the income and substitution effects of dividend growth innovations exactly cancel, regardless of the stochastic process for dividend growth. As a result, the price-dividend ratios y_t and y_t^* are both constant, as can be seen from the variance expressions (25) and (30) when $a_1 = 0$ and $\theta = 0$, respectively. Given that the price-dividend ratios are constant, any variation in the price series must be driven solely by variations in the common stream of exogenous dividends.

Proposition 4. For the special case of *iid* dividend growth ($\rho = \phi = 0$), we have $Var\left[\Delta \log\left(p_t\right)\right] > Var\left[\Delta \log\left(p_t^*\right)\right]$ when $\alpha < 1$, versus $Var\left[\Delta \log\left(p_t\right)\right] \leq Var\left[\Delta \log\left(p_t^*\right)\right]$ when $\alpha \geq 1$.

Proof: When $\rho = \phi = 0$, Proposition 1 implies $a_1 = \theta$ and $a_2 = 0$. From equations (27) and (32), we then have

$$\begin{aligned} Var\left[\Delta \log\left(p_t\right)\right] &= Var\left(x_t\right), \\ Var\left[\Delta \log\left(p_t^*\right)\right] &= \left\{\alpha^2 + \theta^2\left[\frac{1 - \beta\exp\left(\theta\,\overline{x}\right)}{1 + \beta\exp\left(\theta\,\overline{x}\right)}\right]\right\} Var\left(x_t\right), \end{aligned}$$

where $\theta \equiv 1 - \alpha$. By inspection, when $\alpha < 1$, the perfect foresight variance is less than equilibrium variance. When $\alpha = 1$, both variance expressions collapse to $Var\left(x_t\right)$. By inspection, when $\alpha > 1$, the perfect foresight variance exceeds the equilibrium variance. ■

When $\alpha < 1$, Proposition 4 shows $Var\left[\Delta \log\left(p_t\right)\right] > Var\left[\Delta \log\left(p_t^*\right)\right]$, analogous to the risk-neutral result obtained by LeRoy (1984) and Engel (2005). When $\alpha > 1$, the price-change variance inequality is reversed such that $Var\left[\Delta \log\left(p_t\right)\right] < Var\left[\Delta \log\left(p_t^*\right)\right]$. As noted in the introduction, the price-change variance can be interpreted as a measure of the "smoothness" of the underlying price series. According to Proposition 4, a risk aversion coefficient above unity is needed to cause the equilibrium stock price (the forecast) to appear smoother than the discounted sum of subsequent realized dividends (the object being forecasted).

Intuition for the variance inequality reversal in Proposition 4 can be obtained by writing out equations (18) and (19) for the case of *iid* dividend growth, which implies $x_{t+j} = \overline{x} + \varepsilon_{t+j}$ for $j = 1, 2, ...$ We have

$$p_t = d_t E_t\left[\beta\exp\left[\theta\overline{x} + \theta\varepsilon_{t+1}\right] + \beta^2\exp\left(2\theta\overline{x} + \theta\varepsilon_{t+1} + \theta\varepsilon_{t+2}\right) + ...\right] \tag{33}$$

$$p_t^* = d_t\left[\beta\exp\left[\theta\overline{x} + \theta\varepsilon_{t+1}\right] + \beta^2\exp\left(2\theta\overline{x} + \theta\varepsilon_{t+1} + \theta\varepsilon_{t+2}\right) + ...\right]. \tag{34}$$

Since $E_t\exp\left(\theta\varepsilon_{t+j}\right) = \exp\left(\theta^2\sigma_\varepsilon^2/2\right)$ for $j = 1, 2, ...$, the equilibrium price-dividend ratio p_t/d_t is constant in this case. If we neglect the higher-order terms in the above expression for p_t^*, then the corresponding log price changes can be compared directly as follows

$$\begin{aligned} \Delta \log (p_t) &= \log (d_t) - \log (d_{t-1}) \\ &= \overline{x} + \varepsilon_t \end{aligned} \tag{35}$$

$$\begin{aligned} \Delta \log (p_t^*) &\simeq \log (d_t) - \log (d_{t-1}) + \theta \varepsilon_{t+1} - \theta \varepsilon_t \\ &\simeq \Delta \log (p_t) + (1-\alpha)\, \varepsilon_{t+1} - (1-\alpha)\, \varepsilon_t . \end{aligned} \tag{36}$$

