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# AGM-consistency and perfect Bayesian equilibrium. Part I: definition and properties.

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**Abstract**

We provide a general notion of perfect Bayesian equilibrium which can be applied to arbitrary extensive-form games and is intermediate between subgame-perfect equilibrium and sequential equilibrium. The essential ingredient of the proposed definition is the qualitative notion of AGM-consistency, which has an epistemic justification based on the AGM theory of belief revision.

Keywords: belief revision, plausibility order, consistency, subgame-perfect equilibrium, sequential equilibrium, Bayesian updating.

## 1 Introduction

Attempts to refine the notion of Nash equilibrium in extensive-form (or dynamic) games must deal with the issue of *belief revision*: how should a player revise her beliefs when informed that she has to make a choice at an information set to which she initially assigned zero probability? Kreps and Wilson [8] suggested the notion of assessment as a way of expressing the players' beliefs during an arbitrary play of the game. An *assessment* is a pair $(\sigma, \mu)$ where $\sigma$ is a strategy profile and $\mu$ a "system of beliefs", defined as a collection of probability distributions, one for every information set, over the nodes in that information set. The system of beliefs $\mu$ specifies a player's beliefs about past moves, while the strategy profile $\sigma$ provides the initial beliefs as well as beliefs

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about future moves of the opponents conditional on every node. Given the separate roles played by $\sigma$ and $\mu$ in specifying players' beliefs, it is necessary to impose some requirement of compatibility between the two. Kreps and Wilson proposed the notion of "consistency", which we will call KW-consistency. An assessment $(\sigma, \mu)$ is *KW-consistent* if there is an infinite sequence $\langle \sigma^1, ..., \sigma^m, ... \rangle$ of completely mixed strategy profiles such that, letting $\mu_m$ be the unique system of beliefs associated - using Bayes' rule - to $\sigma^m$, $\lim_{m\to\infty}(\sigma^m, \mu_m) = (\sigma, \mu)$. A number of authors have tried to shed light on this topological notion by relating it to more intuitive concepts, such as 'structural consistency' ([9]), 'generally reasonable extended assessment' ([5]), 'stochastic independence' ([2, 7]).[1] Kreps and Wilson then proceeded to define a sequential equilibrium as an assessment which is KW-consistent and sequentially rational.[2] In applications, however, checking the KW-consistency requirement has proved to be rather complex and simpler notions of equilibrium have been sought. A drastic simplification is the notion of weak sequential equilibrium ([10], p. 170), which is defined as a sequentially rational assessment $(\sigma, \mu)$ where the beliefs expressed by $\mu$ are obtained using Bayes' rule at all the information sets that are reached by $\sigma$ with positive probability (while no restrictions are imposed on the beliefs at information sets that are not reached by $\sigma$). However, this notion is too weak in the sense that $(\sigma, \mu)$ can be a weak sequential equilibrium without $\sigma$ being a subgame-perfect equilibrium.[3] Thus attempts have been made to find an intermediate notion between subgame-perfect equilibrium and sequential equilibrium incorporating a simpler requirement than KW-consistency. Such an intermediate notion was proposed by Fudenberg and Tirole [5] and called perfect Bayesian equilibrium. Unfortunately this new notion was defined only for a small subset of extensive-form games (namely the class of multi-stage games with observed actions) and extending it to arbitrary games proved to be difficult.

In this paper we propose a new intermediate notion between subgame-perfect equilibrium and sequential equilibrium, which - in order to avoid introducing a new expression - we will also call perfect Bayesian equilibrium. The advantages of this equilibrium concept are that (1) it is a general notion that can be applied to arbitrary extensive-form games and (2) its main ingredient is a purely qualitative condition - we call it "AGM-consistency" - which is simple, easy to verify and a generalization of KW-consistency. An assessment $(\sigma, \mu)$ is *AGM-consistent* if there is a total pre-order $\precsim$ on the set of histories $H$ - which we call a *plausibility order* - such that: (1) for every information set $I$ the histories that are assigned positive probability by $\mu$ are precisely those that are most plausible in $I$ and (2) at every information set $I$ the choices that are assigned positive probability by $\sigma$ are precisely those that "preserve plausibil-

[1] Perea *et al* [13] offer an algebraic characterization of consistent assessments.

[2] All the definitions will be spelled out in detail later. Sequential rationality requires, for every information set, that the choice prescribed by the relevant component of $\sigma$ be optimal given the corresponding player's beliefs at that information set.

[3] A strategy profile $\sigma$ is a subgame-perfect equilibrium if, for every subgame, the restriction of $\sigma$ to the subgame yields a Nash equilibrium of the subgame. The notion of subgame-perfect equilibrium was introduced by Selten [15].

ity", in the sense that if $h$ is a history in $I$ and $a$ is a choice at $h$ then $ha$ is as plausible as $h$. An attractive feature of the notion of AGM-consistency is that it has an independent epistemic justification based on the so called AGM theory of belief revision introduced by Alchourrón, Gärdenfors and Makinson [1]. Because of space limitations we leave the discussion of the epistemic foundations of AGM-consistency to a companion paper ([4]).

We define an assessment $(\sigma, \mu)$ to be a *perfect Bayesian equilibrium* if it is AGM-consistent, sequentially rational and the probabilities specified by $\mu$ are compatible with Bayes' rule. We show that if $(\sigma, \mu)$ is a perfect Bayesian equilibrium then $\sigma$ is a subgame-perfect equilibrium and that the set of sequential equilibria is a proper subset of the set of perfect Bayesian equilibria.

## 2 Extensive forms, assessments and KW-consistency

We shall use the history-based definition of extensive-form game (see, for example, [12]). If $A$ is a set, we denote by $A^*$ the set of finite sequences in $A$. If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $1 \leq j \leq k$, the sequence $h' = \langle a_1, ..., a_j \rangle$ is called a *prefix* of $h$. If $h = \langle a_1, ..., a_k \rangle \in A^*$ and $a \in A$, we denote the sequence $\langle a_1, ..., a_k, a \rangle \in A^*$ by $ha$.

A *finite extensive form* is a tuple $\left\langle A, H, N, P, \{\approx_i\}_{i \in N} \right\rangle$ whose elements are:

- A finite set of actions $A$.

- A finite set of histories $H \subseteq A^*$ which is closed under prefixes (that is, if $h \in H$ and $h' \in A^*$ is a prefix of $h$, then $h' \in H$). The null history $\langle \rangle$, denoted by $\emptyset$, is an element of $H$ and is a prefix of every history. A history $h \in H$ such that, for every $a \in A$, $ha \notin H$, is called a *terminal history*. The set of terminal histories is denoted by $Z$. Let $D = H \setminus Z$ denote the set of non-terminal or *decision* histories. For every history $h \in H$, we denote by $A(h)$ the set of actions available at $h$, that is, $A(h) = \{a \in A : ha \in H\}$. Thus $A(h) \neq \varnothing$ if and only if $h \in D$. We assume that $A = \bigcup_{h \in D} A(h)$ (that is, we restrict attention to actions that are available at some decision history).

