European University Institute

Department of Economics

Risk attitudes and measures of value for risky lotteries

Michal Lewandowski

Thesis submitted for assessment with a view to obtaining the degree of
Doctor of Economics of the European University Institute

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Declaration of Authorship

I, Michał Lewandowski, declare that this thesis titled, 'Risk attitudes and measures of value for risky lotteries' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Lottery: A tax on people who are bad at math

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Abstract

Department of Economics

Doctor of Philosophy

by Michał Lewandowski

The topic of this thesis is decision-making under risk. I focus my analysis on expected utility theory by von Neumann and Morgenstern [1]. I am especially interested in modeling risk attitudes represented by Bernoulli utility functions that belong to the following classes: Constant Absolute Risk Aversion, Decreasing Absolute Risk Aversion (understood as strictly decreasing) and in particular a subset thereof - Constant Relative Risk Aversion. I build a theory of buying and selling price for a lottery, the concepts defined by Raiffa [2], since such theory proves useful in analyzing a number of interesting issues pertaining to risk attitudes' characteristics within expected utility model. In particular, I analyze the following:

- Chapter 2 - expected utility without consequentialism, buying/selling price gap, preference reversal, Rabin [3] paradox
- Chapter 3 - characterization results for CARA, DARA, CRRA, simple strategies and an extension of Pratt [4] result on comparative risk aversion
- Chapter 4 - riskiness measure and its intuition, extended riskiness measure and its existence, uniqueness and properties

Keywords: decision-making under risk, lottery, gamble, expected utility theory, risk attitudes, CARA, DARA, CRRA, buying and selling price for a lottery

JEL Classification Numbers: D81, D03, C91

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Symbols and notation

$U(\cdot)$	Bernoulli utility function
$\mathbf{x}$	lottery/ gamble
$(x_1, p_1; x_2, p_2; ...; x_n, p_n)$	n-dimensional lottery with values x_i and probabilities p_i
W	wealth
$\min \mathbf{x}$	minimum in the support of $\mathbf{x}$
$S(W, \mathbf{x})$	selling price for a lottery $\mathbf{x}$ at wealth W
$B(W, \mathbf{x})$	buying price for a lottery $\mathbf{x}$ at wealth W
$CE(\mathbf{x})$	certainty equivalent of $\mathbf{x}$
$\succ_C$	direct choice relation
$s(W, \cdot)$	simple strategy of an individual at wealth W
$A_\alpha(x)$, $A(x)$, $ARA(x)$	absolute risk aversion function evaluated at x
$RRA(x)$	relative risk aversion function evaluated at x
$p(W, h)$	probability premium
$\tau(W)$	relative spread between buying and selling price
$U_\alpha(x)$	CRRA utility function
$L(\mathbf{x})$	maximal loss of $\mathbf{x}$
$M(\mathbf{x})$	maximal gain of $\mathbf{x}$
$R_{FH}(\mathbf{x})$	Foster and Hart [5] riskiness measure
$R(\mathbf{x})$, $R^*(\mathbf{x})$	extended riskiness measure
λ^*	an inverse of extended riskiness measure
$\mathbb{R}$	space of real numbers
$\mathbb{R}^+$	space of non-negative real numbers
$\mathbb{R}^{++}$	space of strictly positive real numbers
$\mathbb{Z}$	space of integers
$\mathrm{E}[\cdot]$	expected value operator
$\mathrm{P}[E]$	probability of event E
$f'(x)$, $f''(x)$, $f'''(x)$	the first, the second and the third derivative of function f
$\frac{\partial f(x,y)}{\partial x}$	partial derivative of f with respect to x

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Chapter 1

Introduction

Expected utility theory and related studies on individual risk attitudes are central to many core areas of economic theory mainly due to its great simplicity and normative appeal.

Since 1738 and Bernoulli [6] famous St. Petersburg paradox economists who tried to describe individual preferences in a systematic manner were appealed by the idea of expected utility maximization. It was however not until 1944 and von Neumann and Morgenstern [1] seminal work when they realized what expected utility maximization really entails in terms of real-world primitives. A step in this direction was made when the idea of revealed preference was introduced. Economists assume that people have preference relation over a certain set of consequences and they are entitled to do so since by making choices people reveal their preferences. If a preference relation satisfies certain set of requirements called axioms, then this preference relation can be represented by expected utility. And since preference relation is derived from choices really made, there is a well defined link between expected utility maximization (artificial concept) and choices made by people (real world). Ignoring some technical requirements, the following are the main axioms of expected utility. An individual can rank all alternatives in a choice set only if his preference relation is complete and transitive. This weak order result is due to Cantor (1915)[1]. Hence with a preference relation defined over lotteries that is complete and transitive, there exists an index or utility function representing this preference relation. A great contribution of von Neumann and Morgenstern [1] was to introduce independence axiom. It allows the index representing preference relation over lotteries to be separable (in the summation) across states of the world and linear in probabilities of these states. With independence axiom, one can separate consequences in different states of the world, find a utility index representing these consequences and sum them up as weighted average, where weights are probabilities of these states.

[1] See Kreps [7].

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Twenty years later, Pratt [4] and Arrow [8] showed that within expected utility attitudes towards risk may be represented solely the concavity properties of Bernoulli utility function. They also defined an index representing risk aversion and explored its characteristics and connections to different forms of utility functions.

This thesis is devoted to the study of risk attitudes under expected utility. Throughout the subsequent chapters I will use and analyze two concepts, namely buying price for a lottery and selling price for a lottery. These concepts were defined by Raiffa [2] but they were never analyzed in depth. I will show that these measures and the understanding of their characteristics may help in deeper understanding of expected utility model and risk attitudes of Bernoulli utility function. I will use buying and selling price for a lottery and their derived characteristics in three different contexts in each of the three chapters that follow.

In the second chapter, I join in the debate led between economists who stress the fact that expected utility is a good normative theory of behavior and economists who stress that expected utility is not a good descriptive theory of behavior. Although in principle there is no tension between these two claims, there seems to be a lot of antagonism between researchers representing these claims. For simplicity, I will use "traditional economists" label when referring to the first and "behavioral economists" label when referring to the second of the two groups. Traditional economists believe that expected utility is a good theory since it is very simple, normatively appealing and, what is related to the latter, has sharp easily testable predictions. Behavioral economists on the other hand stress the fact that in many well documented settings people systematically violate expected utility axioms, most notably independence. They believe that expected utility theory is therefore not a good positive theory of choice. They propose other theories which are meant to accommodate behavior not consistent with expected utility. Such theories are designed to describe behavior better. Despite its descriptive advantages, there are some risks involved in behavioral theories. If one model does not explain certain patterns of behavior one can make it more elastic by increasing the degrees of freedom. The risk is that by increasing the degrees of freedom the theory loses its normative power, i.e. there are fewer testable predictions. Moreover, such theory is usually more complicated. Another risk is that behavioral theories are often designed to explain behavior in certain environment. It might turn out that the same arguments which were used to explain behavior in one environment, may give wrong or even opposite conclusion in the other environments.

Nevertheless, I believe that contrary to what some parts of the debate between traditional and behavioral economists might suggest, there is no fundamental tension between traditional and behavioral approach. These two approaches are to some extent independent and they serve different purposes. While traditional approach focuses on a general parsimonious theory that requires people to be consistent and avoid arbitrage, behavioral economists seek for theories that describe people's behavior accurately in certain well defined environment and such description may be used to support policy decisions in this environment.

Behavioral economists often claim that certain kinds of behavior are not consistent with expected utility. I think that equally important to the question what is not consistent with expected utility theory is the question what exactly is consistent with expected utility. In the second chapter of the thesis I will try to answer this type of question in the following context.

Experiments reveal that people report much higher selling price for a lottery than buying price. It is widely believed that expected utility theory cannot accommodate such evidence. Along the lines of Rubinstein [9] I will argue that large gap between selling and buying price is possible within expected utility framework if one abandons the doctrine of consequentialism with lifetime wealth interpretation of wealth. As an alternative to total lifetime wealth I propose to introduce the concept of gambling wealth proposed by Foster and Hart [5]. I discuss ways to test expected utility with gambling wealth and advantages of this approach. In addition, I compare preference reversal involving selling price valuation with the related phenomenon which I call preference reversal B involving buying price valuation. To this end I introduce a concept of buying/selling price reversal, which is a connecting element between the two kinds of preference reversals. Buying/selling price reversal occurs when given two lotteries the decision maker assigns higher selling price and lower buying price for one lottery as compared to another. I show that within expected utility buying/selling price reversal is possible, it is equivalent to preference reversal B and it does not allow arbitrage whereas preference reversal is not possible and it is susceptible to arbitrage.

The third chapter of the thesis is entirely technical. I present systematic characterization results for three widely used risk attitude classes - CARA, DARA and CRRA. I define a concept of simple strategy. Within expected utility framework simple strategy prescribes whether to accept a gamble or not only on the basis of initial wealth level and the gamble itself. In a dynamic context simple strategy corresponds to Markov stationary strategy. I define wealth-invariant, "wealthier-accept more" and scale-invariant simple strategies. I show that each of these simple strategies is equivalent to the corresponding properties of buying and selling price for a lottery and also to the corresponding risk attitudes' classes of Pratt [4], i.e. CARA, DARA and CRRA respectively. Moreover, I discuss the difference between nominal and multiplicative gambles and the way they can be handled. I show how CARA and CRRA utility class may be derived from simple functional equations belonging to the Cauchy family. Also, I show that buying price for a lottery may be used as an index representing greater risk aversion relation equivalently to other alternatives laid out in Pratt [4].

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Finally, in the fourth chapter I analyze measure of riskiness defined by Foster and Hart [5]. I show simple intuition behind this measure and the model which leads to it. I define an extension of this measure based on decreasing absolute risk aversion utility functions. For the more specialized case of constant relative risk aversion I obtain more precise characteristics of an extended riskiness measure. Necessary and sufficient conditions for existence of such measure are given. In a series of propositions I establish the theoretical relation which holds between an extended riskiness measure and Foster and Hart [5] riskiness measure as a special case and buying and selling price for a lottery. I show how these results may be used to make inferences about the riskiness for gambles with negative expectation or gambles containing no losses. To this end I define riskiness for gambles minus a price for it. I show that if a price for a gamble corresponds to buying or selling price for this gamble for some wealth level, the riskiness measure is always well defined and meaningful.

As I mentioned before, parallel to discussing the above topics of interests, I build a theory of buying and selling price. Such theory may be used in a number of other topics which I did not discuss in this thesis.

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Chapter 2

Buying and selling price for risky lotteries and expected utility theory without consequentialism

Abstract

In this chapter I show along the lines of Rubinstein [9] that within expected utility large buying and selling price gap is possible and Rabin [3] paradox may be resolved if only initial wealth is allowed to be small. It implies giving up the doctrine of consequentialism which may be reduced to requiring initial wealth to be total lifetime wealth of the decision maker. Still, even when initial wealth is allowed to be small and interpreted narrowly as gambling wealth, classic preference reversal is not possible within expected utility. I show that only another kind of reversal which I call preference reversal B is possible within expected utility. Preference reversal B occurs when buying price for one lottery is higher than for another, but the latter lottery is chosen in a direct choice. I demonstrate that classic preference reversal is susceptible to arbitrage whereas preference reversal B is not which suggests that the latter reversal is more rational.

Keywords: expected utility, consequentialism, total wealth, gambling wealth, narrow framing, Rabin [3] paradox, preference reversal, WTA/WTP disparity, buying and selling price for a lottery

2.1 Introduction

Willingness-to-accept or selling price for a lottery is a minimal sure amount of money which a person is willing to accept to forego the lottery. Willingness-to-pay or buying price for a lottery on the other hand is a maximal sure amount of money which a person is willing to pay in order to play the lottery. The disparity between willingness to pay (WTP) and willingness to accept (WTA) is a well-known phenomenon that arises in experimental settings - see Thaler [10] or Kahneman et al. [11]. There is strong belief in the literature that this evidence is not consistent with expected utility theory. Along the lines of Rubinstein [12] I will argue that the source of this belief lies in associating expected utility theory with the doctrine of consequentialism, according to which "the decision maker makes all decisions having in mind a preference relation over the same set of final consequences". This association is harmless when considering Constant Absolute Risk Aversion, as in this case decisions whether to accept a given lottery do not depend on wealth. However as many studies confirm people usually exhibit Decreasing Absolute Risk Aversion[1], in which case wealth effects are present.
In practice the doctrine of consequentialism means that the initial wealth underlying any decision whether to accept or reject a given lottery is assumed to be the decision maker's lifetime wealth. It follows that most lotteries under consideration are small relative to initial wealth and therefore, by Rabin [3] argument for any reasonable level of risk aversion expected utility predicts approximate risk neutrality towards such lotteries. In this case, not only is expected utility incapable of accommodating large spreads between buying and selling price, but also it is inconsistent with risk averse behavior for small gambles[2]. Instead of burying expected utility theory I propose to divorce it from the doctrine of consequentialism, i.e. relax the assumption that initial wealth underlying any decision whether to accept a gamble is total lifetime wealth of the decision maker. If initial wealth is allowed to be small, I will show that expected utility is consistent with large buying/selling price spread, i.e. that within expected utility for reasonable[3] levels of risk aversion one can obtain buying/selling price spread of the magnitude consistent with experimental results. Following this finding I will propose an alternative for consequentialism involving narrow framing. Instead of asserting that preferences are always defined over total lifetime wealth, I will assume that preferences over gambling are defined over gambling wealth, i.e this part of the decision maker's total wealth which he designates for taking gambles. The idea is taken from Foster and Hart [5], although the seeds of this approach, and in particular the idea of separating lifetime wealth and something else for different decision problems, are already in Rubinstein [12]. I will

[1] In this paper decreasing absolute risk aversion means strictly decreasing absolute risk aversion.
[2] "Small" here means "small relative to lifetime wealth".
[3] Consistent with experimental evidence.

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propose several ways for testing the hypothesis of gambling wealth.

There are many papers on the disparity between willingness to accept and willingness to pay for risky lotteries. It is part of a vast literature stream on WTA and WTP valuations in general. For example, Schmidt et al. [13] explain WTA/WTP spread for risky lotteries using prospect theory. They propose the third-generation prospect theory, in which, unlike in the previous versions, reference point is allowed to be random. They show that loss aversion in such model implies positive WTA/WTP gap[4]. In general, there have been many accounts for the disparity based on non-expected utility models. My aim in this chapter is not to offer a better explanation. I am even convinced that specific behavioral theories will fit empirical and experimental evidence better than expected utility model which I analyze. My goal is to show, that large spreads between WTA and WTP, due entirely to wealth effects, are possible within expected utility if only wealth is interpreted narrowly as gambling wealth. The advantage of this approach is that expected utility has stronger normative appeal as compared to many behavioral models. And hence, it is useful to know that certain patterns of preferences, or in this case valuations, can be accommodated not only within behavioral models but also within expected utility.

The approach I take in this chapter in general is not novel. As I mentioned before, Rubinstein [12] claims that a lot of recent confusion around expected utility, which led some researchers to question it as a descriptive theory is caused by associating expected utility theory with the assumption of consequentialism - the idea that there is a single preference relation over the set of lotteries with prizes being the "final wealth levels" such that the decision maker at any wealth level W who has vNM preference relation $\succsim_W$ over the set of "wealth changes" derives that preference from $\succsim$ by $L_1 \succsim_W L_2 \iff W+L_1 \succsim W+L_2$, where L_1 and L_2 are lotteries. Also, Cox and Sadiraj [20] argue that the confusion around expected utility in general, and Rabin's paradox in particular, is caused by the failure in the literature to distinguish between expected utility theories, which stands for all models based on a set of axioms among which there is independence axiom, and a specific expected utility model. They show on the basis on Rabin's argument that the expected utility of income model is capable of accommodating evidence which the expected utility of terminal model cannot accommodate. Finally, Palacios-Huerta and Serrano [19] show that in case of Rabin's paradox, it is the assumption of rejecting small gambles over a large range of wealth levels, and not expected utility, that does not match real-world behavior. For more discussion on Rabin's paradox, see section 2.4.2.4.

My approach in this chapter follows the lines of the aforementioned articles. The difference is that these articles focus on Rabin's paradox and I focus here on buying/selling price or WTA/WTP spread.

[4] In fact, by imposing some symmetry conditions on prospect theory utility function in their model, it is possible to show that loss aversion is equivalent to positive WTA/WTP gap

Related to buying/selling price disparity is the issue of preference reversal analyzed by Grether and Plott [14]. There are two lotteries called the \$-bet and the P-bet both of which promise some prize with some probability and nothing otherwise such that the probability of winning is higher for the P-bet but the prize is bigger for the \$-bet. Preference reversal occurs when selling price for the \$-bet is higher than that for the P-bet but the P-bet is preferred to the \$-bet in a direct choice[5]. A related possibility, which I call preference reversal B occurs when buying price for the P-bet is higher than that for the \$-bet and yet the \$-bet is chosen over the P-bet in a direct choice. I will show that traditional preference reversal is susceptible to arbitrage and is not possible within expected utility, whereas preference reversal B is possible within expected utility and it does not allow arbitrage. This result may suggest that traditional preference reversal is less rational that preference reversal B.

Buying and selling price for a lottery are the concepts introduced by Raiffa [2] in the context of expected utility. More popular perhaps are the terms willingness to pay (WTP) and willingness to accept (WTA), respectively, the terms introduced primarily in the context of non-expected utility theories. Except for the fact that buying and selling price terms were introduced in a different context than WTP and WTA, these terms have the same meaning. Since I focus on expected utility model I will henceforth use the former terms.

The structure of this paper is as follows. First, I introduce the model, its assumptions, definitions of buying and selling price for a lottery and buying/selling price reversal. Then I state a couple of technical propositions which describe the shape and properties of buying and selling price for a lottery for different risk attitudes. The subsequent section contains the main theses of the paper. Focusing on constant relative risk aversion class of utility functions, I demonstrate first that expected utility with consequentialism is likely to predict risk neutral behavior towards most gambles and eventually a gap between buying and selling price becomes negligible. Second, I demonstrate that if the doctrine of consequentialism is abandoned and wealth is allowed to move over the whole domain, significant spreads between buying and selling price are possible due to income effects when wealth is sufficiently small. As a next step, I propose an alternative to consequentialism involving narrow framing. Instead of defining wealth as total lifetime wealth of the decision maker I suggest to use gambling wealth which is that part of the decision maker total wealth which he designates for the purpose of taking gambles. I discuss ways to test gambling wealth hypothesis and then I examine the possibility of what I call preference reversal B which I compare to the related concept of traditional preference reversal. I show that whereas preference reversal B is possible within expected

[5] Experimentally, in order to confirm preference reversal one must show that the asymmetry described above occurs more often than the opposite kind of asymmetry, i.e. when the \$-bet is preferred in a direct choice but the P-bet gets higher selling price.

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utility framework with gambling wealth instead of total lifetime wealth and it does not allow arbitrage opportunities, preference reversal allows arbitrage opportunities and is not possible within expected utility. And finally I conclude. The appendix at the end of this chapter contains proofs of the propositions.

2.2 The model

I start with basic assumptions and definitions.

Assumption 2.1. *Preferences obey expected utility axioms. Bernoulli utility function* $U : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable, strictly increasing and strictly concave.*

Definition 2.2. A lottery $\mathbf{x}$ is a real- and finite-valued random variable with finite support. The space of all lotteries will be denoted $\mathcal{X}$. I define the maximal loss of lottery $\mathbf{x}$ as: $\min(\mathbf{x}) = \min \operatorname{supp}(\mathbf{x})$.

The typical lottery will be denoted as $\mathbf{x} \equiv (x_1, p_1; ...; x_n, p_n)$, where $x_i \in \mathbb{R}$, $i \in \{1, 2, ..., n\}$ are outcomes and $p_i \in [0, 1]$ $i \in \{1, 2, ..., n\}$ the corresponding probabilities. Outcomes should be interpreted here as monetary values. Although most results that follow are true for more general lotteries, the finite support assumption is sufficient for the purposes of this paper. Now I define buying and selling price for a lottery given wealth level along the lines of Raiffa [2]. To avoid repetitions, I will henceforth skip statements of the form: "Given utility function U satisfying assumption 2.1, any lottery $\mathbf{x}$ and wealth W...".

Definition 2.3. I define selling price and buying price for a lottery $\mathbf{x}$ at wealth W as functions denoted, respectively, $S(W, \mathbf{x})$ and $B(W, \mathbf{x})$. Provided that they exist, values of these functions will be determined by the following equations:

$$\mathrm{E}U[W + \mathbf{x}] = U[W + S(W, \mathbf{x})] \tag{2.1}$$

$$\mathrm{E}U[W + \mathbf{x} - B(W, \mathbf{x})] = U(W) \tag{2.2}$$

If utility function is defined over the whole real line as is the case for constant absolute risk aversion, buying and selling price as functions of wealth exists for any wealth level by assumption 2.1. If the domain of utility function is restricted to a part of real line as is the case of constant relative risk aversion utility function analyzed here, I will specify later on in the paper on which domain buying and selling price are defined as functions of wealth.

In economic terms, given an individual with initial wealth W whose preferences are

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represented by utility function $U(\cdot)$, $S(W, \mathbf{x})$ is the minimal amount of money which he demands for giving up lottery $\mathbf{x}$. Similarly, $B(W, \mathbf{x})$ is the maximal amount of money which he is willing to pay in order to play lottery $\mathbf{x}$. Additionally I define a concept of buying/selling price reversal.

Definition 2.4. Given two lotteries $\mathbf{x}$ and $\mathbf{y}$ and some wealth level W, define buying/selling price reversal as:

$$S(W, \mathbf{y}) > S(W, \mathbf{x}) \text{ and } B(W, \mathbf{x}) > B(W, \mathbf{y})$$

This kind of preference pattern may be interpreted as follows. For a given initial wealth, an individual's certainty equivalent for lottery $\mathbf{y}$ is higher than for lottery $\mathbf{x}$, and yet he is willing to pay more to play lottery $\mathbf{x}$ than to play lottery $\mathbf{y}$. In other words, an individual exhibiting buying/selling price reversal, may prefer to buy $\mathbf{x}$ than $\mathbf{y}$ if he does not play any lottery initially. When, on the other hand, he does play the lottery initially, he would prefer to sell $\mathbf{x}$ than $\mathbf{y}$.

2.2.1 Buying short and selling short price for a lottery

It is possible to introduce buying short and selling short for a lottery $\mathbf{x}$ at wealth level W denoted, respectively, by $B^S(W, \mathbf{x})$ and $S^S(W, \mathbf{x})$. They satisfy the following equations:

$$\mathrm{E}U[W - \mathbf{x}] = U[W - B^S(W, \mathbf{x})] \tag{2.3}$$

$$\mathrm{E}U[W - \mathbf{x} + S_S(W, \mathbf{x})] = U(W) \tag{2.4}$$

The interpretation of these two measures is the following: $B^S(W, \mathbf{x})$ is the maximal sure amount of money which an individual would pay for not taking a short position in lottery $\mathbf{x}$. In other words if initial position is $W - \mathbf{x}$, $B^S(W, \mathbf{x})$ is the maximal sure amount of money which an individual is willing to pay for $\mathbf{x}$. On the other hand $S^S(W, \mathbf{x})$ is the minimal sure amount of money which an individual would accept for taking a short position in $\mathbf{x}$. In other words, it is the minimal selling price for a lottery which an individual does not have initially.

Notice that buying price $B(W, \mathbf{x})$ and $S^S(W, \mathbf{x})$ are evaluated with respect to the same initial position W. Using Jensen's inequality it is easy to show that for strictly concave utility function and a nondegenerate lottery $\mathbf{x}$:

$$S^S(W, \mathbf{x}), B^S(W, \mathbf{x}) \in (\mathrm{E}[\mathbf{x}], \max(\mathbf{x}))$$

where $\max(\mathbf{x})$ denotes the maximal consequence in the support of lottery $\mathbf{x}$. As shown in proposition 2.5 below, classical buying and selling price for a lottery are on the other

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hand strictly in between $\min(\mathbf{x})$ and $\mathrm{E}[\mathbf{x}]$ for strictly concave utility function. Hence, for strictly concave utility function, both $S^S(W, \mathbf{x})$ and $B^S(W, \mathbf{x})$ are strictly greater than $S(W, \mathbf{x})$ and $B(W, \mathbf{x})$ for any wealth level.

Certain global (i.e. holding for any lottery) properties of buying and selling price as functions of wealth are "mirrored" by the corresponding global properties of buying short and selling short prices for a lottery. This is due to the simple relation which holds between these measures and which is the following:

$$\begin{aligned} S^S(W, \mathbf{x}) &= -B(W, -\mathbf{x}) \\ B^S(W, \mathbf{x}) &= -S(W, -\mathbf{x}) \end{aligned}$$

So, if for example buying and selling price for any no-degenerate lottery are strictly concave and strictly increasing in W as is the case for CRRA utility function[6], then selling short and buying short prices for any non-degenerate lottery will be strictly decreasing and strictly convex in W. If $0 < B(W, \mathbf{x}) < S(W, \mathbf{x})$ as is the case for DARA[7], then $0 < S^S(W, \mathbf{x}) < B^S(W, \mathbf{x})$.

2.3 Preliminary results

Before introducing the main point of this chapter I need a couple of theoretical results which describe properties of buying and selling price for a lottery for different risk attitudes. The most basic property of buying and selling price which is true for any concave strictly increasing utility function is the following:

Proposition 2.5 (Concave)**.** *For any non-degenerate lottery* $\mathbf{x}$ *and any wealth* W *such that buying and selling price exist,* $S(W, \mathbf{x})$ *and* $B(W, \mathbf{x})$ *lie in the interval* $(\min(\mathbf{x}), \mathbf{E}(\mathbf{x}))$*. For a degenerate lottery* $\mathbf{x}$*,* $S(W, \mathbf{x}) = B(W, \mathbf{x}) = x$*.*

Proof. In the appendix. □

Below I state propositions which characterize constant and decreasing absolute risk aversion utility functions in terms of buying and selling price. Proofs of these propositions may be found in chapter 3 of this thesis.[8]. Also I refer to chapter 3 of this thesis for an extensive discussion on multiplicative and nominal gambles, risk aversion notions for the two kinds of gambles, etc.

[6]See results 3.29, 3.31 and 3.20 in chpater 3.
[7]See lemma 3.30 in chapter 3.
[8]Although the first of these results was proved already by Pratt [4], I refer to my proof due to its unified treatment for the next three propositions.

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Proposition 2.6 (CARA)**.** *The following two statements are equivalent:*

i. Bernoulli utility function exhibits CARA

ii. Buying and selling price are independent from wealth and equal i.e.

$$B(W, \mathbf{x}) = S(W, \mathbf{x}) = C_\alpha, \ \forall W$$

where α is absolute risk aversion coefficient and C_α takes real values and depends only on α.

Proposition 2.7 (DARA)**.** *The following two statements are equivalent:*

i. Bernoulli utility function exhibits DARA

ii. buying and selling price are increasing in W

$$B(W, \mathbf{x}) > 0 \iff B(W, \mathbf{x}) < S(W, \mathbf{x})$$

for a non-degenerate lottery $\mathbf{x}$*.*

The above propositions show that in expected utility model a gap between buying and selling price can only arise due to wealth effects. Selling price is higher than buying price for a lottery for which I would be willing to pay positive amount only if absolute risk aversion decreases in wealth. Since I want to focus on CRRA utility functions which is a subclass of DARA utility functions I will additionally state one more proposition, the proof of which may also be found in Lewandowski [15].

Proposition 2.8 (CRRA)**.** *The following two statements are equivalent:*

i. Bernoulli utility function exhibits CRRA

ii. buying and selling price for any lottery are homogeneous of degree one i.e.

$$\begin{aligned} S(\lambda W, \lambda \mathbf{x}) &= \lambda S(W, \mathbf{x}), \ \forall \lambda > 0 \\ B(\lambda W, \lambda \mathbf{x}) &= \lambda B(W, \mathbf{x}), \ \forall \lambda > 0 \end{aligned}$$

2.4 Buying/selling price spread within expected utility framework

In this section I focus on constant relative risk aversion utility class, since it is simple and empirically well validated. For convenience but without loss of generality I normalize

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Bernoulli utility function as follows:

$$U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & 1 \neq \alpha > 0, \quad x > 0 \\ \log x, & \alpha = 1, \quad x > 0 \end{cases} \tag{2.5}$$

Parameter α is required to be bounded. I also focus on non-degenerate lotteries with non-negative values such that outcome zero gets positive probability. This restriction is a matter of convenience as the forthcoming results extend to the case of general lotteries. The following proposition is necessary to establish the domain and the range of buying and selling price for a lottery as functions of wealth for the case of CRRA functions of the above form. Before I state this proposition a couple of remarks might be useful. First, since CRRA utility function used in this section is defined only for positive real numbers I need to be sure that both sides of equations (2.2) and (2.1) defining buying and selling price are well defined. Second, notice that CRRA function of the above form is unbounded from below for $\alpha \geq 1$ and bounded from below for $0 < \alpha < 1$. This is the reason why for $0 < \alpha < 1$ the infimum of $B(W, \mathbf{x})$ and $S(W, \mathbf{x})$ cannot be equal to $\min(\mathbf{x})$, the lower bound given in proposition 2.5. It turns out that there is a certain threshold denoted by $W_L(\mathbf{x}) \in (0, \mathrm{E}[\mathbf{x}])$ such that the infimum of $B(W, \mathbf{x})$ and $S(W, \mathbf{x})$ is equal to $W_L(\mathbf{x}) + \min(\mathbf{x})$ which is greater than $\min(\mathbf{x})$.

