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The Lorenz-maximal core allocations and the kernel in some classes of assignment games

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Abstract: In this paper we prove that in the class of assignment games with a dominant diagonal (Solymosi and Raghavan, 2001), Thompson's fair division point (which is known to be the τ-value) is the unique core allocation that is maximal with respect to the Lorenz dominance relation and it coincides with the Dutta-Ray (1989) solution (or egalitarian solution).

Secondly, strengthening the dominant diagonal condition, a new class of assignment games is introduced where the profit obtained by each agent with her optimal partner is at least twice as much as her potential profit with any other partner. For these assignment games with a 2-dominant diagonal, Thompson's fair division point is also the unique element of the kernel, and thus the nucleolus.

Key words: assignment game, core, kernel, Lorenz domination

JEL: C71, C78

Resum: En aquest treball demostrem que en la classe de jocs d'assignació amb diagonal dominant (Solymosi i Raghavan, 2001), el repartiment de Thompson (que coincideix amb el valor tau) és l'únic punt del core que és maximal respecte de la relació de dominància de Lorenz, i a més coincideix amb la solució de Dutta i Ray (1989), també coneguda com solució igualitària.

En segon lloc, mitjançant una condició més forta que la de diagonal dominant, introduïm una nova classe de jocs d'assignació on cada agent obté amb la seva parella òptima almenys el doble que amb qualsevol altra parella. Per aquests jocs d'assignació amb diagonal 2-dominant, el repartiment de Thompson és l'únic punt del kernel, i per tant el nucleolo.

1 Introduction

Coalitional game theory aims to propose allocation rules that express some notion of fairness or distributive justice in the division of the jointly obtained worth. Egalitarianism is one notion of fairness and it is agreed that a least requirement to say a rule is egalitarian is that it should be maximal according to the Lorenz criterion applied to some subset of outcomes. See for instance Thomson (2011) for a recent application of Lorenz criterion to the ranking of rules for the adjudication of conflicting claims.

The set of Lorenz-maximal core allocations is analyzed as a solution concept for balanced coalitional games in Hougaard et al. (2001) and Arin and Iñarra (2001), combining core selection with egalitarianism as a standard of fairness. This solution is defined as the subset of core allocations that are not Lorenz dominated by another core element.

Before Hougaard et al. (2001), Dutta and Ray (1989) propose a solution concept responding to egalitarianism under participation constraints (core-like constraints). This means to select an egalitarian (Lorenz-maximal) allocation among those that are not blocked by any subcoalition, where the term “block” is used in the sense that the subcoalition can find an allocation according to its egalitarian rule that makes all its members just as well off, and some member strictly better off. Thus, by a recursive procedure, the Lorenz core of each coalition is constructed and then the Lorenz-maximal allocation in the Lorenz core of the grand coalition is the Dutta-Ray solution.

The nucleolus (Schmeidler, 1969) is a single-valued solution concept for coalitional games that responds to the Rawl’s egalitarian criterion applied not only to individuals but to coalitions. According to Maschler et al. (1979), the nucleolus is fair in the sense that it is “the result of an arbitrator’s desire to minimize the dissatisfaction of the most dissatisfied coalition”, where the dissatisfaction of a coalition at a payoff vector is the difference between the worth of the coalition and its total payoff.

Sudhölter and Peleg (1998) apply the Lorenz criterion of egalitarianism over coalitions, since they compare allocations by ordering their distributions of excesses by the Lorenz domination, and find that the (pre)nucleolus is maximal in this Lorenz order. A generalization of this procedure in Arin and Feltkamp (2002) allows to see that the Shapley value also responds to some egalitarian criterion over coalitions.[2]

In the present paper we focus on a particular class of coalitional games that are the assignment games and see that, under certain conditions, different notions of egalitarianism lead to the same allocation rule. The idea of applying Lorenz dominance to the assignment market was posed in Hougaard et al. (2001). More recently, Roth et al. (2005) address distributive justice issues in some matching problems by means of Lorenz dominance.

In an assignment market agents are partitioned in two disjoint sets, let us say buyers and sellers (or firms and workers) and when a member of one group is paired with a member of the other group an additional value is created. From this situation, Shapley and Shubik (1972) introduce a cooperative game where the worth that a coalition can attain is the addition of the values generated by all the pairs in an optimal matching. Since side-payments are allowed, the problem is how to share among the agents the worth of an optimal matching.

The first approach is to look for allocations of the total worth that no coalition can improve upon. Shapley and Shubik prove that this is indeed possible (the core is non-empty) and that to obtain a core allocation it is enough to exclude third-party payments (that is, the paired agents must share exactly the value of their pairing) and impose pairwise stability (in a core allocation no buyer-seller pair can produce together a value higher than their actual join payoff). But the core of an assignment game rarely consists of a single point. It is then necessary to select some core element under some criteria of fairness.

Thompson (1981) defines the "fair division point" as the midpoint between the core allocation that is optimal for all the buyers and the one that is optimal for all the sellers. Thus, it applies symmetry to solve the bargaining problem among the two sides of the market, treated each side as a whole. This fair division point of Thompson is proved in Núñez and Rafels (2002) to coincide with the τ-value, this being a solution concept introduced by Tijs (1981) for arbitrary coalitional games.

On the other hand, Rochford (1984) assumes that each optimally assigned pair of individuals engage in a bargaining problem (*à la* Nash) to determine how to distribute among them the output of their partnership. Each pair is assumed to solve this bargaining problem

[2] See Arin (2007) for a survey on egalitarian allocation rules in coalitional games.

invoking symmetry, the threats reflecting the other opportunities available in the market. Then, the symmetrically pairwise-bargained allocations are those core allocations z such that all partners are in bargained equilibrium. This means that each optimally matched pair splits equally what remains of their join value once each of them has taken her threat, that is, the most she could gain in another partnership after her potential partner is paid according to z. By Rochford (1984) and Driessen (1998), the set of symmetrically pairwise-bargained allocations coincides with the kernel of the assignment game. But again, the kernel may not reduce to a single point. An outstanding element of the kernel is the nucleolus (Schmeidler, 1969).

In fact, for assignment games, the aforementioned solution concepts are also known to depend only on the core. That is, two assignment markets with the same core have the same τ-value, the same kernel and the same nucleolus (Núñez, 2004).

As a consequence of its definition, also the set of Lorenz-maximal core allocations depends only on the core of the game and not on the characteristic function (the worth of coalitions). This is the reason why Hougaard et al. (2001) say that it would be interesting to study the Lorenz-maximal core allocations in assignment games.

It is immediate to realize that, for arbitrary coalitional games, when the allocation that splits equally the worth of the grand coalition among the agents belongs to the core, then it is the unique Lorenz-maximal core allocation. But for assignment games, this equal-share allocation being in the core is quite restrictive, since it implies that in the market all optimally matched pairs attain the same value. Instead, the allocation in which each two optimally paired agents split equally the value of their partnership seems a natural division to be considered in an assignment market, at least when it belongs to the core of the assignment game.

In the class of assignment games with a dominant diagonal (Solymosi and Raghavan, 2001), it is known that the aforementioned allocation coincides with Thompson's fair division point, and thus it lies in the core. The aim of this paper is to analyze when this solution to the assignment game fulfills as many standards of fairness as possible.

In Section 3 we find that, for square assignment games with a dominant diagonal (which means that the join profit of any agent with her optimal partner is the most she could make with any other partner), Thompson's fair division point not only belongs to the core but also coincides with the Dutta-Ray solution, and moreover it Lorenz dominates all other core allocations. In section 4 a class of assignment games is considered where the join profit of any agent with her optimal partner is at least twice as much as her join profit in any other partnership. We conclude that for this class of assignment games, which we name 2-dominant diagonal assignment games, although the core can be considerably ample, the game is somehow "determined" since the kernel and the Lorenz-maximal core allocations coincide and reduce to the nucleolus, which also coincides with Thompson's fair division point and the Dutta-Ray solution.