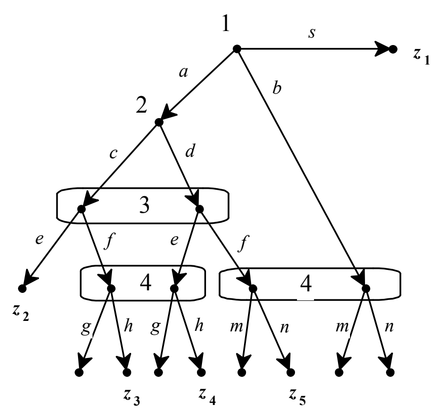
- A finite set $N = \{1, ...n\}$ of players. In some cases there is also an additional, fictitious, player called *chance*.

- A function $P : D \to N \cup \{chance\}$ that assigns a player to each decision history. Thus $P(h)$ is the player who moves at history $h$. A game is said to be *without chance moves* if $P(h) \in N$ for every $h \in D$. For every $i \in N \cup \{chance\}$, let $D_i = P^{-1}(i)$ be the histories assigned to player $i$. Thus $\{D_{chance}, D_1, ..., D_n\}$ is a partition of $D$. We follow Kreps and Wilson [8] in assuming that chance moves occur at most at the beginning of the game, that is, either $D_{chance} = \varnothing$ or $D_{chance} = \{\emptyset\}$ (recall that $\emptyset$ denotes the null history). If $\emptyset$ is assigned to chance, then a probability distribution over $A(\emptyset)$ is given that assigns positive probability to every $a \in A(\emptyset)$.

- For every player $i \in N$, $\approx_i$ is an equivalence relation on $D_i$. The interpretation of $h \approx_i h'$ is that, when choosing an action at history $h \in D_i$, player $i$ does not know whether she is moving at $h$ or at $h'$. The equivalence class of $h \in D_i$ is denoted by $I_i(h)$ and is called an *information set of player* $i$; thus $I_i(h) = \{h' \in D_i : h \approx_i h'\}$. The following restriction applies: if $h' \in I_i(h)$ then $A(h') = A(h)$, that is, the set of actions available to a player is the same at any two histories that belong to the same information set of that player.

- The following property, known as *perfect recall*, is assumed: for every player $i \in N$, if $h_1, h_2 \in D_i$, $a \in A(h_1)$ and $h_1 a$ is a prefix of $h_2$ then for every $h' \in I_i(h_2)$ there exists an $h \in I_i(h_1)$ such that $ha$ is a prefix of $h'$. Intuitively, perfect recall requires a player to remember what she knew in the past and what actions she took previously.[4]

Given an extensive form, one obtains an *extensive game* by adding, for every player $i \in N$, a *utility (*or *payoff) function* $U_i : Z \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers; recall that $Z$ is the set of terminal histories).




An extensive form without chance moves
Figure 1

Figure 1 shows an extensive form without chance moves where $A = \{a, b, s, c, d, e, f, g, h, m, n\}$, $H = D \cup Z$ with (to simplify the notation we write $a$ instead of $\langle \emptyset, a\rangle$, $ac$ instead of $\langle \emptyset, a, c\rangle$, etc.) $D = \{\emptyset, a, b, ac, ad, acf, ade, adf\}$, $Z = \{s, ace, acfg, acfh, adeg, adeh, adfm, adfn, bm, bn\}$, $A(\emptyset) = \{a, b, s\}$, $A(a) = \{c, d\}$, $A(ac) = A(ad) = \{e, f\}$, $A(acf) = A(ade) = \{g, h\}$, $A(adf) = A(b) = \{m, n\}$, $N = \{1, 2, 3, 4\}$, $P(\emptyset) = 1$, $P(a) = 2$, $P(ac) = P(ad) = 3$, $P(acf) = P(ade) = P(adf) = P(b) = 4$, $\approx_1 \; = \{(\emptyset, \emptyset)\}$, $\approx_2 \; = \{(a, a)\}$, $\approx_3 \; = \{(ac, ac), (ac, ad), (ad, ac), (ad, ad)\}$ and $\approx_4 \; = \{(acf, acf), (acf, ade), (ade, acf), (ade, ade), (adf, adf), (adf, b), (b, adf), (b, b)\}$.

[4]For an investigation of the conceptual content of the property of perfect recall see [3].

The information sets containing more than one history (for example, $I_4(b) = \{adf, b\}$) are shown as rounded rectangles. The root of the tree represents the null history $\emptyset$.

**Notation 1** *If $h$ and $h'$ are decision histories, we write $h' \in I(h)$ as a short-hand for "$h' \in I_i(h)$ with $i \in P^{-1}(h)$". Thus $h' \in I(h)$ means that $h$ and $h'$ belong to the same information set (of the relevant player).*

**Remark 2** *In order to simplify the notation in the proofs, we shall assume that no action is available at more than one information set: $\forall h, h' \in H, \forall a \in A$, if $a \in A(h) \cap A(h')$ then $h' \in I(h)$.*[5]

Given an extensive form, a *pure strategy* of player $i \in N$ is a function that associates with every information set of player $i$ an action at that information set, that is, a function $s_i : D_i \to A$ such that (1) $s_i(h) \in A(h)$ and (2) if $h' \in I_i(h)$ then $s_i(h') = s_i(h)$. For example, one of the pure strategies of Player 4 in the extensive form illustrated in Figure 1 is $s_4(acf) = s_4(ade) = g$ and $s_4(adf) = s_4(b) = m$. A *behavior strategy* of player $i$ is a collection of probability distributions, one for each information set, over the actions available at that information set; that is, a function $\sigma_i : D_i \to \Delta(A)$ (where $\Delta(A)$ denotes the set of probability distributions over $A$) such that (1) $\sigma_i(h)$ is a probability distribution over $A(h)$ and (2) if $h' \in I_i(h)$ then $\sigma_i(h') = \sigma_i(h)$. A behavior strategy *profile* is an $n$-tuple $\sigma = (\sigma_1, ..., \sigma_n)$ where, for every $i \in N$, $\sigma_i$ is a behavior strategy of player $i$. Given our assumption that no action is available at more than one information set, without risking ambiguity we shall denote by $\sigma(a)$ the probability assigned to action $a$ by the relevant component of the strategy profile $\sigma$.[6] Note that a pure strategy is a special case of a behavior strategy where each probability distribution is degenerate. A behavior strategy is *completely mixed* at history $h \in D$ if, for every $a \in A(h)$, $\sigma(a) > 0$.

For example, in the extensive form of Figure 1 a possible behavior strategy for Player 1 is $\begin{pmatrix} a & b & s \\ 0 & 0 & 1 \end{pmatrix}$, which we will more simply denote by $s$ (and coincides with a pure strategy of Player 1) and a possible behavior strategy of Player 2 is $\begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, which we will more simply denote by $\left(\frac{1}{3}c, \frac{2}{3}d\right)$ (and is a completely mixed strategy).