Proposition 2.9 (CRRA2)**.** *Given the class of CRRA utility function of the form given by (2.5) the following holds for any non-degenerate lottery* $\mathbf{x}$*: for* $\alpha \geq 1$

- $\lim_{W \to 0} B(W, \mathbf{x}) = \min(\mathbf{x})$
- $\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = \min(\mathbf{x})$

Define $W_L(\mathbf{x}) = U^{-1}[\mathrm{E}U(-\min(\mathbf{x}) + \mathbf{x})]$*. For* $0 < \alpha < 1$

- $\lim_{W \to W_L(\mathbf{x})} B(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$,
- $\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$

Additionally,

$$\forall \alpha > 0 \ \lim_{W \to \infty} B(W, \mathbf{x}) = \lim_{W \to \infty} S(W, \mathbf{x}) = \mathrm{E}[\mathbf{x}] \tag{2.6}$$

Proof. In the appendix. □

The above proposition establishes the domain and the range of buying and selling price for a given lottery $\mathbf{x}$ as functions of wealth for CRRA utility functions which are defined above. Now that I introduced the necessary theoretical results, I proceed to the main message of this chapter.

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2.4.1 Expected utility and consequentialism

Consequentialism is a doctrine that says that an individual makes all decisions according to a preference relation defined over one set of final consequences. In practice it means that initial wealth taken into account when making whatever decision is interpreted as the decision maker's total lifetime wealth. Most lotteries which a person may encounter are small relative to his lifetime wealth. Especially, lotteries used in experiments have values which are small relative to total lifetime wealth of experimental subjects. Therefore to explain certain experimental results it is sufficient to focus on lotteries that have values which are negligible as compared to total lifetime wealth. To represent this fact I assert here that lotteries have bounded values and consequentialism approximately means that wealth tends to infinity. In this case the following result holds:

Proposition 2.10. *Expected utility with consequentialism and CRRA approximately predicts no buying/selling price spread and risk neutrality.*

Proof. The proof follows directly from equation (2.6) in proposition 2.9. To represent the fact that most lotteries are small relative to lifetime wealth, I take any lottery with bounded values and let wealth go to infinity. What happens is that both selling price and buying price tend to $\mathrm{E}[\mathbf{x}]$ and hence the gap between them vanishes. Since the distance $\mathrm{E}[\mathbf{x}] - S(W, \mathbf{x})$ measures risk aversion, it is clear that there is no risk aversion either. □

This proposition is very similar to Rabin [3] calibration theorem confined to CRRA class of utility functions. Reasonable levels of risk aversion for big gambles give rise to risk neutral behavior towards small gambles within expected utility with consequentialism. The difference between Rabin [3] argument is that I claim after Rubinstein [9] that this is due to consequentialism and not due to expected utility.
This negative result immediately rises the issue of what happens if I drop the assumption of consequentialism. To answer this question I proceed in two steps. First, I show that relaxing consequentialism is promising, i.e. large buying/selling price for a lottery for reasonable levels of risk aversion may be obtained. Second, I propose an alternative assumption which could replace the assumption of consequentialism.
In the first step I allow wealth to vary freely. I will therefore analyze buying and selling price for a lottery as functions of wealth. The goal is to see for what values of wealth is the spread between buying and selling price likely to be high. To save on notation, given a fixed lottery $\mathbf{x}$ I shall write $S(W, \mathbf{x}) = S(W)$ and $B(W, \mathbf{x}) = B(W)$. I define relative spread between buying and selling price as follows:

$$\tau(W) = \frac{S(W) - B(W)}{B(W)}$$

The following lemma can be used to infer certain properties of the relative gap between buying and selling price.

Lemma 2.11. *For differentiable decreasing absolute risk aversion utility function, given any non-degenerate lottery* $\mathbf{x}$ *and any wealth level* W*, the following holds:*

- $B'(W) < 1$
- $S'(W - B(W)) = \frac{B'(W)}{1-B'(W)}$ *and hence* $S'(W - B(W)) > B'(W)$
- $B'(W + S(W)) = \frac{S'(W)}{1+S'(W)}$ *and hence* $B'(W + S(W)) < S'(W)$
- $S'(W) = \frac{S(W)-B(W)}{B(W)}$ *and* $B'(W) = \frac{S(W)-B(W)}{S(W)}$ *for small positive* $S(W)$

Proof. In the appendix. □

Observe that the slope of buying price is always smaller than one whereas the slope of selling price can be higher for small values of wealth. Before I state a proposition describing the characteristics of the relative gap between buying and selling price I need the following lemma:

Lemma 2.12. *For CRRA utility function, given any non-degenerate lottery* $\mathbf{x}$*,* $S(W)$ *and* $B(W)$ *are concave functions.*

Proof. See Lewandowski [15]. □

I focus now on the case when $S(W) > B(W) > 0$. The remaining cases can be analyzed similarly. By proposition 2.9, to make sure that $B(W)$ is positive I require that $\min(\mathbf{x})$ cannot be lower than zero. The following proposition suggests that for CRRA utility function the lower the wealth the higher the relative gap between buying and selling price.

Proposition 2.13. *For CRRA utility function and any lottery* $\mathbf{x}$ *with* $\min(\mathbf{x}) \geq 0$*, the relative gap between buying and selling price* $\tau(W)$ *is strictly decreasing in* W*.*

Proof. In the appendix. □

This proposition already gives an explanation of why buying/selling price gap cannot be predicted within expected utility with consequentialism for small experimental lotteries. The reason is that within expected utility, the gap between buying and selling price is the highest for small values of wealth. So if initial wealth is small, expected utility model can accommodate large buying and selling price gap. Obviously, assuming initial wealth

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to be total lifetime wealth of the decision maker is as far as one can go away from this possibility.

Using lemma 2.11 and proposition 2.13 it is possible to infer certain properties of buying and selling price when data on relative gap between buying and selling price is available. Also, in the opposite direction, it is possible to infer properties of the relative gap between buying and selling price when certain properties of buying and selling price are known. Here, I mention just a couple of possibilities:

- the gap is equal to the slope of selling price for small $B(W)$
- for small values of $B(W)$ the gap is equal to $\frac{B'(W)}{1-B'(W)}$ and hence
- the maximal gap depends on the slope of $B(W)$ for small values of $B(W)$

The above mathematical results can be best illustrated on the basis of an example. Let $\mathbf{x}$ be a lottery giving 100 euros or nothing with equal probabilities. The notation I use for such a lottery is $(100, \frac{1}{2}; 0, \frac{1}{2})$. Table 2.1 contains graphs of selling and buying price for lottery $\mathbf{x}$ on the left and relative spread between them as functions of wealth W on the right, each of them for CRRA utility function for three different coefficients of relative risk aversion: 1/2, 1 and 2[9]. Notice that as stated in propositions above buying and selling price are between $\min(\mathbf{x})$ and $\mathrm{E}[\mathbf{x}]$ for $\alpha = 1$ and $\alpha = 2$. For $\alpha = 0.5$ I can calculate $W_L(\mathbf{x})$ as follows:

$$W_L(\mathbf{x}) = \left(\frac{1}{2}\sqrt{100} + \frac{1}{2}\sqrt{0}\right)^2 = 25$$

Hence buying and selling price for $\alpha = 0.5$ are indeed between $W_L(\mathbf{x}) + \min(\mathbf{x})$ and $\mathrm{E}[\mathbf{x}]$. Notice also that buying and selling price are increasing and strictly concave in wealth and that selling price is higher than buying price over the whole domain of buying and selling price. Finally as stated in proposition 2.13 the relative gap indeed is the highest for the minimal value of wealth for which both buying and selling price are defined.

As illustrated by this simple example and stated formally in the propositions, the smaller the wealth the greater the relative gap between buying and selling price. So if wealth is small enough it is possible to obtain the gap between buying and selling price consistent with experimental evidence for reasonable levels of risk aversion. I will summarize this finding in a proposition.

Proposition 2.14. *For levels of risk aversion which are consistent with experimental evidence on risk attitudes there exists levels of wealth such that the expected utility model predicts high relative gap between buying and selling price.*

[9]The CRRA utility function is of the form given in (2.5).

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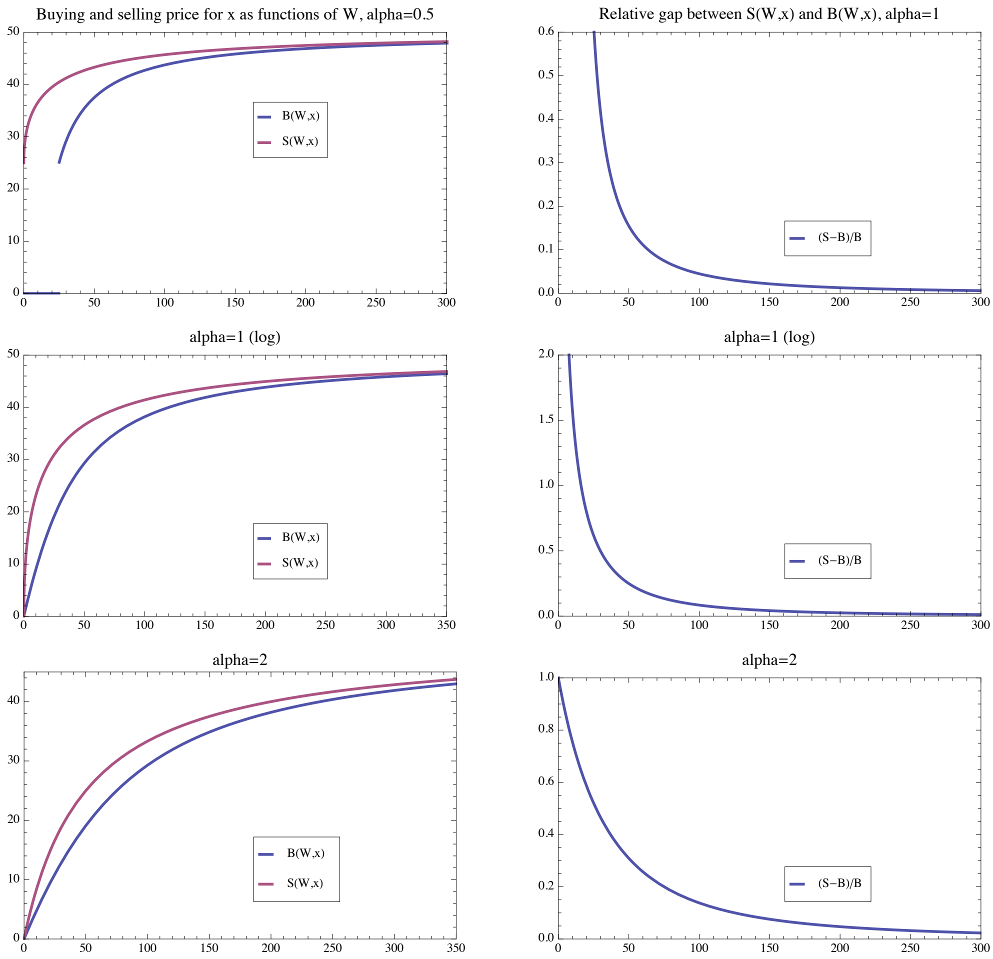
TABLE 2.1: Buying/selling price spread for **x** for CRRA utility function



To illustrate the proposition consider again the above example. For instance, to obtain selling price 30 per cent higher than buying price for the lottery in consideration and for different relative risk aversion coefficients I need wealth levels which are listed in table 2.2.

TABLE 2.2: Selling price 30% higher than buying price

α	W
0.5	35.15
1	43.94
2	51.57

For example to obtain selling price 30% higher than buying price for the lottery $(100, \frac{1}{2}; 0, \frac{1}{2})$ for logarithmic utility function, initial wealth level of almost 44 is necessary. In the next

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subsection I introduce gambling wealth. If one believes that expected utility model accurately predicts behavior 44 would correspond to the calibrated gambling wealth. Assuming that the decision maker exhibits constant relative risk aversion, one can calibrate pairs of wealth and relative risk aversion consistent with any given level of relative gap between selling and buying price for a given lottery.

2.4.2 Expected utility with gambling wealth

I have argued above that expected utility with total wealth interpretation of wealth predicts no gap between buying and selling price and risk neutrality for a wide range of gambles used in experiments. On the other hand I have shown that if small values of wealth are possible one can obtain large gaps between buying and selling price for a lottery for reasonable levels of risk aversion. One way to proceed would be to make wealth a free parameter of the model. Then, if one believes that expected utility is a good descriptive model of behavior, then given the data on risky choices one can calibrate which pairs of risk attitude and wealth level are consistent with the data, as I have illustrated in table 2.2. Unfortunately, by making wealth a free parameter, the model loses much of its predictive power. In particular, it is harder to falsify the model or design testable predictions. Another way to proceed is to give wealth a new interpretation or, even better, to develop a theory of endogenous wealth determination and then to test whether this new interpretation gives better answers than consequentialist interpretation. Since at this point I am unable to offer a theory of endogenous wealth determination, I will only propose a new interpretation of wealth and ways to test it.

2.4.2.1 Gambling wealth

Consequentialism assumption implies that when making any kind of decision people consider and have in mind their lifetime wealth. I think a good alternative assumption is that people frame decisions narrowly and separate them into categories. When they engage themselves in housing decisions they think about housing budget, when they consume they think about consumption budget and when they consider gambling or whether to accept of reject an offered gamble, they consider gambling budget. Of course, personal assignment of different categories, budgets for them and time span for the budgets is a very complex subject and certainly there is plenty of factors which influence such decisions. Therefore I do not aim at a theory of endogenous budget determination. For the purposes of this paper I focus only on gambling category and a budget assigned to it, which I call gambling wealth. Gambling wealth was proposed informally by Foster and Hart [5]. They define gambling wealth as that part of total wealth designated only

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for taking gambles. Alternatively, if $\underline{W}$ is wealth designated for the purposes of living, housing and consumption, then gambling wealth is what is left over.
In the light of the results from previous subsection, one can argue that the idea of gambling wealth and more generally, the idea of separate budgets for different categories of decisions could explain a number of interesting phenomena, for example:

- Agents who gamble more, have higher gambling wealth and therefore buying and selling price gap for a given lottery is smaller than for less experienced individuals
- If an object is treated narrowly the disparity should be higher; if it is integrated into a wider set of objects the disparity should decrease (Hanemann [16])
- The disparity should also be higher for artificial environments such as experiments than for a real market place.

This approach also has potential of explaining why buying/selling price gap is more pronounced when objects of choice are not monetary, e.g. coffee mugs. The more specific or narrowly defined is the object of choice the more pronounced are wealth or income effects since the value of the object is comparable with the money designated for taking such objects.
The attractive feature of all these explanations is that they are all within expected utility framework. The only novel thing is narrow framing with which expected utility model is supplemented. Naturally, a theory of endogenous wealth determination would be much appreciated to make this kind of explanations fully testable. At this point, I may suggest a couple of ways to test gambling wealth hypothesis.
I propose the following experiment design which could shed light on the validity of this approach. The first stage of experiment is to give people small amount of money for trading in gambles and then to elicit buying price and selling price for a given lottery. It is possible to use sealed bid second price auction to elicit the true buying price and Becker et al. [17] procedure to elicit the true selling price for a lottery. In the second stage subjects are given more money for trading in gambles and again buying and selling price is elicited. Alternatively, instead of giving the subjects more money it is possible to scale down or up the lotteries being played. If subjects exhibit constant relative risk aversion it should be equivalent to increasing or decreasing initial wealth - here the characterization results from Lewandowski [15] are useful. If my explanation for the gap between buying and selling price is correct then the gap should decrease when subjects are given more gambling money or if the lotteries are scaled down without changing gambling wealth.
The second experiment design is the following - I show some possible lotteries to the subjects. Then I ask them how much money maximally, they would risk playing these

lotteries. Their answer would correspond to their gambling wealth. Then I again repeat the procedure as in the first experiment design.

Another way to test the approach would be to elicit buying and selling prices for objects from a very narrowly defined set, such as coffee mugs and then extend the set of objects to say all kitchen stuff and again elicit buying and selling prices. The gap between reported buying and selling price in the first case should be bigger than the one in the second case. The reason is that the money designated for trading in coffee mugs is definitely no bigger than money designated to trade in all kitchen stuff. This also would be consistent with results of Hanemann [16]. He argues that buying/selling price gap should be small if there is some substitute on the market and should be bigger if there is no.

Assuming the approach is valid then I propose the following experiment for calibrating gambling wealth. The experiment should be designed to test risk attitudes[10] and at the same time to elicit selling and buying prices for lotteries. Given the data it is then easy to calculate the underlying wealth level. This is then interpreted as gambling wealth. More precisely, given observed buying and selling price for a given lottery I can calculate wealth-relative risk aversion coefficient pair which is consistent with these prices.

Gambling wealth hypothesis is promising. However, until there is no theory of gambling wealth interpretation it can not be fully testable. In the next subsection I discuss another concept which is related to gambling wealth - the concept of pocket cash by Fudenberg and Levine [18]. The advantage of pocket cash idea is that there is a theory of pocket cash determination. I would like to show in what respect pocket cash and gambling wealth are similar and in what respect they differ.

2.4.2.2 Pocket cash

The idea of pocket cash money in the context of gambling decisions is the following. If a small gamble is offered, an individual decides whether to take it or not on the basis of what he has in his pockets, and hence pocket cash will be the relevant wealth level for this decision. If, on the other hand, the same individual is offered a big gamble the values of which exceed significantly what he has in his pockets, the individual decides more carefully taking into account his lifetime wealth. I will introduce now some details of the model.

Fudenberg and Levine [18] develop a dynamic model in which long-run self controls the series of short-run selves. In each period t there are two subperiods:

- bank subperiod

[10] Characterization results from Lewandowski [15] are useful here.

    - consumption is not possible
    - wealth y_t is divided between savings s_t, which remain in the bank, and pocket cash x_t which is carried to the nightclub

- nightclub subperiod
    - consumption $0 \leq c_t \leq x_t$ is determined and $x_t - c_t$ is returned to the bank at the end of the period
    - wealth next period is $y_{t+1} = R(s_t + x_t - c_t)$

The long-run self can implement a^*, the optimum of the problem without self-control, by simply choosing pocket cash $x_t = (1 - a^*)y_t$ to be the target consumption. In this way self-control costs might be avoided.

- At the nightclub in the first period there is a small probability the agent will be offered a choice between several lotteries.
- The model predicts then that:
    - for large gambles risk aversion is relative to wealth
    - for small gambles it is relative to pocket cash

In this way the model can explain Rabin [3] paradox and large buying and selling price gap.

2.4.2.3 Gambling wealth vs. pocket cash

An interesting feature of Fudenberg and Levine [18] approach is the following. Fudenberg and Levine [18] estimate pocket cash to be roughly in the range of 20-100 dollars. This is very similar to the range of gambling wealth necessary to get large and consistent with the evidence buying/selling price gaps as indicated in table 2.2. Even if not supported by the thorough econometric analysis it is striking that two totally different approaches give rise to results of a very similar range.

In spite of the similarities, the two concepts are nevertheless different from each other. To illustrate the difference I will now discuss what testable predictions are obtained in Rabin [3] paradox according to the dual self model with pocket cash and what testable prediction are obtained according to gambling wealth approach.

Rabin [3] calibrated that expected utility model predicts the following:

- if a risk averse agent with wealth ≤ 350000 rejects the lottery $(105, 1/2; -100, 1/2)$

- then he should reject the lottery $(635000, 1/2; -4000, 1/2)$ at wealth level 340000

Denote the first of the above lotteries by lottery 1 and the second by lottery 2. According to Rabin [3] the first statement is plausible and the second is not and hence it is called a paradox.
In the dual-self model it is not true anymore that the decision maker rejects both lotteries. The first lottery is small and hence it is evaluated relative to pocket cash. The second lottery is big and therefore it is evaluated according to total wealth. Suppose that the utility function is logarithmic. Then the following is true:

- lottery 1 small - reject if pocket cash < 2100
- lottery 2 large - accept if total wealth higher than 4035

Now both statements (pocket cash less than 2100 and total wealth higher than 4035) are plausible.
Now consider gambling wealth interpretation. Suppose that utility is logarithmic. If gambling wealth is less than 2100 then the decision maker should

- reject lottery 1
- reject lottery 2

There is nothing paradoxical in rejecting the second lottery since gambling wealth in the amount of 2100 is too little to cover the loss (-4000) which occurs with probability 1/2. No matter how attractive is the second prize, the decision maker cannot afford to take lottery 2.

2.4.2.4 Rabin's paradox in the literature

Although, in this paper, I adopt the lines of Rubinstein [9], and focus on the assumption of consequentialism, there has been other explanations for Rabin's paradox in the literature. Palacios-Huerta and Serrano [19] claim that it is the assumption of rejecting small gambles over a large range of wealth levels, which should be questioned as it does not match real-world behavior. In particular they show that the assumption that an expected utility maximizer turns down a given even-odds gamble with gain and loss for a given range of wealth levels implies that there exists a positive lower bound on the coefficient of absolute risk aversion which can be calculated exactly. This lower bound is an additional assumption imposed on a utility function. [19] show that in Rabin's examples this lower bound turns out to be very high, which is not consistent with empirical evidence. Another paper which addresses Rabin's critique of expected utility is Cox and Sadiraj [20]. They argue that the source of confusion around expected utility lies in a failure to distinguish between expected utility theories, i.e. all models based on a set of axioms with independence axiom being the key axiom, and a specific expected utility model. In a similar spirit to Rubinstein [12], they claim that Rabin in fact criticizes expected utility model of terminal wealth, in which there is a single preference relation over final wealth consequences. They show that expected utility of income model, in which prizes are interpreted as changes in wealth levels, does not exhibit Rabin's paradoxical behavior. In order to enable the dependence of preference over income on initial wealth, they design an expected utility of initial wealth and income model. They demonstrate that such a model can withstand the Rabin's critique if initial wealth is not additive to income in the utility function. Safra and Segal [21] on the other hand point out that paradoxes of the kind considered by Rabin, are not specific to expected utility theory. They show that they can be constructed in non-expected utility theories as well.

2.5 Preference reversal versus buying/selling price reversal

Preference reversal is commonly observed in experiments. Suppose that $A \succ_C B$ denotes "A preferred to B in a direct choice". Using my notation, preference reversal is possible if:

$$S(W, \mathbf{y}) > S(W, \mathbf{x}) \quad \text{and} \quad \mathbf{x} \succ_C \mathbf{y}$$

Preference reversal is not possible within expected utility framework. To see this, note that expected utility implies that $\mathbf{x} \succ_C \mathbf{y}$ which can be equivalently written as $\mathrm{E}U(W + \mathbf{x}) > \mathrm{E}U(W + \mathbf{y})$. By definition of S, this is equivalent to $U(W + S(W, \mathbf{x})) > U(W + S(W, \mathbf{y})$ and since utility function is strictly increasing: $S(W, \mathbf{x}) > S(W, \mathbf{y})$. So expected utility implies the following:

$$S(W, \mathbf{y}) > S(W, \mathbf{x}) \quad \Longleftrightarrow \quad \mathbf{y} \succ_C \mathbf{x} \tag{2.7}$$

On the other hand buying/selling price is possible within expected utility framework:

Proposition 2.15. *For a given decreasing absolute risk aversion utility function and any wealth level W, buying/selling price reversal is possible.*

Proof. In the appendix. □

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By condition (2.7) this proposition implies that expected utility admits the possibility of the following kind of preference reversal:

$$B(W, \mathbf{y}) > B(W, \mathbf{x}) \text{ and } \mathbf{x} \succ_C \mathbf{y}$$

This kind of preference reversal will be referred to as preference reversal B. Preference reversal B is equivalent to buying/selling price reversal within expected utility framework.

Since expected utility theory imposes rather strong consistency assumptions, the result above suggests that the possibility of preference reversal is less rational than the related possibility of buying/selling price reversal or preference reversal B. The following two propositions clarify the meaning of "less rational" beyond the strength of consistency requirements argument.

Proposition 2.16. *Suppose that preferences of the decision maker are continuous, monotonic and that preference reversal pattern is fixed for the range of wealth* $W \in [\underline{w}, \bar{w}]$*. Then arbitrage opportunities exist.*

Proof. In the appendix. □

Hence preference reversal allows arbitrage. On the other hand buying/selling price reversal or preference reversal B does not allow arbitrage.

Proposition 2.17. *Buying/selling price reversal does not allow arbitrage.*

Proof. In the appendix. □

The analysis shows that buying/selling price reversal or preference reversal B is more rational than traditional preference reversal in two respects - it is consistent with expected utility and it does not allow arbitrage.

Preference reversal B or buying/selling price reversal occur within expected utility theory. However it does not mean that they have to be meaningful. If buying/selling price gap is small, then these two reversals are not meaningful i.e. they can occur theoretically but the scope for their occurrence is negligible. For these reversals to be meaningful, it is necessary for buying/selling price gap to be non-negligible. Testing of preference reversal B might be therefore relevant only if wealth is interpreted narrowly, either as gambling wealth or pocket cash. It is not relevant if the doctrine of consequentialism is maintained. I will illustrate this fact in the following example.

Example 2.1. *Suppose utility function is CRRA with relative risk aversion coefficient of* 2*, the \$-bet (denote it by* $\mathbf{x}$*) gives* \$100 *or* \$0 *with equal probabilities and the P-bet* *(denote it by* $\mathbf{y}$*) gives* \$40 *with probability* 3/4 *and* \$0 *otherwise. The picture below graphs buying and selling prices for these two lotteries as functions of wealth:*

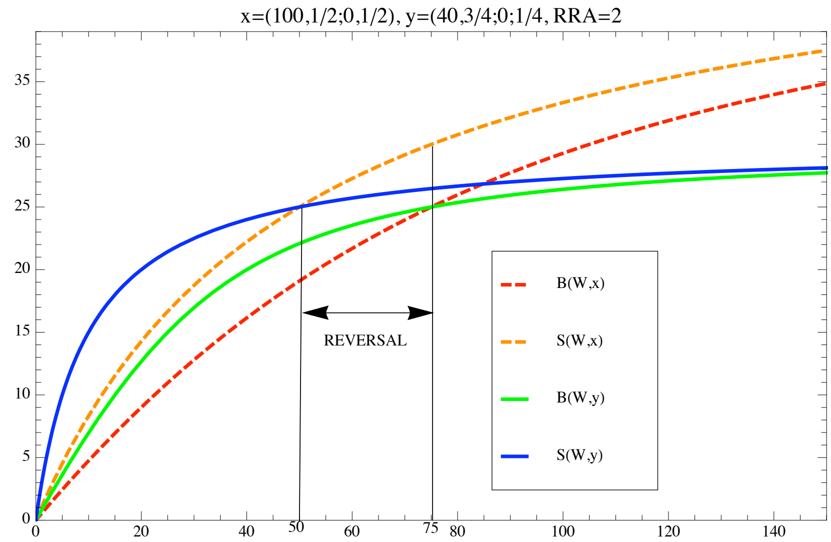

FIGURE 2.1: Wealth region for buying/seeling price reversal

In the above example, there is an interval $(50, 75)$ of wealth for which buying/selling price reversal (and hence also preference reversal B) occurs[11]. This is the common pattern that buying/selling price reversal occurs only at small wealth and only in the limited interval of wealth. The reason is that for such reversal to occur the \$-bet has to have higher variance and higher expected value. Then since as wealth becomes large, buying and selling price approach expected value of a lottery, these prices for the \$-bet have to increase above those of the P-bet. For smaller values of wealth, the CRRA decision maker would become very risk averse, so he will be solely preoccupied by the gamble's variance. Therefore, both selling and buying price for the \$-bet are below those of the P-bet. Technically speaking, notice that if W^* denotes wealth level at which selling price of $\mathbf{x}$ and $\mathbf{y}$ are equal, i.e. $S(W^*, \mathbf{x}) = S(W^*, \mathbf{y}) = S^*$, then by lemma 3.23 in chapter 3, it also holds that $B(W^* + S^*, \mathbf{x}) = B(W^* + S^*, \mathbf{y})$, so that $B(W, \mathbf{x})$ crosses $B(W, \mathbf{y})$ at $W = W^* + S^*$. Hence the interval for which buying/selling price reversal occurs is of length S^* exactly.

2.6 Concluding remarks

Expected utility theory by von Neumann and Morgenstern [1] imposes a set of consistency assumptions on choices among lotteries. The theory is used in a large part of

[11]The following holds $S(50, \mathbf{y}) = S(50, \mathbf{x})$ and $B(75, \mathbf{y}) = B(75, \mathbf{x})$.