where I have made use of the definition $\theta \equiv 1 - \alpha$. The above expressions imply

$$\begin{aligned} Var\left[\Delta \log (p_t^*)\right] &\simeq Var\left[\Delta \log (p_t)\right] + 2\,(1-\alpha)^2 \sigma_\varepsilon^2 - 2\,(1-\alpha) \underbrace{Cov\left[\Delta \log (p_t)\,,\, \varepsilon_t\right]}_{\sigma_\varepsilon^2}, \\ &\simeq Var\left[\Delta \log (p_t)\right] - 2\,(1-\alpha)\,\alpha\, \sigma_\varepsilon^2, \qquad \text{when } \rho = \phi = 0. \end{aligned} \tag{37}$$

The two variance statistics are equal when $\alpha = 1$. When $\alpha < 1$, the covariance term involving the current innovation ε_t serves to shrink the magnitude of $Var\left[\Delta \log (p_t^*)\right]$ relative to $Var\left[\Delta \log (p_t)\right]$, whereas the variance of the future innovation ε_{t+1} always serves to magnify $Var\left[\Delta \log (p_t^*)\right]$ relative to $Var\left[\Delta \log (p_t)\right]$. The negative influence of ε_t dominates the positive influence of ε_{t+1} when $\alpha < 1$. The differential impact of current versus future innovations is similar to the effect noted earlier in describing the intuition for Proposition 1. Recall that the above approximation neglects the variance impact of the higher order terms which involve the future innovations ε_{t+2}, $\varepsilon_{t+3}, ...$ etc. But when $\alpha < 1$, the stochastic discount factors applied to these future innovations decay gradually in equation (34). By taking the log-difference of p_t^* in equation (34) to obtain $\Delta \log (p_t^*)$, the terms involving the future innovations tend to cancel out, rendering the above approximation valid.

5 Quantitative Analysis

Figure 1 plots the volatility (standard deviation in percent) of $\Delta \log (p_t)$ and $\Delta \log (p_t^*)$ as a function of the risk aversion coefficient α. For each value of α, I calibrate the subjective time discount factor β so as to achieve $E(y_t) = 26.6$ in the model, consistent with the average value of the price-dividend ratio for the S&P 500 stock index going back to 1871.[9] When α exceeds a value of about 2.8, achieving the target value of $E(y_t)$ in the model requires a value

[9] Cochrane (1992) employs a similar calibration procedure. For a given discount factor, he chooses the risk coefficient α to match the mean price-dividend ratio in the data.

of β that is greater than unity. Nevertheless, for all values of α examined, the mean value of the stochastic discount factor $E\left[\beta\left(c_{t+1}/c_t\right)^{-\alpha}\right]$ remains below unity.[10]

Given that the Lucas model implies $c_t = d_t$ in equilibrium, I calibrate the stochastic process for x_t in equation (14) using U.S. annual data for the growth of real consumption from 1890 to 2004.[11] Given a value for ϕ, the remaining parameters are set to match the mean, standard deviation, and autocorrelation of consumption growth in the data using the moment equations (15) through (17). For the case of a stochastic trend in log dividends ($\phi = 0$), the parameter values are $\overline{x} = 0.0206$, $\sigma_\varepsilon = 0.0354$, and $\rho = -0.1$. For the case of a deterministic trend ($\phi = 1$), the parameter values are $\overline{x} = 0.0206$, $\sigma_\varepsilon = 0.0338$, and $\rho = 0.8$.

The top panel in Figure 1 shows the results for a stochastic trend in log dividends ($\phi = 0$) while the bottom panel shows the results for a deterministic trend ($\phi = 1$). In each panel, the volatility of $\Delta\log(p_t)$ (solid blue line) is compared to the volatility of $\Delta\log(p_t^*)$ (dotted red line). The solid green line at 18 percent marks the standard deviation of changes in the logarithm of the real S&P 500 stock price index going back to 1871.