A *system of beliefs*, is a collection of probability distributions, one for every information set, over the elements of that information set, that is, a function $\mu : D \to \Delta(H)$ such that (1) if $h \in D_i$ then $\mu(h)$ is a probability distribution over $I_i(h)$ and (2) if $h \in D_i$ and $h' \in I_i(h)$ then $\mu(h) = \mu(h')$. Without risking

[5] See Footnote 17 in the Appendix for an explanation of how the proofs would have to be written without this convention.

[6] If $h \in D_i$ and $\sigma_i$ is the *ith* component of $\sigma$, then $\sigma_i(h)$ is a probability distribution over $A(h)$ and if $a \in A(h)$ then $\sigma_i(h)(a)$ is the probability assigned to action $a$ by $\sigma_i(h)$. We denote $\sigma_i(h)(a)$ more simply by $\sigma(a)$.

ambiguity we shall denote by $\mu(h)$ the probability assigned to history $h$ by the system of beliefs $\mu$.[7]

Note that a completely mixed behavior strategy profile yields, using Bayes' rule, a unique system of beliefs.

An *assessment* is a pair $(\sigma, \mu)$ where $\sigma$ is a behavior strategy profile and $\mu$ is a system of beliefs. An assessment represents the beliefs of the players: the strategy profile $\sigma$ yields the initial beliefs as well as conditional beliefs about the future, while the system of beliefs $\mu$ gives conditional beliefs about the past. For example, consider the following assessment for the extensive form of Figure 1: $\sigma = \left(s, \left(\frac{1}{3}c, \frac{2}{3}d\right), f, (h, n)\right)$ and $\mu = \left(\left(\frac{1}{2}ac, \frac{1}{2}ad\right), acf, \left(\frac{1}{4}adf, \frac{3}{4}b\right)\right)$ and consider Player 3. By $\sigma$, the initial beliefs of Player 3 (that is, before the game begins) is that Player 1 will play $s$ and thus the outcome of the game will be $z_1$; in particular, Player 3 believes that she will not be asked to move. If she were asked to move, then, by $\mu$, she would assign equal probability to the event that Player 2 chose $c$ and the event that Player 2 chose $d$ and, by $\sigma$, she would believe that her choice of $e$ would lead to either outcome $z_2$ or outcome $z_4$ (with equal probability) and that her choice of $f$ would lead to either outcome $z_3$ or outcome $z_5$ (with equal probability).

Given that $\sigma$ and $\mu$ play separate roles in the representation of the players' initial beliefs and disposition to change those beliefs, it is natural to impose a requirement of compatibility between the two. Kreps and Wilson [8] proposed the following notion of compatibility, which they called consistency; we will call it KW-consistency. An assessment $(\sigma, \mu)$ is *KW-consistent* if there is an infinite sequence $\langle \sigma^1, ..., \sigma^m, ... \rangle$ of completely mixed strategy profiles such that, letting $\mu_m$ be the unique system of beliefs associated - using Bayes' rule - to $\sigma^m$,[8] $\lim_{m\to\infty}(\sigma^m, \mu_m) = (\sigma, \mu)$.

In this paper we provide an independently motivated and qualitative notion of compatibility, which we call *AGM-consistency*, and use it to define a notion of equilibrium which is intermediate between subgame-perfect equilibrium and sequential equilibrium.

An epistemic justification of the notion of AGM-consistency based on the AGM theory of belief revision [1] and a more in-depth analysis of the relationship between AGM-consistency and KW-consistency are provided in a companion paper ([4]).

[7] A more precise notation would be $\mu(h)(h)$: if $h \in D_i$ then $\mu(h)$ is a probability distribution over $I_i(h)$ and, for every $h' \in I(h)$, $\mu(h) = \mu(h')$ so that $\mu(h)(h) = \mu(h')(h)$. With slight abuse of notation we denote this common probability by $\mu(h)$.

[8] That is, for every $h \in H$, $\mu_m(h) = \dfrac{\prod_{a\in h} \sigma^m(a)}{\sum_{x\in I(h)} \prod_{a\in x} \sigma^m(a)}$, where $a \in h$ means that action $a$ occurs in history $h$. Since $\sigma^m$ is completely mixed, $\sigma^m(a) > 0$ for every $a \in A$.

# 3 AGM-consistency

Given a set $H$, a total pre-order on $H$ is a binary relation $\precsim \; \subseteq H \times H$ which is complete ($\forall h, h' \in H$, either $h \precsim h'$ or $h' \precsim h$) and transitive ($\forall h, h', h'' \in H$, if $h \precsim h'$ and $h' \precsim h''$ then $h \precsim h''$).

**Definition 3** *Given an extensive form, a* plausibility order *is a total pre-order on the set of histories $H$ that satisfies the following properties:* $\forall h \in D$,

1. $h \precsim ha, \quad \forall a \in A(h)$,
2. $\exists a \in A(h)$ *such that* $ha \precsim h$ *and,*
   $\forall a \in A(h)$, *if* $ha \precsim h$ *then* $h'a \precsim h'$, $\forall h' \in I(h)$.
3. *If the empty history $\emptyset$ is assigned to chance, then* $\forall a \in A(\emptyset)$, $a \precsim \emptyset$.

If $h \precsim h'$ we say that history $h$ is *at least as plausible* as history $h'$. Property 1 says that adding an action to a history $h$ cannot yield a more plausible history than $h$ itself. Property 2 says that at every decision history $h$ there is some action $a$ such that adding $a$ to $h$ yields a history which is at least as plausible as $h$ and, furthermore, any such action $a$ performs the same role with any other history that belongs to the same information set. Property 3 says that all the actions assigned to chance (if any) are "plausibility preserving" (see Remark 4 below).

We write $h \sim h'$ (with the interpretation that $h$ is *as plausible* as $h'$) as a short-hand for "$h \precsim h'$ and $h' \precsim h$" and we write $h \prec h'$ (with the interpretation that $h$ is *more plausible* than $h'$) as a short-hand for "$h \precsim h'$ and $h' \not\precsim h$".

**Remark 4** *It follows from Property 1 of Definition 3 that, for every $h, h' \in H$, if $h'$ is a prefix of $h$ then $h' \precsim h$.*[9] *Furthermore, by Properties 1 and 2, for every decision history $h$, there is at least one action $a$ at $h$ such that $h \sim ha$, that is, $ha$ is as plausible as $h$; furthermore, if $h'$ belongs to the same information set as $h$, then $h' \sim h'a$. We call such actions* plausibility preserving.