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economic theory, including the famous Nash existence theorem. However there is a lot of mainly experimental evidence that people often violate von Neumann and Morgenstern [1] axioms, in particular the most crucial among them - independence. In response to this evidence economists started to question expected utility theory and investigate other models of choice which describe human behavior better. However, since these new theories usually have lower consistency requirements being imposed on the admissible choice, they necessarily also have lower prediction power and less scope for testable predictions. Moreover, they also have weaker normative appeal, since the decision makers violating expected utility axioms are vulnerable to money pumps. It is therefore an important issue to identify patterns of choices and behavior which are consistent with expected utility and contrast them with those which are impossible within expected utility. In order to perform this task it is important to identify expected utility theory in its bare form and in particular separate it from the doctrine of consequentialism. More precisely, it is necessary to abandon the common practice of interpreting wealth variable as total wealth position common to all decisions.

If one is willing to accept that wealth underlying gambling decisions is separated from total wealth so that gambling decisions are framed narrowly, important implications can be derived. If gambling wealth is small enough, which should be tested in an experiment, then selling price for a lottery can be significantly greater than buying price without going beyond expected utility model and the extent of this difference can be as high as the one found in experiments. Also, the famous Rabin [3] paradox can be resolved, suggesting that expected utility is not guilty here, but rather the doctrine of consequentialism.

Still, traditional preference reversal is not possible even if wealth is allowed to be small. If expected utility is to be regarded as a positive theory, it is definitely a negative result. However, if one is willing to accept expected utility as a good normative theory, then the same result is very useful. It informs us then, that preference reversal is not rational. It is confirmed further by the result proved in the paper, that individuals exhibiting preference reversal are susceptible to arbitrage under certain mild conditions. The same kind of arbitrage, which I prefer to call strong arbitrage, is not possible within expected utility. What might be interesting is that another kind of preference reversal, which I call preference reversal B and which involves buying price in place of selling price and otherwise is the same as the traditional preference reversal, is possible within expected utility and is not vulnerable to arbitrage as shown in the paper. What it could suggest if one is willing to treat expected utility as a good normative theory, is that preference reversal B is perhaps "more rational" than traditional preference reversal. An interesting thing to do in the future would be to check whether people exhibit preference reversal B as frequently as they exhibit the traditional preference reversal and if not, then check why this is so.

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2.7 Appendix

In what follows I will need the following lemma:

Lemma 2.18. *For any lottery* $\mathbf{x}$ *and any wealth level* W*, the following holds:*

$$S[W, \mathbf{x} - B(W, \mathbf{x})] = 0 \quad (2.8)$$

$$S[W - B(W, \mathbf{x}), \mathbf{x}] = B(W, \mathbf{x}) \quad (2.9)$$

$$B[W + S(W, \mathbf{x}), \mathbf{x}] = S(W, \mathbf{x}) \quad (2.10)$$

The proof is directly from definitions. For details, see Lewandowski [15].

2.7.1 Proof of proposition 2.5

Proposition 2.19 (Concave)**.** *For any concave and strictly increasing utility function and a non-degenerate lottery* $\mathbf{x}$ *the following holds:*

$$\min(\mathbf{x}) < B(W, \mathbf{x}) < \mathrm{E}[\mathbf{x}]$$

$$\min(\mathbf{x}) < S(W, \mathbf{x}) < \mathrm{E}[\mathbf{x}]$$

Proof. Notice first, that for degenerate lottery $\mathbf{x} = x$, equations (2.1) and (2.2)imply the following:

$$W + S(W, x) = W + x$$

$$W + x - B(W, x) = W$$

And so $S(W, x) = B(W, x) = x$. From now on I will focus on a non-degenerate lottery $\mathbf{x}$. I will prove the proposition only for the case of selling price. For buying price the proof is similar. I define $S \equiv S(W, \mathbf{x})$. Suppose $\min_{i \in \{1,\dots,n\}} x_i \geq S$. Then notice that:

$$U(W + x_i) \geq U\left(W + \min_{i \in \{1,\dots,n\}} x_i\right) \geq U(W + S)$$

with strict inequality for any $x_i \neq \min_{i \in \{1,\dots,n\}} x_i$. Since lottery $\mathbf{x}$ is non-degenerate there exists at least one $x_i \neq \min_{i \in \{1,\dots,n\}} x_i$ Hence

$$\sum_{i=1}^{n} p_i U(W + x_i) > U(W + S)$$

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So S cannot be the selling price - a contradiction.
Suppose now that $S \geq \mathrm{E}[\mathbf{x}]$. By strict Jensen's inequality

$$\mathrm{E}U[W+\mathbf{x}] < U[W+\mathrm{E}[\mathbf{x}]] \leq U(W+S)$$

So S cannot be the selling price - a contradiction. So I have shown that indeed $\min_{i\in\{1,\ldots,n\}} x_i < S(W,\mathbf{x}) < \mathrm{E}[\mathbf{x}]$.

□

Hence for lotteries with bounded values buying and selling price are bounded below by the minimal prize of the lottery and bounded above by the expected value of the lottery.

2.7.2 Proof of proposition 2.9

Note first that U_α is unbounded from below if $\alpha \geq 1$ and bounded from below if $\alpha < 1$.

$$\lim_{x\to 0} U_\alpha(x) = \begin{cases} -\frac{1}{1-\alpha}, & 0<\alpha<1 \\ -\infty, & \alpha \geq 1 \end{cases} \tag{2.11}$$

By proposition 2.5 buying and selling price are necessarily greater than $\min(\mathbf{x})$. For $\alpha \geq 1$ the utility function is unbounded from below, therefore from the definition it follows that: $\lim_{W\to 0} B(W,\mathbf{x}) = \min(\mathbf{x})$ and $\lim_{W\to -\min(\mathbf{x})} S(W,\mathbf{x}) = \min(\mathbf{x})$. On the other hand for $0<\alpha<1$ the utility function is bounded from below. Additionally, $W - B(W,\mathbf{x})$ is strictly increasing in W since $\frac{\partial B(W,\mathbf{x})}{\partial W} < 1$. Therefore the lower bound for the domain of $B(W,\mathbf{x})$ as a function of W is given by $W_L(\mathbf{x})$ such that:

$$\mathrm{E}U(-\min(\mathbf{x})+\mathbf{x}) = U(W_L(\mathbf{x}))$$

It follows that $\lim_{W\to W_L(\mathbf{x})} B(W,\mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$. Similarly, the lower bound for the domain of $S(W,\mathbf{x})$ as a function of W is $-\min(\mathbf{x})$ and hence: $\lim_{W\to -\min(\mathbf{x})} S(W,\mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$ since

$$\begin{aligned} \mathrm{E}U(-\min(\mathbf{x})+\mathbf{x}) &= U(-\min(\mathbf{x}) + S(-\min(\mathbf{x}),\mathbf{x})) \\ &= U(-\min(\mathbf{x}) + \min(\mathbf{x}) + U^{-1}(\mathrm{E}U(-\min(\mathbf{x})+\mathbf{x}))) \end{aligned}$$

Proof. Now I prove the following statement:

$$\forall \alpha > 0, \lim_{W\to\infty} B(W,\mathbf{x}) = \lim_{W\to\infty} S(W,\mathbf{x}) = \mathrm{E}[\mathbf{x}]$$

Note that the Absolute Risk Aversion for CRRA utility function has the form $A_\alpha(W) = \frac{\alpha}{W}$. Hence as W goes to infinity and α is bounded (no extreme risk aversion) $A_\alpha(W)$ tends to zero. This implies risk neutrality and hence $\lim_{W\to\infty} S(W,\mathbf{x}) = \lim_{W\to\infty} B(W,\mathbf{x}) = \mathrm{E}[\mathbf{x}]$ irrespective of relative risk aversion coefficient. □

2.7.3 Proof of lemma 2.11

I prove first that $B'(W) < 1$. From the definition of buying price using implicit function formula:

$$\frac{dB}{dW} = 1 - \frac{U'(W)}{\mathrm{E}U'(W + \mathbf{x} - B(W,\mathbf{x}))}$$

Since utility function is strictly increasing it must be that $\frac{dB}{dW} < 1$.
Now I prove that $S'(W - B(W)) = \frac{B'(W)}{1-B'(W)}$ and $S'(W - B(W)) > B'(W)$
From lemma 2.18 equation (2.9), using chain rule of differentiation, I have $B'(W) = S'(W - B(W))(1 - B'(W))$ Rearranging gives

$$S'(W - B(W)) = \frac{B'(W)}{1 - B'(W)}$$

Since $0 < B'(W) < 1$ by the above argument and proposition 2.7, I obtain $S'(W - B(W)) > B'(W)$.
Similarly I prove that $B'(W + S(W)) = \frac{S'(W)}{1+S'(W)}$ and $B'(W + S(W)) < S'(W)$.
Using equation (2.10) from lemma 2.18, I have $S'(W) = B'(W + S(W))(1 + S'(W))$ and hence

$$B'(W + S(W)) = \frac{S'(W)}{1 + S'(W)}$$

Since $S'(W) > 0$ by proposition 2.7, I get $B'(W + S(W)) < S'(W)$
Now I will prove that $S'(W) = \frac{S(W)-B(W)}{B(W)}$ for small positive $S(W)$.
And by proposition 2.7 $S(W) > B(W) > 0$. So when $S(W)$ is small and positive, then also $B(W)$ is small and positive. By lemma 2.18 equation (2.9), $S(W - B(W)) = B(W)$. For small $B(W)$ using first order Taylor expansion $B(W) = S(W) - \frac{dS}{dW}B(W)$ and hence it follows that

$$S'(W) = \frac{S(W) - B(W)}{B(W)}$$

Similarly, by lemma 2.18 equation (2.10), $B(W + S(W)) = S(W)$. Hence, for small $S(W)$ using first order Taylor expansion $S(W) = B(W) + \frac{dB}{dW}S(W)$ and it follows that

$$B'(W) = \frac{S(W) - B(W)}{S(W)}$$

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2.7.4 Proof of proposition 2.13

Without loss of generality I assume that $\min(\mathbf{x}) = 0$. Fix $\mathbf{x}$ such that $\min(\mathbf{x}) = 0$. By proposition 2.9, $B(W)$ and $S(W)$ are positive and hence by proposition 2.7 $\tau(W)$ is positive over the whole range. Notice that range of $\tau(W)$ is determined by proposition 2.9. If the domain of $S(W)$ is denoted D_S and the domain of $B(W)$ is denoted D_B, then the domain of $\tau(W)$ is just $D_S \cap D_B = D_B$. In particular, for $\alpha \geq 1$ the domain of $\tau(W)$ is the interval $(0, \infty)$ and for $\alpha \in (0, 1)$, the domain is the interval $(W_L(\mathbf{x}), \infty)$, where $W_L(\mathbf{x})$ is defined as in proposition 2.9. To prove the proposition I have to check whether the following expression is negative:

$$\tau'(W) = \frac{S(W)}{B(W)} \left[\frac{S'(W)}{S(W)} - \frac{B'(W)}{B(W)} \right] \tag{2.12}$$

From lemma 2.18 I have the following equations:

$$\begin{aligned} B(W) &= S(W - B(W)) \\ S(W) &= B(W + S(W)) \end{aligned}$$

For the proof first order effects are not sufficient, but it turns out second order effects are. Therefore, by Taylor expansion of the second order I get from the above equations:

$$\begin{aligned} B(W) &= S(W) - S'(W)B(W) + S''(W)B^2(W) \\ S(W) &= B(W) + B'(W)S(W) + B''(W)S^2(W) \end{aligned}$$

I only need to check the difference from equation (2.12) which I can rewrite as follows using the above Taylor expansions:

$$\begin{aligned} \frac{S'(W)}{S(W)} - \frac{B'(W)}{B(W)} &= \frac{\frac{S(W)-B(W)}{B(W)} + S''(W)B(W)}{S(W)} - \frac{\frac{S(W)-B(W)}{S(W)} - B''(W)S(W)}{B(W)} \\ &= \frac{S''(W)B^2(W) + B''(W)S^2(W)}{S(W)B(W)} < 0 \end{aligned}$$

where the last inequality follows from the fact that both $B(W)$ and $S(W)$ are concave (by lemma 2.12) and nonnegative (by proposition 2.9).

2.7.5 Proof of proposition 2.15

Take any non-degenerate lottery $\mathbf{y}$ with $S(W, \mathbf{y}) > B(W, \mathbf{y})$. Such a lottery exists by proposition 2.7. I can find a sequence of real numbers which all are greater than $B(W, \mathbf{y})$ and smaller than $S(W, \mathbf{y})$. I can then treat these numbers as a support for a new lottery $\mathbf{x}$. I assign probabilities to each of this numbers such that they sum to one and are positive for at least two of these numbers (such that the resulting lottery is non-degenerate). Suppose I choose n such numbers. By proposition 2.5 I can now conclude that:

$$S(W,\mathbf{y}) > \max_{i\in\{1,...,n\}} x_i > \mathrm{E}[\mathbf{x}] > S(W,\mathbf{x}) > B(W,\mathbf{x}) > \min_{i\in\{1,...,n\}} x_i > B(W,\mathbf{y})$$

And hence the result is proved.

2.7.6 Proof of proposition 2.16

Suppose that at any wealth $W \in [\underline{w}, \bar{w}]$ the decision maker prefers lottery $\mathbf{x}$ to lottery $\mathbf{y}$ in a direct choice but assigns higher certainty equivalent to lottery $\mathbf{y}$. Given such pattern of preferences it is easy to design an arbitrage strategy that extracts at least $W - \underline{w}$ from this decision maker. Suppose $W \in [\underline{w}, \bar{w}]$ is an initial wealth. Construct a sequence $W_i,\ i \in \{1, 2, ..., n\}$ such that:

- $W_0 = W$
- $W_i = W_0 - \sum_{k=1}^{i} \epsilon_k, \quad \epsilon_i > 0 \quad i \in 1, 2, ..., n$
- $W_n \geq \underline{w}, \qquad W_{n+1} < \underline{w}$
- for i even (including 0) $W_{i+1} + \mathbf{x} \succ W_i + \mathbf{y}$
- for i odd: $CE(W_i + \mathbf{x}) < CE(W_{i+1} + \mathbf{y})$

Notice that such a sequence exists by monotonicity and continuity of preferences and by properties of real numbers. Assume w.l.o.g. that $W_0 + \mathbf{y} \succ W_0$. The arbitrage strategy is now the following:

0) Take $\mathbf{y}$
1) Exchange $\mathbf{y}$ for $\mathbf{x}$ and pay me ϵ_1
2) Exchange $\mathbf{x}$ for $CE(W_1 + \mathbf{x}) - W_1$
3) Exchange $CE(W_1 + \mathbf{x})$ for $CE(W_1 + \mathbf{y})$ and pay me ϵ_2
4) Exchange $CE(W_2 + \mathbf{y}) - W_2$ for $\mathbf{y}$
5) Exchange $\mathbf{y}$ for $\mathbf{x}$ and pay me ϵ_3
6) Exchange $\mathbf{x}$ for $CE(W_3 + \mathbf{x}) - W_3$

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7) Exchange $CE(W_3 + \mathbf{x})$ for $CE(W_3 + \mathbf{y})$ and pay me ϵ_4

8) Exchange $CE(W_4 + \mathbf{y}) - W_4$ for $\mathbf{y}$

....

The above arbitrage strategy extracts the amount of wealth equal to $W - \underline{w}$ from the decision maker.

2.7.7 Proof of proposition 2.17

In what follows I will try to construct an arbitrage strategy to exploit the decision maker and show that it is not possible. Given DARA utility function U, take $\mathbf{x}$ such that $B(W, \mathbf{x}) < 0$. I will examine only this case since in the other cases the proof is trivial.

Suppose first, the decision maker initially has non-random position W. If the price b for the lottery is bigger than $B(W, \mathbf{x})$, the decision maker will not buy it. Hence, a price which is a part of an arbitrage strategy must be smaller than $B(W, \mathbf{x})$. Given such price b, the decision maker buys the lottery. His new position is $W + \mathbf{x} - b$. If the price s is smaller than $S(W - b, \mathbf{x})$, then the decision maker does not want to sell. Hence a price which is a part of an arbitrage strategy must be bigger than $S(W - b, \mathbf{x})$. By proposition 2.7, I know that S is strictly increasing and $b < B(W, \mathbf{x})$. Therefore:

$$s > S(W - b, \mathbf{x}) > S(W - B(W, \mathbf{x}), \mathbf{x}) = B(W, \mathbf{x}) > b$$

where the equality follows from lemma 2.18 equation (2.9).

Suppose now, that the decision maker initially has a random position $W + \mathbf{x}$. By the same argument as above the price s, which is a part of an arbitrage strategy has to be greater that $S(W, \mathbf{x})$, otherwise the decision maker would not sell the lottery $\mathbf{x}$. After selling the lottery, the decision maker's new position is $W + s$. The price b which is a part of an arbitrage strategy has to be smaller than $B(W + s, \mathbf{x})$. By lemma 2.11, I know that $\frac{\partial B(W,\mathbf{x})}{\partial W} \leq 1$ for all $W \geq 0$. Hence:

$$s - S(W, \mathbf{x}) > B(W + s, \mathbf{x}) - B(W + S(W, \mathbf{x}), \mathbf{x})$$

By lemma 2.18 equation (2.10), I know that $B(W + S(W, \mathbf{x}), \mathbf{x}) = S(W, \mathbf{x})$, and hence:

$$s > B(W + s, \mathbf{x}) > b$$

That proves that with decision maker's initial position equal to either W or $W + \mathbf{x}$, all arbitrage strategies have the property that $s > b$. However, this cannot be an arbitrage

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strategy since it makes negative profit equal to $b - s$. This proves that there are no arbitrage strategies.

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Chapter 3

Risk attitudes, buying and selling price for a lottery and simple strategies

Abstract

In this chapter I introduce a concept of simple strategy and define three kinds of such strategies. For three classes of utility functions - CARA, DARA and CRRA I state and prove equivalent characterizations in terms of the corresponding simple strategy characteristics, the corresponding properties of buying and selling price and the corresponding functional equations for utility function. I also prove an extension of famous Pratt [4] theorem on comparative risk aversion. More specifically, I show that buying price for a lottery can be used alternatively to other measures including selling price to compare risk aversion across individuals. Additionally a number of propositions on both selling and buying price for a lottery and CRRA utility class are proved.

Keywords: characterization, comparative risk aversion, simple strategy, Pratt [4], Arrow [8] coefficient of risk aversion, CARA, DARA, CRRA

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3.1 Introduction

In this paper I will analyze four ways to represent risk attitudes within expected utility, corresponding to the behavior of the following functions:

- absolute and relative risk aversion
- buying and selling price for a lottery
- simple strategy
- Bernoulli utility

The first of the aforementioned representations is based on local risk attitudes defined by Pratt [4] and Arrow [8]. Within this representation three classes of individual risk attitudes and the associated Bernoulli utility functions will be discussed, namely constant absolute risk aversion (CARA), decreasing absolute risk aversion (DARA)[1] and a subset thereof - constant relative risk aversion (CRRA). The second representation is given by the properties of buying and selling price for a lottery, the concepts defined by Raiffa [2]. Buying price for a lottery is a maximal sure amount which the decision maker is willing to pay for a lottery. Selling price for a lottery is a minimal sure amount which the decision maker is willing to accept to forgo a lottery. The alternative names which are often used in non-expected utility theories and experimental work are willingness to pay (WTP) for a lottery and willingness to accept (WTA) for a lottery, respectively. Within this representation I will consider buying and selling price properties in separation as well as the way they are linked together. In case of CRRA I will analyze both the properties of classic buying and selling price as well as the properties of buying and selling price designed for multiplicative gambles. I will show in what sense CARA and buying and selling price for nominal gambles are analogous to CRRA and buying and selling price for multiplicative gambles. Buying and selling price representation might shed light on a recent experimental evidence documenting large spreads between elicited buying and selling prices for the same lottery as well as preference reversal phenomenon. I analyze these issues in the accompanying paper Lewandowski [22]. The third representation involves the concept of a simple strategy. Simple strategy recommends whether to accept a given gamble or not only on the basis of the gamble itself and the initial wealth which the decision maker is endowed with prior to taking the decision. In a dynamic setting simple strategy corresponds to the notion of Markov stationary strategy. Wealth-invariant simple strategy is a strategy which for any gamble does not depend on initial wealth. Among wealth-varying simple strategies I focus on "wealthier-accept

[1] In this paper DARA means strictly decreasing absolute risk aversion.

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more" simple strategy for which the acceptance set increases when initial wealth increases. And finally, a special case of "wealthier-accept more" simple strategy is scale invariant simple strategy which does not depend on scale. If a gamble is accepted at some wealth level, then if the gamble's outcomes and wealth are multiplied by some positive factor, the new gamble will be accepted at a new wealth level.
And finally, I will show that CARA and CRRA class of utility functions can be derived from functional equations belonging to the Cauchy family.
I will prove three equivalence results for these four representations valid within the framework of expected utility. Each of the three classes of Bernoulli utility function will be shown equivalent to the corresponding properties of buying and selling prices and to the corresponding simple strategy. Additionally, I will show that CARA and CRRA classes of utility function can be derived from Cauchy functional equations. The following diagram illustrates the idea behind these results:

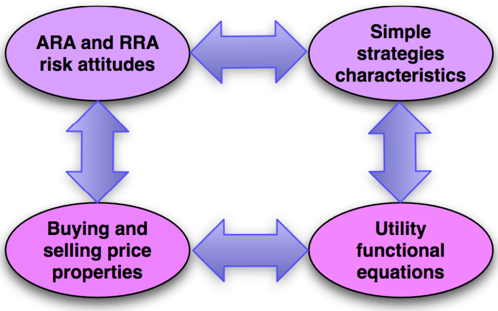

FIGURE 3.1: Diagrammatic representation of the equivalency results

The first three representations described above will be dealt with in section 3.4, whereas the last representation involving Cauchy functional equations will be analyzed in section 3.5 Even though some parts of these results are well known I decided to put them all together both for the sake of completeness as for the sake of their novel formulation and the unifying method of proof. For example, the equivalence between CARA, wealth invariance and constant buying and selling price has been around since Pratt [4], however the way it's stated in this paper is much more straightforward and ready to use. The concept of simple strategy is a novel component introduced by Foster and Hart [5] in the context of constant relative risk aversion class[2]. "Wealthier-accept more" and wealth-invariant simple strategies are defined for the purpose of this paper.
The result, which is perhaps not so well known among economists is the one for constant relative risk aversion utility class. For example, in section 4 Proposition 1 Barberis and Huang [23] restrict certainty equivalent functional, which they denote $\mu(\cdot)$, to the case of constant relative risk aversion utility functions. They claim that "the same method of proof used in Proposition 1 can also be applied to **other** explicitly defined **forms of**

[2]Foster and Hart [5] introduced the concept of homogeneous simple strategy.

$\mu(\cdot)$, whether **expected utility** or not, that satisfy the homogeneity property[3]". In the paper I show that except for CRRA there are no other expected utility forms of certainty equivalent that satisfy homogeneity property and hence the statement Barberis and Huang [23] make is not correct. The result concerning CRRA as well as other characterization results in this paper may help in clarifying some of the imprecise statements from the literature such as the one cited above.

The three characterization results in this paper show that the same message can be delivered in four different ways depending on the needs and on the context. For example to assume CRRA is equivalent to assume positive homogeneity of buying and selling prices, and also equivalent to assuming scale-invariance of simple strategy. Moreover, CRRA utility function satisfies a simple functional equation. The formulation in terms of simple strategies makes the notion of CRRA more intuitive since it is expressed directly in terms of the decision maker's actions. Therefore, when testing the hypothesis of CRRA, it might be useful to test instead whether simple strategy is homogeneous. Alternatively, if the experimenter has access to data on buying and selling price, it might be more straightforward to test homogeneity of these. Thus the characterization results form the bridge between different formulations.

Another result of this paper shows that buying price for a lottery can be used to compare risk aversion of two agents in an equivalent way as selling price for a lottery and other methods laid out in Pratt [4] famous theorem[4]. This result is an extension of Pratt [4] theorem which characterizes comparative risk aversion. It might be useful in testing comparative risk aversion when the data on individual buying prices is available whereas the data on individual selling prices is not. Also, using buying price has one technical advantage over using selling price. Buying price exhibits the so called delta property[5] whereas selling price in general does not[6]. It means that calibration process in case of buying price might be much easier than in case of selling price.

The paper is divided into three parts. In section 3.2 I introduce definitions and assumptions of the model in a formal way. In section 3.3 I introduce the idea of nominal and multiplicative gambles and the way risk aversion is incorporated for these two concepts. In section 3.4 I state four main results. In section 3.5 I state additional results which introduce Cauchy functional equations and their equivalence with CRRA and CARA utility class and describe certain theoretical properties of buying and selling price. These properties are invoked in the other chapters. In section 3.7 main results are proved together with a number of auxiliary lemmas and propositions.

[3]I.e. positive homogeneity of certainty equivalence.

[4]Selling price for a lottery is the negative or risk compensation used in Pratt [4].

[5]For details see appendix lemma 3.36.

[6]The exception is CARA class for which selling price exhibits delta property.

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3.2 The model

In this section I introduce the assumptions, definitions of buying and selling price and the notion of simple strategies.

Assumption 3.1. *Preferences obey expected utility axioms. Bernoulli utility function* $U : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable, strictly increasing and strictly concave.*

Definition 3.2. A lottery $\mathbf{x}$ is a real- and finite-valued random variable with finite support. I define a maximal loss of lottery $\mathbf{x}$ as $\min(\mathbf{x}) = \min \operatorname{supp}(\mathbf{x})$. Wealth W is a real number.

Although most of the results that follow are true for more general lotteries, the finite support assumption is adopted for the sake of simplicity. Now I define buying and selling price for a lottery given wealth level along the lines of Raiffa [2]. To avoid repetitions, I will henceforth skip statements of the form: "Given utility function U satisfying assumption 3.1, any lottery $\mathbf{x}$ and wealth W...".

Definition 3.3. I define selling price $S(W, \mathbf{x})$ and buying price $B(W, \mathbf{x})$ for a lottery $\mathbf{x}$ at wealth W as follows:

$$\mathrm{E}U[W + \mathbf{x}] = U[W + S(W, \mathbf{x})] \tag{3.1}$$

$$\mathrm{E}U[W + \mathbf{x} - B(W, \mathbf{x})] = U(W) \tag{3.2}$$

The domain of S and B, i.e. all admissible pairs $(W, \mathbf{x})$, is assumed to be such that the left-hand side and the right-hand side of the above equations are well defined, given the domain of U[7]. The space of such admissible pairs will be denoted by $\mathcal{X}$. Notice that functions B and S are then well defined by assumption 3.1.

Definition 3.4. An individual's simple strategy $s : \mathcal{X} \to \{1, 0\}$ assigns to each admissible pair $(W, \mathbf{x})$ either value 1 or 0, representing "Accept $\mathbf{x}$ at W" or "Reject $\mathbf{x}$ at W", respectively.

Since the aim in this chapter is to link the concept of simple strategy to expected utility maximization, I impose the following non-triviality assumption:

[7]If the utility function is not defined over the whole real line as is the case of CRRA utility function, one has to make sure first that both sides of the above equations are well defined and second that the equality has a solution. For details see Lewandowski [22].

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Assumption 3.5. *For any non-degenerate lottery* $\mathbf{x}$ *and wealth* $W > 0$*, there exists a unique* $p^*_{W,\mathbf{x}} \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$ *such that:*

$$\begin{aligned} s(W, \mathbf{x} - p) &= 1,\ p \leq p^*_{W,\mathbf{x}} \\ s(W, \mathbf{x} - p) &= 0,\ p > p^*_{W,\mathbf{x}} \end{aligned}$$

This assumption asserts that there are other lotteries which are accepted beyond those with no losses and there are other lotteries which are rejected beyond those with negative expectation. This assumption imposes monotonicity of preferences and risk aversion. Notice that in expected utility setup given wealth W and lottery $\mathbf{x}$, $p^*_{W,\mathbf{x}} = B(W, \mathbf{x})$.

Definition 3.6. Simple strategy is wealth-invariant if

$$s(W_1, \mathbf{x}) = 1 \iff s(W_2, \mathbf{x}) = 1,\ \forall W_1, W_2 \tag{3.3}$$

And the above holds for all $\mathbf{x}$ that are accepted.

It follows that if simple strategy is wealth-varying (i.e. not wealth-invariant) then there exists lottery $\mathbf{x}$ and two different wealth levels W_1, W_2 such that:

$$s(W_1, \mathbf{x}) = 1 \ \wedge \ s(W_2, \mathbf{x}) = 0$$

To understand the difference between wealth-varying and wealth-invariant simple strategy suppose there is a lottery with only positive outcomes. Any individual who prefers less to more will accept such lottery irrespective of initial wealth level. It does not mean however that the underlying simple strategy is wealth-invariant. That is the reason why wealth-varying simple strategy is defined using the existence quantifier and not the universal quantifier.
I introduce now two kinds of wealth-varying simple strategies:

Definition 3.7. Wealth-varying simple strategy is of "wealthier-accept more" type if

$$s(W_1, \mathbf{x}) = 1 \ \wedge \ s(W_2, \mathbf{x}) = 0 \Rightarrow \ W_1 > W_2 \tag{3.4}$$

Definition 3.8. Wealth-varying simple strategy is scale-invariant or homogeneous if

$$s(W, \mathbf{x}) = 1 \iff s(\lambda W, \lambda \mathbf{x}) = 1,\ \forall W,\ \forall \lambda > 0 \tag{3.5}$$

And this holds for all $\mathbf{x}$ that are accepted.