2 Definitions and notations

Let $N = \{1,2,...,n\}$ denote a finite set of players, and 2^N the set of all possible coalitions or subsets of N. The cardinality of coalition S is denoted by $|S|$. Given two coalitions S and T, $S \subseteq T$ denotes inclusion while $S \subset T$ denotes strict inclusion.

A *cooperative game in coalitional form* (a game) is a pair (N,v), where $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$, is the characteristic function which assigns to each coalition S the worth $v(S)$ it can attain. If no confusion arises, a game (N,v) is denoted by simply v.

Given a game (N,v), a payoff vector is $x \in \mathbb{R}^N$, where x_i stands for the payoff to player $i \in N$. The restriction of a payoff vector x to a coalition S is denoted by $x_{|S}$. Given two payoff vectors $x,y \in \mathbb{R}^N$, we write $y \geq x$ if $y_i \geq x_i$ for all $i \in N$, and $y > x$ whenever $y \geq x$ and moreover there exists $i \in N$ such that $y_i > x_i$. An *imputation* is a payoff vector x that is efficient, $\sum_{i \in N} x_i = v(N)$, and individually rational, $x_i \geq v(\{i\})$ for all $i \in N$. The set of all imputations of a game (N,v) is denoted by $I(v)$, and when $I(v) \neq \emptyset$ the game is said to be *essential*. The *excess* of a coalition S at an imputation $x \in I(v)$ is $v(S) - \sum_{i \in S} x_i$.

A *solution concept* defined on the set of games with player set N is a rule that assigns to each such game a subset of efficient payoff vectors. The best known set-solution concept for coalitional games is the core. The *core* of the game is the set of payoff vectors that are efficient and coalitionally rational, that is, $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$. A game with a non-empty core is a *balanced game*. Typically, the core of a game contains infinitely many payoff vectors. This forces to select some particular core allocation following some idea of fairness. Among others, given a game (N,v), a well known single–valued core selection is *the nucleolus* (Schmeidler, 1969). The nucleolus of a game (N,v) is the imputation $\nu(v)$ that lexicographically minimizes, over the set of imputations, the vector of excesses arranged in non-increasing order.

The *kernel* (Davis and Maschler, 1965), is another set-solution concept for cooperative games. The kernel, $\mathscr{K}(v)$, of an essential cooperative game (N,v) is always nonempty and it contains the nucleolus. Thus, for balanced games, there always exist imputations in the intersection of the core and the kernel which could be selected in front of other core allocations if we agree with the standard of fairness that supports the kernel.

For *zero–monotonic games*,[3] as it is the case of assignment games, the kernel can be described by

$$\mathscr{K}(v) = \{z \in I(v) \mid s^v_{ij}(z) = s^v_{ji}(z) \text{ for all } i,\, j \in N\,,\, i \neq j\}\,,$$

where the maximum surplus $s^v_{ij}(z)$ of player i over another player j with respect to the imputation z is defined by

$$s^v_{ij}(z) = \max\left\{ v(S) - \sum_{k\in S} z_k \,\middle|\, S \subseteq N\,,\, i \in S\,,\, j \notin S \right\}.$$

We will just write $s_{ij}(z)$ when no confusion regarding the game v can arise. Then, for zero-monotonic games, the kernel can be viewed as those imputations for which any two players are equally powerful concerning their mutual threats.

A different standard of fairness is that provided by the Lorenz domination. This binary relation compares payoff vectors by means of how evenly distributed are the agents' payoffs, and is favorable to agents with lower payoffs. Formally, for any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1,\ldots,\hat{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-increasing order, that is, $\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_n$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that *y weakly Lorenz dominates x*, denoted by $y \succeq_L x$, if $\sum_{j=1}^k \hat{y}_j \leq \sum_{j=1}^k \hat{x}_j$, for all $k \in \{1,\ldots,n\}$. We say that *y Lorenz dominates x*, denoted by $y \succ_L x$, if at least one of the above inequalities is strict.

[3]A game (N,v) is zero-monotonic if for any pair of coalitions S,T, $S \subset T \subseteq N$ it holds $v(S) + \sum_{i\in T\setminus S} v(\{i\}) \leq v(T)$.

Given a subset of payoff vectors, $A \subseteq \mathbb{R}^N$, we denote by $E(A)$ the set of *Lorenz-maximal elements* in A, that is, $E(A) = \{x \in A \mid \text{there is no } y \in A \text{ such that } y \succ_L x\}$. If (N,v) is a balanced game, Hougaard et al. (2001) and Arin and Iñarra (2001) propose the Lorenz-maximal core elements, $L(v) = E(C(v))$, as a set-solution concept. Since $C(v)$ is a non-empty compact set, $L(v)$ is non-empty.

But Dutta and Ray (1989) argue that if we pursue egalitarianism as standard of fairness, we should not restrict to core allocations since an allocation outside the core is disregarded because it is blocked by some viable coalition, but we do not require the blocking payoff vector to be egalitarian (in the sense of Lorenz undomination) for the blocking coalition.

To this end, they define the *Lorenz cores* of coalitions. The Lorenz core of a singleton coalition is $L(\{i\}) = \{v(i)\}$. Now suppose that the Lorenz cores for all coalitions of cardinality k or less have been defined, where $1 \leq k < n$. The Lorenz core of a coalition $S \subseteq N$ of cardinality $k+1$ is defined by

$$L(S) = \left\{ x \in \mathbb{R}^S \;\middle|\; \begin{array}{l} \sum_{i \in S} x_i = v(S) \text{ and there is no } T \subset S \text{ and} \\ y \in E(L(T)) \text{ such that } y > x_{|T} \end{array} \right\}, \tag{1}$$

where recall that $y > x_{|T}$ means that $y_i \geq x_i$ for all $i \in T$, with at least one of these inequalities being strict. It follows from this definition that the Lorenz core of coalition N contains the core: $C(v) \subseteq L(N)$.

Given (N,v) a game, the *Dutta-Ray solution* is $DR(v) = E(L(N))$. Duta and Ray (1989) prove that $DR(v)$ is either empty or a singleton. Moreover, for convex games,[4] the Dutta-Ray solution exists (an algorithm to calculate it is provided), it belongs to the core and Lorenz dominates any other core allocation.

The assignment model

A two-sided assignment market (M, M', A) is defined by a finite set of buyers M, a finite set of sellers M', and a nonnegative matrix $A = \left(a_{ij}\right)_{(i,j) \in M \times M'}$. The real number a_{ij} represents the profit obtained by the mixed-pair $(i,j) \in M \times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers, and $n = m + m'$ is the cardinality of $N = M \cup M'$. Any

[4] A game (N,v) is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, for all $S, T \subseteq N$.

subset $R \subseteq M \cup M'$ defines a submarket $(M \cap R, M' \cap R, A_{|R})$, where $A_{|R}$ is the restriction of matrix A to the rows and columns that correspond to agents in R.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M_0' \subseteq M'$, such that $|M_0| = |M_0'| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all matchings is denoted by $\mathscr{M}(M,M')$. If $m = m'$, the assignment market is said to be square.

A *matching* $\mu \in \mathscr{M}(M,M')$ is *optimal* for the assignment market (M,M',A) if for all $\mu' \in \mathscr{M}(M,M')$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathscr{M}_A^*(M,M')$.