**Definition 5** *Fix an extensive-form. An assessment $(\sigma, \mu)$ is* AGM-consistent *if there exists a plausibility order $\precsim$ on $H$ such that:*

*(i) the actions that are assigned positive probability by $\sigma$ are precisely the plausibility-preserving actions:* $\forall h \in D, \forall a \in A(h)$,

$$\sigma(a) > 0 \text{ if and only if } h \sim ha, \qquad (P1)$$

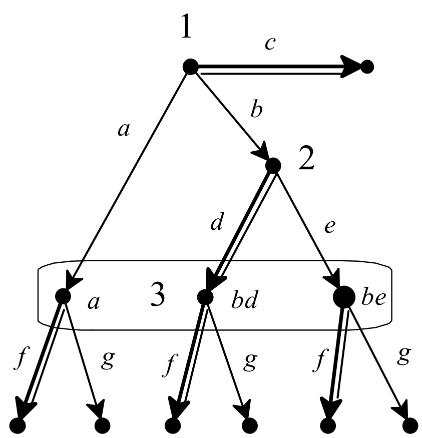
*(ii) the histories that are assigned positive probability by $\mu$ are precisely those that are most plausible within the corresponding information set:* $\forall h \in D$,

$$\mu(h) > 0 \text{ if and only if } h \precsim h', \forall h' \in I(h). \qquad (P2)$$

[9] In fact, if $h'$ is a prefix of $h$ then $h = h'a_1...a_m$ for some (possibly none) $a_1, ..., a_m \in A$, so that, by Property 1, $h' \precsim h'a_1 \precsim h'a_1a_2 \precsim ... \precsim h'a_1...a_m = h$ and thus, by transitivity of $\precsim$, $h' \precsim h$.

*If $\precsim$ satisfies properties P1 and P2 with respect to $(\sigma, \mu)$, we say that $\precsim$* rationalizes $(\sigma, \mu)$.

The notion of AGM-consistency imposes natural restrictions on assessments. Consider, for example, the extensive form of Figure 2 and any assessment $(\sigma, \mu)$ where $\sigma = (c, d, f)$ (highlighted by double edges) and $\mu$ assigns positive probability to history $be$. Any such assessment is *not* AGM-consistent. In fact, if there were a plausibility order $\precsim$ that satisfied Definition 5, then, by *P1*, $b \sim bd$ (since $\sigma(d) = 1 > 0$) and $b \prec be$ (since $\sigma(e) = 0$)[10] and, by *P2*, $be \precsim bd$ (since - by hypothesis - $\mu$ assigns positive probability to $be$). By transitivity of $\precsim$, from $b \sim bd$ and $b \prec be$ it follows that $bd \prec be$, yielding a contradiction.



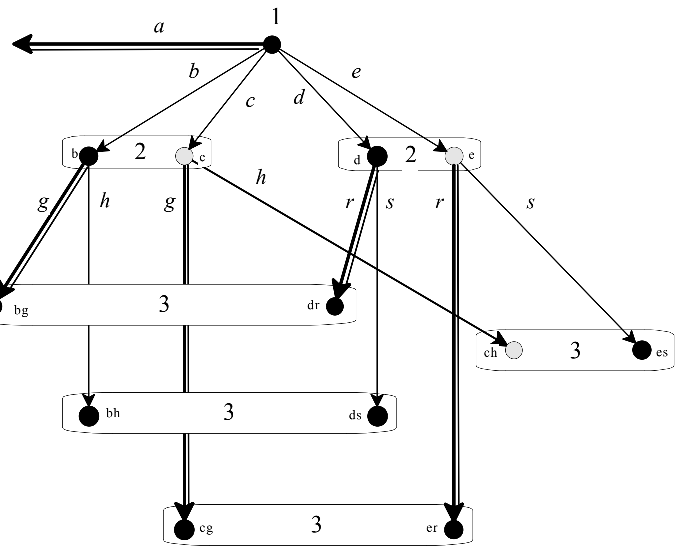

Any assessment $(\sigma, \mu)$ where $\sigma = (c, d, f)$ and $\mu$ assigns positive probability to history $be$ is not AGM-consistent.
Figure 2

On the other hand, consider the partial extensive form of Figure 3 (in order to simplify the figure, the choices of Player 3 have been omitted) and an arbitrary assessment of the form $(\sigma, \mu)$ where $\sigma = (a, g, r)$ (highlighted by double edges) and $\mu$ assigns positive probability to the black nodes and zero probability to the gray nodes (thus $\mu(c) = \mu(e) = \mu(ch) = 0$, while every other history is assigned positive probability). Any such assessment is AGM-consistent. In fact it is rationalized by the following plausibility order ($h$ is on the same row as $h'$ if and only if $h \sim h'$ and $h$ is above $h'$ if and only if $h \prec h'$ ):

$$\begin{pmatrix} \emptyset, a \\ b, bg, d, dr \\ bh, ds \\ c, cg, e, er \\ es \\ ch \end{pmatrix}$$

[10] By definition of plausibility order, $b \precsim be$ and by *P1* it is not the case that $b \sim be$ because $e$ is not assigned positive probability by $\sigma$. Thus $b \prec be$.

Note that there may be several plausibility orders that rationalize a given assessment (for instance, an alternative plausibility order to the one given above is obtained by switching the rows $(bh, ds)$ and $(c, cg, e, er)$).




Any assessment $(\sigma, \mu)$ where $\sigma = (a, g, r)$ and $\mu$ assigns zero probability to, and only to, histories $c$, $e$ and $ch$ is AGM-consistent.
Figure 3

Note that, for every extensive form, the set of plausibility orders is non-empty and, for every plausibility order, there is at least one assessment which is rationalized by that plausibility order.[11]

We now show that the notion of AGM-consistency generalizes the notion of KW-consistency, in the sense that *the set of KW-consistent assessments is a proper subset of the set of AGM-consistent assessments.*

Given a plausibility order $\precsim$ on the set of histories $H$, a function $F : H \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of non-negative integers) is an *integer-valued representation* of $\precsim$ if $F(\emptyset) = 0$ (recall that $\emptyset$ denotes the null history) and, $\forall h, h' \in H$, $h \precsim h'$ if and only if $F(h) \leq F(h')$. Since $H$ is finite, the set of integer-valued representations of $\precsim$ is non-empty.[12]

[11] If $h \in D_i$ let $\sigma_i(h)$ be an arbitrary probability distribution whose support coincides with the set of plausibility preserving actions at $h$ and if $I$ is an information set then let $\mu$ assign positive probability to all and only the most plausible histories in $I$.

[12] A natural integer-valued representation is the following. Define $H_0 = \{h \in H : h \precsim x, \ \ \forall x \in H\}$, $H_1 = \{h \in H \setminus H_0 : \ h \precsim x, \ \ \forall x \in H \setminus H_0\}$ and, in general, for every integer $k \geq 1$, $H_k = \{h \in H \setminus H_0 \cup ... \cup H_{k-1} : \ h \precsim x, \ \ \forall x \in H \setminus H_0 \cup ... \cup H_{k-1}\}$. Since $H$ is finite, there is an $m \in \mathbb{N}$ such that $\{H_0, ..., H_m\}$ is a partition of $H$ and, for every $j, k \in \mathbb{N}$, with $j < k \leq m$, and for every $h, h' \in H$, if $h \in H_j$ and $h' \in H_k$ then $h \prec h'$. Define $F : H \to \mathbb{N}$ as follows: $F(h) = k$ if and only if $h \in H_k$. The function $F$ so defined is an integer-valued representation of $\precsim$.