I am interested in analyzing risk attitudes. It is convenient to define two kinds of utility function transformations which do not alter the underlying risk attitudes.

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Lemma 3.9. *If U is a utility function, relative and absolute risk aversion function is unique up to the following transformation of U:*

$$\mathcal{A} : \{v(x) = aU(\delta x) + b\}$$

where $\delta = \{1, -1\}$, $a, b \in \mathbb{R}$ and $a > 0$.
Furthermore, for $a < 0$, relative and absolute risk aversion changes only its sign.

Proof. Since Bernoulli utility function is unique only up to affine transformation $au(\cdot)+b$, $A > 0$ represents the same risk attitudes as $u(\cdot)$. Furthermore, relative and absolute risk aversion obtained from $u(x)$ and $u(-x)$ is the same in sign and magnitude and that from $-u(x)$ and $-u(-x)$ is of opposite sign but the same magnitude. □

Observe that if $u(x)$ is increasing and concave, then $u(-x)$ is decreasing and concave, $-u(x)$ is decreasing and convex, and $-u(-x)$ is increasing and convex. The absolute value of Arrow, Pratt risk aversion measures is however the same for all these functions. The use of such transformations will prove useful when characterizing different classes of risk attitudes by means of Cauchy family functional equations.

3.3 Nominal and multiplicative gambles

3.3.1 Nominal gambles and wealth invariance vs multiplicative gambles and scale invariance

Suppose an individual with wealth W faces a choice whether to accept or reject gamble $\mathbf{x}$. The consequences of $\mathbf{x}$ are monetary. Consider two different objectives this individual might have:

a. wealth from accepting $\mathbf{x}$ should increase on average in nominal terms

b. return from $\mathbf{x}$ should be positive on average

Let's define random return from $\mathbf{x}$ given wealth $W > 0$ as $\mathbf{h} = 1 + \frac{\mathbf{x}}{W}$. Gamble $\mathbf{x}$ is called a nominal gamble since its units are expressed in nominal terms. Gamble $\mathbf{h}$ is called a multiplicative gamble and it is dimensionless. It is assumed that the maximal loss of $\mathbf{x}$ is strictly smaller than W so that gamble $\mathbf{h}$ takes only positive values. Aumann and Serrano [24] suggest that financial instruments may be regarded as such multiplicative gambles. Notice that nominal gamble does not depend on wealth and the acceptance of such gamble using the first of the above criteria does not depend

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on initial wealth W. On the other hand multiplicative gamble does depend on initial wealth and the acceptance of such gamble using the second of the above criteria also depends on initial wealth. However, what the multiplicative gamble is invariant to is scale. No matter in what units I measure consequences[8], or, alternatively, whether I multiply both initial wealth and the nominal gamble $\mathbf{x}$, by the same positive factor, the resulting multiplicative gamble remains unchanged. Notice that the two criteria above do not invoke any arguments on risk aversion. In fact, the first criterion amounts to risk neutrality in a classical sense. Similarly, I think it is useful to think of the second criterion as risk neutrality for multiplicative gambles. I would like to show below that the two widely used classes of utility functions, CARA and CRRA, are generalizations of the above two criteria, respectively - generalizations in the sense of introducing risk aversion, specific to nominal gambles and wealth invariance in the first case and specific to multiplicative gambles and scale invariance in the second case.

First, notice that the first criterion above is equivalent to evaluating the arithmetic mean of a nominal gamble:

$$\text{accept } \mathbf{x} \iff \mathrm{E_a}(\mathrm{W} + \mathbf{x}) \geq \mathrm{W} \iff \mathrm{E_a}(\mathbf{x}) \geq 0 \tag{3.6}$$

where $\mathrm{E_a}$ denotes arithmetic mean operator.[9] The second criterion, on the other hand, is equivalent to the following:

$$\text{accept } \mathbf{h} \iff W \times \mathrm{E_g}(\mathbf{h}) \geq \mathrm{W} \iff \mathrm{E_g}(\mathbf{h}) \geq 1 \tag{3.7}$$

where $\mathrm{E_g}$ denotes geometric mean operator. The detailed explanation why the above is true may be found in the appendix at the end of this chapter.

Since $\mathbf{h}$ takes only positive values, the condition on the right of (3.7) may be rewritten as:

$$\begin{aligned} \log \mathrm{E_g}(\mathbf{h}) &\geq 0 \\ \mathrm{E_a} \log(\mathbf{h}) &\geq 0 \\ \mathrm{E_a} \log\left(1 + \frac{\mathbf{x}}{W}\right) &\geq 0 \\ \mathrm{E_a} \log(W + \mathbf{x}) &\geq \log W \end{aligned}$$

The conclusion from this analysis is very interesting: the second criterion saying that the return from $\mathbf{x}$ should be positive on average is equivalent to expected utility from accepting $\mathbf{x}$ is at least as high as from rejecting $\mathbf{x}$, where the Bernoulli utility function is

[8] Both, values of $\mathbf{x}$ and initial wealth W.

[9] Generally, in the whole thesis, if not explicitly stated otherwise E denotes arithmetic mean operator. Here, different notation is used to stress the difference to geometric mean operator $\mathrm{E_g}$, which will also be used.

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logarithmic. So a kind of risk neutrality for multiplicative gambles (the second criterion) is equivalent to logarithmic risk aversion in classical sense i.e. for nominal gambles. This fact is a basis for Foster and Hart [5] paper which will be analyzed in the next chapter of this thesis. I would like to demonstrate now that the same transformation of the two criteria a. and b. discussed in this section lead to CARA and CRRA class, respectively. The goal is to introduce risk aversion into criteria a. and b. As noted before, criteria a. and b. are equivalent to the following two conditions, respectively:

$$\begin{aligned} \mathrm{E_a}\mathbf{x} &\geq 0 \\ \mathrm{E_a}\log(\mathbf{h}) &\geq 0 \end{aligned} \tag{3.8}$$

Consider, CARA utility function $U(x) = \frac{1-e^{-\beta x}}{\beta}$. By definition, exchanging x with $U(x)$ in the first of the above equation gives rise to expected utility decision making with CARA Bernoulli utility function. Now consider exchanging $\log x$ with $U(\log x)$ in the second case:

$$U(\log x) = \frac{1 - e^{-\beta \log x}}{\beta} = \frac{1 - x^{-\beta}}{\beta}$$

Now if I define $\beta = -(1-\alpha)$ I obtain:

$$U(\log x) = \frac{x^{1-\alpha} - 1}{1 - \alpha}$$

And hence equation (3.8) will change to:

$$\mathrm{E_a}\left[\frac{\mathbf{h}^{1-\alpha} - 1}{1-\alpha}\right] \geq 0 \tag{3.9}$$

And this is expected utility decision making with CRRA Bernoulli utility function.

3.3.2 Buying and selling price for multiplicative gambles

In the previous section, I introduced the concepts of buying and selling price for a lottery. These concepts were specifically designed to deal with nominal gambles. It is possible to define similar concepts for multiplicative gambles. If $\mathbf{x}$ is a nominal gamble and $W > L(\mathbf{x})$ is initial wealth, denote $\mathbf{h}$ as multiplicative gamble and write $\mathbf{h} = \frac{W+\mathbf{x}}{W}$.

Definition 3.10. Given utility function: $U : R^+ \rightarrow R$, a multiplicative gamble $\mathbf{h}$, I define selling return price $s(W, \mathbf{h})$ and buying return price $b(W, \mathbf{h})$ for a multiplicative gamble $\mathbf{h}$ at wealth W as follows:

$$\mathrm{E}U[W\mathbf{h}] = U[Ws(W,\mathbf{h})] \tag{3.10}$$

$$\mathrm{E}U\left[\frac{W\mathbf{h}}{b(W,\mathbf{h})}\right] = U(W) \tag{3.11}$$

The interpretation of these two measures is the following. Selling return price $s(W,\mathbf{h})$ is the minimal sure return which an individual whose preferences are represented by U would demand to forgo random return $\mathbf{h}$. On the other hand, buying return price $b(W,\mathbf{h})$ is the maximal sure return which an individual is willing to forgo for the right to play gamble $\mathbf{h}$. It is easy to show in as similar way to that in lemma 3.24 that for non-degenerate $\mathbf{h}$, both $b(W,\mathbf{h})$ and $s(W,\mathbf{h})$ lie in the interval $\left(1-\frac{L(\mathbf{x})}{W}, 1+\frac{\mathrm{E}[\mathbf{x}]}{W}\right)$. Since it is assumed that $W > L(\mathbf{x})$, it is guaranteed that all the arguments in the above two equations are non-negative and hence by monotonicity and continuity of U, the two equations are well defined and there exist unique selling and buying return prices. There is a simple relationship between selling price for nominal lottery $\mathbf{x}$ and selling return price for multiplicative gamble:

$$s(W,\mathbf{h}) = 1 + \frac{S(W,\mathbf{x})}{W}$$

An analogous relationship between $b(W,\mathbf{h})$ and $B(W,\mathbf{x})$ is however more complex and, in general, can only be given in an implicit form:

$$\mathrm{E}U\left(\frac{W\mathbf{h}}{b(W,\mathbf{h})}\right) = \mathrm{E}U(W\mathbf{h} - B(W,\mathbf{x}))$$

3.4 Results

The first result below is essentially just a reformulation and combination of results appearing in Pratt [4] and Raiffa [2]. I restate it in a convenient form for completeness. The proofs of all the following results, including the first one are provided in section 3.7.

Proposition 3.11 (CARA)**.** *The following three statements are equivalent:*

i. strategy is wealth-invariant

ii. Bernoulli utility function exhibits CARA

iii. buying and selling price are independent from wealth and equal i.e.

$$B(W,\mathbf{x}) = S(W,\mathbf{x}) = C_\alpha, \ \forall W \tag{3.12}$$

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where α is absolute risk aversion coefficient and C_α takes real values and depends only on α.

Proposition 3.12 (DARA)**.** *The following three statements are equivalent:*

i. strategy is wealthier - accept more

ii. Bernoulli utility function exhibits DARA

iii. buying and selling price are increasing in W and

$$B(W, \mathbf{x}) > 0 \iff B(W, \mathbf{x}) < S(W, \mathbf{x})$$

Proposition 3.13 (CRRA)**.** *The following four statements are equivalent:*

i. strategy is scale-invariant

ii. Bernoulli utility function exhibits CRRA

iii. buying and selling price for any lottery are homogeneous of degree one i.e.

$$S(\lambda W, \lambda \mathbf{x}) = \lambda S(W, \mathbf{x}), \ \forall \lambda > 0 \tag{3.13}$$

$$B(\lambda W, \lambda \mathbf{x}) = \lambda B(W, \mathbf{x}), \ \forall \lambda > 0 \tag{3.14}$$

iv. buying and selling return prices for any multiplicative lottery are independent from wealth and equal i.e.

$$b(W, \mathbf{h}) = s(W, \mathbf{h}) = C_\beta, \ \forall W \tag{3.15}$$

where β is relative risk aversion coefficient and C_β takes real values and depends only on β. Additionally,

$$b(W, \lambda \mathbf{h}) = \lambda b(W, \mathbf{h}), \ \forall \lambda > 0 \tag{3.16}$$

$$s(W, \lambda \mathbf{h}) = \lambda s(W, \mathbf{h}), \ \forall \lambda > 0 \tag{3.17}$$

In the last proposition concerning CRRA class of utility function, I have added an additional item which characterizes buying and selling return prices. As suggested by Roberto Serrano, it is useful to see that buying and selling return price in case of CRRA share the same characteristics with buying and selling price in case of CARA. In particular, conditions (3.12) and (3.15) make it clear that in case of CRRA buying and selling return price are equal to each other and independent from wealth the same way as in case of CARA buying and selling price are equal to each other and independent from

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wealth. Conditions (3.16) and (3.17) are on the other hand specific for buying and selling return prices in case of CRRA. These conditions state that buying and selling return prices are both homogeneous in a gamble. Since buying and selling return prices were designed to deal with multiplicative gambles and CRRA class is a scale-invariant, these conditions are perhaps more intuitive then the analogous conditions (3.13) and (3.14) which concern buying and selling price, the concepts designed for nominal gambles.
The above three propositions characterize three widely used risk attitude classes of utility function. Certain parts of these propositions are already known in the literature and some are novel. The advantage lies in putting all these results together and offering a unifying way to prove them. In applied work these results should be especially useful since they allow to interchangeably use the notions of risk attitude classes of utility function represented by absolute and relative risk aversion, the corresponding simple strategies defined above and properties of buying and selling price for a lottery. It should help in testing of risk attitudes, as it might be simpler to test either buying and selling price for a lottery or simple strategies depending on the context. Experiments should be designed as naturally as possible. Subjects are reluctant to engage in considerations regarding abstract notions. Here, the advantage of a simple strategy notion is apparent due to its direct reference to actions. In other contexts on the other hand, where trading atmosphere is to be created in experimental settings, buying and selling price for a lottery might be more appropriate. Needless to say, the three equivalent characterizations of risk attitudes classes of utility function should be of advantage both in theoretical as well as applied work. Notice further, that a number of useful observation might be made after careful examination of the above results. For example, since CRRA is a subclass of DARA, it is therefore the case that homogeneous simple strategy is a subset of "wealthier-accept more" simple strategies. Such conclusion is not obvious without the propositions above.
The next result that follows is an extension of Pratt [4] famous result on comparative risk aversion. It establishes an equivalence between buying and selling price as an index for greater-risk aversion relation.

Proposition 3.14. *For two different utility functions $U_1(\cdot)$ and $U_2(\cdot)$ with non-increasing absolute risk aversion, let $S_1(W,\mathbf{x})$, $B_1(W,\mathbf{x})$ and $S_2(W,\mathbf{x})$, $B_2(W,\mathbf{x})$ be the corresponding buying and selling price functions. The following equivalence holds:*

$$\forall W\ \forall \mathbf{x}:\ \exists\, \delta > 0\ |x_i| < \delta\ \forall i \in \{1, ..., n\}$$
$$S_1(W,\mathbf{x}) > S_2(W,\mathbf{x}) \quad \Longleftrightarrow \quad B_1(W,\mathbf{x}) > B_2(W,\mathbf{x})$$

This proposition may be useful as well in both theoretical and empirical work. Since buying price for a lottery exhibits delta property no matter what the risk attitude, and selling price in general does not[10] it might be simpler to use buying price as an index of comparative risk aversion since the proposition says that one can use the two indices interchangeably. In empirical settings, one might have data only on buying price for a lottery and not on selling price. In this case inferences regarding selling price for a lottery and hence absolute risk aversion are still possible due to the above result.

3.5 Additional results

3.5.1 Additional characterization results

The following results give further insights into the nature of widely used risk attitudes classes. By means of a couple of simple functional equations one can give alternative proofs to the first equivalence results (i. $\iff$ ii.) in propositions 3.11, 3.12 and 3.13. The equations I will analyze belong to a Cauchy family of functional equations:

a. $v(x+y) = v(x) + v(y)$

b. $v(x+y) = v(x)v(y)$

c. $v(xy) = v(x) + v(y)$

d. $v(xy) = v(x)v(y)$

It is useful to treat the above functions as transformations described in lemma 3.9 of the corresponding utility functions. It proves much easier to work with transformations and not directly with utility functions since these transformations are chosen to satisfy the simplest functional equation with a given concavity/convexity properties. By lemma 3.9 I am then entitled to transform function v into a utility function U which satisfies any desired properties - e.g. normalized conveniently, increasing and concave without changing the risk attitudes properties.
I will now show that the above four functional equations are equivalent characterizations of risk neutral, CARA, logarithmic and CRRA preferences respectively.

Proposition 3.15. *All twice continuously differentiable functions $v : \mathbb{R} \to \mathbb{R}$ that satisfy the following functional equation: $v(x+y) = v(x) + v(y)$ for all x, y belonging to the domain of v, are of the following form:*

$$v(x) = cx, \qquad c \in \mathbb{R}$$

[10]Selling price exhibits delta property only in case of CARA.

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Proof. For $x = y = 0$, $v(0) = v(0) + v(0)$. It implies that $v(0) = 0$. Rearrange the equation and divide by y:

$$\frac{v(x+y) - v(x)}{y} = \frac{v(y)}{y}$$

Letting y tend to zero and using Hospital rule on the right hand side, I obtain:

$$v'(x) = \frac{v'(0)}{1}$$

I define $c = v'(0)$ and integrate both sides from 0 to x of the above equation to obtain $v(x) = cx$ as required. $\square$

Proposition 3.16. *All twice continuously differentiable functions $v : \mathbb{R} \to \mathbb{R}$ that satisfy the following functional equation: $v(x+y) = v(x)v(y)$ for all x, y belonging to the domain of v, are of the following form:*

$$v(x) = e^{cx}, \qquad c \in \mathbb{R}$$

Proof. For $y = 0$, $v(x) = v(x)v(0)$. It implies that $v(0) = 1$. Using the equation I can write:

$$\frac{v(x+y) - v(x)}{y} = v(x)\frac{v(y) - 1}{y}$$

Now I let y on both sides go to zero. Using Hospital rule on the right hand side I obtain:

$$v'(x) = v(x)\frac{v'(0)}{1}$$

Define $c = v'(0)$ and rearrange to obtain:

$$[\log v(x)]' = c$$

Now integrate both sides from 0 to x and exponentiate on both sides to obtain:

$$v(x) = e^{cx} \quad \square$$

Proposition 3.17. *All twice continuously differentiable functions $v : \mathbb{R}^{++} \to \mathbb{R}$ that satisfy the following functional equation: $v(xy) = v(x) + v(y)$ for all x, y belonging to the domain of v, are of the following form:*

$$v(x) = c \log x, \qquad c \in \mathbb{R}$$

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Proof. For $y = 1$, $v(x) = v(x) + v(1)$. It implies that $v(1) = 0$. I define $y = 1 + h$. Now using the equation I can write:

$$\frac{v(x(1+h)) - v(x)}{xh} = \frac{1}{x}\frac{v(1+h)}{h}$$

Now let h tend to zero on both sides and apply Hospital rule on the right hand side:

$$v'(x) = \frac{1}{x}\frac{v'(1)}{1}$$

Now define $c = v'(1)$ and integrate both sides from 1 to x to obtain:

$$v(x) = c \log x \quad \square$$

Proposition 3.18. *All twice continuously differentiable functions $v : \mathbb{R}^{++} \to \mathbb{R}$ that satisfy the following functional equation: $v(xy) = v(x)v(y)$ for all x, y belonging to the domain of v, are of the following form:*

$$v(x) = x^c, \qquad c \in \mathbb{R}$$

Proof. For $y = 1$, $v(x) = v(x)v(1)$. It implies that $v(1) = 1$. I define $y = 1 + h$. Now using the equation I can write:

$$\frac{v(x(1+h)) - v(x)}{xh} = \frac{v(x)}{x}\frac{v(1+h) - 1}{h}$$

Now let h tend to zero on both sides and apply Hospital rule on the right hand side:

$$v'(x) = \frac{v(x)}{x}\frac{v'(1)}{1}$$

Now define $c = v'(1)$ and rearrange to obtain:

$$[\log v(x)]' = c\frac{1}{x}$$

Integrate both sides from 1 to x and rearrange to obtain:

$$v(x) = x^c \quad \square$$

It is worth noting that the above propositions could be stated in a stronger form. To prove that any of the four functional equations implies the corresponding function, one does not need to assume that v is twice continuously differentiable. It is true for any continuous or monotonic functions. Furthermore, it requires a full proof only for the

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first of the four functional equations, as the others may be reduced to it by using appropriate transformation of v. This is also the reason why all the four equations belong to one family of Cauchy equations. Suppose function v satisfies $v(x+y) = v(x)v(y)$. Define a transformation of v, namely $g(x) = \log v(x)$. It is straightforward to see that $g(x)$ satisfies $g(x+y) = g(x) + g(y)$, so it has to be that $g(x) = cx$ and going back to the original function $v(x) = e^{cx}$. Similarly, the corresponding transformation of v which satisfies $v(xy) = v(x) + v(y)$ is $g(x) = v(e^x)$ and the corresponding transformation of v which satisfies $v(xy) = v(x)v(y)$ is $g(x) = \log v(e^x)$.

To see how the above functional equations connect to expected utility decision making, suppose that W is initial wealth, $\mathbf{x}$ is a lottery to be chosen and U is a utility function which is transformed from the corresponding function v satisfying one of the four functional equations. Define the following expression $a(W, \mathbf{x}) = \mathrm{E}U(W + \mathbf{x}) - U(W)$. The list below corresponds to the four functional equations above:

a. $a(W, \mathbf{x}) = \mathrm{E}U(\mathbf{x})$

b. $a(W, \mathbf{x}) = U(W)(\mathrm{E}U(\mathbf{x}) - 1)$

c. $a(\lambda W, \lambda \mathbf{x}) = \mathrm{E}U(W + \mathbf{x}) - U(W)$

d. $a(\lambda W, \lambda \mathbf{x}) = U(\lambda)[\mathrm{E}U(W + \mathbf{x}) - U(W)]$

The conclusions are the following. In the first case, corresponding to linear utility function, the utility from accepting lottery $\mathbf{x}$ does not depend on W. In the second case, corresponding to CARA (without linear) utility function, the acceptance of $\mathbf{x}$ does not depend on W but the utility value from accepting $\mathbf{x}$ depends on W. In case c., corresponding to logarithmic utility function, the utility from accepting $\lambda\mathbf{x}$ at λW does not depend on scale λ. In case d., corresponding to CRRA (without log) utility function, the acceptance of $\lambda\mathbf{x}$ at λW does not depend on scale λ but the utility value from accepting it does.

It can therefore be proposed to call linear utility function - totally wealth invariant, logarithmic utility function- totally scale invariant, other than linear CARA functions - acceptance wealth invariant, and other than logarithmic CRRA functions - acceptance scale invariant.

The characterization of CRRA utility function may be supplemented by the following lemma:

Lemma 3.19. *Given twice continuously differentiable utility function* $u : \mathbb{R}^{++} \to \mathbb{R}$ *and* $\theta, \alpha \in \mathbb{R},\ \theta > 0,\ \alpha \neq 0$*, the following holds:*

$$u(\theta x) = \theta^{\alpha} u(x) \iff u(x) = Ax^{\alpha}$$

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where $A > 0$ is a constant.

Proof. Sufficiency is straightforward.
Let's prove necessity. Suppose $\theta = 1 + \frac{h}{x}$. Then

$$\lim_{h \to 0} \frac{(1+\frac{h}{x})^\alpha - 1}{h} u(x) = u'(x)$$

Using Hospital's rule

$$\lim_{h \to 0} \frac{\alpha(1+\frac{h}{x})^{\alpha-1}\frac{1}{x}}{1} u(x) = u'(x)$$

And this is equivalent to:

$$\frac{u'(x)}{u(x)} = \frac{\alpha}{x}$$

Define $u(1) = A$, then

$$\int_1^x [\log u(t)]' dt = \alpha \int_1^x [\log t]' dt$$

Hence

$$\log \frac{u(x)}{A} = \alpha \log x$$

And finally

$$u(x) = Ax^\alpha \quad \square$$

Notice that since Bernoulli utility function is unique up to affine transformation, utility function of the form $u(x) = Ax^\alpha$, for $A > 0$ and $\alpha \neq 0$ is equivalent to utility function of the form $u(x) = ax^\alpha + b$, where $a > 0$ and $b \in \mathbb{R}$. Hence for any CRRA utility function[11] there exists an equivalent utility function that is homogeneous of some degree different than zero.

3.5.2 Additional results on buying and selling price

This chapter is part of the research project developing a theory of buying and selling price for a lottery. Therefore, apart from results which may be useful for their own sake, the following are some additional results describing properties of buying and selling price. In particular it turns out that buying and selling price for a lottery $\mathbf{x}$ are concave in W for CRRA utility functions.

Proposition 3.20. *For any wealth W and any non-degenerate lottery $\mathbf{x}$, such that $B(W, \mathbf{x})$ and $S(W, \mathbf{x})$ are well defined, the following holds for CRRA utility function:*

[11] Except for the logarithm, but since it is a limiting case of a power utility function one can ignore it.

- *$B(W, \mathbf{x})$ and $S(W, \mathbf{x})$ are strictly concave in W*
- *$B(\theta W, \mathbf{x}) + B((1-\theta)W, \mathbf{y}) < B(W, \mathbf{x}+\mathbf{y})$, $\forall \theta \in (0,1)$*
- *$S(\theta W, \mathbf{x}) + S((1-\theta)W, \mathbf{y}) < S(W, \mathbf{x}+\mathbf{y})$, $\forall \theta \in (0,1)$*

Proof. In section 3.7. □

Below I examine the additivity properties of buying and selling price. It turns out that buying price is sub-additive for all strictly increasing and strictly concave utility functions and selling price is sub-additive only for CRRA subclass of such utility functions.

Proposition 3.21. *Suppose lottery $\mathbf{x}$ has at least two values in the support. Let U be a strictly increasing and strictly concave function. For any W and any lottery $\mathbf{x}$ such that buying and selling price are well defined and $n \in \mathbb{Z}$, $n > 1$, the following holds:*

$$B(W, n\mathbf{x}) < nB(W, \mathbf{x}) \tag{3.18}$$

Proof. In section 3.7. □

For buying price the result holds for all concave functions[12]. For selling price an equivalent result does not hold in general for all concave functions. To see this consider the following utility function:

$$U(x) = \begin{cases} 2x & \text{for } x < 1 \\ \frac{1}{2}x + \frac{3}{2} & \text{for } x \geq 1 \end{cases}$$

This is clearly a continuous weakly concave function. Now consider the following lottery: $\mathbf{x} = (5, \frac{1}{2}; 0, \frac{1}{2})$, i.e. a lottery which gives 5 or 0 with equal probabilities. Simple calculation delivers that the selling price[13] for this lottery is equal to 1. Now, let's consider another lottery $\mathbf{y} = 2\mathbf{x} = (10, \frac{1}{2}; 0, \frac{1}{2})$. Selling price for this lottery is equal to $\frac{7}{2}$. Hence, I have $S(0, 2\mathbf{x}) = \frac{7}{2} > 2 = 2S(0, \mathbf{x})$.
It is clear therefore that the result equivalent to proposition 3.21 for selling price does not hold in general. However it does hold for certain classes of utility functions. Below I show that it holds for the CRRA class:

Proposition 3.22. *Suppose lottery $\mathbf{x}$ has at least two values in the support. Let U be a strictly increasing and strictly concave CRRA function. For any W and any lottery $\mathbf{x}$*

[12]Even if the function is not strictly concave, the result is still true if I change strict inequality to weak inequality in equation (3.18)above.
[13]Without loss of generality I consider $W = 0$.