Shapley and Shubik (1972) associate to any assignment market (M,M',A) a cooperative game in coalitional form, with player set $N = M \cup M'$ and characteristic function w_A, defined by: for $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\left\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathscr{M}(S,T)\right\}$, $\mathscr{M}(S,T)$ being the set of matchings between S and T. The core of the assignment game is always non-empty, and it is enough to impose coalitional rationality for one-player coalitions and mixed-pair coalitions:

$$Core(w_A) = \left\{ (u,v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \;\middle|\; \begin{array}{l} \sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(N), \\ u_i + v_j \geq a_{ij}, \text{ for all } (i,j) \in M \times M' \end{array} \right\}, \quad (2)$$

where $\mathbb{R}_+$ stands for the set of non-negative real numbers. It follows from (2) that, if μ is an optimal matching, unassigned agents receive zero payoff and, moreover,

$$\text{if } (i,j) \in \mu \text{ then } u_i + v_j = a_{ij}. \quad (3)$$

There exists a *buyers-optimal core allocation*, $(\overline{u}, \underline{v})$, where each buyer attains her maximum core payoff and each sellers his minimum one, and a *sellers-optimal core allocation*, $(\underline{u}, \overline{v})$, with the converse situation. By Roth and Sotomayor (1990),

$$\overline{u}_i = w_A(N) - w_A(N \setminus \{i\}) \text{ for all } i \in M, \text{ and}$$
$$\overline{v}_j = w_A(N) - w_A(N \setminus \{j\}) \text{ for all } j \in M'.$$

Notice that, if μ is an arbitrary optimal matching of (M,M',A), we obtain from (3) that $\underline{u}_i = a_{i\mu(i)} - \overline{v}_{\mu(i)}$ for all $i \in M$ assigned by μ and $\underline{v}_j = a_{\mu^{-1}(j)j} - \overline{u}_{\mu^{-1}(j)}$ for all $j \in M'$ assigned by μ, while agents unmatched by μ have a null minimum core payoff. The *fair division point* is defined by Thompson (1981) as the midpoint between these two extreme core allocations, and it is proved in Núñez and Rafels (2002) to coincide with the τ-value[5] of the assignment game:

$$\tau(w_A) = \frac{1}{2}(\overline{u}, \underline{v}) + \frac{1}{2}(\underline{u}, \overline{v}).$$

As for the kernel of assignment games, it turns out that it is always included in the core, $\mathscr{K}(w_A) \subseteq C(w_A)$, (Driessen, 1998). Moreover, if $(u,v) \in C(w_A)$, then *(i)* $s_{ij}(z) = 0$ whenever $i, j \in M$ or $i, j \in M'$, and *(ii)* if $i \in M$ and $j \in M'$, then $s_{ij}(z)$ is always attained at the excess of some individual coalition or mixed–pair coalition:

$$s_{ij}(u,v) = \max_{k \in M' \setminus \{j\}} \{-u_i, a_{ik} - u_i - v_k\}.$$

As a consequence, given $(u,v) \in C(w_A)$, we get that $(u,v) \in \mathscr{K}(w_A)$ if and only if $s_{ij}(u,v) = s_{ji}(u,v)$ for all (i,j) belonging to all the optimal matchings, since the remaining equalities hold trivially (see Rochford, 1984).

All the solution concepts so far reviewed for the assignment game are tightly related to the core, in the sense that two assignment games with the same core have the same τ-value, the same kernel and the same nucleolus (Núñez, 2004).

By adding dummy players, that is, null rows or columns in the assignment matrix, we can assume from now on, and without loss of generality, that the number of sellers equals the number of buyers, and in this way the assignment matrix is square. When necessary, we denote the ith seller by i', so that it can be distinguished from the ith buyer, which is denoted by i.

A square assignment game is said to have a *dominant diagonal* (Solymosi and Raghavan, 2001) if, once fixed an optimal matching on the main diagonal, $\mu = \{(i,i) \mid i \in M\}$, it holds, for all $i \in M$,

$$a_{ii} \geq \max\{a_{ij}, a_{ji}\}, \quad \text{for all } j \in M \setminus \{i\}. \tag{4}$$

This means that the matrix entries related to optimal pairs by μ are row and column maxima. If an assignment game has a dominant diagonal, then the payoff vectors $((a_{ii})_{i \in M}, 0) \in$

[5] The *τ-value* is a single-valued solution defined by Tijs (1981) for arbitrary coalitional games. It is a compromise solution that may lie outside the core, although for assignment games it always selects a core allocation.

$\mathbb{R}^M \times \mathbb{R}^{M'}$ belongs to the core and thus it coincides with the buyers-optimal core allocation $(\overline{u}, \underline{v})$. Similarly, $(\underline{u}, \overline{v}) = (0, (a_{ii})_{i \in M}) \in \mathbb{R}^M \times \mathbb{R}^{M'}$. In fact, an assignment game has a dominant diagonal if and only if the minimum core payoff of each agent is zero. As a consequence, the property of having a dominant diagonal does not depend on the selected optimal matching.

The class of assignment games with a dominant diagonal is a quite large class of assignment games that has specific properties such as the stability of the core, where we refer to stability in the sense of von Neumann and Morgenstern.

3 Dominant diagonal assignment games and Lorenz-maximal core allocations

The study of the Lorenz-maximal core allocations of the assignment game was posed in Hougaard et al. (2001). For arbitrary assignment games, the set of Lorenz-maximal core allocations may contain infinitely many points. The next example illustrates this situation.

Example 1. *Let us consider a family of assignment markets with set of buyers* $M = \{1, 2\}$, *set of sellers* $M' = \{1', 2'\}$ *and assignment matrices*

$$A^{\varepsilon} = \begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix}$$

where $0 < \varepsilon < \frac{1}{2}$.

For all $0 < \varepsilon < \frac{1}{2}$, *there exists only one optimal matching* $\mu = \{(1,1),(2,2)\}$. *The core is the triangle with vertices*

$$A = (\overline{u}, \underline{v}) = (1, 1-\varepsilon; 0, 0)\,, B = (\underline{u}, \overline{v}) = (\varepsilon, 0; 1-\varepsilon, 1-\varepsilon) \text{ and } C = (1, 0; 0, 1-\varepsilon),$$

and its projection to the space of the buyers' payoffs is depicted in Figure 1

We claim that the set of Lorenz-maximal core allocations $L(w_A)$ *is the segment with extreme points*

$$D = \left(\frac{1}{2}, \frac{1}{2}-\varepsilon; \frac{1}{2}, \frac{1}{2}\right) \text{ and } E = \left(\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{2}; \frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}\right).$$

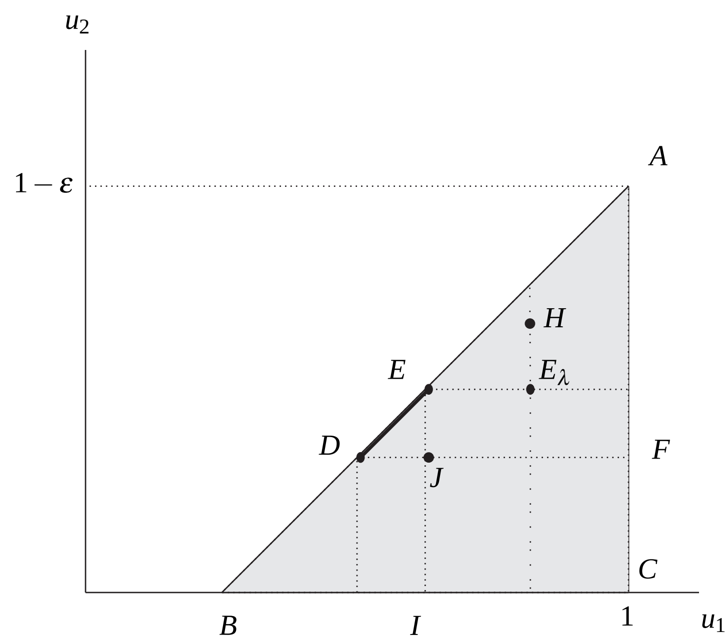

Figure 1:

Notice that the upper extreme point E coincides with the Thompson's fair division point.