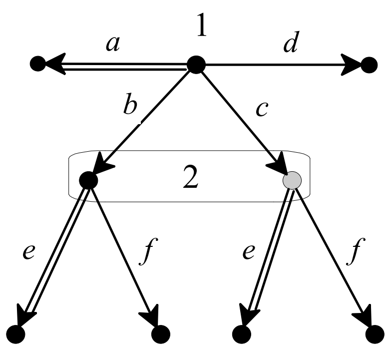
**Definition 6** *A plausibility order* $\precsim$ *on the set of histories* $H$ *is* choice-measurable *if it has at least one integer-valued representation* $F$ *that satisfies the following property:* $\forall h \in D$, $\forall h' \in I(h)$, $\forall a \in A(h)$,

$$F(ha) - F(h) = F(h'a) - F(h'). \tag{CM}$$

Note that not every integer-valued representation of a choice measurable plausibility order need satisfy Property $CM$. For example, consider the plausibility order and the two integer-valued representations $F_1$ and $F_2$ shown in Figure 4$a$. $F_1$ is the representation described in Footnote 12 and does not satisfy Property *CM*, since $c \in I(b)$ and $F_1(bf) - F_1(b) = 2 - 1 = 1$ while $F_1(cf) - F_1(c) = 5 - 3 = 2$. On the other hand, $F_2$ does satisfy *CM*.

| $\precsim$ | $F_1$ | $F_2$ |
|---|---|---|
| $\emptyset,\ a$ | 0 | 0 |
| $b,\ be$ | 1 | 1 |
| $bf$ | 2 | 3 |
| $c,\ ce$ | 3 | 4 |
| $d$ | 4 | 5 |
| $cf$ | 5 | 6 |

($a$) A plausibility order and two integer-valued representations of it.




($b$) An assessment rationalized by the choice-measurable plausibility order shown on the left.

Figure 4

The following proposition (which follows from Lemma A.1 in Kreps and Wilson [8], p. 887) provides a characterization of KW-consistency in terms of rationalizability by a choice-measurable plausibility order. The proof is given in the Appendix.

**Proposition 7** *Fix an extensive form.*

*(1) If* $(\sigma, \mu)$ *is a KW-consistent assessment then there is a choice-measurable plausibility order that rationalizes it (and thus* $(\sigma, \mu)$ *is AGM consistent).*

*(2) Conversely, if* $\precsim$ *is a choice measurable plausibility order, then there exists a KW-consistent assessment* $(\sigma, \mu)$ *which is rationalized by* $\precsim$.

Proposition 7 makes it easy to check whether an assessment is KW-consistent. For example, consider again the extensive form of Figure 3 and an arbitrary assessment $(\sigma, \mu)$ where $\sigma = (a, g, r)$ (highlighted by double edges) and $\mu$ assigns positive probability to the black nodes and zero probability to the gray nodes (thus $\mu(c) = \mu(e) = \mu(ch) = 0$, while every other history is assigned positive probability). We saw above that any such assessment is AGM-consistent.

On the other hand, no such assessment is KW-consistent, because there is no choice-measurable plausibility order that rationalizes it. To see this, let $\precsim$ be a plausibility order that rationalizes the assessment under consideration and let $F$ be an arbitrary integer-valued representation of $\precsim$. By *P2* of Definition 5, $es \prec ch$ and thus

$$F(es) < F(ch). \tag{1}$$

By *P1* of Definition 5, $c \sim cg$ and $e \sim er$ and by *P2* $cg \sim er$. Thus, by transitivity of $\precsim$, $c \sim e$ so that $F(c) = F(e)$. Hence, by (1),

$$F(es) - F(e) < F(ch) - F(c). \tag{2}$$

Similarly, by *P1*, $b \sim bg$ and $d \sim dr$ and, by *P2*, $bg \sim dr$. Thus, by transitivity of $\precsim$, $b \sim d$ so that

$$F(b) = F(d). \tag{3}$$

By *P2*, $bh \sim ds$ and thus $F(bh) = F(ds)$. Hence, by (3),

$$F(bh) - F(b) = F(ds) - F(d). \tag{4}$$

Now, if $F(ch) - F(c) = F(bh) - F(b)$ (as required by Property $CM$ of Definition 6) then, by (2) and (4), $F(es) - F(e) < F(ds) - F(d)$, violating Property $CM$.

# 4 Perfect Bayesian equilibrium

We now define a notion of perfect Bayesian equilibrium which is very general, in the sense that it can be applied to every finite extensive-form game.[13] The essential ingredient is the qualitative notion of AGM-consistency to which we add a Bayesian updating requirement for probabilities and sequential rationality.

Given a total pre-order $\precsim$ on $H$ and a subset $I \subseteq H$ we denote by $Min_{\precsim}\, I$ the set of most plausible histories in $I$, that is,

$$Min_{\precsim}\, I = \{h \in I : h \precsim h', \forall h' \in I\}.$$

**Definition 8** *Let* $\precsim$ *by a plausibility order that rationalizes the assessment* $(\sigma, \mu)$*. We say that* $(\sigma, \mu)$ *is* Bayesian relative to $\precsim$ *if for every* $\precsim$*-equivalence class* $E$ *there exists a probability measure* $\nu_E : H \to [0, 1]$ *such that:*

*(i)* $Supp(\nu_E) = E$.

*(ii) If* $h, h' \in E$ *and* $h$ *is a prefix of* $h'$*, that is,* $h' = ha_1 ..., a_m$*, then*

$$\nu_E(h') = \nu_E(h) \times \sigma(a_1) \times \ ... \ \times \ \sigma(a_m).$$

[13] Previous definitions of perfect Bayesian equilibrium were proposed for restricted classes of extensive-form games (e.g. the class of multi-stage games with observed actions: see [5]).

*(iii) For every information set $I$ such that $Min_{\precsim} I \subseteq E$ and for every $h \in I$,*

$$\mu(h) = \nu_E(h|I) = \frac{\nu_E(h)}{\nu_E(I)}.$$

Property (i) requires the support of $\nu_E$ to coincide with $E$ (thus $\nu_E(x) = 0$ if and only if $x \in H \backslash E$). Property (ii) requires $\nu_E$ to be consistent with the strategy profile $\sigma$ in the sense that if $h' = ha_1 ..., a_m$ then the probability of $h'$ (according to $\nu_E$) is equal to the probability of $h$ multiplied by the probabilities (according to $\sigma$) of the actions that lead from $h$ to $h'$.[14] Property (iii) requires the system of beliefs $\mu$ to satisfy Bayes' rule in the sense that if history $h$ belongs to information set $I$ then $\mu(h)$ (the probability assigned to $h$ by $\mu$) is the probability of $h$ conditional on $I$ using the probability measure $\nu_E$, where $E$ is the equivalence class of the most plausible elements of $I$.[15]