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such that buying and selling price are well defined and $n \in \mathrm{Z}$, $n > 1$, *the following holds:*

$$S(W, n\mathbf{x}) < nS(W, \mathbf{x})$$

Proof. In section 3.7. □

I may use the above results for analyzing selling and buying price for several lotteries with given dependence structure. For illustration, suppose utility function is CRRA and let's take a sequence of $n > 1$ identically distributed lotteries $\mathbf{x_i} = (-x, 1/2; x, 1/2)$, where $x > 0$ and $i \in \{1, 2, ..., n\}$. I am interested in finding a buying and selling price for a sum of such lotteries: $\mathbf{y} = \sum_{i=1}^{n} \mathbf{x_i}$. Such a sum is a new lottery which is not identified until I specify the joint distribution between lotteries $\mathbf{x_i}$. Let's focus on two benchmark cases of the joint distribution - one with maximal positive linear correlation given marginals and one with maximal negative linear correlation given marginals. In the first case, lottery $\mathbf{y}$ takes the following form:

$$\mathbf{y} = (-nx, 1/2; nx, 1/2)$$

and in the second case lottery $\mathbf{y}$ takes the following form:

$$\begin{aligned} &\mathbf{y} = (-x, 1/2; x, 1/2) && \text{if } n \text{ odd} \\ &\mathbf{y} = (0) && \text{if } n \text{ even} \end{aligned} \tag{3.19}$$

Applying previous results I obtain:

- perfect positive correlation given marginals:

$$B\left(W, \sum_{i=1}^{n} \mathbf{x_i}\right) < \sum_{i=1}^{n} B(W, \mathbf{x_i}) = nB(W, \mathbf{x_i}) \text{ for } i \in \{1, 2, ..., n\}$$
$$S\left(W, \sum_{i=1}^{n} \mathbf{x_i}\right) < \sum_{i=1}^{n} S(W, \mathbf{x_i}) = nS(W, \mathbf{x_i}) \text{ for } i \in \{1, 2, ..., n\}$$

- perfect negative correlation given marginals:

$$n \ \text{ odd} \begin{cases} B(W, \sum_{i=1}^{n} \mathbf{x_i}) = B(W, \mathbf{x_i}) & \text{for } i \in \{1, 2, ..., n\} \\ S(W, \sum_{i=1}^{n} \mathbf{x_i}) = S(W, \mathbf{x_i}) & \text{for } i \in \{1, 2, ..., n\} \end{cases}$$
$$n \ \text{ even} \begin{cases} B(W, \sum_{i=1}^{n} \mathbf{x_i}) = 0 \\ S(W, \sum_{i=1}^{n} \mathbf{x_i}) = 0 \end{cases}$$

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3.6 Concluding remarks

In this chapter for three different widely used risk attitudes classes I have attempted to present characterizations in terms of:

- simple strategy
- buying/selling price for a lottery
- Pratt [4], Arrow [8] measures of risk aversion

I stated and proved in a systematic way equivalencies of these characterizations for CARA, DARA and CRRA. These results can be useful both as technical help as well as a useful guide in empirical work. Not all of these results are new. It is however useful to put all of these results together and to offer a systematic proof of them. A simple strategy concept is a novel way to formalize existing intuition. Although parts of these results are well known in the literature, other parts turn out to be not sufficiently acknowledged and one can find statements in the literature which confirm it.
Another result in this chapter is an extension to Pratt [4] famous theorem on comparative risk aversion. It incorporates buying price as an alternative way to compare risk aversion across individuals. This result also might be useful both in theoretical and empirical work.
In section "Additional results" an interesting fact about CRRA utility function class is proved, namely that for any CRRA utility function except for the logarithm, which can be ignored as a limiting case, there exists an equivalent utility function that is homogeneous of some degree different than zero. Other results in this section develop further analysis of buying and selling price properties such as concavity and additivity.

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3.7 Proofs

I will need a couple of lemmas.

Lemma 3.23. *For any lottery* $\mathbf{x}$ *and any wealth level* W*, the following holds:*

$$S[W, \mathbf{x} - B(W, \mathbf{x})] = 0 \tag{3.20}$$

$$S[W - B(W, \mathbf{x}), \mathbf{x}] = B(W, \mathbf{x}) \tag{3.21}$$

$$B[W + S(W, \mathbf{x}), \mathbf{x}] = S(W, \mathbf{x}) \tag{3.22}$$

Proof. First I prove (3.20). Define $\mathbf{y} = \mathbf{x} - B(W, \mathbf{x})$. Using equations (3.2) and (3.1)

$$\begin{aligned} U(W) &= \mathrm{E}U[W + (\mathbf{x} - B(W, \mathbf{x}))] \\ &= \mathrm{E}U[W + \mathbf{y}] \\ &= U[W + S(W, \mathbf{y})] \\ &= U[W + S(W, \mathbf{x} - B(W, \mathbf{x}))] \end{aligned}$$

And condition (3.20) follows. Now I prove (3.21). Define $V = W - B(W, \mathbf{x})$. Using equations (3.2) and (3.1)

$$\begin{aligned} U(W) &= \mathrm{E}U[(W - B(W, \mathbf{x})) + \mathbf{x}] \\ &= \mathrm{E}U[V + \mathbf{x}] \\ &= U[V + S(V, \mathbf{x})] \\ &= U[W - B(W, \mathbf{x}) + S(W - B(W, \mathbf{x}), \mathbf{x})] \end{aligned}$$

And condition (3.21) follows. Now I prove (3.22). Define $V = W + S(W, \mathbf{x})$. Using equations (3.2) and (3.1)

$$\begin{aligned} \mathrm{E}U(W + \mathbf{x}) &= U[W + S(W, \mathbf{x})] \\ &= U(V) \\ &= \mathrm{E}U[V + \mathbf{x} - B(V, \mathbf{x})] \\ &= \mathrm{E}U[W + S(W, \mathbf{x}) + \mathbf{x} - B(W + S(W, \mathbf{x}), \mathbf{x})] \end{aligned}$$

And so condition (3.22) is proved. □

Lemma 3.24. *For any non-degenerate lottery* $\mathbf{x}$ *and any wealth* W *such that buying and selling price exist,* $S(W, \mathbf{x})$ *and* $B(W, \mathbf{x})$ *lie in the interval* $(\min(\mathbf{x}), \mathrm{E}(\mathbf{x}))$*. For a degenerate lottery* $\mathbf{x}$*,* $S(W, \mathbf{x}) = B(W, \mathbf{x}) = x$*.*

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Proof. Notice first, that for degenerate lottery $\mathbf{x} = x$, equations (3.1) and (3.2) imply the following:

$$\begin{aligned} W + S(W,x) &= W + x \\ W + x - B(W,x) &= W \end{aligned}$$

And so $S(W,x) = B(W,x) = x$. From now on I will focus on a non-degenerate lottery $\mathbf{x}$. I will prove the lemma only for selling price. For buying price the proof is similar. For simplicity I define $S \equiv S(W,\mathbf{x})$. The proof is by contradiction. Suppose $\min_{i\in\{1,\ldots,n\}} x_i \geq S$. Then:

$$U(W + x_i) \geq U\left(W + \min_{i\in\{1,\ldots,n\}} x_i\right) \geq U(W+S)$$

with strict inequality for any $x_i \neq \min_{i\in\{1,\ldots,n\}} x_i$. Since lottery $\mathbf{x}$ is non-degenerate there exists at least one $x_i \neq \min_{i\in\{1,\ldots,n\}} x_i$ Hence

$$\sum_{i=1}^{n} p_i U(W + x_i) > U(W+S)$$

So S cannot be the selling price - a contradiction.
Suppose now that $S \geq \mathrm{E}[\mathbf{x}]$. By strict Jensen's inequality:

$$\mathrm{E}U[W + \mathbf{x}] < U[W + \mathrm{E}[\mathbf{x}]] \leq U(W+S)$$

So S cannot be the selling price - a contradiction. □

Lemma 3.25. *Given a twice continuously differentiable utility function U, the following holds:*

$$ARA(W) = \lim_{h\to 0^+} \frac{4}{h}\left(p(W,h) - \frac{1}{2}\right) \tag{3.23}$$

where $ARA(W) = -\frac{U''(W)}{U'(W)}$ is an absolute risk aversion coefficient and $p(W,h)$ is a probability premium defined implicitly by:

$$p(W,h)U(W+h) + (1 - p(W,h))U(W-h) = U(W) \tag{3.24}$$

Moreover, for $h = \lambda W$, the following is obtained:

$$RRA(W) = \lim_{\lambda\to 0^+} \frac{4}{\lambda}\left(p(W,\lambda W) - \frac{1}{2}\right) \tag{3.25}$$

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Proof. Rewriting equation (3.24) by using second order Taylor expansion of U around W, the following is obtained for small h:

$$p(W,h)[U(W) + U'(W)h + \frac{1}{2}U''(W)h^2]$$
$$+(1 - p(W,h))[U(W) - U'(W)h + \frac{1}{2}U''(W)h^2] \approx U(W)$$

And after simplifying:

$$\frac{1}{2}U''(W)h + U'(W)(2p(W,h) - 1) \approx 0$$

Or

$$ARA(W) = \lim_{h\to 0^+} \frac{4}{h}\left(p(W,h) - \frac{1}{2}\right)$$

as was to be shown. If I set $h = \lambda W$, equation (3.25) immediately follows. □

3.7.1 Proof of proposition 3.11

The proof will be split into two lemmas.

Lemma 3.26. *Simple strategy of an individual is wealth invariant if and only if he exhibits CARA.*

Proof. Similar technique to that used in this proof was used in Aumann and Kurz [25]. If the decision maker's Bernoulli utility function is U, then wealth-invariant strategy can be described alternatively by the following condition:

$$\mathrm{E}U(W_1 + \mathbf{x}) \geq U(W_1) \iff \mathrm{E}U(W_2 + \mathbf{x}) \geq U(W_2), \quad \forall W_1, W_2 \tag{3.26}$$

(Necessity)
CARA utility functions take the following form $U(x) = Ae^{-ax} + B$, where $A < 0$, $a \geq 0$ and B are arbitrary constants (such that utility is strictly increasing). It is straightforward to verify that CARA utility functions correspond to wealth-invariant strategies.
(Sufficiency)
Given utility function U, consider two lotteries $\mathbf{x_i} \equiv (h, p(W_i,h); -h, 1-p(W_i,h))$, where $W_i, h > 0,\ i \in \{1,2\}$, such that:

$$\mathrm{E}U(W_i + \mathbf{x_i}) = U(W_i) \tag{3.27}$$

Contrary to what is to be shown, assume that $A(W_1) > A(W_2)$, where $A(W)$ is absolute risk aversion function. By lemma 3.25 equation (3.23), for h sufficiently small I know that $p(W_1, h) > p(W_2, h)$. Let q be between the two probability premiums: $p(W_1, h) > q > p(W_2, h)$. I define another lottery $\mathbf{y} \equiv (h, q; -h, 1-q)$. By definition of a probability premium utility function U "rejects" lottery $\mathbf{y}$ at wealth W_1 and "accepts" it at wealth W_2, which contradicts wealth invariance. □

Lemma 3.27. *Given any W_1 and W_2, the following holds:*

$$B(W_1, \mathbf{x}) = S(W_2, \mathbf{x}) \ \forall W_1, W_2 \quad \Longleftrightarrow \quad \text{the strategy is wealth} - \text{invariant}$$

where B and S are buying and selling price function, respectively[14].

Proof. First step) First, I will prove that S is independent of W iff the strategy is wealth-invariant.
(Neccessity)
If the strategy is wealth invariant then by condition (3.26) it follows that:

$$\mathrm{E}U(W_1 + \mathbf{x}) = U(W_1) \iff \mathrm{E}U(W_2 + \mathbf{x}) = U(W_2), \quad \forall W_1, W_2 \tag{3.28}$$

Let's denote $S(W_i, \mathbf{x}) = S_i \ i \in \{1, 2\}$ and assume $W_1 \neq W_2$. From the definition of selling price:

$$\mathrm{E}U(W_1 + \mathbf{x}) = U(W_1 + S_1)$$

Let's define $V_1 = W_1 + S_1$, $V_2 = W_2 + S_2$ and $\mathbf{y} = \mathbf{x} - S_1$. By equation (3.28) I know that:

$$\mathrm{E}U(V_1 + \mathbf{y}) = U(V_1) \iff \mathrm{E}U(V_2 + \mathbf{y}) = U(V_2)$$

Hence by substituting $\mathbf{y} = \mathbf{x} - S_1$ and $V_2 = W_2 + S_2$ into the RHS of the above condition:

$$\mathrm{E}U(W_2 + S_2 + \mathbf{x} - S_1) = U(W_2 + S_2)$$

And by definition of S_2 I know that S_1 has to be equal to S_2: $S(W_1, \mathbf{x}) = S(W_2, \mathbf{x})$
(Sufficiency)
Suppose that the strategy is not wealth-invariant. Then:

$$\exists W_1, W_2, \mathbf{x} : \quad \mathrm{E}U(W_1 + \mathbf{x}) \geq U(W_1) \ \text{ and } \ \mathrm{E}U(W_2 + \mathbf{x}) < U(W_2),$$

Notice that by strict monotonicity of U it follows that $S(W_1, \mathbf{x}) \geq 0$ and $S(W_2, \mathbf{x}) < 0$ which contradicts the fact that S is wealth-invariant.
Second step) Now it is sufficient to prove that S is equal to B iff S is independent of W.
(Necessity)

[14] Notice that the condition on the left-hand side of the above equivalence is the same as condition iii. in the statement of proposition 3.11.

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If S is independent of W then $S(W, \mathbf{x}) = S(W', \mathbf{x})$ for any W and W'. Take $W' = W - B(W, \mathbf{x})$ and any W. Then from lemma 3.23 equation (3.21) I have that: $B(W, \mathbf{x}) = S(W - B(W, \mathbf{x}), \mathbf{x})$. And from the fact that S is independent of wealth, I obtain $S(W, \mathbf{x}) = B(W, \mathbf{x})$. Since W was arbitrary B is also wealth independent and necessity is proved.
(Sufficiency)
Take any W and fix it. Suppose $S(W', \mathbf{x}) = B(W, \mathbf{x})$ for any W'. Then obviously S has to be independent of wealth. This finishes the proof. □

Taken together lemma 3.26 and lemma 3.27 establish proposition 3.11.

3.7.2 Proof of proposition 3.12

The proof is split into three parts. The first part is the following.

Lemma 3.28. *If simple strategy of an individual is of "wealthier-accept more" type then he exhibits DARA.*

Proof. I will prove the contrapositive of the above statement. Suppose $A(W_1) \leq A(W_2)$ for any $W_1 < W_2$. If $A(W_1) = A(W_2)$ then by proposition 3.11 simple strategy is wealth-invariant so it cannot be of "wealthier-accept more" type. Suppose then that $A(W_1) < A(W_2)$ for some given $W_1 < W_2$. Consider two lotteries: $\mathbf{x}_i = (h, p(W_i, h); -h, 1 - p(W_i, h))$, $i = \{1, 2\}$ such that:

$$\mathrm{E}U(W_i + \mathbf{x_i}) = U(W_i)$$

Then by lemma 3.25, $p(W_1, h) < p(W_2)$ for h small enough. I construct another lottery $\mathbf{y} = (h, q; -h, 1 - q)$ such that $p(W_1, h) < q < p(W_2)$. Then by construction U "rejects" lottery $\mathbf{y}$ at wealth level W_2 and "accepts" it at wealth level W_1 which means that the strategy cannot be of "wealthier-accept more" type. □

The second part of the proof consists of three results.

The following result is corollary to Pratt [4] theorem. The proof is due to LeRoy and Werner [26].

Corollary 3.29. *For a strictly increasing and twice differentiable utility function U with continuous second derivative, the following holds:*

- *$S(W, \mathbf{x})$ is increasing/constant/decreasing in W for every $\mathbf{x}$ iff $A(W)$ is decreasing/constant/increasing in W*

where $A(W)$ denotes absolute risk aversion as a function of W.

Proof. I will prove only increasing S case. The rest is similar. Given a utility function $U_1(W)$, define another utility function $U_2(W) = U_1(W + \Delta)$, where $\Delta \geq 0$. I can then apply Pratt [4] theorem: $S_1(W, \mathbf{x}) < S_2(W, \mathbf{x}) = S_1(W+\Delta, \mathbf{x}) \;\Leftrightarrow\; A_1(W) > A_2(W) = A_1(W + \Delta)$. Since Δ was arbitrary the corollary is proved. □

Lemma 3.30. *If utility function is of DARA type, then the following holds:*

$$B(W, \mathbf{x}) > 0 \iff B(W, \mathbf{x}) < S(W, \mathbf{x})$$

Proof. Using equation (3.21) of lemma 3.23 and corollary 3.29 to Pratt [4] theorem, the following is obtained:

$$0 < B(W, \mathbf{x}) = S[W - B(W, \mathbf{x}), \mathbf{x}] < S(W, \mathbf{x}) \quad \square$$

Lemma 3.31. *For a strictly increasing and twice differentiable utility function $U(\cdot)$ with continuous second derivative, the following holds:*

- *$B(W, \mathbf{x})$ is increasing/constant/decreasing in W for every $\mathbf{x}$ iff $A(W)$ is decreasing/constant/increasing in W*

where $A(W)$ denotes absolute risk aversion as a function of W.

Proof. By corollary 3.29, it suffices to show the following:

- $S(W, \mathbf{x})$ is increasing/constant/decreasing in W for any $\mathbf{x}$ iff $B(W, \mathbf{x})$ is increasing/constant/decreasing in W for any $\mathbf{x}$.

I will show only the "increasing part". The rest is similar.
(⇒) The proof is by contradiction. Take $\mathbf{x}$ such that, if $W_1 < W_2$ then $B(W_1, \mathbf{x}) \geq B(W_2, \mathbf{x})$. Fix this $\mathbf{x}$. Since $S(W, \mathbf{x})$ is increasing in W for any $\mathbf{x}$, I have:

$$S[W_2, \mathbf{x} - B(W_1, \mathbf{x})] > S[W_1, \mathbf{x} - B(W_1, \mathbf{x})] = S[W_2, \mathbf{x} - B(W_2, \mathbf{x})] = 0 \qquad (3.29)$$

where I made use of lemma 3.23 equation (3.20). By lemma 3.36, equation (3.38), I obtain from above::

$$S(W_2 - B(W_1, \mathbf{x}), \mathbf{x}) - B(W_1, \mathbf{x}) > S(W_2 - B(W_2, \mathbf{x}), \mathbf{x}) - B(W_2, \mathbf{x})$$

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And hence after rearranging and using the assumption:

$$\begin{aligned} 0 &\leq B(W_1, \mathbf{x}) - B(W_2, \mathbf{x}) \\ &< S(W_2 - B(W_1, \mathbf{x}), \mathbf{x}) - S(W_2 - B(W_2, \mathbf{x}), \mathbf{x}) \\ &\leq 0 \end{aligned}$$

Which is a contradiction and hence the "if" part of the lemma is proved.
($\Leftarrow$) Again by contradiction. Take $\mathbf{x}$ such that, if $W_1 < W_2$ then $S(W_1, \mathbf{x}) \geq S(W_2, \mathbf{x})$. Fix this $\mathbf{x}$. By lemma 3.23 equation (3.21) I have:

$$\begin{aligned} B(W_1, \mathbf{x}) &= S(W_1 - B(W_1, \mathbf{x}), \mathbf{x}) \\ &\geq S(W_2 - B(W_1, \mathbf{x}), \mathbf{x}) \\ &\geq S(W_2 - B(W_2, \mathbf{x}), \mathbf{x}) = B(W_2, \mathbf{x}) \end{aligned}$$

where the first inequality follows from our assumption and second inequality follows from the fact that $B(W, \mathbf{x})$ is increasing in W for any $\mathbf{x}$. Hence $B(W_1, \mathbf{x}) \geq B(W_2, \mathbf{x})$ - a contradiction. □

Combining corollary 3.29 and lemmas 3.30 and 3.31, it is established that when utility is DARA then buying and selling price are increasing in W and that

$$B(W, \mathbf{x}) > 0 \iff B(W, \mathbf{x}) < S(W, \mathbf{x})$$

The third step in proving proposition 3.12 is the following:

Lemma 3.32. *If selling price is increasing in W then strategy is of "wealthier-accept more" type.*

Proof. Let's focus on lotteries the acceptance of which, given preferences, depends on wealth level. That is there exists two different wealth levels W_1, W_2 such that $S(W_1, \mathbf{x}) < 0$ (reject) and $S(W_2, \mathbf{x}) > 0$ (accept). Since S is increasing in W, it must be that $W_1 < W_2$. □

3.7.3 Proof of proposition 3.13

Now I prove proposition 3.13. The proof is split into three lemmas.

Lemma 3.33. *Simple strategy of an individual is scale invariant if and only if he exhibits CRRA.*

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CRRA. If the decision maker's Bernoulli utility function is U, then homogeneous strategy can be described alternatively by the following condition:

$$\mathrm{E}U(W+\mathbf{x}) \geq U(W) \iff \mathrm{E}U(\lambda W+\lambda\mathbf{x}) \geq U(\lambda W), \quad \forall\lambda>0 \tag{3.30}$$

(Necessity)
All CRRA utility functions belong to the following family: $U(x) = Ax^a + B$, where $A > 0$, $1 \neq a \geq 0$ and B are arbitrary constants (such that utility is strictly increasing) and $U(x) = A\log x + B$ for $a = 1$, where $A \geq 0$ and B are arbitrary constants. It is easy to verify that indeed, CRRA class of utility functions represents homogeneous strategies.
(Sufficiency)
Given utility function U, consider lottery $\mathbf{x} \equiv (Wh, p(W,h); -Wh, 1-p(W,h))$, where $W, h > 0$, such that:

$$\mathrm{E}U(W+\mathbf{x}) = U(W) \tag{3.31}$$

Contrary to what is to be shown, assume that $R(\lambda W) > R(W)$, for $\lambda > 0$ and $\lambda \neq 1$. Define $\mathbf{x}' \equiv (Wh, p(\lambda W,h); -Wh, 1-p(\lambda W,h))$, such that:

$$\mathrm{E}U(\lambda W+\lambda\mathbf{x}') = U(\lambda W) \tag{3.32}$$

For h sufficiently small, by proposition 3.25 equation (3.25), I know that: $p(\lambda W,h) > p(W,h)$. Therefore, I can find q such that: $p(\lambda W,h) > q > p(W,h)$. Define $\mathbf{y} \equiv (Wh, q; -Wh, (1-q))$.
By equation (3.32), since $p(\lambda W,h) > q$:

$$\mathrm{E}U(\lambda W+\lambda\mathbf{x}) < U(\lambda W)$$

By equation (3.31), and since $q > p(W,h)$, I have:

$$\mathrm{E}U(W+\mathbf{y}) > U(W)$$

which is a contradiction. □

Lemma 3.34. *Given any $\lambda > 0$, the following holds:*

$$\begin{matrix} S(\lambda W,\lambda\mathbf{x}) = \lambda S(W,\mathbf{x}) \\ B(\lambda W,\lambda\mathbf{x}) = \lambda B(W,\mathbf{x}) \end{matrix} \iff \text{strategy is homogeneous.}$$

where B and S are buying and selling price function, respectively.

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Proof. (Necessity)
I will prove it only for selling price S. For buying price proof is similar. To avoid heavy notation, I define $S_\lambda \equiv S(\lambda W, \lambda \mathbf{x})$ and $S \equiv S(W, \mathbf{x})$. If the decision maker's Bernoulli utility function is U, then scale-invariant strategy can be described alternatively by the following condition:

$$\mathrm{E}U(W + \mathbf{x}) = U(W) \quad \Longleftrightarrow \quad \mathbf{E}U(\lambda W + \lambda \mathbf{x}) = U(\lambda W), \quad \forall \lambda > 0 \tag{3.33}$$

By definition of S_λ:

$$\mathrm{E}U(\lambda W + \lambda \mathbf{x}) = U(\lambda W + S_\lambda)$$

Define $V \equiv W + \frac{1}{\lambda} S_\lambda$ and $\mathbf{y} \equiv \mathbf{x} - \frac{1}{\lambda} S_\lambda$. Then I can rewrite the above equation as:

$$\mathrm{E}U(\lambda V + \lambda \mathbf{y}) = U(\lambda V)$$

By condition (3.33), I have:

$$\mathbf{E}U(V + \mathbf{y}) = U(V) \quad \Longleftrightarrow \quad \mathbf{E}U(\lambda V + \lambda \mathbf{y}) = U(\lambda V), \quad \forall \lambda > 0$$

And hence

$$\mathrm{E}U(W + \mathbf{x}) = U(W + \frac{1}{\lambda} S_\lambda)$$

So, by definition of S, it has to be that: $\frac{1}{\lambda} S_\lambda = S$ or by returning to the original notation:

$$S(\lambda W, \lambda \mathbf{x}) = \lambda S(W, \mathbf{x}), \quad \forall \lambda > 0$$

(Sufficiency)
Suppose the strategy is not homogeneous. Then there exists $\lambda > 0$, such that:

$$\mathrm{E}U(W + \mathbf{x}) \geq U(W) \text{ and } \mathrm{E}U(\lambda W + \lambda \mathbf{x}) < U(\lambda W) \tag{3.34}$$

It follows that $S(W, \mathbf{x}) \geq 0$ and $S(\lambda W, \lambda \mathbf{x}) < 0$, by strict monotonicity of U and the fact that λ is positive. Hence it is not possible that this λ, $S(\lambda W, \lambda \mathbf{x}) = \lambda S(W, \mathbf{x})$. A contradiction. □

Lemma 3.35. *Simple strategy is scale-invariant if and only if buying and selling return prices for any multiplicative lottery are independent from wealth and equal i.e.*

$$b(W, \mathbf{h}) = s(W, \mathbf{h}) = C_\beta, \ \forall W \tag{3.35}$$

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where β is relative risk aversion coefficient and C_β takes real values and depends only on β. Additionally,

$$b(W, \lambda\mathbf{h}) = \lambda b(W, \mathbf{h}) \tag{3.36}$$

$$s(W, \lambda\mathbf{h}) = \lambda s(W, \mathbf{h}) \tag{3.37}$$

Proof. If a strategy is scale-invariant, it is easy to see from the definitions of buying and selling return prices (3.11) and (3.10), that all the conditions of $b(W, \mathbf{h})$ and $s(W, \mathbf{h})$ above are satisfied. Similarly in the other direction, it is easy to see that if the conditions above are satisfied, the strategy must be scale-invariant due to the nature of multiplicative gambles and the way they are handled in conditions (3.11) and (3.10) defining buying and selling return prices. □

3.7.4 Proof of proposition 3.14

I will first prove two additional lemmas and then the proposition. The first proposition states that buying price exhibits the so called delta property whereas selling price in general does not.

Lemma 3.36. *For any lottery* $\mathbf{x}$ *and any wealth level* W *and for* $\Delta \in \mathbb{R}$*, the following holds:*

$$S(W, \mathbf{x} + \Delta) = S(W + \Delta, \mathbf{x}) + \Delta \tag{3.38}$$

$$B(W, \mathbf{x} + \Delta) = B(W, \mathbf{x}) + \Delta \tag{3.39}$$

Proof. From the definition of selling price:

$$\begin{aligned} \mathrm{E}U(W + \mathbf{x} + \Delta) &= U[W + S(W, \mathbf{x} + \Delta)] \\ &= U[W + \Delta + S(W + \Delta, \mathbf{x})] \end{aligned}$$

And hence equation (3.38) holds. From the definition of buying price:

$$\begin{aligned} U(W) &= \mathrm{E}U[W + (\mathbf{x} + \Delta) - B(W, \mathbf{x} + \Delta)] \\ &= \mathrm{E}U[W + \mathbf{x} - B(W, \mathbf{x})] \end{aligned}$$

And hence equation (3.39) holds. □

A function $F(W, \mathbf{x})$ exhibits delta property if $F(W, \mathbf{x} + \Delta) = F(W, \mathbf{x}) + \Delta$ for $\Delta \in \mathbb{R}$. Thus, the buying price exhibits delta property, while selling price in general does not -

see equations (3.38) and (3.39) above. There is however a special class of utility functions for which selling price exhibits the delta property, namely the class of constant absolute risk aversion (CARA). Notice from equation (3.38), that selling price would obey the delta property if only $S(W+\Delta,\mathbf{x}) = S(W,\mathbf{x})$ for $\Delta \in \mathbb{R}$. That means that selling price would be independent of wealth level W. And indeed, as could easily be checked, selling price for CARA utility is independent of wealth. In fact, the stronger result by Raiffa [2] holds- CARA utility is equivalent to selling price exhibiting the delta property.

Lemma 3.37. *For differentiable DARA utility functions, given any non-degenerate lottery* $\mathbf{x}$ *and any wealth level* W*, the following holds:*

$$0 < \frac{\partial B(W,\mathbf{x})}{\partial W} < 1$$

Proof. From the definition of buying, selling price and the fact that they are both increasing in wealth, it follows that:

$$\frac{\partial B(W,\mathbf{x})}{\partial W} = \frac{\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x})) - U'(W)}{\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x}))} > 0$$

The result follows. □

Proposition 3.38. *For two different utility functions* $U_1(\cdot)$ *and* $U_2(\cdot)$ *with decreasing absolute risk aversion (DARA), any wealth level* W *and any non-degenerate random variable* $\mathbf{x}$ *with bounded values, I define corresponding selling and buying prices* $S_1(W,\mathbf{x})$, $B_1(W,\mathbf{x})$ *and* $S_2(W,\mathbf{x})$, $B_2(W,\mathbf{x})$. *The following equivalence holds:*

$$\forall W \ \forall \mathbf{x} : \ \exists \ \delta > 0 \ |x_i| < \delta \ \forall i \in \{1,...,n\}$$
$$S_1(W,\mathbf{x}) > S_2(W,\mathbf{x}) \quad \Longleftrightarrow \quad B_1(W,\mathbf{x}) > B_2(W,\mathbf{x})$$

Proof. (⇒) By contradiction. Fix $\mathbf{x}$ with bounded values and W for which the following holds: $B_1(W,\mathbf{x}) \le B_2(W,\mathbf{x})$. By lemma 3.23 equation (3.21), I obtain:

$$\begin{aligned} S_1(W-B_1(W,\mathbf{x}),\mathbf{x}) &\le S_2(W-B_2(W,\mathbf{x}),\mathbf{x}) \\ &\le S_2(W-B_1(W,\mathbf{x}),\mathbf{x}) \end{aligned}$$

where the second inequality follows from the fact that S_2 is increasing in the first argument and $B_1(W,\mathbf{x}) \le B_2(W,\mathbf{x})$. This is a contradiction since I have found $V = W - B_1(W,\mathbf{x})$ and $\mathbf{x}$ for which $S_1(V,\mathbf{x}) \le S_2(V,\mathbf{x})$. Thus the first part of the proposition is proved.