To check that indeed $L(w_A) = [D,E]$, we will make use of a remark in Hougaard et al. (2001) that is based on Theorem 108 in Hardy et al. (1934) and states that, when we ask whether a payoff vector Lorenz dominates another payoff vector with its same efficiency level, we can omit those components that are coincident for the two vectors. We then proceed in four steps:

1. *Let $P = (u_1, u_2; 1-u_1, 1-\varepsilon-u_2)$, with $\frac{1}{2} \leq u_1 \leq 1$ be an arbitrary point in the segment $[D,A]$, that is $u_2 = u_1 - \varepsilon$. If $P' = (u_1', u_2; 1-u_1', 1-\varepsilon-u_2)$, with $u_1 < u_1'$, is an arbitrary core element in the same horizontal line as P, we have that $P \succ_L P'$, since from $u_1' > \frac{1}{2}$ we get $u_1' > 1-u_1'$ and then apply the remark in Hougaard et al. (2001).*

2. *Let $E_\lambda = \left(\lambda, \frac{1-\varepsilon}{2}; 1-\lambda, \frac{1-\varepsilon}{2}\right)$, with $\frac{1+\varepsilon}{2} < \lambda \leq 1$, be a core element in the same horizontal line as E. Any other core allocation in the same vertical line as E_λ is Lorenz dominated by E_λ. Formally, if $H = (\lambda, u_2; 1-\lambda, 1-\varepsilon-u_2)$ where $0 \leq u_2 \leq \lambda-\varepsilon$ and $u_2 \neq \frac{1-\varepsilon}{2}$, then E_λ Lorenz dominates H. To see this simply apply the remark in Hougaard et al. (2001) and notice that $\left(\frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}\right) \succ_L (u_2, 1-\varepsilon-u_2)$ for*

any $0 \le u_2 \le \lambda - \varepsilon$ *and* $u_2 \neq \frac{1-\varepsilon}{2}$.

Notice that a consequence of 1 and 2 is that the allocation E Lorenz dominates all other core allocations in the convex hull of E, I, C and A. Moreover, from 1, allocations in $[D,E]$ *Lorenz dominate those other in the triangle with vertices D, E and J*

3. *An argument analogous to the previous one shows that D Lorenz dominates all the core elements in the convex hull of D, B, F and C different from itself.*

4. *Finally, two different points in the segment* $[D,E]$ *do not Lorenz dominate one another. To this end, notice that an arbitrary point in this segment is*

$$\begin{aligned} z(\lambda) &= \lambda\left(\frac{1}{2},\frac{1}{2}-\varepsilon;\frac{1}{2},\frac{1}{2}\right)+(1-\lambda)\left(\frac{1+\varepsilon}{2},\frac{1-\varepsilon}{2};\frac{1-\varepsilon}{2},\frac{1-\varepsilon}{2}\right) \\ &= \left(\frac{1}{2}+\frac{\varepsilon}{2}(1-\lambda),\frac{1}{2}-\frac{\varepsilon}{2}(1+\lambda);\frac{1}{2}+\frac{\varepsilon}{2}(\lambda-1),\frac{1}{2}+\frac{\varepsilon}{2}(\lambda-1)\right) \end{aligned}$$

with $0 \le \lambda \le 1$. *Then,* $\widehat{z(\lambda)} = \left(\frac{1}{2}+\frac{\varepsilon}{2}(1-\lambda),\frac{1}{2}+\frac{\varepsilon}{2}(\lambda-1),\frac{1}{2}+\frac{\varepsilon}{2}(\lambda-1),\frac{1}{2}-\frac{\varepsilon}{2}(1+\lambda)\right)$, *and notice that* $\widehat{z(\lambda)}_1$ *is a decreasing function of* λ *while* $\widehat{z(\lambda)}_1+\widehat{z(\lambda)}_2+\widehat{z(\lambda)}_3 = \frac{3}{2}+\frac{\varepsilon}{2}(\lambda-1)$ *is an increasing function of* λ.

The main result in this section is that if the assignment game $(M \cup M', w_A)$ has a dominant diagonal (see (4)), then the set of Lorenz-maximal core allocations reduces to only one point. Moreover, this allocation will be proved to be the Thompson's fair division point (or the τ-value) and also the Dutta-Ray solution. Notice that the above example shows that our first statement is tight in the sense that if the assignment matrix fails to have a dominant diagonal, even if it is by an small amount $\varepsilon > 0$, then the set of Lorenz-maximal core allocations may not be a singleton.

Theorem 2. *Let* $(M \cup M', w_A)$ *be a square assignment game with a dominant diagonal. Then, the set of Lorenz-maximal core allocations* $L(w_A)$ *reduces to Thompson's fair division point* $\tau(w_A)$. *Moreover, the Dutta-Ray solution exists and*

$$L(w_A) = \{\tau(w_A)\} = DR(w_A).$$

Proof. We assume, without loss of generality, that $\mu = \{(1,1),(2,2),\ldots,(m,m)\}$ is optimal and $a_{11} \geq a_{22} \geq \ldots \geq a_{mm}$. Under the assumption of dominant diagonal, the optimal core allocations for each side of the market are $(\overline{u},\underline{v}) = ((a_{ii})_{i\in M},0)$ and $(\underline{u},\overline{v}) = (0,(a_{ii})_{i\in M})$ and thus $\tau(w_A)_i = \tau(w_A)_{i'} = \frac{a_{ii}}{2}$ for all $i \in M$. To simplify notation, let us write $\tau = \tau(w_A)$. Then, the vector obtained from τ by rearranging its coordinates in non-increasing order is $\widehat{\tau} = \left(\frac{a_{11}}{2},\frac{a_{11}}{2},\ldots,\frac{a_{mm}}{2},\frac{a_{mm}}{2}\right)$. In order to prove that τ is the unique Lorenz-maximal core allocation we first prove that vector τ weakly Lorenz dominates every other point in the core.

Let it be $z = (u,v) \in C(w_A)$. To see that $\widehat{\tau}_1 + \cdots + \widehat{\tau}_i \leq \widehat{z}_1 + \cdots + \widehat{z}_i$ for all $i \in \{1,2,\ldots,2m\}$, let us first consider the case in which i is even, that is, $i = 2k$ for some $k \in \{1,2,\ldots,m\}$. Then,

$$\begin{aligned} \widehat{\tau}_1 + \cdots + \widehat{\tau}_{2k} &= \frac{a_{11}}{2} + \frac{a_{11}}{2} + \cdots + \frac{a_{kk}}{2} + \frac{a_{kk}}{2} \\ &= a_{11} + \cdots + a_{kk} \\ &= u_1 + v_1 + \cdots + u_k + v_k \\ &\leq \widehat{z}_1 + \widehat{z}_2 + \cdots + \widehat{z}_{2k-1} + \widehat{z}_{2k}, \end{aligned} \tag{5}$$

where the last equality follows from (3) and the inequality from the definition of vector $\hat{z}$, where $z = (u,v)$.

Let us now consider the case in which i is odd. Notice that for $i = 1$, we have $\widehat{\tau}_1 = \frac{a_{11}}{2} \leq \widehat{z}_1$. Otherwise, if $\widehat{z}_1 < \frac{a_{11}}{2}$, we have both $u_1 < \frac{a_{11}}{2}$ and $v_1 < \frac{a_{11}}{2}$, which contradicts $z = (u,v) \in C(w_A)$.

For $i = 2k+1$ with $k \in \{1,2,\ldots,m-1\}$, from (5) it follows

$$\widehat{\tau}_1 + \cdots + \widehat{\tau}_{2k} \leq \widehat{z}_1 + \ldots + \widehat{z}_{2k}.$$

Assume

$$\widehat{\tau}_1 + \cdots + \widehat{\tau}_{2k} + \widehat{\tau}_{2k+1} > \widehat{z}_1 + \ldots + \widehat{z}_{2k} + \widehat{z}_{2k+1}. \tag{6}$$

Then $\widehat{\tau}_{2k+1} > \widehat{z}_{2k+1}$. On the other hand, again from (5) and the assumption that the market is square, we have $\widehat{\tau}_1 + \cdots + \widehat{\tau}_{2k+2} \leq \widehat{z}_1 + \cdots + \widehat{z}_{2k+2}$. This last inequality, together with (6), implies $\widehat{\tau}_{2k+2} \leq \widehat{z}_{2k+2}$. But then, since $\widehat{\tau}_{2k+1} = \widehat{\tau}_{2k+2}$, we have $\widehat{\tau}_{2k+2} = \widehat{\tau}_{2k+1} > \widehat{z}_{2k+1} \geq \widehat{z}_{2k+2} \geq \widehat{\tau}_{2k+2}$, which is a contradiction. Hence, for all $i \in \{1,2,\ldots,m\}$ we have $\widehat{\tau}_1 + \cdots + \widehat{\tau}_i \leq \widehat{z}_1 + \cdots + \widehat{z}_i$, and we conclude that $\tau \succeq_L z$ which means that $\tau(w_A)$ is Lorenz-maximal in the core.