Both panels illustrate the reversal in the price-change variance inequality as the risk aversion coefficient crosses unity. These are the same basic patterns described in Proposition 4 for the special case of *iid* dividend growth. Numerical experiments with the model confirm that a single variance inequality reversal occurs at $\alpha = 1$ when the parameters of the dividend growth process are set reasonably close to those implied by the U.S. data calibration. It turns out, however, that if dividend growth exhibits strong positive serial correlation (in contrast to U.S. data), then a second variance inequality reversal occurs at higher levels of risk aversion. For example, when $\phi = 0$ and $\rho \gtrsim 0.6$, a second variance inequality reversal occurs as α increases above unity, yielding the result $Var\left[\Delta\log(p_t)\right] > Var\left[\Delta\log(p_t^*)\right]$. The value of α at the second reversal depends on the value of ρ. Given the intuition from the risk neutral case, it is perhaps not surprising that the dividend growth AR(1) parameter can influence the direction of the variance inequality in the model with risk aversion.[12]

The top panel ($\phi = 0$) shows that the volatility of $\Delta\log(p_t)$ in the model remains well below the U.S. data value of 18 percent for plausible values of risk aversion. This is because the calibration for this case implies $\rho \simeq \phi = 0$ such that consumption growth is close to *iid,* resulting in a nearly constant price-dividend ratio y_t in equilibrium. In contrast, the $\phi = 1$ case

[10] Kocherlakota (1990) shows that a well-defined competitive equilibrium with positive interest rates can still exist in growth economies when $\beta > 1$.

[11] Long-run annual data for U.S. real consumption, real dividends, and real stock prices are from Robert Shiller's website: http://www.econ.yale.edu/~shiller/.

[12] A similar reversal pattern occurs when plotting the variance of log returns, as shown by Lansing and LeRoy (2010).

shown in the bottom panel generates much more volatility in $\Delta \log(p_t)$ and can actually match the U.S. data value when the risk aversion coefficient α is around 5.5. A deterministic trend in dividends generates much more volatility in the equilibrium price-dividend ratio because the representative investor knows that a high current realization of dividends will be reversed in the future as dividends return to the trend. It should be noted, however, that the $\phi = 1$ case cannot be calibrated to match the moments of U.S. consumption growth after World War II because observed consumption growth exhibits positive autocorrelation, whereas the $\phi = 1$ case admits only negative autocorrelation in growth rates.[13]

6 Conclusion

This paper demonstrates that the direction of a theoretical variance inequality involving the *change* in equilibrium stock prices (or log stock prices) relative to the *change* in their perfect foresight counterpart can be reversed, depending on the values of some key parameters of the underlying asset pricing model. In the risk neutral case, the direction of the theoretical variance inequality depends on the values assigned to the dividend AR(1) parameter ρ and the investor's subjective time discount factor β. In the model with risk aversion, the direction of the variance inequality depends on the value assigned to the coefficient of relative risk aversion α. Overall, the results demonstrate that a generalized version of the present-value model does not impose theoretical bounds on price-change volatility.

[13] From equation (17) with $\phi = 1$ we have $Corr(x_t, x_{t-1}) = -(1-\rho)/2$.

A Appendix: Fundamental Equilibrium Solution

A.1 Proof of Proposition 2

Iterating ahead the conjectured law of motion for z_t and taking the conditional expectation yields

$$E_t z_{t+1} = \exp\left[a_0 + \rho a_1 (x_t - \overline{x}) - a_1 \phi\, \varepsilon_t + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]. \tag{A.1}$$

Substituting the above expression into the first order condition (22) and then taking logarithms yields

$$\begin{aligned} \log(z_t) = F(x_t,\, \varepsilon_t) &= \log(\beta) + \theta x_t \\ &+ \log\left\{\exp\left[a_0 + \rho a_1 (x_t - \overline{x}) - a_1 \phi\, \varepsilon_t + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] + 1\right\}, \\ &\simeq a_0 + a_1 (x_t - \overline{x}) + a_2\, \varepsilon_t, \end{aligned} \tag{A.2}$$

where the Taylor-series coefficients $a_0 \equiv E[\log(z_t)]$, a_1, and a_2 are given by

$$a_0 = F(\overline{x},\, 0) = \log(\beta) + \theta\,\overline{x} + \log\left\{\exp\left[a_0 + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] + 1\right\} \tag{A.3}$$

$$a_1 = \left.\frac{\partial F}{\partial x_t}\right|_{\overline{x},\,0} = \theta + \frac{\rho a_1 \exp\left[a_0 + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]}{\exp\left[a_0 + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] + 1}. \tag{A.4}$$

$$a_2 = \left.\frac{\partial F}{\partial \varepsilon_t}\right|_{\overline{x},\,0} = \frac{-a_1 \phi \exp\left[a_0 + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]}{\exp\left[a_0 + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] + 1} \tag{A.5}$$