(⇐) By contradiction. Suppose $S_1(V,\mathbf{x}) \le S_2(V,\mathbf{x})$ for some V and some $\mathbf{x}$ with bounded values. Take lottery $\mathbf{y} : \mathbf{y} = \mathbf{x}$. Take wealth level $W : V = W - B_1(W,\mathbf{x})$. Such wealth

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level exists for any $V \in \mathbb{R}$. To prove this, I define a function $W : \mathbb{R} \to \mathbb{R}$ taking values $V(W) = W - B_1(W, \mathbf{x})$. This function is a bijection and takes values in the whole real line $(-\infty, +\infty)$. First, by the fact that lottery $\mathbf{x}$ has bounded values I know by lemma 3.25 that $B_1(W, \mathbf{x})$ is also bounded. On the other hand W is not bounded. Hence, $V(W)$ is also not bounded. Second, by lemma 3.37, I know that $\frac{\partial B_1(W,\mathbf{x})}{\partial W} < 1$ and thus $V'(W) > 0$. Therefore, $V(W)$ is both surjection and injection and hence bijection. This proves that for any $V \in \mathbb{R}$, there exists a unique W such that $V = W - B_1(W, \mathbf{x})$. If this holds for any V, then it holds for some V such that given some $\mathbf{x}$, the following holds $S_1(V, \mathbf{x}) \le S_2(V, \mathbf{x})$. I will now show that for lottery $\mathbf{y}$ and wealth level W, $B_1(W, \mathbf{y}) \le B_2(W, \mathbf{y})$. In fact, using lemma 3.23 equation (3.21):

$$\begin{aligned}
B_1(W, \mathbf{y}) &= B_1(W, \mathbf{x}) \\
&= S_1(W - B_1(W, \mathbf{x}), \mathbf{x}) \\
&= S_1(V, \mathbf{x}) \\
&\le S_2(V, \mathbf{x}) \\
&= S_2(W - B_1(W, \mathbf{x}), \mathbf{x}) \\
&< S_2(W - B_2(W, \mathbf{x}), \mathbf{x}) \\
&= B_2(W, \mathbf{x}) = B_2(W, \mathbf{y})
\end{aligned}$$

where the last inequality holds due to the fact that $B_1(W, \mathbf{x}) > B_2(W, \mathbf{x})$ for all W and for all $\mathbf{x}$ with bounded values and S_2 is increasing in the first argument. A contradiction. Hence the proposition is proved. □

3.7.5 Proof of proposition 3.20

The proof is split into two lemmas.

Lemma 3.39. *For any $W_1 \neq W_2$ and for all $\theta \in (0, 1)$ and any non-degenerate lottery $\mathbf{x}$, the following holds for constant relative risk aversion utility function:*

$$S(\theta W_1 + (1-\theta)W_2, \mathbf{x}) > \theta S(W_1, \mathbf{x}) + (1-\theta)S(W_2, \mathbf{x})$$
$$B(\theta W_1 + (1-\theta)W_2, \mathbf{x}) > \theta B(W_1, \mathbf{x}) + (1-\theta)B(W_2, \mathbf{x})$$

provided that both sides are well defined.

Proof. I will show that for all $\theta \in (0, 1)$ and for all $W_1 \neq W_2$, the following holds:

$$S(\theta W_1 + (1-\theta)W_2, \mathbf{x}) > \theta S(W_1, \mathbf{x}) + (1-\theta)S(W_2, \mathbf{x})$$

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I define $S_i = S(W_i, \mathbf{x})$, where $i = 1, 2$. By the property of homogeneity the following follows from the definition:

$$\mathrm{E}U\left(\frac{\theta W_1 + (1-\theta)W_2 + \mathbf{x}}{\theta(W_1+S_1) + (1-\theta)(W_2+S_2)}\right) > 1$$

Define $\lambda = \frac{\theta(W_1+S_1)}{\theta(W_1+S_1)+(1-\theta)(W_2+S_2)}$. Then, by concavity of U:

$$\begin{aligned} 1 &= \lambda \mathrm{E}U\left(\frac{W_1+\mathbf{x}}{W_1+S_1}\right) + (1-\lambda)\mathrm{E}U\left(\frac{W_2+\mathbf{x}}{W_2+S_2}\right) \\ &< \mathrm{E}U\left(\lambda\frac{W_1+\mathbf{x}}{W_1+S_1} + (1-\lambda)\frac{W_2+\mathbf{x}}{W_2+S_2}\right) \\ &= \mathrm{E}U\left(\frac{\theta W_1 + (1-\theta)W_2 + \mathbf{x}}{\theta(W_1+S_1) + (1-\theta)(W_2+S_2)}\right) \end{aligned}$$

Similarly for the buying price case, define $B_i = B(W_i, \mathbf{x})$, where $i = 1, 2$. I will prove that:

$$\mathrm{E}U\left(\frac{\theta(W_1-B_1) + (1-\theta)(W_2-B_2) + \mathbf{x}}{\theta W_1 + (1-\theta)W_2}\right) > 1$$

Define $\lambda = \frac{\theta W_1}{\theta W_1 + (1-\theta)W_2}$

$$\begin{aligned} 1 &= \lambda \mathrm{E}U\left(\frac{W_1-B_1+\mathbf{x}}{W_1}\right) + (1-\lambda)\mathrm{E}U\left(\frac{W_2-B_2+\mathbf{x}}{W_2}\right) \\ &< \mathrm{E}U\left(\lambda\frac{W_1-B_1+\mathbf{x}}{W_1} + (1-\lambda)\frac{W_2-B_2+\mathbf{x}}{W_2}\right) \\ &= \mathrm{E}U\left(\frac{\theta(W_1-B_1) + (1-\theta)(W_2-B_2) + \mathbf{x}}{\theta W_1 + (1-\theta)W_2}\right) \quad \square \end{aligned}$$

Lemma 3.40. *For CRRA utility function, the following holds* $\forall \theta \in (0, 1)$

$$S(\theta W, \mathbf{x}) + S((1-\theta)W, \mathbf{y}) < S(W, \mathbf{x}+\mathbf{y})$$
$$B(\theta W, \mathbf{x}) + B((1-\theta)W, \mathbf{y}) < B(W, \mathbf{x}+\mathbf{y})$$

provided that both sides are well defined.

Proof. Let's start with the selling price. Define $S_1 = S(\theta W, \mathbf{x})$ and $S_2 = S((1-\theta)W, \mathbf{y})$. The proof is similar to the proof of concavity of S and B in W. Define $\lambda = \frac{\theta W + S_1}{W + S_1 + S_2}$. Note that $1 - \lambda = \frac{(1-\theta)W}{W+S_1+S_2}$. Then it follows from homogeneity and the definition that:

$$\begin{aligned} 1 &= \lambda \mathrm{E}U\left(\frac{\theta W+\mathbf{x}}{\theta W+S_1}\right) + (1-\lambda)\mathrm{E}U\left(\frac{(1-\theta)W+\mathbf{y}}{(1-\theta)W+S_2}\right) \\ &< \mathrm{E}U\left(\lambda\frac{\theta W+\mathbf{x}}{\theta W+S_1} + (1-\lambda)\frac{(1-\theta)W+\mathbf{y}}{(1-\theta)W+S_2}\right) \\ &= \mathrm{E}U\left(\frac{W+\mathbf{x}+\mathbf{y}}{W+S_1+S_2}\right) \end{aligned}$$

Similarly with the buying price. Define $B_1 = B(\theta W, \mathbf{x})$ and $B_2 = B((1-\theta)W, \mathbf{y})$.

$$\begin{aligned} 1 &= \theta \mathrm{E}U\left(\frac{\theta W + \mathbf{x} - B_1}{\theta W}\right) + (1-\theta)\mathrm{E}U\left(\frac{(1-\theta)W + \mathbf{y} - B_2}{(1-\theta)W}\right) \\ &< \mathrm{E}U\left(\frac{W + \mathbf{x} + \mathbf{y} - B_1 - B_2}{W}\right) \quad \square \end{aligned}$$

3.7.6 Proof of propositions 3.21 and 3.22

Notice that for a concave function f, the following holds: $f(W + nx) < nf(W + x)$. Using this fact and the definition of a buying price:

$$\begin{aligned} & \sum_{i=1}^{m} p_i \frac{1}{n}[U(W + n(x_i - B(W, \mathbf{x}))) - U(W)] \\ < & \sum_{i=1}^{m} p_i [U(W + x_i - B(W, \mathbf{x})) - U(W)] = 0 \\ = & \sum_{i=1}^{m} p_i \frac{1}{n}[U(W + nx_i - B(W, n\mathbf{x})) - U(W)] \end{aligned}$$

This implies that $B(W, n\mathbf{x}) < nB(W, \mathbf{x})$.

Now I prove proposition 3.22. According to proposition 3.20, for CRRA utility function and given that $n > 0$, I have: $S(nW, n\mathbf{x}) = nS(W, \mathbf{x})$. And since CRRA functions belong to the class of DARA functions, I know that S is increasing in W. Hence $S(W, n\mathbf{x}) < S(nW, n\mathbf{x}) = nS(W, \mathbf{x})$.

3.8 Appendix

In section 3.3 I introduced two criteria for accepting monetary gamble $\mathbf{x}$:

a. wealth from accepting $\mathbf{x}$ should increase on average in nominal terms

b. return from $\mathbf{x}$ should be positive on average

I claimed that these two criteria are equivalent to the following:

$$\begin{aligned} \text{accept } \mathbf{x} \quad &\Longleftrightarrow \quad \mathrm{E_a}(\mathrm{W} + \mathbf{x}) \geq \mathrm{W} \quad \Longleftrightarrow \quad \mathrm{E_a}(\mathbf{x}) \geq 0 \\ \text{accept } \mathbf{h} \quad &\Longleftrightarrow \quad W \times \mathrm{E_g}(\mathbf{h}) \geq \mathrm{W} \quad \Longleftrightarrow \quad \mathrm{E_g}(\mathbf{h}) \geq 1 \end{aligned}$$

where $\mathbf{h} = 1 + \frac{\mathbf{x}}{W}$, and $W > L(\mathbf{x})$.

I will explain it on the basis of two simple lotteries. Let $\mathbf{x} = (x, p; y, 1-p)$ and $\mathbf{h} =$

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$(h, q; k, 1-q)$. Suppose the decision maker with initial wealth W accepts a sequence of n independent gambles of the form $\frac{\mathbf{x}}{n}$. Suppose that $\frac{x}{n}$ occurs i times in the sequence. Then his final wealth will be: $W + \frac{i}{n}x + \frac{n-i}{n}y$. By the law of large numbers as n becomes large, $\frac{i}{n}$ tends to p, so that final wealth may be written as:

$$W + [px + (1-p)y] = W + \mathrm{E_a}\mathbf{x}$$

Now suppose that the decision maker with initial wealth W accepts a sequence of n independent multiplicative gambles of the form $(\mathbf{h})^{\frac{1}{n}}$. Suppose that $h^{\frac{1}{n}}$ occurs i times in the sequence. Then his final wealth will be: $Wh^{\frac{i}{n}}k^{\frac{n-1}{n}}$. By the law of large numbers as n becomes large, $\frac{i}{n}$ tends to q, so that final wealth may be written as:

$$Wh^q k^{1-q} = W \times \mathrm{E_g}\mathbf{h}$$

Chapter 4

Gambles with prices, operational measure of riskiness and buying and selling price for risky lotteries

Abstract

In this chapter I analyze the operational measure of riskiness defined by Foster and Hart [5]. I give simple intuition behind their main result. Then I extend the concept of riskiness measure in two respects - I define a generalized riskiness measure based on decreasing absolute risk aversion utility function. I derive necessary and sufficient conditions for existence of such measure for DARA and CRRA class of utility functions. In addition, I show the way how to compare riskiness of gambles with negative expectation or with nonnegative outcomes only. To this end I use properties of buying and selling price for a lottery and their relations to riskiness measure. In particular, I show how buying and selling price for a lottery concepts may be used complementary to the concept of riskiness measure.

Keywords: operational measure of riskiness, extended measure of riskiness

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4.1 Introduction

In economics there is more consensus over how to define risk aversion than how to define risk. Aumann and Serrano [24] used this startling matter of fact to define a concept of risk derived from risk aversion, i.e. risk is what risk averters hate. This ingenious approach and nice axiomatic treatment led to defining economic index of riskiness. Since it looked like the index was measured in dollars but there was no theoretical support for this claim at the time, Dean Foster and Sergiu Hart started to work on giving the index operational interpretation. This plan did not succeed but instead Foster and Hart [5] came up with another way to measure riskiness which bears a lot of similarity to Aumann and Serrano [24] index and has a nice operational interpretation. They define measure of riskiness for a gamble as an amount of initial wealth below which the decision maker should reject the gamble if he wishes to guarantee no-bankruptcy making consistent choices in the long run[1]. In this paper I want to provide simple intuition behind Foster and Hart [5] measure of riskiness. In particular, I want to show why their result holds and which assumptions are crucial for that. I will discuss assumptions behind the model, in particular the assumption of homogeneous simple strategies. Simple strategies are strategies whether to accept a given gamble or not taking into account only current wealth level and the gamble in question. In a dynamic setting it corresponds to Markov stationary strategy concept. Simple strategies are homogeneous if they are scale-invariant - the decision whether to accept a gamble at a given wealth level should not change when both the gamble outcomes and wealth level are rescaled by some positive factor. I will show that in the expected utility framework this assumption is equivalent to constant relative risk aversion utility function representing the individual's preferences. I will argue that the assumption of scale-invariant simple strategies is crucial for the main result of Foster and Hart [5] even though the statement of alternative result without this assumption seems similar in mathematical terms. I will try to show that the assumption of scale-invariant simple strategies makes the result of Foster and Hart [5] particularly strong by imposing strong consistency requirements.

Another restriction of the model by Foster and Hart [5] is necessary for the existence of the riskiness measure. Gambles are assumed to have positive expectation and negative outcomes with positive probability. To be precise this assumption is not necessary for existence of the riskiness measure as the authors argue that the riskiness measure can be extended so that all gambles with non-positive expectation have riskiness equal to infinity and all gambles with no losses get zero riskiness. However the fact is that only gambles satisfying the assumption can be compared in terms of their riskiness and gambles which do not satisfy this assumption cannot. This is to say that the assumption

[1] If this statement is not clear at this point, it will be clarified later on when the exact formulation and the nature of the consistency requirements are outlined.

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restricts the set of gambles to those for which riskiness measure is meaningful. I want to show the way to compare riskiness of the gambles which do not satisfy this assumption. For this purpose I will link the concept of riskiness measure with the concept of buying and selling price for a lottery as defined by Raiffa [2]. I will analyze riskiness measure for gambles with prices. The advantage of this approach is that the riskiness measure in this case is well defined even if it is not well defined for gambles without prices. A couple of theoretical results will make it clear in what sense measuring riskiness for gambles without prices can sometimes be inferred from the riskiness measure for gambles with prices.

The riskiness measure of Foster and Hart [5] can be linked to expected utility maximization. It is rather straightforward to see from the definition[2] that the riskiness measure of a gamble is equal to the value of initial wealth for which the logarithmic utility maximizer is indifferent between taking the lottery and not taking the lottery. Logarithmic utility function is a member of constant relative risk aversion class of utility functions. I extend the definition of the riskiness measure as follows - given some utility function and a gamble, the riskiness measure is the value of wealth for which the decision maker whose preferences are represented by this utility function is indifferent between taking and not taking the gamble. For the case of decreasing absolute risk aversion and the narrower case of constant relative risk aversion I show what are the necessary and sufficient conditions for existence of such riskiness measure. Obviously for functions other than logarithm it is not necessarily the case that the simple strategy corresponding to such utility function guarantees no-bankruptcy so strictly speaking to allow other utility functions is not really an extension of the riskiness measure by Foster and Hart [5]. However, it turns out that for CRRA utility functions the extended riskiness measure is increasing in relative risk aversion coefficient. It means that for utility functions with relative risk aversion coefficient higher that 1 (the limiting case equivalent to logarithm), the extended riskiness measure in fact guarantees no-bankruptcy. Since these utility functions "reject" more than logarithmic utility function, it is the case that the riskiness measure of Foster and Hart [5] is the lowest and hence least restrictive among all riskiness measures which guarantee no-bankruptcy. This result appears already in Foster and Hart [5] but in an implicit form and therefore I state it in the paper explicitly.

This paper is organized as follows. In section 4.2 I introduce the model and its assumptions and in particular I introduce the concepts of buying and selling price for a lottery and the riskiness measure of Foster and Hart [5]. In section 4.3 I show some intuition behind the riskiness measure. In section 4.4 I extend the definition of the riskiness measure and give the necessary and sufficient conditions for existence of such measure. In section 4.5 I show in what relation to each other buying and selling price for a lottery and the extended riskiness measure are. I also discuss how this relation can be helpful

[2]See equation (4.4).

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in comparing riskiness measures of gambles that do not have positive expectation and negative outcomes. In section 4.6 I demonstrate equivalence between expected utility decision making, riskiness measure based decision making and buying/selling price based decision making. Finally, I conclude. Most of the results which I refer to in the paper and which have been proved in another paper have been placed in the appendix at the end.

4.2 The model

Below I define the class of utility functions and the space of lotteries which I will focus on in this paper.

Assumption 4.1. *Preferences obey expected utility axioms. Bernoulli utility function* $U : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable, strictly increasing and strictly concave.*

Definition 4.2. A lottery $\mathbf{x}$ is a real- and finite-valued random variable with finite support. The space of all lotteries will be denoted $\mathcal{X}$. I define the maximal loss of lottery $\mathbf{x}$ as: $\min(\mathbf{x}) = \min \operatorname{supp}(\mathbf{x})$.

The typical lottery will be denoted as $\mathbf{x} \equiv (x_1, p_1; ...; x_n, p_n)$, where $x_i \in \mathbb{R}$ $i \in \{1, 2, ..., n\}$ are outcomes and $p_i \in [0, 1]$ $i \in \{1, 2, ..., n\}$ are the corresponding probabilities. Outcomes should be interpreted here as monetary values. Although most of results that follow are true for more general lotteries, the finite support assumption is sufficient for the purposes of this paper. Now I define buying and selling price for a lottery given wealth level along the lines of Raiffa [2]. To avoid repetitions, I will henceforth skip statements of the form: "Given utility function U satisfying assumption 4.1, any lottery $\mathbf{x}$ and wealth W...".

4.2.1 Buying and selling price for a lottery

Here and in the next sections I introduce the key concepts of the paper.

Definition 4.3. I define selling price and buying price for a lottery $\mathbf{x}$ at wealth W as functions denoted, respectively, $S(W, \mathbf{x})$ and $B(W, \mathbf{x})$. Provided that they exist, values of these functions will be determined by the following equations:

$$\mathrm{E}U[W + \mathbf{x}] = U[W + S(W, \mathbf{x})] \tag{4.1}$$

$$\mathrm{E}U[W + \mathbf{x} - B(W, \mathbf{x})] = U(W) \tag{4.2}$$

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If utility function is defined over the whole real line as is the case for constant absolute risk aversion, buying and selling price as functions of wealth exists for any wealth level by assumption 4.1. If the domain of utility function is restricted to a part of real line as is the case of constant relative risk aversion utility function analyzed here, the domain of buying and selling price for a lottery is also restricted. I will focus mostly on the case of constant relative risk aversion (CRRA) utility function normalized conveniently[3] so that it takes the following form:

$$U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & 0 < \alpha < 1, \quad x \geq 0 \\ \log x, & \alpha = 1, \quad x > 0 \\ \frac{x^{1-\alpha}-1}{1-\alpha}, & 1 < \alpha, \quad x > 0 \end{cases} \tag{4.3}$$

Observe that

$$\lim_{\alpha \to 1} \frac{x^{1-\alpha} - 1}{1 - \alpha} \overset{H}{=} \lim_{\alpha \to 1} \frac{-\log x}{-1} = \log x$$

The proposition below establishes the domain and the range of selling and buying price for a lottery if the utility function takes the above form. The statement and the proof is due to Lewandowski [22].

Proposition 4.4 (CRRA2)**.** *Given the class of CRRA utility function used in the section the following holds for any non-degenerate lottery* $\mathbf{x}$*: for* $\alpha \geq 1$

- $\lim_{W \to 0} B(W, \mathbf{x}) = \min(\mathbf{x})$
- $\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = \min(\mathbf{x})$

Define $W_L(\mathbf{x}) = U^{-1}[\mathrm{E}U(-\min(\mathbf{x}) + \mathbf{x})]$*. For* $0 < \alpha < 1$

- $\lim_{W \to W_L(\mathbf{x})} B(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$,
- $\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$

Additionally,

$$\forall \alpha > 0 \ \lim_{W \to \infty} B(W, \mathbf{x}) = \lim_{W \to \infty} S(W, \mathbf{x}) = \mathrm{E}[\mathbf{x}]$$

As for the intuition behind selling and buying price for the lottery, in economic terms, given an individual with initial wealth W whose preferences are represented by utility function $U(\cdot)$, $S(W, \mathbf{x})$ is the minimal amount of money which he demands for giving

[3] Normalization is done without loss of generality since cardinal utility function is unique only up to affine transformation. That means that I can choose the slope and the shifting constant in a given point without changing the Pratt [4] risk attitudes characteristics.

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up lottery $\mathbf{x}$. Similarly, $B(W, \mathbf{x})$ is the maximal amount of money which he is willing to pay in order to play lottery $\mathbf{x}$.

Buying and selling price exhibit certain properties, many of which are enumerated in the appendix. Proofs of these properties may be found in Lewandowski [15]. I will make use of these properties when I establish connections between buying and selling price and riskiness measure.

4.2.2 Riskiness measure

Foster and Hart [5] define operational measure of riskiness as follows. The initial wealth is $W_1 > 0$. At every period $t = 1, 2, ...$, the decision maker with wealth W_t is offered a gamble $\mathbf{x_t}$. He may accept or reject the gamble. His wealth next period is $W_{t+1} = W_t + \mathbf{x_t}$ if he accepts and $W_{t+1} = W_t$ if he rejects. Simple strategy of the decision maker whether to accept gamble $\mathbf{x_t}$ at time t or not is assumed to be stationary Markov strategy - it depends only on the gamble $\mathbf{x_t}$ and current wealth level W_t. Simple strategy is homogeneous or scale-invariant if "accept $\mathbf{x}$ at W" implies "accept $\lambda\mathbf{x}$ at λW", for any $\lambda > 0$. For characterization results concerning simple strategies and in particular homogeneous simple strategies consult Lewandowski [15]. If borrowing is not allowed, bankruptcy occurs when wealth converges to zero as time goes to infinity. A given strategy s yields no-bankruptcy for the process $(\mathbf{x_t})_{t=1,2,...}$ and the initial wealth W_1 if probability of bankruptcy is zero, i.e. $\mathrm{P}\left[\lim_{t\to\infty} W_t = 0\right] = 0$. Strategy guarantees no-bankruptcy if it yields no-bankruptcy for every process $(\mathbf{x_t})_{t=1,2,...}$ and every initial wealth level W_1. The technical assumptions state that gambles are assumed to be finite-valued, with finite support and such that $\mathrm{E}[\mathbf{x}] > 0$ and $\mathrm{P}[\mathbf{x} < 0] > 0$, where $\mathrm{P}[E]$ denotes a probability of an event E (positive expected value and losses are possible). The stochastic process $(\mathbf{x_t})_{t=1,2,...}$ is assumed to be finitely generated.

The main theorem of Foster and Hart [5] states the following.

Theorem 4.5 (Foster and Hart [5]). *For every gamble $\mathbf{x}$ there exists a unique real number $R_{FH}(\mathbf{x}) > 0$ such that: a homogeneous strategy s guarantees no-bankruptcy if and only if for every gamble $\mathbf{x}$ and wealth $W > 0$,*

$$W < R_{FH}(\mathbf{x}) \;\Rightarrow\; s \text{ rejects } \mathbf{x} \text{ at } W$$

Moreover, $R_{FH}(\mathbf{x})$ satisfies the following equation

$$\mathrm{E}\left[\log\left(1 + \frac{\mathbf{x}}{R_{FH}(\mathbf{x})}\right)\right] = 0 \tag{4.4}$$

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Foster and Hart [5] call $R_{FH}(\mathbf{x})$ the measure of riskiness of $\mathbf{x}$.
As I mentioned in the introduction, there is a link between the riskiness measure and expected utility maximizing individuals. Consider an expected-utility maximizer with utility function U:

$$\text{accept } \mathbf{x} \text{ at } W \iff \mathrm{E}U(W+\mathbf{x}) \geq U(W) \tag{4.5}$$

Notice that for logarithmic utility function I can rewrite condition on the RHS of (4.5) in relative - instead of absolute - terms, as follows:

$$\mathrm{E}\left[\log\left(1+\frac{\mathbf{x}}{W}\right)\right] \geq 0$$

It is clear that the index $R_{FH}(\mathbf{x})$ has the property that the logarithmic utility rejects $\mathbf{x}$ if $W < R_{FH}(\mathbf{x})$ and accepts $\mathbf{x}$ if $W \geq R_{FH}(\mathbf{x})$. Hence by the theorem above logarithmic utility represents a strategy that is among those which guarantee bankruptcy. In the next section I will provide further intuition behind the riskiness measure and discuss assumptions underlying it.

4.3 Riskiness measure - its assumptions and intuition behind

Notice that simple strategies in the theorem are assumed to be homogeneous. It turns out as proved in Lewandowski [15] that in expected utility setting homogeneous simple strategy is equivalent to utility function being of constant relative risk aversion (CRRA) type. Within this class logarithmic utility function guarantees no bankruptcy as shown above. Even more is true as will be shown in the next section - in CRRA class all utility functions with relative risk aversion coefficient $\alpha \geq 1$ guarantee no-bankruptcy. Logarithmic utility function is the least restrictive (rejects the least) among all CRRA utility functions that guarantee no-bankruptcy[4]. To understand the intuition behind this result is quite simple. Suppose CRRA utility function is normalized conveniently and it takes the form given in (4.3).Notice the following fact about this class of functions:

$$\lim_{x\to 0} U_\alpha(x) = \begin{cases} -\infty & \text{for } \alpha \geq 1 \\ A(\alpha) > -\infty & \text{for } 0 < \alpha < 1 \end{cases}$$

Within CRRA class, logarithmic function is a function with the smallest relative risk aversion among those which "assign" $-\infty$ index to bankruptcy $x = 0$. Intuitively, if utility value for bankruptcy is finite as is the case for $0 < \alpha < 1$ then for any initial

[4] It is also independently shown in Foster and Hart [5].

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wealth W it is possible to construct a sequence of gambles such that the decision maker who makes decisions represented by this utility function goes bankrupt with positive probability. Consider a decision maker whose decisions are represented by a CRRA utility function $U(x)$ for which $\lim_{x\to 0} U(x) = A > -\infty$. Suppose his initial wealth level is $W > 0$. I construct a gamble $(-W, p, M, 1-p)$, $1 > p > 0$ ($-W$ with probability p and M with probability $1-p$) such that $pA + (1-p)U\left(1 + \frac{M}{W}\right) \geq 0$. It is possible to construct such a gamble because CRRA utility function is unbounded above. Hence the decision maker will accept this gamble at wealth level W. In one step only the probability that this decision maker goes bankrupt (his wealth is zero) is $p > 0$. As long as $\lim_{x\to 0} U(x) = A > -\infty$, it is possible to construct gambles that make an individual whose preferences are represented by $U(x)$ bankrupt in one step with positive probability.

The situation is different for $\alpha \geq 1$. Here $\lim_{x\to 0} U_\alpha(x) = -\inf$. An individual with initial wealth W whose preferences are represented by such utility function U will never accept a finite-valued gamble that makes him bankrupt with positive probability. It follows that there does not exists a finite sequence of gambles that such individual would accept and which would make him bankrupt with positive probability. What about the infinite sequence of gambles? Here is a useful illustration: Suppose the decision maker has initial wealth $W > 0$ and his preferences can represented by logarithmic utility function. Suppose further that he is offered an infinite sequence of multiplicative gambles of the following form: $\left(\frac{1}{2}, \epsilon; 2^{\frac{\epsilon}{1-\epsilon}}, 1-\epsilon\right)$, where $\epsilon > 0$. Notice first that the decision maker will be indifferent between accepting and rejecting such gamble: $\epsilon \log \frac{1}{2} + (1-\epsilon)\log 2^{\frac{\epsilon}{1-\epsilon}} = \log 1$. Assume that the gambles in this infinite sequence are perfectly positively correlated.[5] That means that after accepting n such gambles the decision maker's wealth may be written as $\left(W\frac{1}{2^n}, \epsilon; W2^{n\frac{\epsilon}{1-\epsilon}}, 1-\epsilon\right)$. Let's write the "next" multiplicative gamble in nominal terms: $\left(-\frac{1}{2}W\frac{1}{2^n}, \epsilon; +W2^{n\frac{\epsilon}{1-\epsilon}}, 1-\epsilon\right)$. Notice that as n goes to infinity wealth tends to zero with positive probability ϵ. However, in order to achieve this the gambles in the sequence become infinite valued: $\lim_{n\to\infty} W2^{n\frac{\epsilon}{1-\epsilon}} = \infty$

The above illustration shows that in case of logarithmic utility function [6], one needs infinite-valued gambles so that accepting these gambles leads to bankruptcy with positive probability and yet they are accepted.