To show uniqueness, suppose there exists some other $z = (u,v) \in C(w_A)$ such that $\widehat{\tau} = \widehat{z}$. Also, for our convenience, let us write $\tau = (\tau^u, \tau^v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$, where, for all $i \in \{1,2,\ldots,m\}$, τ_i^u denotes the payoff to the ith buyer and τ_i^v denotes the payoff to the ith seller. Then, by the assumption $\widehat{\tau} = \widehat{z}$, we have $\widehat{\tau}_1 = \frac{a_{11}}{2} = \widehat{z}_1$. Let us see that in fact $\widehat{z}_1 = u_1$. Otherwise $\widehat{z}_1 = \frac{a_{11}}{2} > u_1$ and then $v_1 > \frac{a_{11}}{2}$, which implies $\widehat{z}_1 > \frac{a_{11}}{2}$ and this is a contradiction. Thus, $\tau_1^u = \frac{a_{11}}{2} = \widehat{z}_1 = u_1$. Since $u_1 + v_1 = a_{11}$ and $u_1 = \frac{a_{11}}{2}$, we have $v_1 = \frac{a_{11}}{2} = \tau_1^v$. But then, $\widehat{\tau}_3 = \widehat{z}_3 = \frac{a_{22}}{2}$. As before, if $\widehat{z}_3 = \frac{a_{22}}{2} > u_2$ then $v_2 > \frac{a_{22}}{2}$, which implies $\widehat{z}_3 > \frac{a_{22}}{2}$, that is a contradiction. Then, $\widehat{z}_3 = u_2 = \frac{a_{22}}{2}$ and thus $\tau_2^u = u_2$ and consequently $\tau_2^v = v_2$. By repetition of the same argument, we obtain $(\tau^u, \tau^v) = (u,v) = z$ and thus prove that the Thompson's fair division point is the unique Lorenz-maximal core allocation.

Let us now prove that, under the dominant diagonal assumption, Thompson's fair division point also coincides with the Dutta-Ray solution.

Let $P = \{S_1, S_2, \ldots, S_l\}$ be the partition of $N = M \cup M'$ defined as follows:

$$
\begin{aligned}
S_1 &= \{i \in N \mid \tau_i \geq \tau_k \text{ for all } k \in N\}, \\
S_2 &= \{i \in N \setminus S_1 \mid \tau_i \geq \tau_k \text{ for all } k \in N \setminus S_1\}, \\
&\vdots \\
S_l &= \{i \in N \setminus (S_1 \cup \cdots \cup S_{l-1}) \mid \tau_i \geq \tau_k \text{ for all } k \in N \setminus (S_1 \cup \cdots \cup S_{l-1})\}.
\end{aligned}
$$

If $l = 1$, then $\tau_i = \frac{w_A(N)}{|N|}$ for all $i \in N$, and thus $DR(w_A) = \{\tau\}$. If $l > 1$, notice that, for all $k, r \in \{1,2,\ldots,l\}$ and all $i \in S_k$ and $j \in S_r$ it holds:

a) if $k = r$, then $\tau_i = \tau_j$ (i.e. $a_{ii} = a_{jj}$),

b) if $k < r$, then $\tau_i > \tau_j$ (i.e. $a_{ii} > a_{jj}$),

c) the restriction of vector τ to coalition S_k is $\left(\frac{w_A(S_k)}{|S_k|}, \ldots, \frac{w_A(S_k)}{|S_k|}\right)$.

To see statement c), notice that by its own definition and the dominant diagonal condition, for all $k \in \{1,2,\ldots,l\}$, S_k is composed of a subset of pairs in the optimal matching $\mu = \{(i,i) \mid i \in M\}$. Thus, the restriction of the optimal matching μ to S_k is optimal in the submarket with set of agents S_k and as a consequence $\tau_i = \frac{a_{ii}}{2} = \frac{w_A(S_k)}{|S_k|}$ for all $i \in S_k$.

Since $\tau \in C(w_A)$, we can guarantee that $\tau \in L(N)$. If τ were not the Dutta-Ray solution, then there would exist $z \in L(N)$ such that $z \succ_L \tau$, that is to say

$$\begin{array}{rcl} \widehat{z}_1 & \leq & \widehat{\tau}_1 = \frac{a_{11}}{2}, \\ \widehat{z}_1 + \widehat{z}_2 & \leq & \widehat{\tau}_1 + \widehat{\tau}_2 = a_{11}, \\ \cdots & \cdots & \cdots \\ \widehat{z}_1 + \widehat{z}_2 + \cdots + \widehat{z}_{2m} & \leq & \widehat{\tau}_1 + \widehat{\tau}_2 + \cdots + \widehat{\tau}_{2m}, \end{array} \tag{7}$$

with at least one strict inequality.

Notice that, since τ and z are both efficient, if $z_j \geq \tau_j$ for all $j \in N$ then we would have $z = \tau$, in contradiction with $z \succ_L \tau$. As a consequence, the set $J = \{j \in N \mid z_j < \tau_j\}$ must be non-empty. Take then $q^* = \min\{k \in \{1,2,\ldots,l\} \mid J \cap S_k \neq \emptyset\}$ and choose any $j^* \in J \cap S_{q^*}$.

We claim that

$$z_j \leq \tau_j \text{ for all } j \in S_{q^*}. \tag{8}$$

Indeed, if $q^* = 1$, for all $j \in S_1$ it follows from (7) that $z_j \leq \widehat{z}_1 \leq \widehat{\tau}_1 = \tau_j$. If $q^* > 1$, from $z_j \leq \tau_j$ for all $j \in S_1$ and the definition of j^* and J we get $z_j = \tau_j$ for all $j \in S_1$. Then, from (7) we have $\hat{z}_{|S_1|+1} \leq \hat{\tau}_{|S_1|+1}$. The repetition of the same argument leads to $z_j = \tau_j$ for all $j \in S_1 \cup \cdots \cup S_{q^*-1}$. Then, taking into account (7), a) and b) we obtain, for all $j \in S_{q^*}$,

$$z_j \leq \hat{z}_{|S_1 \cup \cdots \cup S_{q^*-1}|+1} \leq \hat{\tau}_{|S_1 \cup \cdots \cup S_{q^*-1}|+1} = \tau_j.$$

So, let it be $T = S_{q^*}$. Then, by c), $\tau_i = \frac{w_A(T)}{|T|}$ for all $i \in T$ implies $\tau(T) = w_A(T)$ and together with $\tau \in C(w_A)$ implies that $\tau_{|T}$ belongs to the core of the subgame $(T, w_{A_{|T}})$, which is known to be included in the Lorenz-core of coalition T, that is, $\tau_{|T} \in C(w_{A_{|T}}) \subseteq L(T)$. But $\tau_{|T}$ is the equal-split allocation of the subgame $(T, w_{A_{|T}})$ and thus $\{\tau_{|T}\} = E(L(T))$. Then, since $\tau_{j^*} > z_{j^*}$, where $j^* \in T$, and by (8) $\tau_i \geq z_i$ for all $i \in T$, we conclude that $z \notin L(N)$ and reach thus a contradiction. This means that τ is the Dutta-Ray solution $DR(w_A) = \{\tau(w_A)\}$. $\square$

It is known that, for arbitrary assignment games, the Dutta-Ray solution may not exist, since an example is provided in page 621 in Dutta and Ray (1989). Thus, under the dominant diagonal assumption, not only the existence of the Dutta-Ray solution of the assignment game is guaranteed, but we also obtain that it lies in the core. In this sense, the behavior of the Dutta-Ray solution in the class of assignment games with a dominant diagonal resembles that in the class of convex games (it belongs to the core and Lorenz dominates any other core allocation), although dominant diagonal assignment games are in general far from being convex games.