Solving equation (A.3) for a_0 yields

$$a_0 = \log\left\{\frac{\beta \exp(\theta\,\overline{x})}{1 - \beta \exp\left[\theta\,\overline{x} + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right]}\right\}, \tag{A.6}$$

which can be substituted into equations (A.4) and (A.5) to yield the following system of nonlinear equations that determines a_1 and a_2:

$$a_1 = \theta + \rho a_1 \beta \exp\left[\theta\,\overline{x} + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] \tag{A.7}$$

$$a_2 = -a_1 \phi \beta \exp\left[\theta\,\overline{x} + \tfrac{1}{2}(a_1 + a_2)^2 \sigma_\varepsilon^2\right] \tag{A.8}$$

Solving equation (A.7) for a_1 yields the expression shown in Proposition 1. When $\rho \neq 0$, the above equations can be combined to obtain the following explicit expression for a_2

$$a_2 = \phi\left(\theta - a_1\right)/\rho, \qquad (\rho \neq 0)\,, \tag{A.9}$$

which can be substituted back into equation (A.7). There are two solutions, but only one solution satisfies the condition $\beta \exp\left[\theta\,\overline{x} + \frac{1}{2}\left(a_1 + a_2\right)^2 \sigma_\varepsilon^2\right] < 1$. ■

A.2 Asset Pricing Moments

This section briefly outlines the derivation of equations (24) through (27). Equation (24) follows directly from equation (23) by taking the unconditional expectation of $\log\left(y_t\right)$. We then have

$$\log\left(y_t\right) - E\left[\log\left(y_t\right)\right] \;=\; a_1\rho\left(x_t - \overline{x}\right) \;-\; a_1\phi\,\varepsilon_t, \tag{A.10}$$

which in turn implies

$$Var\left[\log\left(y_t\right)\right] \;=\; \left(a_1\rho\right)^2 Var\left(x_t\right) \;+\; \left(a_1\phi\right)^2 \sigma_\varepsilon^2 \;-\; 2\left(a_1\right)^2 \rho\,\phi\,\underbrace{Cov\left(x_t,\,\varepsilon_t\right)}_{=\sigma_\varepsilon^2}. \tag{A.11}$$

The above expression can be simplified to obtain equation (25).

Taking the unconditional expectation of the log price change (20) yields equation (26). Substituting for y_t and y_{t-1} from the fundamental equilibrium solution (23) yields

$$\begin{aligned}\Delta\log\left(p_t\right) - E\left[\Delta\log\left(p_t\right)\right] \;&=\; \left(1 + a_1\rho\right)\left(x_t - \overline{x}\right) \;-\; a_1\rho\left(x_{t-1} - \overline{x}\right)\\ &\qquad -a_1\phi\,\varepsilon_t \;+\; a_1\phi\,\varepsilon_{t-1}.\end{aligned} \tag{A.12}$$

Taking the square of the above expression and then taking the unconditional expectation yields equation (27).

B Appendix: Perfect Foresight Solution

B.1 Log-linearized Law of Motion

Taking logarithms of the nonlinear law of motion (19) yields

$$\begin{aligned}\log\left(y_t^*\right) \;&=\; G\left[x_{t+1},\,\log\left(y_{t+1}^*\right)\right] \;=\; \log\left(\beta\right) + \theta x_{t+1} + \log\left\{\exp\left[\log\left(y_{t+1}^*\right)\right] + 1\right\}\\ &\simeq\; b_0 + b_1\left(x_{t+1} - \overline{x}\right) + b_2\,\left[\log\left(y_{t+1}^*\right) - b_0\right],\end{aligned} \tag{B.1}$$