It is worth noting that Foster and Hart [5] have also another theorem in which they relax the assumption of homogeneous simple strategies. It seems however that this theorem is much weaker than the one with homogeneous simple strategies. It still says that to guarantee no-bankruptcy it is necessary to reject gamble $\mathbf{x}$ at wealth level W if $W < R_{FH}(\mathbf{x})$. This time however it is required only if wealth is close to zero already. If wealth is higher other strategies are sufficient such as: reject $\mathbf{x}$ at wealth W

[5] For all other correlation the argument works even better.

[6] Actually, for all CRRA functions with $\alpha \geq 1$.

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if $W < \min(\mathbf{x}) + \epsilon$, for $\epsilon > 0$ and small.

Apart from homogeneity assumption, the riskiness measure is defined only for gambles with positive expectation and possible losses. In the subsequent sections I will show a way to infer something about the riskiness of a gamble even if the gamble does not allow losses or/and has non-positive expectation.

4.4 Extended riskiness measure

In this section I define an extended riskiness measure and analyze conditions which are necessary and sufficient for existence of such measure. I discuss first the more general case of decreasing absolute risk aversion and then I focus on a subset of this, namely constant relative risk aversion.

4.4.1 Existence, uniqueness and no-bankruptcy for DARA

I focus on decreasing absolute risk aversion class of utility functions. Following Yaari [27] I define the acceptance set $A_{\mathbf{x}} \equiv \{W : \mathrm{E}U(W + \mathbf{x}) > U(W)\}$ of wealth levels for which an individual with preferences represented by utility function U facing the lottery $\mathbf{x}$ strictly prefers to accept this lottery. Dybvig and Lippman [28] proved the following result:

Theorem 4.6 (Dybvig and Lippman [28])**.** *Let U be a strictly increasing concave utility function with continuous second derivative. Then absolute risk aversion A is decreasing if and only if for each gamble $\mathbf{x}$, $A_{\mathbf{x}}$ is an interval of the form $(\theta_{\mathbf{x}}, +\infty)$, where $-\infty \leq \theta_{\mathbf{x}} \leq +\infty$.*

The theorem is adjusted for the purposes of this paper. Define function $\phi(W) = \mathrm{E}U(W + \mathbf{x}) - U(W)$. Below I present my proof of this result as it is shorter and more straightforward than the original.

Proof. Notice that since U is continuous, function ϕ is continuous as well. Hence exactly one of the three possibilities can occur:

- $\phi(W) > 0, \ \forall W$, in which case $A_{\mathbf{x}} = (-\infty, +\infty)$
- $\phi(W) < 0, \ \forall W$, in which case $A_{\mathbf{x}} = (+\infty, +\infty)$
- function ϕ crosses zero axis

In the last case I will show that function ϕ crosses zero axis exactly once. Suppose function ϕ crosses zero axis at W^*, i.e. $\phi(W^*) = 0$. From the definition of selling price, it is clear that $S(W^*, \mathbf{x}) = 0$. Using the corollary 4.28 to Pratt [4] theorem which can be found in the appendix, since for DARA utility S is increasing in W, it must be that for $W > W^*$, $S(W, \mathbf{x}) > 0$ and for $W < W^*$, $S(W, \mathbf{x}) < 0$. And hence there can be exactly one such W^* for which function ϕ crosses zero axis. □

The theorem above makes it clear that DARA utility means that wealthier people accept more gambles. It also shows that if there exists number W^* for which $\phi(W^*) = 0$, it must necessarily be unique. Therefore, it makes sense to define, whenever it exists, $R(\mathbf{x}) = W^*$ as an extended riskiness measure. There are two conditions which are necessary for existence of an extended riskiness measure for all functions which are concave and strictly increasing.

Proposition 4.7. *For all utility functions which are concave and strictly increasing and given a lottery* $\mathbf{x}$*, the following are necessary conditions for existence of* $R(\mathbf{x})$*:*

a. $\mathrm{E}[\mathbf{x}] > 0$

b. $\mathrm{P}[\mathbf{x} < 0] > 0$

Proof. To see that these two are the necessary conditions for existence of an extended riskiness measure, note that if $\mathrm{E}[\mathbf{x}] \leq 0$, then by Jensen's inequality $U(R(\mathbf{x})) = \mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) < U[R(\mathbf{x}) + \mathrm{E}(\mathbf{x})] \leq U(R(\mathbf{x}))$, which is a contradiction. If on the other hand losses are not possible and $\mathrm{P}[\mathbf{x} < 0] = 0$ then $\phi(W) > 0,\ \forall W$ so that $R(\mathbf{x})$ does not exist. □

Suppose now that the outcome space is restricted to strictly positive real numbers, the intuition being that zero represents bankruptcy or the worst possible outcome. In this case the riskiness measure, if it exists, can take values in the interval $(L(\mathbf{x}), +\infty)$, where $L(\mathbf{x})$ is defined as the maximal loss of $\mathbf{x}$ and is equal to $-\min(\mathbf{x})$. The following are the necessary and sufficient conditions for the existence of an extended riskiness measure:

Proposition 4.8. *Given DARA utility function* $U : (0, +\infty) \to \mathbb{R}$ *and a lottery* $\mathbf{x}$ *satisfying conditions a. and b. stated above, the necessary and sufficient conditions for* $R(\mathbf{x}) > L(\mathbf{x})$ *to exist are:*

- $\lim_{W \to L(\mathbf{x})^+} \phi(W) < 0$
- $\lim_{W \to +\infty} \phi(W) \geq 0$

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Proof. Notice first that due to condition b. above, $L(\mathbf{x}) > 0$. Therefore expression $\lim_{W \to L(\mathbf{x})^+} \phi(W)$ from the first condition above is well defined. Now let's write the definitions of selling price for two wealth levels W and V:

$$\mathrm{E}U(V + \mathbf{x}) - U[V + S(V, \mathbf{x})] = 0$$
$$\mathrm{E}U(W + \mathbf{x}) - U[W + S(W, \mathbf{x})] = 0$$

If $V > W$ and utility is DARA then by corollary 4.28 to Pratt [4] theorem I have:

$$\mathrm{E}U(V + \mathbf{x}) - U[V + S(W, \mathbf{x})] > 0$$
$$\mathrm{E}U(W + \mathbf{x}) - U[W + S(V, \mathbf{x})] < 0$$

If $R(\mathbf{x}) = W$ then $S(W, \mathbf{x}) = 0$ and $\phi(V) > 0$ and if $R(\mathbf{x}) = V$, then $S(V, \mathbf{x}) = 0$ and $\phi(W) < 0$. That means that $R(\mathbf{x}) > L(\mathbf{x})$ exists, if $\lim_{W \to L(\mathbf{x})^+} \phi(W) < 0$.
The second condition has to be satisfied due to the same reasons for which the proof of theorem 4.6 is true. Since extended riskiness measure is unique and function ϕ has to be increasing when evaluated at the extended riskiness measure, the value of function ϕ at wealth going to infinity has to be not less than zero. □

The above two conditions which are both necessary and sufficient for existence of an extended riskiness measure are not very informative for the general case of decreasing absolute risk aversion. Therefore I will provide below a pair of more informative conditions, the difference being that these conditions are sufficient but not necessary for existence of an extended measure of riskiness:

- $\lim_{x \to 0^+} U(x) = -\infty$
- $\lim_{x \to +\infty} A(x) = 0$

where $A(x)$ is absolute risk aversion function evaluated at x. To see that these conditions are sufficient for existence of an extended measure of riskiness, observe that if $\lim_{x \to 0^+} U(x) = -\infty$ and $\mathrm{P}[\mathbf{x} < 0] > 0$ then $\lim_{W \to L(\mathbf{x})^+} \phi(W) = -\infty$ due to the fact that lottery $\mathbf{x}$ is bounded-valued. Notice further that $\lim_{x \to +\infty} A(x) = 0$ means that the decision maker becomes risk neutral when he gets extremely rich. Since expected value of the lottery is assumed to be positive, $\mathrm{E}[\mathbf{x}] > 0$, therefore an extremely rich individual will accept this lottery meaning that $\lim_{W \to +\infty} \phi(W) \geq 0$.
It is worth noting that condition $\lim_{x \to 0} A(x) = +\infty$ is not sufficient to ensure $\lim_{W \to L(\mathbf{x})^+} \phi(W) < 0$ and hence to ensure that an extended riskiness measure exists. One needs stronger requirement of $\lim_{x \to 0^+} U(x) = -\infty$, which shall be apparent in the next subsection where CRRA class of utility functions is analyzed. There are CRRA

utility function, namely the ones for which relative risk aversion α is smaller than certain cutoff α^* such that $\lim_{x\to 0} A(x) = +\infty$ and yet $\lim_{W\to L(\mathbf{x})^+} \phi(W) > 0$ meaning that an extended riskiness measure does not exists. Observe however that in the next subsection it proves beneficial to use variable $\frac{1}{W}$ instead of W so that the comparison of the two cases must be done with caution.

Before I will proceed to the next subsection, I want to demonstrate that for a certain class of DARA utility functions which are not necessarily CRRA, no-bankruptcy is guaranteed. First I will need the following lemma, which is also of interest for its own sake. Without loss of generality[7] assume that utility function U satisfies the following: $U(1) = 0$ and $U'(1) = 1$. Given such utility function U define relative risk aversion function as $RRA(x) = -\frac{U''(x)x}{U'(x)}$. For utility function which is denoted U_i I will use notation RRA_i for the corresponding relative risk aversion function. Then the following lemma is true.

Lemma 4.9. *For some $\delta > 0$, suppose that $RRA_i(y) > RRA_j(y)$ for all y such that $|y| < \delta$. Then $U_i(y) < U_j(y)$ whenever $y \neq 1$ and $|y| < \delta$*

Proof. First, let me say that the proof is very similar to that used in lemma 2 of Aumann and Serrano [24]. They prove a similar proposition for absolute risk aversion.
Let $|y| < \delta$. If $y > 1$, then

$$\begin{aligned}
\log U_i'(y) &= \log U_i'(y) - \log U_i'(1) \\
&= \int_1^y [\log U_i'(z)]' dz \\
&= \int_1^y \frac{U_i''(z)}{U_i'(z)} dz \\
&= -\int_1^y \frac{RRA_i(z)}{z} dz \\
&< -\int_1^y \frac{RRA_j(z)}{z} dz = \log U_j'(y)
\end{aligned}$$

If $0 < y < 1$, then

$$\begin{aligned}
\log U_i'(y) &= \log U_i'(y) - \log U_i'(1) \\
&= -\int_y^1 [\log U_i'(z)]' dz \\
&= -\int_y^1 \frac{U_i''(z)}{U_i'(z)} dz \\
&= \int_y^1 \frac{RRA_i(z)}{z} dz \\
&> \int_y^1 \frac{RRA_j(z)}{z} dz = \log U_j'(y)
\end{aligned}$$

[7] Cardinal utility function is unique only up to affine transformation.

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Hence $\log U_i'(y) \lesseqgtr \log U_j'(y)$, when $y \gtreqless 1$. It follows that $U_i'(y) \lesseqgtr U_j'(y)$, when $y \gtreqless 1$. If $y > 1$, then

$$U_i(y) = \int_1^y U_i'(z)dz < \int_1^y U_j'(z)dz = U_j(y)$$

If $0 < y < 1$, then

$$U_i(y) = -\int_y^1 U_i'(z)dz < -\int_y^1 U_j'(z)dz = U_j(y)$$

And hence the lemma is proved. □

Equipped with lemma 4.9 I can now demonstrate for which DARA utility functions in general the condition of no-bankruptcy is guaranteed.

Proposition 4.10. *For all bounded-valued lotteries and for all DARA utility functions for which $RRA(x) \geq 1$, $\forall x \in D$, where $RRA(x)$ is relative risk aversion function evaluated at x and D is the utility function's domain, no-bankruptcy is guaranteed.*

Proof. No-bankruptcy is guaranteed for logarithmic utility function for which relative risk aversion coefficient is equal to one. Take a DARA utility function U for which relative risk aversion is not less than one for all arguments in the domain of U. For any wealth level W I can normalize U without loss of generality so that $U(W) = \log(W)$. By lemma 4.9, since $RRA(y) \geq 1$ for all finite y, it is true that $U(y) \leq \log(y)$ and by normalization $U(W) = \log(W)$. It follows that if logarithmic utility function "rejects" a lottery $\mathbf{x}$, utility U also "rejects" this lottery. And hence it also guarantees no-bankruptcy. □

4.4.2 Existence, uniqueness and no-bankruptcy for CRRA

I focus now on the CRRA function which is conveniently normalized[8]:

$$U(x,\alpha) = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & \text{for } 1 \neq \alpha > 0 \\ \log x, & \text{for } \alpha = 1 \end{cases}$$

where $x \in [0, \infty]$. I want to define a measure R for lottery $\mathbf{x}$ for CRRA utility function. This measure should satisfy the following condition:

$$\frac{1}{1-\alpha}\mathrm{E}\left(1 + \frac{\mathbf{x}}{R(\mathbf{x})}\right)^{1-\alpha} - \frac{1}{1-\alpha} = 0 \tag{4.6}$$

[8]See (4.3).

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for a given lottery $\mathbf{x}$ and coefficient α. I want to ensure that such measure is well defined and unique. As already proved in the previous subsection measure of riskiness is unique if it exists. The necessary conditions are already provided in the former subsection and in particular, I will focus only on non-degenerate n-dimensional lotteries $\mathbf{x}$ with bounded values[9] such that $\mathrm{P}[\mathbf{x} < 0] > 0$ and $\mathrm{E}(\mathbf{x}) > 0$. Furthermore, I will restrict attention only to wealth levels W, such that $W \geq L(\mathbf{x}) > 0$. The fact that $L(\mathbf{x}) > 0$ follows from the fact that $\mathbf{x}$ may take negative values. Define lottery $\mathbf{y} = 1 + \frac{\mathbf{x}}{W}$. Notice that this lottery takes only non-negative values. It takes the lowest value of zero for $x_i = -L(\mathbf{x})$ for some $i \in \{1, ..., n\}$, since $W \geq L(\mathbf{x})$.

Notice that for the function form above, the following is true: $U(1) = 0$, $U'(y) = y^{-\alpha}$ and $U'(1) = 1$. Suppose there are two different CRRA utility functions with relative risk aversion coefficients equal to α_i and α_j, respectively. Suppose further that $\alpha_i > \alpha_j$. Then from lemma 4.9 I know that $U(y, \alpha_i) < U(y, \alpha_j)$, for $y \in [0, \delta)$, some $\delta > 0$ and $y \neq 1$. Hence,

$$\begin{aligned} & \frac{1}{1-\alpha_i}\mathrm{E}\left(1 + \frac{\mathbf{x}}{R(\mathbf{x})}\right)^{1-\alpha_i} - \frac{1}{1-\alpha_i} \\ < \quad & \frac{1}{1-\alpha_j}\mathrm{E}\left(1 + \frac{\mathbf{x}}{R(\mathbf{x})}\right)^{1-\alpha_j} - \frac{1}{1-\alpha_j} \end{aligned}$$

Let's define the following function:

$$\begin{aligned} & \phi(\lambda, \alpha) = \frac{1}{1-\alpha}\sum_{i=1}^{n} p_i[1 + \lambda x_i]^{1-\alpha} - \frac{1}{1-\alpha} \\ & 0 \leq \lambda \leq \frac{1}{L(\mathbf{x})}, \quad x_i \in [-L(\mathbf{x}), +M(\mathbf{x})] \end{aligned} \tag{4.7}$$

where $M(\mathbf{x})$ is the maximal gain in $\mathbf{x}$ and $L(\mathbf{x})$ is the maximal loss of $\mathbf{x}$, both assumed to be finite.

I want to find out whether this function has a unique $\lambda > 0$, for which this function is equal to zero, given α, and whether it has a unique α for which the function is equal to zero, given that $\lambda = \frac{1}{L(\mathbf{x})}$. It turns out that the answer to both questions is positive, as I will demonstrate below.

[9] The following condition holds: there exists $\delta > 0$ such that $|x_i| < \delta \quad \forall i \in \{1, ..., n\}$.

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Lemma 4.11. *The following properties characterize function ϕ:*

$$\begin{aligned}
&\phi(0,\alpha) = 0\\
&\frac{\partial\phi(\lambda,\alpha)}{\partial\lambda} = \sum_{i=1}^{n} p_i[x_i(1+\lambda x_i)^{-\alpha}]\\
&\left.\frac{\partial\phi(\lambda,\alpha)}{\partial\lambda}\right|_{\lambda=0} = \sum_{i=1}^{n} x_i p_i = \mathrm{E}[\mathbf{x}] > 0\\
&\frac{\partial^2\phi(\lambda,\alpha)}{\partial^2\lambda} = \alpha\sum_{i=1}^{n} p_i x_i^2(1+\lambda x_i)^{-\alpha-1} < 0 \quad \text{for} \quad \alpha > 0\\
&\lim_{\lambda\to\frac{1}{L(\mathbf{x})}}\lim_{\alpha\to 1}\phi(\lambda,\alpha) = -\infty\\
&\lim_{\lambda\to\frac{1}{L(\mathbf{x})}}\phi(\lambda,0) = \lim_{\lambda\to\frac{1}{L(\mathbf{x})}}\sum_{i=1}^{n} p_i(1+\lambda x_i) - 1 = \frac{1}{L(\mathbf{x})}\mathrm{E}[\mathbf{x}] > 0
\end{aligned} \tag{4.8}$$

Furthermore $\lim_{\lambda\to\frac{1}{L(\mathbf{x})}}\phi(\lambda,\alpha)$ is a continuous function of α and it is strictly monotonic in α (see lemma 4.9). Therefore the following result holds:

Proposition 4.12. *Given function $\phi(\lambda,\alpha)$ and a random variable $\mathbf{x}$ with n values denoted by x_i for $i = 1,...,n$, where $\mathrm{E}(\mathbf{x}) > 0$ and $\mathrm{P}[\mathbf{x} < 0] > 0$, the following is true. Denote $L = L(\mathbf{x})$ and $M = M(\mathbf{x})$.*

$$\exists!\alpha^* < 1 : \begin{cases} \alpha < \alpha^* & \phi(\frac{1}{L},\alpha) > 0\\ \alpha = \alpha^* & \phi(\frac{1}{L},\alpha) = 0\\ \alpha > \alpha^* & \phi(\frac{1}{L},\alpha) < 0 \end{cases}$$

Furthermore, suppose I take $\alpha > \alpha^*$ and fix it. Then:

$$\exists!\lambda^* : \begin{cases} \lambda < \lambda^* & \phi(\lambda,\alpha) > 0\\ \lambda = \lambda^* & \phi(\lambda,\alpha) = 0\\ \lambda > \lambda^* & \phi(\lambda,\alpha) < 0 \end{cases}$$

Proof. Follows from the above stated properties of a function ϕ (lemma 4.11). □

The above proposition states that riskiness measure for CRRA is defined for $\alpha \geq \alpha^*$, where α^* depends on a lottery. In this case the riskiness measure is unique. For different α's from the set of α's satisfying $\alpha > \alpha^*$ I get different λ^*, which is the inverse of the riskiness measure. Let's define a function $\lambda^*(\alpha)$, where $\alpha > \alpha^*$ and $\phi(\lambda^*(\alpha),\alpha) = 0$. I have the following proposition:

Proposition 4.13. *The function $\lambda^*(\alpha)$ is decreasing in α.*

Proof. Suppose $\alpha_1 > \alpha_2$ and that $\alpha_1 > \alpha^*$. Then

$$\begin{aligned} 0 &= \phi(\lambda^*(\alpha_1), \alpha_1) \\ &= \frac{1}{1-\alpha_1}\sum_{i=1}^{n} p_i(1+\lambda^*(\alpha_1)x_i)^{1-\alpha_1} - \frac{1}{1-\alpha_1} \\ &< \frac{1}{1-\alpha_2}\sum_{i=1}^{n} p_i(1+\lambda^*(\alpha_1)x_i)^{1-\alpha_2} - \frac{1}{1-\alpha_2} = \phi(\lambda^*(\alpha_1), \alpha_2) \end{aligned}$$

Hence:

$$\phi(\lambda^*(\alpha_1), \alpha_2) > 0$$
$$\phi(\lambda^*(\alpha_2), \alpha_2) = 0$$

Since $\phi(\lambda, \alpha)$ is concave in λ and $\phi(\frac{1}{L}, \alpha) < 0$, I conclude that $\lambda^*(\alpha_2) > \lambda^*(\alpha_1)$. □

The above proposition states that the higher is α, the relative risk aversion coefficient, the higher is riskiness measure, which is the inverse of $\lambda^*(\alpha)$. It confirms a conjecture that since rejecting for wealth being below riskiness measure based on $\alpha = 1$ (Foster and Hart [5] riskiness measure) guarantees no bankruptcy, also rejecting for wealth below riskiness measure based on $\alpha > 1$ guarantees no bankruptcy, as it means more rejection. To illustrate the above propositions and clarify the meaning of the different concepts and variables, look at the graph below:

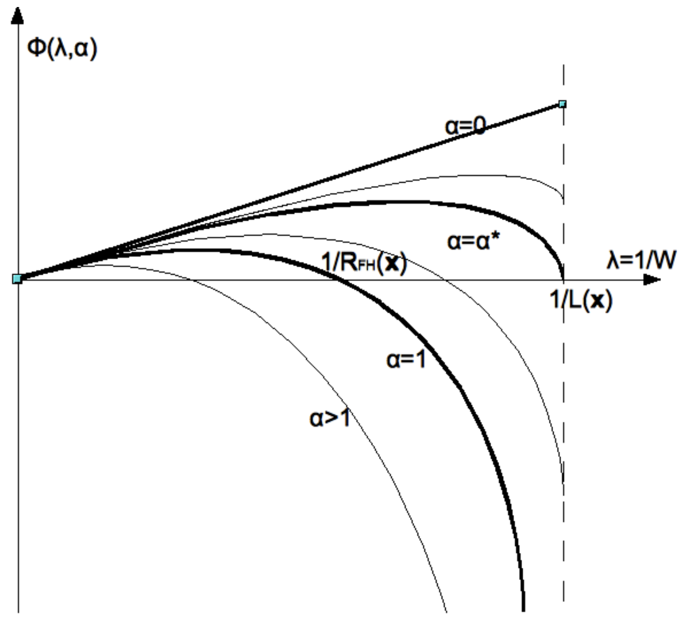

FIGURE 4.1: An extended riskiness measure for CRRA utility

This graph depicts the shape of $\phi(\lambda, \alpha)$ function for different values of relative risk aversion α within the CRRA class of utility functions. For α between 0 and α^* an

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extended riskiness measure is not defined since in this case function ϕ does not cross the zero axis. An extended riskiness measure is defined if $\alpha \geq \alpha^*$. Furthermore, it is also clear from the picture that an extended riskiness measure for values of α greater than 1 is necessarily greater than $R_{FH}(\mathbf{x})$ and hence rejecting $\mathbf{x}$ at wealth smaller than the extended riskiness measure in this case also guarantees no-bankruptcy.

4.5 Relation between riskiness measure and buying and selling price for a lottery

This section provides a link between the concepts of buying and selling price for a lottery and riskiness measure. The first lemma below demonstrates that although the riskiness measure for gambles with negative expectation or gambles without losses is not meaningful, it becomes meaningful and well defined if buying price or selling price for such gamble is subtracted from this gamble.

Lemma 4.14. *Given a non-degenerate lottery* $\mathbf{x}$ *and wealth level* W*, such that* $B(W, \mathbf{x})$ *and* $S(W, \mathbf{x})$ *exist, both* $R(\mathbf{x} - B(W, \mathbf{x}))$ *and* $R(\mathbf{x} - S(W, \mathbf{x}))$ *are well defined.*

Proof. I have to prove that $R(\mathbf{x} - S(W, \mathbf{x}))$ and $R(\mathbf{x} - B(W, \mathbf{x}))$ exist for any lottery $\mathbf{x}$. Notice that by proposition 4.12, the riskiness measure $R(\mathbf{x})$ exists for a lottery $\mathbf{x}$ and $\alpha \in [\alpha^*, +\infty)$ if and only if $\mathrm{E}(\mathbf{x}) > 0$ and $\mathrm{P}[\mathbf{x} < 0] > 0$. I have to check these two conditions for lotteries $\mathbf{x} - S(W, \mathbf{x})$ and $\mathbf{x} - B(W, \mathbf{x})$. Notice that for an arbitrary lottery $\mathbf{x}$, the following holds by proposition 4.27:

$$\begin{aligned} \mathrm{E}[\mathbf{x} - B(W, \mathbf{x})] &> 0 \\ \mathrm{E}[\mathbf{x} - S(W, \mathbf{x})] &> 0 \end{aligned}$$

Notice further that by proposition 4.27, $-L(\mathbf{x}) < B(W, \mathbf{x})$ and $-L(\mathbf{x}) < S(W, \mathbf{x})$. Since all the values in the support of $\mathbf{x}$ get positive probability

$$\begin{aligned} \mathrm{P}[\mathbf{x} - B(W, \mathbf{x}) < 0] &> 0 \\ \mathrm{P}[\mathbf{x} - S(W, \mathbf{x}) < 0] &> 0 \end{aligned}$$

Hence the two necessary conditions for $R(\mathbf{x} - S(W, \mathbf{x}))$ and $R(\mathbf{x} - B(W, \mathbf{x}))$ to exists are satisfied. For $\alpha \geq 1$ these conditions are also sufficient for existence of such measures

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by proposition 4.12. For $0 < \alpha < 1$, the following is true by proposition 4.4[10]:

$$\lim_{W \to W_L(\mathbf{x})} B(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$$
$$\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$$

where $W_L(\mathbf{x}) = U^{-1}[\mathrm{E}U(-\min(\mathbf{x}) + \mathbf{x})]$.
Hence $W_L(\mathbf{x}) + \min(\mathbf{x})$ is the minimal value for B and S when $0 < \alpha < 1$. By definition of $W_L(\mathbf{x})$, the following holds:

$$W_L(\mathbf{x}) = R(\mathbf{x} - (W_L\mathbf{x} + \min(\mathbf{x})))$$

So the riskiness measure for $\mathbf{x} - S$ and $\mathbf{x} - B$ for the lowest possible value of S and B (which is equal) is well defined and its value is $W_L(\mathbf{x})$. For higher values of S and B approaching (but not reaching) $\mathrm{E}[\mathbf{x}]$ the riskiness measure is well defined by proposition 4.12 and its value increases. □

The opposite direction of the above lemma is as follows:

Lemma 4.15. *If $R(\mathbf{x} - \delta)$ is well defined, where $\delta \in \mathbb{R}$, then*

$$\delta = S(R(\mathbf{x} - \delta) - \delta, \mathbf{x}) = B(R(\mathbf{x} - \delta), \mathbf{x}) \tag{4.9}$$

Proof. The relationship in 4.9 follows from the definitions of R, S and B. Again for $\alpha \geq 1$, values of S and B in the equation above are well defined if $R(\mathbf{x} - \delta)$ is well defined. For $0 < \alpha < 1$ on the other hand the following is true. The riskiness measure is defined for $\alpha > \alpha^*$[11]. By proposition 4.12, $\mathrm{E}U_{\alpha^*}(L(\mathbf{x}) + \delta + \mathbf{x}) = U_{\alpha^*}(L(\mathbf{x}) + \delta)$. Hence $W_L(\mathbf{x})$ defined in proposition 4.4 for utility function U_{α^*} is equal to $L(\mathbf{x}) + \delta = -\min(\mathbf{x}) + \delta$. And thus the lowest values of buying and selling price when utility function is U_{α^*} are:

$$S(L(\mathbf{x}), \mathbf{x}) = B(L(\mathbf{x}) + \delta, \mathbf{x}) = \delta$$

If buying and selling price are well defined for the lowest wealth levels, they are also well defined for higher wealth levels by proposition 4.4, and hence the lemma is proved. □

The following proposition establishes a simple link between buying and selling price for a lottery and the riskiness measure for this lottery.

[10] Notice that $L(\mathbf{x})$ used in proposition 4.12 is the same as $-\min(\mathbf{x})$ used in proposition 4.4
[11] It is not indicated but α^* is lottery-dependent.