From Theorem 2 we can also deduce a characterization of Thompson's fair division point for arbitrary assignment games. It is known from Núñez and Rafels (2009) that the core of an arbitrary assignment game is the translation, by the vector $(\underline{u},\underline{v})$ of minimum core payoffs, of the core of another assignment game, say $(M\cup M', w_{A^e})$, that has a dominant diagonal: $C(w_A)=\{(\underline{u},\underline{v})\}+C(w_{A^e})$. The analogous translation property trivially holds for Thompson's fair division point: $\tau(w_A)=(\underline{u},\underline{v})+\tau(w_{A^e})$.

By Theorem 2, $\tau(w_{A^e})$ is the unique Lorenz-maximal core allocation and the Dutta-Ray solution of the game $(M\cup M', w_{A^e})$. But this result cannot be translated to $\tau(w_A)$ since, as pointed out by Dutta and Ray (1989), the Lorenz domination is not preserved by translation. However, we can say that $\tau(w_A)$ is the unique core element that allocates in an egalitarian way what remains of $w_A(M\cup M')$ after having paid the vector of minimum rights $(\underline{u},\underline{v})$.

More precisely, for any arbitrarily fixed vector $m\in\mathbb{R}^N$, let us define the m-Lorenz domination in $\mathbb{R}^N$ by: for all $x,y\in\mathbb{R}^N$, $x\succ_{L_m} y$ if and only if $x-m\succ_L y-m$. Then, the following corollary holds.

Corollary 3. *Let $(M\cup M', w_A)$ be a square assignment game, $m=(\underline{u},\underline{v})$ the vector of minimum core payoffs and $\tau(w_A)$ the Thompson's fair division point. Then, $\tau(w_A)\succ_{L_m}(u,v)$ for all $(u,v)\in C(w_A)$.*

We have seen in this section that, for assignment games with a dominant diagonal, the Thompson's fair division point fulfills the fairness standards of the Dutta-Ray solution and the Lorenz undomination. What about the standards represented by the kernel and the nucleolus?

4 The kernel of 2-dominant diagonal assignment games

Even in the case of assignment games with a dominant diagonal, the kernel may contain infinitely many allocations. Take for instance a square glove market with $m \geq 2$, where the kernel is known to coincide with the core segment. In this example Thompson's fair division belongs to the kernel, but it is not difficult to find instances of assignment games with a dominant diagonal and such that Thompson's fair division point is not in the kernel.

Example 4. *Let it be the* 2×2 *assignment game defined by the matrix*

$$A = \begin{pmatrix} 8 & 5 \\ 3 & 8 \end{pmatrix}.$$

The above market has a dominant diagonal but Thompson's fair division point is $\tau(w_A) = (u_1, u_2; v_1, v_2) = (4,4;4,4)$ and does not belong to the kernel, since

$$\begin{aligned} s_{11'}(\tau(w_A)) &= \max\{-u_1, a_{12} - u_1 - v_2\} = -3, \\ s_{1'1}(\tau(w_A)) &= \max\{-v_1, a_{21} - u_2 - v_1\} = -4. \end{aligned}$$

In fact, in this example, the kernel reduces to one point, $\mathscr{K}(w_A) = \{(4\frac{1}{3}, 3\frac{2}{3}; 3\frac{2}{3}, 4\frac{1}{3})\}$, and as a consequence this is also the nucleolus of the game.

In Rochford (1984) the payoff vectors in the kernel of the assignment game are characterized as the set of fixed points of a function defined in terms of the threats (which can be interpreted as the bargaining ranges outside the core). An alternative characterization is given in Driessen (1999) by applying Brouwer's Fixed Point Theorem to a function defined in terms of the length of bargaining ranges within the core. To obtain our result we need to work a little on the function used in Driessen's characterization.

Let $(M \cup M', w_A)$ be a square assignment game and $C_u(w_A) = \{u \in \mathbb{R}^M \mid \text{there exists } z = (u,v) \in C(w_A)\}$ be the projection of $C(w_A)$ to the set of the buyers' payoffs. For each $u \in C_u(w_A)$ we denote by $z_u = (u,v)$ the unique element in $C(w_A)$ that projects to u. Then, $C_u(w_A)$ is a complete lattice with respect to the usual order in $\mathbb{R}^M$, that is, given $u, u' \in R^M$, $u \leq u'$ if and only if $u_i \leq u'_i$ for all $i \in M$.

Let us now consider the function $f : C_u(w_A) \longrightarrow C_u(w_A)$ defined, for all $u \in C_u(w_A)$ and all $i \in M$, by

$$f_i(u) = u_i + \frac{1}{2}\left[s_{i\mu(i)}(z_u) - s_{\mu(i)i}(z_u)\right], \tag{9}$$

where μ is an arbitrarily fixed optimal matching of (M,M',A). Then, by Driessen (1999), if $z=(u,v)\in C(w_A)$, we have

$$z=(u,v)\in \mathscr{K}(w_A) \quad \Leftrightarrow \quad f(u)=u.$$

If we assume, without loss of generality, that the optimal matching μ is placed on the main diagonal, $\mu=\{(i,i)\mid i\in M\}$, then we can write, for any $z=(u,v)\in C(w_A)$,

$$\begin{aligned}
s_{ii'}(z) &= \max_{k\in M\setminus\{i\}}\{-u_i, a_{ik}-u_i-v_k\} = -u_i+\max_{k\in M\setminus\{i\}}\{0, a_{ik}-a_{kk}+u_k\},\\
s_{i'i}(z) &= \max_{k\in M\setminus\{i\}}\{-v_i, a_{ki}-u_k-v_i\} = -a_{ii}+u_i+\max_{k\in M\setminus\{i\}}\{0, a_{ki}-u_k\}
\end{aligned}$$

and obtain, for any $u\in C_u(w_A)$,

$$\begin{aligned}
f_i(u) &= u_i+\frac{1}{2}\left[s_{ii'}(z_u)-s_{i'i}(z_u)\right] && (10)\\
&= \frac{1}{2}a_{ii}-\frac{1}{2}\max_{k\in M\setminus\{i\}}\{0, a_{ki}-u_k\}+\frac{1}{2}\max_{k\in M\setminus\{i\}}\{0, a_{ik}-a_{kk}+u_k\}. && (11)
\end{aligned}$$

It is not difficult to check (and it is left to the reader) that f is a non-decreasing function on $C_u(w_A)$, with respect to the usual order on $\mathbb{R}^M$. Then, as a result of Tarski's Fixed Point Theorem[6] the kernel $\mathscr{K}(w_A)$ of the assignment game is a complete lattice.

Now, in order to guarantee the inclusion of Thompson's fair division point in the kernel, we strengthen the dominant diagonal condition.

Definition 5. *A square assignment game $(M\cup M', w_A)$ is said to have a* 2-dominant diagonal *if and only if, once placed an optimal matching μ on the main diagonal, for all $i\in M$, it holds*

$$a_{ii}\geq 2\max\{a_{ij}, a_{ji}\} \textit{ for all } j\in M\setminus\{i\}. \qquad (12)$$

If an optimal matching is placed on the main diagonal, a square assignment game has a 2-dominant diagonal if and only if each agent achieves at least twice as much profit with his optimal partner as with any other possible partner. Notice that, trivially, if an assignment game has a 2-dominant diagonal, then it also has a dominant diagonal. Moreover,

[6] If $(S,\leq)$ is a complete lattice and $f: S\longrightarrow S$ an increasing function, then, the set E of fixed points of f is non-empty and $(E,\leq)$ is also a complete lattice.