**Definition 9** *An assessment $(\sigma, \mu)$ is* Bayesian AGM-consistent *if it is rationalized by a plausibility order $\precsim$ on the set of histories $H$ and it is Bayesian relative to $\precsim$.*

Consider, for example, the extensive form of Figure 3 and the following assessment:

$$\sigma = (a, (g, r)) \text{ and } \mu = \left(b, (\tfrac{1}{2}bg, \tfrac{1}{2}dr), (\tfrac{1}{2}bh, \tfrac{1}{2}ds), (\tfrac{1}{2}cg, \tfrac{1}{2}er), d, es\right).$$

Then $(\sigma, \mu)$ is rationalized by

$$\precsim \;=\; \begin{pmatrix} \emptyset, a \\ b, bg, d, dr \\ bh, ds \\ c, cg, e, er \\ es \\ ch \end{pmatrix}$$

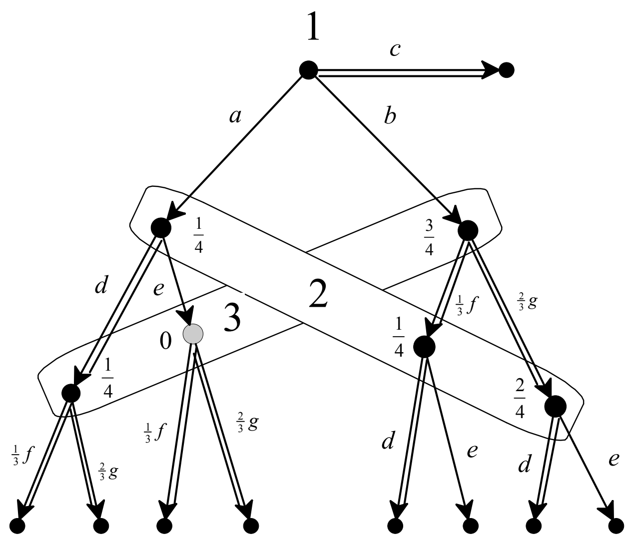
and it is Bayesian relative to $\precsim$: for every $\precsim$-equivalence class $E$ let $\nu_E$ to be the uniform measure over $E$. Take, for instance, $E = \{b, bg, d, dr\}$. Then $\nu_E$ assigns probability $\frac{1}{4}$ to every element of $E$ and zero to every other history. Property (ii) of Definition 8 is satisfied, because $b, bg \in E$, $b$ is a prefix of $bg$ and $\nu_E(bg) = \frac{1}{4} = \nu_E(b) \times \sigma(g) = \frac{1}{4} \times 1 = \frac{1}{4}$ (and similarly for $d$ and $dr$). Property (iii) is also satisfied. For instance, $\mu(bg) = \frac{1}{2} = \frac{\nu_E(bg)}{\nu_E(\{bg,dr\})} = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}}$ and $\mu(b) = 1 = \frac{\nu_E(b)}{\nu_E(\{b,c\})} = \frac{\frac{1}{4}}{\frac{1}{4}+0}$.

As another example, consider the extensive form of Figure 5 and the following assessment:

[14] Note that if $h, h' \in E$ and $h' = ha_1...a_m$, then $\sigma(a_j) > 0$, for all $j = 1, ..., m$. In fact, since $h' \sim h$ every action $a_j$ is plausibility preserving and therefore, by Property $P1$ of Definition 5, $\sigma(a_j) > 0$.

[15] Note that the $\precsim$-equivalence class that contains $Min_{\precsim}\ I$ is not necessarily a subset of $I$. For example, if $H = \{h_1, ..., h_4\}$ and $\precsim$ is given by $h_1 \prec h_2 \sim h_3 \prec h_4 \sim h_5$ and $I = \{h_2, h_4\}$ then $Min_{\precsim}\ I = \{h_2\}$ and the equivalence class that contains $h_2$ is $\{h_2, h_3\}$. .

$$\sigma = \left(c, d, (\tfrac{1}{3}f, \tfrac{2}{3}g)\right),\ \mu = \left((\tfrac{1}{4}a, \tfrac{1}{4}bf, \tfrac{2}{4}bg), (\tfrac{1}{4}ad, \tfrac{3}{4}b)\right).$$




A Bayesian AGM-consistent assessment
Figure 5

Then $(\sigma, \mu)$ is rationalized by

$$\precsim \ = \begin{pmatrix} \emptyset, c \\ a, b, ad, bf, bg, adf, adg, bfd, bgd \\ ae, aef, aeg, bfe, bge \end{pmatrix}$$

and it is Bayesian relative to $\precsim$. First of all, note that, letting $I_2 = \{a, bf, bg\}$ be the information set of Player 2 and $I_3 = \{ad, ae, b\}$ the information set of Player 3, we have that $Min_{\precsim} I_2 = I_2$ and $Min_{\precsim} I_3 = \{ad, b\}$ and the equivalence class that contains these two sets is $E = \{a, b, ad, bf, bg, adf, adg, bfd, bgd\}$. Then all we need to specify is a probability measure over $E$. Let

$$\nu_E = \begin{pmatrix} a & b & ad & bf & bg & adf & adg & bfd & bgd \\ \frac{3}{36} & \frac{9}{36} & \frac{3}{36} & \frac{3}{36} & \frac{6}{36} & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{6}{36} \end{pmatrix}.$$

Then all the properties of Definition 8 are satisfied. For instance, concerning Property (ii), $b$ is a prefix of $bf$ and $\nu_E(bf) = \frac{3}{36} = \nu_E(b) \times \sigma(f) = \frac{9}{36} \times \frac{1}{3}$, and, concerning Property (iii), $\mu(ad) = \frac{1}{4} = \frac{\nu_E(ad)}{\nu_E(I_3)} = \frac{\frac{3}{36}}{\frac{3}{36}+0+\frac{9}{36}}$.

The last ingredient of the definition of perfect Bayesian equilibrium is the standard requirement of sequential rationality. An assessment $(\sigma, \mu)$ is sequentially rational if, for every player $i$ and every information set $I_i$ of player $i$, player $i$'s expected payoff, given her beliefs at $I_i$ (as specified by $\mu$) and the strategy profile $\sigma$, cannot be increased by unilaterally changing her choice at $I_i$

and possibly at information sets of hers that follow $I_i$.[16] In order to define sequential rationality precisely we need to introduce more notation. Recall that $Z$ denotes the set of terminal histories and, for every player $i$, $U_i : Z \to \mathbb{R}$ denotes player $i$'s von Neumann-Morgenstern utility function. Given a decision history $h$, let $Z(h)$ denote the terminal histories which have $h$ as a prefix. Let $\mathbb{P}_{h,\sigma}$ denote the probability distribution over $Z(h)$ induced by the strategy profile $\sigma$, starting from history $h$ (that is, if $z$ is a terminal history and $z = ha_1...a_m$ then $\mathbb{P}_{h,\sigma}(z) = \prod_{j=1}^{m} \sigma(a_j)$). Let $I_i$ be an information set of player $i$ and let $u_i(I_i|\sigma,\mu) = \sum\limits_{h \in I_i} \mu(h) \sum\limits_{z \in Z(h)} \mathbb{P}_{h,\sigma}(z) U_i(z)$ be player $i's$ expected payoff at $I_i$ if $\sigma$ is played, given her beliefs at $I_i$ (as specified by $\mu$). We say that player $i$'s strategy $\sigma_i$ is *sequentially rational at* $I_i$ if $u_i(I_i|(\sigma_i,\sigma_{-i}),\mu) \geq u_i(I_i|(\tau_i,\sigma_{-i}),\mu)$ for every strategy $\tau_i$ of player $i$ (where $\sigma_{-i}$ denotes the strategy profile of the players other than $i$, that is, $\sigma_{-i} = (\sigma_1,...,\sigma_{i-1},\sigma_{i+1},...,\sigma_n)$). An assessment $(\sigma,\mu)$ is *sequentially rational* if, for every player $i$ and for every information set $I_i$ of player $i$, $\sigma_i$ is sequentially rational at $I_i$.