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Proposition 4.16. *Given wealth level $W \geq 0$, CRRA utility function U with RRA coefficient α in the interval $(\alpha^*, +\infty)$, where α^* satisfies $\phi(\frac{1}{L(\mathbf{x})}, \alpha^*) = 0$, and any non-degenerate lottery $\mathbf{x}$, the following relations hold:*

$$W = R(\mathbf{x} - S(W,\mathbf{x})) - S(W,\mathbf{x}) \quad (4.10)$$

$$W = R(\mathbf{x} - B(W,\mathbf{x})) \quad (4.11)$$

Proof. By lemma 4.14, measures $R(\mathbf{x} - S(W,\mathbf{x}))$ and $R(\mathbf{x} - B(W,\mathbf{x}))$ are well defined for an arbitrary non-degenerate lottery $\mathbf{x}$. Now, it follows from definitions of $R(\mathbf{x})$ and $S(W,\mathbf{x})$, $B(W,\mathbf{x})$ that:

$$\begin{aligned} \mathrm{E}\left[U\left(R(\mathbf{x} - B(W,\mathbf{x})) + \mathbf{x} - B(W,\mathbf{x})\right)\right] &= U(R(\mathbf{x} - B(W,\mathbf{x}))) \\ \mathrm{E}\left[U\left(R(\mathbf{x} - S(W,\mathbf{x})) + \mathbf{x} - S(W,\mathbf{x})\right)\right] &= U(R(\mathbf{x} - S(W,\mathbf{x}))) \end{aligned}$$

Therefore, it has to be that $W = R(\mathbf{x} - S(W,\mathbf{x})) - S(W,\mathbf{x})$ and $W = R(\mathbf{x} - B(W,\mathbf{x}))$. □

For the next proposition I will need two lemmas. They establish certain delta properties of the riskiness measure.

Lemma 4.17. *Given lottery $\mathbf{x}$ and $\Delta \in \mathbb{R}$ such that riskiness for $\mathbf{x}$ and $\mathbf{x} + \Delta$ exists, the following holds:*

$$R(\mathbf{x} + \Delta) \leq R(\mathbf{x}) - \Delta \iff \Delta \geq 0$$

Proof. From the definition of R

$$\begin{aligned} 0 &= \mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) - U(R(\mathbf{x})) \\ 0 &= \mathrm{E}U(R(\mathbf{x} + \Delta) + \mathbf{x} + \Delta) - U(R(\mathbf{x} + \Delta)) \end{aligned}$$

Since U is increasing, $\Delta \geq 0$ if and only if

$$\mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) - U(R(\mathbf{x})) \leq \mathrm{E}U(R(\mathbf{x}) - \Delta + \mathbf{x} + \Delta) - U(R(\mathbf{x}) - \Delta)$$

And hence

$$\begin{aligned} 0 &= \mathrm{E}U(R(\mathbf{x} + \Delta) + \mathbf{x} + \Delta) - U(R(\mathbf{x} + \Delta)) \\ &\leq \mathrm{E}U(R(\mathbf{x}) - \Delta + \mathbf{x} + \Delta) - U(R(\mathbf{x}) - \Delta) \end{aligned}$$

Thus by proposition 4.12

$$R(\mathbf{x} + \Delta) \leq R(\mathbf{x}) - \Delta \quad \square$$

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Lemma 4.18. *Given lottery* $\mathbf{x}$ *and* $\Delta \in \mathbb{R}$ *such that riskiness for* $\mathbf{x}$ *and* $\mathbf{x}+\Delta$ *exists, the following holds:*

$$R(\mathbf{x}+\Delta) \leq R(\mathbf{x}) \iff \Delta \geq 0$$

Proof. "Only if" part follows from lemma 4.17. "If" part can be proved as follows: By definition of R.

$$\begin{aligned} 0 &= \mathrm{E}U(R(\mathbf{x})+\mathbf{x}) - U(R(\mathbf{x})) \\ 0 &= \mathrm{E}U(R(\mathbf{x}+\Delta)+\mathbf{x}+\Delta) - U(R(\mathbf{x}+\Delta)) \end{aligned}$$

By proposition 4.12, and since $R(\mathbf{x}+\Delta) \leq R(\mathbf{x})$, $\mathrm{E}U(R(\mathbf{x})+\mathbf{x}+\Delta) - U(R(\mathbf{x})) \geq 0$. And since utility is increasing it must be that $\Delta \geq 0$. □

Note that lemma 4.17 and lemma 4.18 both imply that it is impossible for $R(\mathbf{x}+\Delta)$ to be between $R(\mathbf{x}) - \Delta$ and $R(\mathbf{x})$.
Now I can state a series of main results of this section which establish a well defined connection between riskiness measure and buying and selling price for a lottery.

Proposition 4.19. *Given wealth* W *and two lotteries* $\mathbf{x}$ *and* $\mathbf{y}$*, if there exist wealth levels* W_1, W_2 *such that* $S(W_1,\mathbf{x}) = S(W,\mathbf{y})$ *and* $S(W_2,\mathbf{y}) = S(W,\mathbf{x})$*. Then:*

$$S(W,\mathbf{x}) \geq S(W,\mathbf{y})$$

$$\iff$$

$$R(\mathbf{y}-S(W,\mathbf{x})) \geq R(\mathbf{x}-S(W,\mathbf{x})) \geq R(\mathbf{y}-S(W,\mathbf{y})) \geq R(\mathbf{x}-S(W,\mathbf{y})) \quad (4.12)$$

Proof. The requirement that there exist wealth levels W_1, W_2 such that $S(W_1,\mathbf{x}) = S(W,\mathbf{y})$ and $S(W_2,\mathbf{y}) = S(W,\mathbf{x})$ guarantees that $R(\mathbf{x}-S(W,\mathbf{y}))$ and $R(\mathbf{y}-S(W,\mathbf{x}))$ are well defined. This is so due to lemma 4.14. Also, $R(\mathbf{x}-S(W,\mathbf{x}))$ and $R(\mathbf{y}-S(W,\mathbf{y}))$ are well defined due to this lemma. I will now prove the equivalency stated in the proposition sequentially for the three inequalities in (4.12).
$S(W,\mathbf{x}) \geq S(W,\mathbf{y}) \iff R(\mathbf{y}-S(W,\mathbf{y})) \geq R(\mathbf{x}-S(W,\mathbf{y}))$
Let $\Delta = S(W,\mathbf{x}) - S(W,\mathbf{y})$. By lemma 4.17, $\Delta \geq 0$ if and only if

$$R(\mathbf{x}-S(W,\mathbf{x})) - R(\mathbf{x}-S(W,\mathbf{x})+\Delta) \geq \Delta$$

And after substituting the definition of Δ

$$R(\mathbf{x}-S(W,\mathbf{x})) - R(\mathbf{x}-S(W,\mathbf{y})) \geq S(W,\mathbf{x}) - S(W,\mathbf{y})$$

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By proposition 4.16, this is in turn equivalent to the following:

$$R(\mathbf{x} - S(W, \mathbf{x})) - R(\mathbf{x} - S(W, \mathbf{y})) \geq R(\mathbf{x} - S(W, \mathbf{x})) - R(\mathbf{y} - S(W, \mathbf{y}))$$

And after simplifying

$$R(\mathbf{y} - S(W, \mathbf{y})) \geq R(\mathbf{x} - S(W, \mathbf{y}))$$

Which is what I had to prove. Now notice that the second inequality $R(\mathbf{x} - S(W, \mathbf{x})) \geq R(\mathbf{y} - S(W, \mathbf{y}))$ is equivalent to $S(W, \mathbf{x}) \geq S(W, \mathbf{y})$ by proposition 4.16. It leaves the one remaining inequality to be proved.

$\underline{S(W, \mathbf{x}) \geq S(W, \mathbf{y}) \iff R(\mathbf{y} - S(W, \mathbf{x})) \geq R(\mathbf{x} - S(W, \mathbf{x}))}$

Let $\Delta = S(W, \mathbf{y}) - S(W, \mathbf{x})$. Lemma 4.17 can be restated as follows:

$$R(\mathbf{x} + \Delta) \geq R(\mathbf{x}) - \Delta \iff \Delta \leq 0$$

And hence replacing $\mathbf{x}$ with $\mathbf{y} - S(W, \mathbf{y})$

$$R(\mathbf{y} - S(W, \mathbf{y})) - R(\mathbf{y} - S(W, \mathbf{y}) + \Delta) \leq \Delta$$

And after substituting the definition of Δ

$$R(\mathbf{y} - S(W, \mathbf{y})) - R(\mathbf{y} - S(W, \mathbf{x})) \leq S(W, \mathbf{y}) - S(W\mathbf{x})$$

By proposition 4.16 this is equivalent to

$$R(\mathbf{y} - S(W, \mathbf{y})) - R(\mathbf{y} - S(W, \mathbf{x})) \leq R(\mathbf{y} - S(W, \mathbf{y})) - R(\mathbf{x} - S(W, \mathbf{x}))$$

And after simplifying

$$R(\mathbf{y} - S(W, \mathbf{x})) \geq R(\mathbf{x} - S(W, \mathbf{x}))$$

This finishes the proof. □

The above proposition establishes that selling price for lottery $\mathbf{x}$ is not lower than the selling price for another lottery $\mathbf{y}$ at some wealth level if and only if the riskiness measure of $\mathbf{y} - S(W, \mathbf{x})$ is not lower than $\mathbf{x} - S(W, \mathbf{x})$ and $\mathbf{y} - S(W, \mathbf{y})$ is not lower than $\mathbf{x} - S(W, \mathbf{y})$. A similar proposition is obtained for buying price for a lottery.

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Proposition 4.20. *Given wealth W and two lotteries $\mathbf{x}$ and $\mathbf{y}$, if there exist wealth levels W_1, W_2 such that $B(W_1,\mathbf{x}) = B(W,\mathbf{y})$ and $B(W_2,\mathbf{y}) = B(W,\mathbf{x})$. Then:*

$$B(W,\mathbf{x}) \geq B(W,\mathbf{y})$$

$$\Longleftrightarrow$$

$$R(\mathbf{y} - B(W,\mathbf{x})) \geq R(\mathbf{x} - B(W,\mathbf{x})) = R(\mathbf{y} - B(W,\mathbf{y})) \geq R(\mathbf{x} - B(W,\mathbf{y}))$$

Proof. As in the previous proposition, all the riskiness measures are well defined due to lemma 4.14 and the assumption that there exist wealth levels W_1, W_2 such that $S(W_1,\mathbf{x}) = S(W,\mathbf{y})$ and $S(W_2,\mathbf{y}) = S(W,\mathbf{x})$ hold.
By proposition 4.16, $R(\mathbf{x} - B(W,\mathbf{x})) = R(\mathbf{y} - B(W,\mathbf{y}))$. The two remaining inequalities can be proved by using lemma 4.18.

$$\begin{aligned} R(\mathbf{y} - B(W,\mathbf{x})) \geq R(\mathbf{y} - B(W,\mathbf{y})) \quad &\Longleftrightarrow \quad B(W,\mathbf{x}) \geq B(W,\mathbf{y}) \\ &\Longleftrightarrow \quad R(\mathbf{x} - B(W,\mathbf{x})) \geq R(\mathbf{x} - B(W,\mathbf{y})) \end{aligned}$$ □

The two propositions above establish a link between selling and buying price for a lottery and the riskiness measure for gambles with prices. Even if riskiness measure for a given gamble is not meaningful due to the fact that the gamble has negative expectation or no losses, it is still meaningful and well-defined for gambles with prices, i.e. for gambles constructed by subtracting buying or selling price from the original gamble.
The above two proposition can be extended beyond their local (for a given wealth level) meaning. The following corollary to these propositions states a global result on extended riskiness with prices in relation to selling (and as it will turn out also buying) price for a lottery:

Corollary 4.21. *Let $\mathbf{x}$ and $\mathbf{y}$ be any two lotteries. Suppose that s is a scalar which satisfies:*

$$s \in [\max\{\min_W S(W,\mathbf{x}), \min_W S(W,\mathbf{y})\}, \min\{\max_W S(W,\mathbf{x}), \max_W S(W,\mathbf{y})\}] \tag{4.13}$$

Then the following equivalence holds:

$$R(\mathbf{y} - s) \geq R(\mathbf{x} - s) \quad \Longleftrightarrow \quad s \in \{s : s = S(W,\mathbf{x}), W \in \mathcal{W}\}$$

where $\mathcal{W} = \{W : S(W,\mathbf{x}) \geq S(W,\mathbf{y})\}$.

First, the above corollary could alternatively be stated in terms of buying price. To see this, observe the following. If W^* is wealth level at which selling prices of $\mathbf{x}$ and $\mathbf{y}$ cross, i.e. $S(W^*,\mathbf{x}) = S(W^*,\mathbf{y}) = S^*$, then by lemma 3.23, it holds that: $S^* =$

$B(W^*+S^*,\mathbf{x}) = B(W^*+S^*,\mathbf{y})$. Now, let's define an equivalent of set $\mathcal{W}$ for the case of buying price: $\mathcal{V} = \{W : B(W,\mathbf{x}) \geq B(W,\mathbf{y})\}$. The sets $\mathcal{W}$ and $\mathcal{V}$ are obviously different. However, by the argument I have just given, the sets $\{s : s = S(W,\mathbf{x}), W \in \mathcal{W}\}$ and $\{s : s = B(W,\mathbf{x}), W \in \mathcal{V}\}$ are the same. This is why the proposition above can be stated both in terms of selling price as well as in terms of buying price.

Condition (4.13) guarantees that s has to be in the range of both $S(W,\mathbf{x})$ and $S(W,\mathbf{y})$ as functions of wealth. In case of CRRA utility function, since proposition 4.4 establishes exactly what the range of buying and selling price is, this condition can be written explicitly. Suppose, for instance, that utility function is CRRA with coefficient of relative risk aversion greater than 1. In this case condition (4.13) takes the following form: $s \in [\max\{\min(\mathbf{x}), \min(\mathbf{y})\}, \min\{\mathrm{E}[\mathbf{x}], \mathrm{E}[\mathbf{y}]\}]$.

Cases in which the interval in (4.13) is empty, are not interesting since either $\mathbf{x}$ is unambiguously better than $\mathbf{y}$[12], or the other way around. Of course, in such a case it is possible to establish a similar (to the one above) proposition, where riskiness of lotteries with different prices would be compared, i.e. the riskiness of $\mathbf{x} - s_1$ and $\mathbf{y} - s_2$, where $s_1 \neq s_2$. However, this would be a rather different exercise to the one I wish to pursue in this section.

The next two propositions inform us what can be inferred about the riskiness measure from the global properties of selling and buying price as functions of wealth. These results are in fact special cases of the above corollary. However it is useful to state them and prove independently.

Proposition 4.22. *Given lotteries* $\mathbf{x}$ *and* $\mathbf{y}$ *and DARA utility function for which "riskiness measures"* $R(\mathbf{x})$ *and* $R(\mathbf{y})$ *are well defined, the following holds:*

$$B(W,\mathbf{y}) > B(W,\mathbf{x}) \quad \forall W \implies R(\mathbf{x}) > R(\mathbf{y})$$

Proof. Suppose not. Then $R(\mathbf{x}) \leq R(\mathbf{y})$. By the proposition 4.16 equation (4.11), given any $\mathbf{x}$ for which R is defined and unique I have for $W = R(\mathbf{x})$:

$$R(\mathbf{x}) = R(\mathbf{x} - B(R(\mathbf{x}),\mathbf{x}))$$

By the uniqueness of $R(\mathbf{x})$ I get that $B(R(\mathbf{x}),\mathbf{x}) = 0$. From the fact that B is increasing in wealth for DARA utility (proposition 4.31) I have:

$$B(R(\mathbf{y}),\mathbf{y}) = 0 = B(R(\mathbf{x}),\mathbf{x}) \leq B(R(\mathbf{y}),\mathbf{x})$$

This proves that $\exists W$, such that $B(W,\mathbf{y}) \leq B(W,\mathbf{x})$. □

[12] All the values in $\mathbf{x}$ are higher than those in $\mathbf{y}$.

Proposition 4.23. *Given lotteries* $\mathbf{x}$ *and* $\mathbf{y}$ *and DARA utility function for which "riskiness measures"* $R(\mathbf{x})$ *and* $R(\mathbf{y})$ *are well defined. Then:*

$$S(W,\mathbf{y}) > S(W,\mathbf{x}) \ \ \forall W \implies R(\mathbf{x}) > R(\mathbf{y})$$

Proof. Suppose not. Then $R(\mathbf{x}) \leq R(\mathbf{y})$. From the proof of proposition 4.22 I know that given any $\mathbf{x}$ for which R is defined and unique $B(R(\mathbf{x}),\mathbf{x}) = 0$. Hence, by proposition 4.29, I know that $S(R(\mathbf{x}),\mathbf{x}) = 0$. From the the fact that S is increasing in wealth for DARA utility (corollary 4.28)I have:

$$S(R(\mathbf{y}),\mathbf{y}) = 0 = S(R(\mathbf{x}),\mathbf{x}) \leq S(R(\mathbf{y}),\mathbf{x})$$

This proves that $\exists W$, such that $S(W,\mathbf{y}) \leq S(W,\mathbf{x})$. □

The reverse direction in the above two propositions at the same time cannot be true, at least in the case of DARA utilities. To see it I can use the proposition 2.15 from the second chapter of this thesis which states that preference reversal for DARA utilities is possible. That means that given any DARA utility there exist two lotteries $\mathbf{x}$ and $\mathbf{y}$ such that the following holds:

$$S(W,\mathbf{y}) > S(W,\mathbf{x}) > B(W,\mathbf{x}) > B(W,\mathbf{y})$$

If the reverse direction in both propositions 4.22 and 4.23 was true, the above pattern would not be possible. And hence the reverse direction of both propositions cannot be true. Whether the reverse direction in one case is true and in another is not or whether the reverse direction in both cases is not true remains unknown.
Below I wish to examine further connections between riskiness measure and buying and selling price for a lottery. The proposition below is an extension to Pratt [4] famous theorem on comparative risk aversion. It shows that riskiness measure can be used along with buying price and selling price to compare risk aversion across individuals.

Proposition 4.24. *Given two CRRA utility functions* U_1, U_2 *with RRA coefficients* α_1 *and* α_2, *respectively, both in the interval* $(\alpha^*, +\infty)$, *where* α^* *satisfies* $\phi(\frac{1}{L(\mathbf{x})}, \alpha^*) = 0$, *and any non-degenerate lottery* $\mathbf{x}$, *such that* $R_1(\mathbf{x})$ *and* $R_2(\mathbf{x})$ *exists, the following holds:*

$$R_1(\mathbf{x}) > R_2(\mathbf{x}) \iff B_1(W,\mathbf{x}) < B_2(W,\mathbf{x}) \ \forall W \iff S_1(W,\mathbf{x}) < S_2(W,\mathbf{x}) \ \forall W$$

where R_i, B_i *and* S_i *are, respectively, the riskiness measure, the buying price and the selling price corresponding to utility function* U_i.

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Proof. The second equivalence above is a special case (CRRA) of proposition 4.33 and hence was already proved there. I need to prove the first equivalence.
($\Longleftarrow$)
I start by assuming $B_1(W,\mathbf{x}) < B_2(W,\mathbf{x})\ \forall W$. By lemma 4.14 I know that $R_i(\mathbf{x} - S_i(W,\mathbf{x}))$ and $R_i(\mathbf{x} - B_i(W,\mathbf{x}))$ exist. By proposition 4.16 I know that $W = R(\mathbf{x} - B(W,\mathbf{x}))$. Furthermore, I know that $B(R(\mathbf{x}),\mathbf{x}) = 0$. Therefore:

$$\begin{aligned} R_1(\mathbf{x}) &= R_1(\mathbf{x} - B_1(R_1(\mathbf{x}),\mathbf{x})) \\ &= R_2(\mathbf{x} - B_2(R_1(\mathbf{x}),\mathbf{x})) \\ &> R_2(\mathbf{x}) \end{aligned}$$

Since $B_2(R_1(\mathbf{x}),\mathbf{x}) > B_1(R_1(\mathbf{x}),\mathbf{x}) = 0$, the last inequality follows from lemma 4.18. That the riskiness measure $R_2(\mathbf{x} - B_2(R_1(\mathbf{x}),\mathbf{x}))$ is well defined follows from the similar argument as in the proof of lemma 4.14. Since $\mathbf{x}$ was arbitrary, the above implication holds generally.
($\Longrightarrow$)
I start by assuming $R_1(\mathbf{y}) > R_2(\mathbf{y})$ for all $\mathbf{y}$, such that R_1 and R_2 are defined. This holds in particular for lottery $\mathbf{y} = \mathbf{x} - B_1(W,\mathbf{x})$, for some W. It follows from proposition 4.16, that:

$$\begin{aligned} W = R_2(\mathbf{x} - B_2(W,\mathbf{x})) &= R_1(\mathbf{x} - B_1(W,\mathbf{x})) \\ &> R_2(\mathbf{x} - B_1(W,\mathbf{x})) \end{aligned}$$

And hence I know that $R_2(\mathbf{x} - B_2(W,\mathbf{x})) > R_2(\mathbf{x} - B_1(W,\mathbf{x}))$. By lemma 4.18, I conclude that $B_1(W,\mathbf{x}) < B_2(W,\mathbf{x})$. Since wealth W was arbitrary, as well as lottery $\mathbf{x}$, the proof is finished. □

I proved that one can use buying and selling price for a lottery as well as riskiness measure as equivalent ways to express absolute risk aversion. Although the proof is only valid for the CRRA case, the proposition is true whenever the existence of riskiness measure for the appropriate lotteries is guaranteed.

4.6 Expected utility decision-making using riskiness measure and buying and selling price for a lottery

Finally in this section, I will show how one can make decisions based on the concepts of buying and selling price for a lottery or riskiness measure. It shall come as no surprise that no matter with help of what concepts decisions are made within expected utility

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theory, they give rise to equivalent decision criteria.
Consider two situations.

- Case A: A decision maker with wealth $W \geq 0$ considers buying a nondegenerate lottery $\mathbf{x}$ for a price $b \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$
- Case B decision maker with wealth $W \geq 0$ participating in a nondegenerate lottery $\mathbf{x}$ considers selling lottery $\mathbf{x}$ for a price $s \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$

Proposition 4.25. *Given utility function U and lottery $\mathbf{x}$, such that the arguments of U are in the domain of U amd $S(W, \mathbf{x})$ and $B(W, \mathbf{x})$ are both well defined, the following criteria for decision making are equivalent:*

- *Expected utility criterion:*
  - *Case A: Buy $\mathbf{x}$ if $\mathrm{E}U(W + \mathbf{x} - b) \geq U(W)$, otherwise don't buy.*
  - *Case B: Sell $\mathbf{x}$ if $\mathrm{E}U(W + \mathbf{x}) \leq U(W + s)$, otherwise don't sell.*
- *Buying/selling price criterion:*
  - *Case A: Buy $\mathbf{x}$ if $B(W, \mathbf{x}) \geq b$, otherwise don't buy.*
  - *Case B: Sell $\mathbf{x}$ if $S(W, \mathbf{x}) \leq s$, otherwise don't sell.*
- *Riskiness measure criterion:*
  - *Case A: Buy $\mathbf{x}$ if $W \geq R(\mathbf{x} - b)$*
  - *Case B: Sell $\mathbf{x}$ if $W \leq R(\mathbf{x} - s) - s$*

Proof. The proof follows from the respective definitions and hence is omitted. Notice that since $b, s \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$, $R(\mathbf{x} - b)$ and $R(\mathbf{x} - s)$ are well defined by lemma 4.14. □

4.7 Concluding remarks

In this chapter I analyzed riskiness measure as introduced by Foster and Hart [5]. I gave simple intuition behind their result and I tried to make some steps towards extending this measure in two respects - first to define an extended riskiness measure based on DARA utility functions and derive necessary and sufficient conditions for existence and uniqueness of such measure for DARA and CRRA class of utility functions. Obviously, for the more specialized case of CRRA utility functions more exact conditions are obtained than for the more general case of DARA utilities. I also tried to extend the domain of riskiness measure. For gambles with non-positive expectation or no losses I proposed a way to compare their riskiness by subtracting prices from them. If the riskiness ordering is unchanged over the whole range of prices for which the lottery minus the price exists is unchanged, something can be inferred about the riskiness of a gamble without prices. To this end a number of useful properties relating buying and selling price for a lottery and riskiness measure were established and should be useful also for their own sake. An extension of Pratt [4] famous result on comparative risk aversion involving riskiness measure along with buying and selling price for a lottery was stated and proved. Finally a simple link between decision-making using riskiness measure and decision-making using buying and selling price was developed.

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Appendix

Lemma 4.26. *Given any lottery* $\mathbf{x}$ *and wealth level* W*, the following three relations between buying price and selling price hold:*

$$S[W, \mathbf{x} - B(W, \mathbf{x})] = 0 \tag{4.14}$$

$$S[W - B(W, \mathbf{x}), \mathbf{x}] = B(W, \mathbf{x}) \tag{4.15}$$

$$B[W + S(W, \mathbf{x}), \mathbf{x}] = S(W, \mathbf{x}) \tag{4.16}$$

Proposition 4.27. *For any non-degenerate lottery* $\mathbf{x}$ *and any wealth* W *such that buying and selling price exist,* $S(W, \mathbf{x})$ *and* $B(W, \mathbf{x})$ *lie in the interval* $(\min(\mathbf{x}), \mathrm{E}(\mathbf{x}))$*. For a degenerate lottery* $\mathbf{x}$*,* $S(W, \mathbf{x}) = B(W, \mathbf{x}) = x$*.*

The following is a corollary to Pratt [4] famous theorem of comparative risk aversion.

Corollary 4.28. *For a strictly increasing and twice differentiable utility function* U *with continuous second derivative, the following holds:*

- $S(W, \mathbf{x})$ *is increasing/constant/decreasing in* W *for every* $\mathbf{x}$ *iff* $A(W)$ *is decreasing/constant/increasing in* W

Proposition 4.29. *For any lottery* $\mathbf{x}$ *and any wealth* W*, for utilities with decreasing absolute risk aversion (DARA) the following equivalence holds:*

$$B(W, \mathbf{x}) > 0 \iff B(W, \mathbf{x}) < S(W, \mathbf{x})$$

Proposition 4.30. *For any lottery* $\mathbf{x}$ *and any wealth level* W *and for* $\Delta \in \mathbb{R}$*, the following holds:*

$$B(W, \mathbf{x} + \Delta) = B(W, \mathbf{x}) + \Delta \tag{4.17}$$

$$S(W, \mathbf{x} + \Delta) = S(W + \Delta, \mathbf{x}) + \Delta \tag{4.18}$$

Notice that for DARA utility function and $B(W, \mathbf{x}) > 0$ the above result together with proposition 4.29 implies the following:

$$S(W, \mathbf{x} + \Delta) - B(W, \mathbf{x} + \Delta) = S(W + \Delta, \mathbf{x}) - B(W, \mathbf{x}) > S(W, \mathbf{x}) - B(W, \mathbf{x})$$

Proposition 4.31. *For a strictly increasing and twice differentiable utility function* U *with continuous second derivative, the following holds:*

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- *$B(W, \mathbf{x})$ is increasing/constant/decreasing in W for every $\mathbf{x}$ iff $A(W)$ is decreasing/constant/increasing in W*

Lemma 4.32. *For differentiable DARA utility functions, given any n-dimensional non-degenerate lottery $\mathbf{x}$ and any wealth level W, the following holds:*

- $\mathrm{E}U'(W+\mathbf{x}) - U'(W+S(W,\mathbf{x})) > 0$
- $\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x})) - U'(W) > 0$
- $\mathrm{E}U'(R(\mathbf{x})+\mathbf{x}) - U'(R(\mathbf{x})) > 0$
- $0 < \frac{\partial B(W,\mathbf{x})}{\partial W} < 1$

Proof. From the definition of buying, selling price and the fact that they are both increasing in wealth, it follows that:

$$\frac{\partial S(W,\mathbf{x})}{\partial W} = \frac{\mathrm{E}U'(W+\mathbf{x}) - U'(W+S(W,\mathbf{x}))}{U'(W+S(W,\mathbf{x}))} > 0$$
$$\frac{\partial B(W,\mathbf{x})}{\partial W} = \frac{\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x})) - U'(W)}{\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x}))} > 0$$

All of the properties above follow immediately. □

Proposition 4.33. *For two different utility functions U_1 and U_2, any wealth level W and any n-dimensional non-degenerate random variable $\mathbf{x}$ with bounded values, I define corresponding selling and buying prices $S_1(W,\mathbf{x})$, $B_1(W,\mathbf{x})$ and $S_2(W,\mathbf{x})$, $B_2(W,\mathbf{x})$. The following equivalence holds:*

$$\forall W \ \forall \mathbf{x}: \ \exists\, \delta > 0 \ |x_i| < \delta \ \forall i \in \{1,...,n\}$$
$$S_1(W,\mathbf{x}) > S_2(W,\mathbf{x}) \iff B_1(W,\mathbf{x}) > B_2(W,\mathbf{x})$$

Proposition 4.34. *The following two statements are equivalent:*

i. Bernoulli utility function exhibits CRRA

ii. buying and selling price for any lottery are homogeneous of degree one i.e.

$$S(\lambda W, \lambda \mathbf{x}) = \lambda S(W,\mathbf{x}), \ \forall \lambda > 0$$
$$B(\lambda W, \lambda \mathbf{x}) = \lambda B(W,\mathbf{x}), \ \forall \lambda > 0$$

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