Definition 5 does not depend on the optimal matching μ that is placed on the diagonal.[7]

It is straightforward to see that if the assignment game has a 2-dominant diagonal, then $\tau(w_A) = \left(\left(\frac{a_{ii}}{2}\right)_{i\in M}, \left(\frac{a_{ii}}{2}\right)_{i\in M} \right)$ belongs to the kernel. Indeed, for all $i \in M$,

$$\begin{aligned} s_{ii'}(\tau(w_A)) &= \max_{k\in M\setminus\{i\}} \left\{ -\frac{a_{ii}}{2}, a_{ik} - \frac{a_{ii}}{2} - \frac{a_{kk}}{2} \right\} = -\frac{a_{ii}}{2}, \\ s_{i'i}(\tau(w_A)) &= \max_{k\in M\setminus\{i\}} \left\{ -\frac{a_{ii}}{2}, a_{ki} - \frac{a_{kk}}{2} - \frac{a_{ii}}{2} \right\} = -\frac{a_{ii}}{2}. \end{aligned}$$

It turns out that, when the assignment game has a 2-dominant diagonal, then $\tau(w_A)$ is in fact the unique allocation in the kernel.

Theorem 6. *Let $(M \cup M', w_A)$ be a square assignment game with a 2-dominant diagonal and $\mu = \{(i,i) \mid i \in M\}$ an optimal matching. Then, the kernel reduces to Thompson's fair division point, that is*

$$\mathscr{K}(w_A) = \left\{ \left(\left(\frac{a_{ii}}{2}\right)_{i\in M}, \left(\frac{a_{ii}}{2}\right)_{i\in M} \right) \right\}.$$

Proof. Let it be $z = (u,v) \in \mathscr{K}(w_A)$, $0 \in \mathbb{R}^M$ and $a = (a_{ii})_{i\in M} \in \mathbb{R}^M$. Then, $0 \le u \le a$. Besides, the fact that u is a fixed point of the function f defined in (9) implies $f^n(u) = u$ for all $n \ge 1$. Thus, since f is non-decreasing, we have $f^n(0) \le u \le f^n(a)$ and we only have to prove that both sequences $\{f^n(0)\}_{n\ge1}$ and $\{f^n(a)\}_{n\ge1}$ converge to $\left(\frac{a_{ii}}{2}\right)_{i\in M}$.

Let us first consider the sequence $\{f^n(0)\}_{n\ge1}$.

[7] We claim that if $(M \cup M', w_A)$ has a 2-dominant diagonal with respect to μ and there exists another optimal matching μ', then, for all $i \in M$, either $\mu(i) = \mu'(i)$ or $a_{i\mu(i)} = 0$. If $L = \{i \in M \mid \mu(i) = \mu'(i)\}$ we can restrict to the submarket with set of agents $(M \setminus L) \cup (M' \setminus \mu(L))$. So, let us assume without loss of generality that $\mu(i) \ne \mu'(i)$ for all $i \in M$. Let then be $K = \{i \in M \mid a_{i\mu(i)} = 0\}$. If $K = M$ we are done. Otherwise notice that from the dominant diagonal condition, we have $a_{ij} = 0$ for all $i \in K$ and $j \in M'$, and $a_{ij} = 0$ if $j \in \mu(K)$ and $i \in M$. As a consequence, the restrictions of μ and μ' to $(M \setminus K) \times M'$ are both optimal matchings for the submarket $(M \setminus K, M', A_{|(M\setminus K)\cup M'})$. But then, taking into account the 2-dominant diagonal condition, we have

$$\sum_{i\in M\setminus K} a_{i\mu(i)} \ge 2 \sum_{i\in M\setminus K} a_{i\mu'(i)} = 2 \sum_{i\in M\setminus K} a_{i\mu(i)},$$

which implies that $a_{i\mu(i)} = 0$ for all $i \in M \setminus K$, in contradiction with $K \ne M$. Once proved the claim it easily follows that $(M \cup M', w_A)$ also has a 2-dominant diagonal with respect to μ'.

Notice that for $n = 1$, by substitution in (11), and taking into account the 2-dominant diagonal condition, we obtain, for all $i \in M$,

$$f_i^1(0) = \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \{a_{ki}\} \geq \frac{a_{ii}}{2} - \frac{1}{2} \max_{(l,j) \in M \times M'} \{a_{lj}\}. \tag{13}$$

Now we prove by induction on $n \geq 2$ that, for all $i \in M$,

$$f_i^n(0) = \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^{n-1}(0)\right\}, \tag{14}$$

and

$$f_i^n(0) \geq \frac{a_{ii}}{2} - \frac{1}{2^n} \max_{(l,j) \in M \times M'} \{a_{lj}\}. \tag{15}$$

Let us first prove (14) and (15) for $n = 2$.

Again by substitution in (11),

$$\begin{aligned} f_i^2(0) &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^1(0)\right\} + \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ik} - a_{kk} + f_k^1(0)\right\} && (16) \\ &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^1(0)\right\} && (17) \end{aligned}$$

where the second equality follows because the 2-dominant diagonal condition implies, for all $k \in M \setminus \{i\}$,

$$a_{ik} - a_{kk} + f_k^1(0) = a_{ik} - a_{kk} + \frac{a_{kk}}{2} - \frac{1}{2} \max_{l \in M \setminus \{k\}} \{a_{lk}\} \leq 0.$$

Moreover, for all $i \in M$, from equation (17), (13) and the 2-dominant diagonal condition,

$$\begin{aligned} f_i^2(0) &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^1(0)\right\} \\ &\geq \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - \frac{a_{kk}}{2} + \frac{1}{2} \max_{(l,j) \in M \times M'} \{a_{lj}\}\right\} \geq \frac{a_{ii}}{2} - \frac{1}{2^2} \max_{(l,j) \in M \times M'} \{a_{lj}\}. \end{aligned}$$

Let us assume, by induction hypothesis, that, for all $k \in M$ and all $n \geq 3$, $f_k^{n-1}(0) = \frac{a_{kk}}{2} - \frac{1}{2} \max_{l \in M \setminus \{k\}} \left\{0, a_{lk} - f_l^{n-2}(0)\right\}$ and $f_k^{n-1}(0) \geq \frac{a_{kk}}{2} - \frac{1}{2^{n-1}} \max_{(l,j) \in M \times M'} \{a_{lj}\}$. First, for all $i \in M$ and taking into account the 2-dominant diagonal condition,

$$\begin{aligned} f_i^n(0) &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^{n-1}(0)\right\} + \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ik} - a_{kk} + f_k^{n-1}(0)\right\} \\ &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^{n-1}(0)\right\} \\ &\quad + \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ik} - a_{kk} + \frac{a_{kk}}{2} - \frac{1}{2} \max_{l \in M \setminus \{k\}} \left\{0, a_{lk} - f_l^{n-2}(0)\right\}\right\} \\ &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^{n-1}(0)\right\}. \end{aligned}$$

Notice that the above equation implies that $f_i^n(0) \leq \frac{a_{ii}}{2}$ for all $i \in M$ and all $n \geq 3$.

Also,

$$\begin{aligned} f_i^n(0) &= \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - f_k^{n-1}(0)\right\} \\ &\geq \frac{a_{ii}}{2} - \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ki} - \frac{a_{kk}}{2} + \frac{1}{2^{n-1}} \max_{(l,j) \in M \times M'} \{a_{lj}\}\right\} \\ &\geq \frac{a_{ii}}{2} - \frac{1}{2^n} \max_{(l,j) \in M \times M'} \{a_{lj}\}, \end{aligned}$$

where the first inequality follows from the induction hypothesis and the second one from the 2-dominant diagonal condition.

Now, from $\frac{a_{ii}}{2} - \frac{1}{2^n} \max_{(l,j) \in M \times M'} \{a_{lj}\} \leq f_i^n(0) \leq \frac{a_{ii}}{2}$, for all $n \geq 1$, it follows that, for all $i \in M$, $\lim_{n \to \infty} f_i^n(0) = \frac{a_{ii}}{2}$.