**Definition 10** *An assessment* $(\sigma,\mu)$ *is a* perfect Bayesian equilibrium *if it is Bayesian AGM-consistent and sequentially rational.*

The following propositions are proved in the Appendix.

**Proposition 11** *If* $(\sigma,\mu)$ *is a perfect Bayesian equilibrium then* $\sigma$ *is a subgame-perfect equilibrium.*

Note that not every subgame-perfect equilibrium is part of an assessment which is a perfect Bayesian equilibrium. That is, the notion of perfect Bayesian equilibrium is a *strict refinement of subgame-perfect equilibrium.* This can be shown with the aid of the extensive form of Figure 2. Turn it into a game by adding the following payoffs:

| | $af$ | $ag$ | $bdf$ | $bdg$ | $bef$ | $beg$ | $c$ |
|---|---|---|---|---|---|---|---|
| $U_1$ | 0 | 2 | 0 | 0 | 0 | 0 | 1 |
| $U_2$ | 0 | 2 | 1 | 0 | 0 | 0 | 0 |
| $U_3$ | 0 | 2 | 0 | 1 | 1 | 0 | 0 |

Let $\sigma = (c,d,f)$. Then $\sigma$ is a Nash equilibrium and thus - since there are no proper subgames - also a subgame-perfect equilibrium. Add to $\sigma$ a system of beliefs $\mu$ that makes $(\sigma,\mu)$ a sequentially rational assessment . Then it must be that $\mu(be) > 0$ (since, for player 3, $g$ is strictly better than $f$ at both history $a$ and history $bd$). As showed above, such an assessment is not AGM-consistent.

[16] There are two definitions of sequential rationality: the *weakly local* one - which is the one adopted here - according to which at an information set a player can contemplate changing her choice not only there but possibly also at subsequent information sets of hers and a *strictly local* one, according to which at an information set a player contemplates changing her choice only there. If the definition of perfect Bayesian equilibrium (Definition 10 below) is modified by imposing the strictly local definition of sequential rationality, then an extra condition needs to be added to Definition 8, namely the "pre-consistency" condition on $\mu$ identified in [6] and [14] as being necessary and sufficient for the equivalence of the two notions. For simplicity we have chosen the weakly local definition.

**Proposition 12** *If $(\sigma, \mu)$ is a sequential equilibrium then it is a perfect Bayesian equilibrium.*

Note that not every perfect Bayesian equilibrium is a sequential equilibrium (this can be shown with the aid of the extensive form of Figure 3). Thus *sequential equilibrium is a strict refinement of perfect Bayesian equilibrium.*

Kreps and Wilson [8] proved that every finite extensive-form game has at least one sequential equilibrium. From this existence result and Proposition 12 one thus obtains the following.

**Corollary 13** *Every finite extensive-form game has at least one perfect Bayesian equilibrium.*

# 5 Conclusion

Due to space limitations we leave to a companion paper ([4]) the epistemic justification of AGM-consistency (and thus of perfect Bayesian equilibrium) based on the AGM theory of belief revision ([1]) as well as a more in-depth investigation of the relationship between AGM-consistency and KW-consistency (and thus between perfect Bayesian equilibrium and sequential equilibrium). In particular, *the notion of AGM-consistency does not rule out correlation in a player's beliefs about the choices of her opponents*, while KW-consistency has several independence properties built into it. Some of these independence properties can be expressed in purely qualitative terms (that is, as properties of the underlying plausibility order) and used to refine the notion of AGM-consistency and thus of perfect Bayesian equilibrium. A simple characterization of the backward-induction solution(s) in perfect information games in terms of AGM-consistency can also be obtained.

# A Appendix

Although Proposition 7 follows easily from a result in Kreps and Wilson ([8], Lemma A.1, p. 887) we provide a proof for completeness. First some preliminary definitions and lemmas.

**Remark 14** *Recall our assumption that $A = \bigcup_{h \in D} A(h)$. Thus, for every $a \in A$ there is an $h \in D$ such that $a \in A(h)$. Recall also the assumption that no action is available at more than one information set, that is, if $h, h' \in H$ are such that $A(h) \cap A(h') \neq \varnothing$ then $h' \in I(h)$.*

**Notation 15** *For every $h \in H$ and $a \in A$ we write $a \in h$ if there exists an $h' \in H$ such that $h'a$ is a prefix of $h$ (thus $a \in A(h')$); that is, if $a$ is an action that occurs in history $h$.*

**Definition 16** *An* $A$-weighting *is a function $\lambda : A \to \mathbb{N}$ such that, for every $h \in D$, there is at least one $a \in A(h)$ with $\lambda(a) = 0$. Furthermore, if the null history $\emptyset$ is assigned to chance, then $\lambda(a) = 0$ for every $a \in A(\emptyset)$.*

**Lemma 17** *Let $\precsim$ be a choice measurable plausibility order. Let $F : H \to \mathbb{N}$ be a integer-valued representation that satisfies Property $CM$ of Definition 6. Define $\lambda_{\precsim} : A \to \mathbb{N}$ as follows: $\lambda_{\precsim}(a) = F(ha) - F(h)$ for some $h$ such that $a \in A(h)$. Then*

*(i) $\lambda_{\precsim}$ is an $A$-weighting;*

*(ii) if $\Lambda_{\precsim} : H \to \mathbb{N}$ is defined by $\Lambda_{\precsim}(\emptyset) = 0$ and, for $h \in H \backslash \{\emptyset\}$, $\Lambda_{\precsim}(h) = \sum_{a \in h} \lambda_{\precsim}(a)$ then, for every $h \in H$, $\Lambda_{\precsim}(h) = F(h)$.*

**Proof.** (i) First of all, $\lambda_{\precsim}$ is well defined since if $h$ and $h'$ are such that $a \in A(h) \cap A(h')$ then, by Remark 14, $h' \in I(h)$[17] and thus, Property $CM$ of Definition 6, $F(ha) - F(h) = F(h'a) - F(h')$. By Property 1 of Definition 3, $F(ha) \geq F(h)$ and thus, since $F$ is integer-valued, $\lambda_{\precsim}(a) \in \mathbb{N}$. By Property 2 of Definition 3, for every $h \in D$ there is an $a \in A(h)$ such that $ha \sim h$ and thus $F(ha) = F(h)$ so that $\lambda_{\precsim}(a) = 0$.