where the Taylor-series coefficients $b_0 \equiv E\left[\log\left(y_t^*\right)\right]$, b_1, and b_2 are given by

$$b_0 = G\left(\overline{x},\, b_0\right) = \log\left(\beta\right) + \theta\,\overline{x} + \log\left[\exp\left(b_0\right) + 1\right], \tag{B.2}$$

$$b_1 = \left.\frac{\partial G}{\partial x_t}\right|_{\overline{x},\, b_0} = \theta, \tag{B.3}$$

$$b_2 = \left.\frac{\partial G}{\partial \log\left(y_{t+1}^*\right)}\right|_{\overline{x},\, b_0} = \frac{\exp\left(b_0\right)}{\exp\left(b_0\right)+1}. \tag{B.4}$$

Solving equation (B.2) for b_0 yields

$$b_0 = \log\left\{\frac{\beta\exp\left(\theta\,\overline{x}\right)}{1-\beta\exp\left(\theta\,\overline{x}\right)}\right\}, \tag{B.5}$$

which can be substituted into equation (B.4) to obtain $b_2 = \beta\exp\left(\theta\,\overline{x}\right)$.

B.2 Asset Pricing Moments

This section briefly outlines the derivation of equations (29) through (32). Equation (29) follows directly from equation (B.5), where $b_0 \equiv E\left[\log\left(y_t^*\right)\right]$. From equation (B.1), we then have

$$\log\left(y_t^*\right) - E\left[\log\left(y_t^*\right)\right] = \theta\left(x_{t+1}-\overline{x}\right) + \beta\exp\left(\theta\,\overline{x}\right)\left\{\log\left(y_{t+1}^*\right) - E\left[\log\left(y_t^*\right)\right]\right\}, \tag{B.6}$$

which in turn implies

$$Var\left[\log\left(y_t^*\right)\right] = \frac{\theta^2 Var\left(x_t\right) + 2\theta\beta\exp\left(\theta\,\overline{x}\right)\, Cov\left[\log\left(y_t^*\right),\, x_t\right]}{1-\beta^2\exp\left(2\theta\,\overline{x}\right)}. \tag{B.7}$$

The next step is to compute $Cov\left[\log\left(y_t^*\right),\, x_t\right]$ which appears in equation (B.7). Starting from equation (B.6), we have

$$Cov\left[\log\left(y_t^*\right),\, x_t\right] = \theta Cov\left(x_t,\, x_{t-1}\right) + \beta\exp\left(\theta\,\overline{x}\right)\underbrace{Cov\left[\log\left(y_{t+1}^*\right),\, x_t\right]}_{=Cov\left[\log\left(y_t^*\right),\, x_{t-1}\right]}, \tag{B.8}$$

$$Cov\left[\log\left(y_t^*\right),\, x_{t-1}\right] = \theta\underbrace{Cov\left(x_t,\, x_{t-2}\right)}_{=\rho Cov\left(x_t,\, x_{t-1}\right)} + \beta\exp\left(\theta\,\overline{x}\right)\underbrace{Cov\left[\log\left(y_{t+1}^*\right),\, x_{t-1}\right]}_{=Cov\left[\log\left(y_t^*\right),\, x_{t-2}\right]}, \tag{B.9}$$

where we use repeated substitution to eliminate $Cov\left[\log\left(y_t^*\right),\, x_{t-j}\right]$ for $j = 1, 2, ...$ Applying a transversality condition yields

$$\begin{aligned} Cov\left[\log\left(y_t^*\right),\, x_t\right] &= \theta Cov\left(x_t,\, x_{t-1}\right) \sum_{j=0}^{\infty} \left[\rho\beta \exp\left(\theta\, \overline{x}\right)\right]^j \\ &= \frac{\theta Cov\left(x_t,\, x_{t-1}\right)}{1 - \rho\beta \exp\left(\theta\, \overline{x}\right)}. \end{aligned} \tag{B.10}$$

where the infinite sum converges provided that $\rho\beta \exp\left(\theta\, \overline{x}\right) < 1$. Substituting equation (B.11) into equation (B.7), together with $Cov\left(x_t,\, x_{t-1}\right) = Corr\left(x_t,\, x_{t-1}\right) \times Var\left(x_t\right)$ and then simplifying yields equation (30).