Let us now consider the sequence $\{f^n(a)\}_{n \geq 1}$. Similarly, making use of (11) and the 2-dominant diagonal condition, we prove that $f_i^1(a) = \frac{1}{2} a_{ii} + \frac{1}{2} \max_{k \in M \setminus \{i\}} \{a_{ik}\}$ and, for all $n \geq 2$,

$$f_i^n(a) = \frac{a_{ii}}{2} + \frac{1}{2} \max_{k \in M \setminus \{i\}} \left\{0, a_{ik} - a_{kk} + f_k^{n-1}(a)\right\}, \tag{18}$$

and

$$f_i^n(a) \leq \frac{a_{ii}}{2} + \frac{1}{2^n} \max_{(l,j) \in M \times M'} \{a_{lj}\}, \tag{19}$$

and then deduce that, for all $i \in M$, $\lim_{n \to \infty} f_i^n(a) = \frac{a_{ii}}{2}$.

Finally, from $f_i^n(0) \leq u_i \leq f_i^n(a)$, taking limits as n goes to infinity, we obtain, for all $i \in M$,

$$\frac{a_{ii}}{2} = \lim_{n \to \infty} f_i^n(0) \leq u_i \leq \lim_{n \to \infty} f_i^n(a) = \frac{a_{ii}}{2},$$

which implies $u_i = \frac{a_{ii}}{2}$ for all $i \in M$. $\square$

The above result, that is, the reduction of the kernel to a single point, is tight in the sense that it cannot be guaranteed for assignment games satisfying $a_{ii} \geq k \max\{a_{ij}, a_{ji}\}$ for all $j \in M \setminus \{i\}$ and $k < 2$, being $\mu = \{(i,i) \mid i \in M\}$ an optimal matching. This is shown by the next example.

Example 7. *Let us consider a family of assignment markets with set of buyers* $M = \{1,2\}$, *set of sellers* $M' = \{1', 2'\}$ *and, for each* $0 < \varepsilon \leq 8$, *assignment matrix*

$$A^{\varepsilon} = \begin{pmatrix} 8 & 4 \\ 4+\varepsilon & 8 \end{pmatrix}.$$

We show that in all these markets the kernel does not reduce to a unique point.

Let us see that the segment with extreme points $B=\left(4-\frac{\varepsilon}{2},4;4+\frac{\varepsilon}{2},4\right)$ and $C=\left(4,4+\frac{\varepsilon}{2};4,4-\frac{\varepsilon}{2}\right)$ is contained in the kernel. Let, for $\lambda\in[0,1]$,

$$x_{\lambda}=\left(4-\frac{\varepsilon}{2}\lambda,4+\frac{\varepsilon}{2}(1-\lambda);4+\frac{\varepsilon}{2}\lambda,4-\frac{\varepsilon}{2}(1-\lambda)\right)$$

be an arbitrary element in this segment. Then,

$$\begin{aligned} s_{11'}(x_{\lambda}) &= \max\left\{-4+\frac{\varepsilon}{2}\lambda,-4+\frac{\varepsilon}{2}\right\}=-4+\frac{\varepsilon}{2}, \\ s_{1'1}(x_{\lambda}) &= \max\left\{-4-\frac{\varepsilon}{2}\lambda,-4+\frac{\varepsilon}{2}\right\}=-4+\frac{\varepsilon}{2}. \end{aligned}$$

and

$$\begin{aligned} s_{22'}(x_{\lambda}) &= \max\left\{-4-\frac{\varepsilon}{2}(1-\lambda),-4+\frac{\varepsilon}{2}\right\}=-4+\frac{\varepsilon}{2}, \\ s_{2'2}(x_{\lambda}) &= \max\left\{-4+\frac{\varepsilon}{2}(1-\lambda),-4+\frac{\varepsilon}{2}\right\}=-4+\frac{\varepsilon}{2}, \end{aligned}$$

which implies $x_{\lambda}\in\mathscr{K}(w_A)$ for all $\lambda\in[0,1]$. Thus, the condition of 2-dominant diagonal cannot be weakened and still guarantee the reduction of the kernel to a single point.

In fact, it can be proved that the kernel of the above example reduces to the segment $[B,C]$.

Since for all $0<\varepsilon<4$ the above assignment game has a dominant diagonal, by Theorem 2 the set of Lorenz-maximal core allocations reduces to the Thompson's fair division point that is (4,4;4,4) and does not belong to the kernel. Hence, this example also shows that, for arbitrary assignment markets, $L(w_A)$ and $\mathscr{K}(w_A)$ may be disjoint.

Let us remark to end this section that, for assignment games with a 2-dominant diagonal, it has been established in the paper (Theorems 2 and 6) that the best-known core-based solutions, that is to say the kernel, the Lorenz-maximal core allocations, the nucleolus, the Dutta-Ray solution and the τ-value recommend the same allocation, which is Thompson's fair division point. Thus, for this class of assignment games the Thompson's fair division is supported by the different standards of fairness of all these game-theoretical solutions.

Corollary 8. *Let $(M\cup M',w_A)$ be a square assignment game with a 2-dominant diagonal. Then*

$$L(w_A)=\mathscr{K}(w_A)=\{\nu(w_A)\}=\{\tau(w_A)\}=DR(w_A).$$

5 Concluding remarks

In the first part of the paper (Section 3) we have proved that, for those assignment games with a dominant diagonal, the Thompson's fair division is the unique Lorenz-maximal core allocation and coincides with the Dutta-Ray solution. The proof relies on the fact that, as a consequence of the dominant diagonal assumption, Thompson's fair division point is the allocation where, given an optimal matching μ, each pair optimally matched by μ shares equally the join profit achieved, that is, $u_i = v_{\mu(i)} = \frac{a_{i\mu(i)}}{2}$ for all $i \in M$, and this payoff vector lies in the core.

But there are instances where this allocation $\left(\left(\frac{a_{i\mu(i)}}{2}\right)_{i\in M}, \left(\frac{a_{i\mu(i)}}{2}\right)_{i\in M}\right)$ also lies in the core, although the assignment game does not have a dominant diagonal. In these cases, the same proof as in Theorem 2 guarantees that this is the unique Lorenz-maximal core allocation and coincides with the Dutta-Ray solution, but may not coincide with Thompson's fair division point.

This is the situation in the following example with set of buyers $M = \{1,2\}$, set of sellers $M' = \{1',2'\}$, and assignment matrix

$$A = \begin{pmatrix} 9 & 8 \\ 7 & 7 \end{pmatrix}.$$

There is only one optimal matching, $\mu = \{(1,1'),(2,2')\}$ and notice that the market does not have a dominant diagonal. Thompson's fair division point is $\tau(w_A) = (5, 3\frac{1}{2}; 4, 3\frac{1}{2})$. The allocation where each optimal pair shares equally the achieved profit is $(4\frac{1}{2}, 3\frac{1}{2}; 4\frac{1}{2}, 3\frac{1}{2})$ and it can be checked that it belongs to the core. Thus, the Dutta-Ray solution is $DR(w_A) = \{(4\frac{1}{2}, 3\frac{1}{2}; 4\frac{1}{2}, 3\frac{1}{2})\}$, that differs from Thompson's fair division point. Notice that this example also shows that, for arbitrary assignment games, Thompson's fair division point need not be a Lorenz-maximal core allocation: in this example $DR(w_A) \succ_L \tau(w_A)$.

However, by Corollary 3 we know that if we subtract the vector of minimum core payoffs, $m = (\underline{u}, \underline{v}) = (1,0;0,0)$, from both allocations, then $\tau(w_A) - (\underline{u}, \underline{v}) \succ_L DR(w_A) - (\underline{u}, \underline{v})$, that is, $\tau(w_A) \succ_{L_m} DR(w_A)$. In this sense, and in the case of arbitrary assignment markets, some egalitarian principle, once the minimal rights have been paid, supports Thompson's fair division point.

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