Taking the unconditional expectation of the perfect foresight log price change (21) yields equation (31). Substituting for y_{t-1}^* using the approximate law of motion (B.6) yields

$$\Delta \log\left(p_t^*\right) - E\left[\Delta \log\left(p_t^*\right)\right] = \alpha\left(x_t - \overline{x}\right) + \left[1 - \beta \exp\left(\theta\, \overline{x}\right)\right] \left\{\log\left(y_t^*\right) - E\left[\log\left(y_t^*\right)\right]\right\}. \tag{B.11}$$

Taking the square of the above expression, followed by the unconditional expectation and then once again making use of equation (B.10) yields equation (32).

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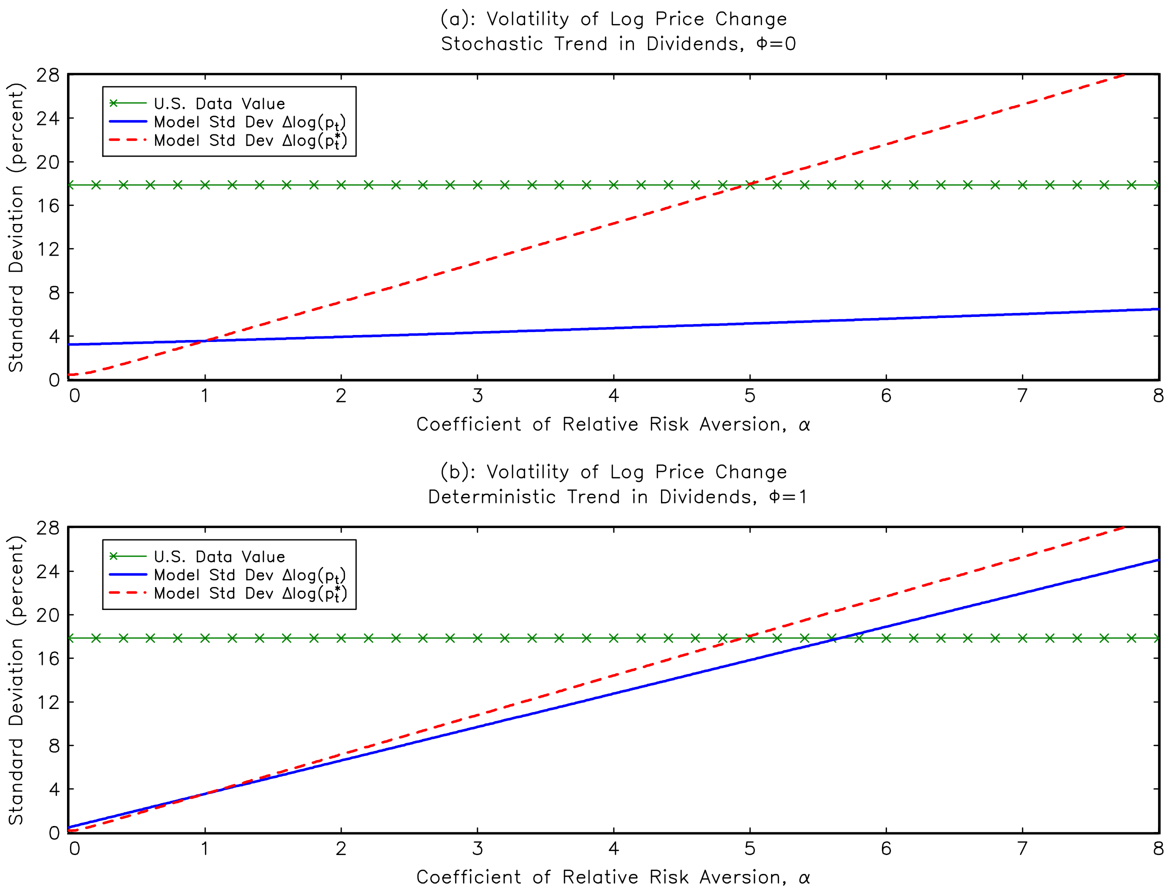

Figure 1: Variance inequality reversal occurs at $\alpha = 1$ for both dividend specifications.