(ii) Fix an arbitrary $h \in H$. If $h = \emptyset$ then $\Lambda_{\precsim}(\emptyset) = F(\emptyset) = 0$. If $h = \langle \emptyset, a_1, ..., a_m \rangle$ $(m \geq 1)$, then $\Lambda_{\precsim}(h) = \underbrace{[F(\emptyset a_1) - F(\emptyset)]}_{=\lambda_{\precsim}(a_1)} + \underbrace{[F(\emptyset a_1 a_2) - F(\emptyset a_1)]}_{=\lambda_{\precsim}(a_2)} + ... + \underbrace{[F(h) - F(\emptyset a_1 a_2 ... a_{m-1})]}_{=\lambda_{\precsim}(a_m)} = -F(\emptyset) + F(h) = F(h)$ (since $F(\emptyset) = 0$). $\blacksquare$

The following result is proved in Kreps and Wilson ([8], Lemma A.1, p. 887; we have re-written the result in terms of the notation used in this paper and slightly reworded it).

**Lemma 18** *Fix an extensive form.*

*(a) If $(\sigma, \mu)$ is a KW-consistent assessment then there exists an $A$-weighting $\lambda : A \to \mathbb{N}$ such that, $\forall h \in D, \forall a \in A(h)$, (i) $\lambda(a) = 0$ if and only if $\sigma(a) > 0$, and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ for all $h' \in I(h)$ (where $\Lambda(h) = \sum_{a \in h} \lambda(a)$).*

*(b) If $\lambda : A \to \mathbb{N}$ is an $A$-weighting, then there exists a KW-consistent assessment $(\sigma, \mu)$ such that, $\forall h \in D, \forall a \in A(h)$, (i) $\sigma(a) > 0$ if and only if $\lambda(a) = 0$ and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ for all $h' \in I(h)$.*

**Proof of Proposition 7**. (1) Let $(\sigma, \mu)$ be a KW-consistent assessment. By (a) of Lemma 18 there exists an $A$-weighting $\lambda : A \to \mathbb{N}$ such that, $\forall h \in D, \forall a \in A(h)$, (i) $\lambda(a) = 0$ if and only if $\sigma(a) > 0$, and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ for all $h' \in I(h)$. Define the following total pre-order on the set of histories $H$: $h \precsim h'$ if and only if $\Lambda(h) \leq \Lambda(h')$. Then $\precsim$ is a plausibility order. In fact, if $h \in D$ and $a \in A(h)$ then $\Lambda(ha) = \Lambda(h) + \lambda(a) \geq \Lambda(h)$ (since $\lambda(a) \geq 0$), so that $h \precsim ha$. Furthermore, by definition of $A$-weighting, for every $h \in D$ there is an $a \in A(h)$ such that $\lambda(a) = 0$ and thus $\Lambda(ha) = \Lambda(h) + \lambda(a) = \Lambda(h)$, so that $ha \precsim h$. It is also clear that $\precsim$ is choice measurable (the function $\Lambda$

[17] Without the assumption that no action is available at more than one information set, the domain of the weigthing function $\lambda$ would have to be the set of action-history pairs $(a, h)$ with $a \in A(h)$. This would make the notation more complicated, but the proofs would go through.

provides an integer-valued representation of $\precsim$ that satisfies Property *CM* of Definition 6) and that $\precsim$ rationalizes $(\sigma, \mu)$ (Definition 5).

(2) Let $\precsim$ be a choice measurable plausibility order on the set of histories $H$. By Lemma 17 one can derive from $\precsim$ an $A$-weighting $\lambda_{\precsim} : A \to \mathbb{N}$ such that the associated function $\Lambda_{\precsim} : H \to \mathbb{N}$ defined by $\Lambda_{\precsim}(h) = \sum_{a \in h} \lambda_{\precsim}(a)$ is such that $h \precsim h'$ if and only if $\Lambda_{\precsim}(h) \leq \Lambda_{\precsim}(h')$. Thus, by (b) of Lemma 18, there exists a consistent assessment $(\sigma, \mu)$ such that, $\forall h \in D, \forall a \in A(h)$, (i) $\sigma(a) > 0$ if and only if $\lambda(a) = 0$ (hence if and only if $h \sim ha$) and (ii) $\mu(h) > 0$ if and only if $\Lambda(h) \leq \Lambda(h')$ (hence if and only if $h \precsim h'$) for all $h' \in I(h)$. Thus $(\sigma, \mu)$ is rationalized by $\precsim$. ■

Before we prove Proposition 11 we need to give a precise definition of subgame and of subgame-perfect equilibrium. Let $G = \langle A, H, N, P, \{\approx_i\}_{i \in N} \rangle$ be an extensive-form game and let $h_0 \in D$. We say that $G' = \langle A', H', N', P', \{\approx'_i\}_{i \in N'} \rangle$ is a *subgame of* $G$ *with root* $h_0$ if: (1) $I(h_0) = \{h_0\}$ (that is, the information set that contains $h_0$ consists of $h_0$ only), (2) $H' = \{h \in H : h_0$ is a prefix of $h\}$ (that is, $H'$ is the subset of $H$ consisting of histories that have $h_0$ as a prefix), (3) $\forall h' \in H'$, $\forall i \in N$, $\forall h \in H$, if $h \approx_i h'$ then $h \in H'$ (that is, the information sets of $G$ that contain an element of $H'$ are entirely included in $H'$), (4) $A' = \{a \in A : a \in A(h')$ for some $h' \in H'\}$ (that is, $A'$ is the subset of $A$ consisting of those actions that are available at histories in $H'$), (5) $N' = \{i \in N : P^{-1}(i) \subseteq H'\}$ (that is, $N'$ is the subset of players that are assigned to histories in $H'$), (6) $P'$ is the restriction of $P$ to $H'$ and (7) $\forall i \in N'$, $\approx'_i$ is the restriction of $\approx_i$ to $H'$.

Let $G$ be a game and $\sigma$ a profile of behavior strategies. Then $\sigma$ is a *subgame-perfect equilibrium* of $G$ if, for every subgame $G'$ of $G$, the restriction of $\sigma$ to $G'$ is a Nash equilibrium of $G'$.

Next we introduce some notation. Fix a game $G$ and a profile of behavior strategies $\sigma$. Let $h, h' \in H$ with $h'$ a prefix of $h$, that is, $h = h'a_1...a_m$ for some actions $a_1, ..., a_m \in A$. Define $\mathbb{Q}_{h',\sigma}(h) = \sigma(a_1) \times ... \times \ \sigma(a_m)$ (that is, $\mathbb{Q}_{h',\sigma}(h)$ is the probability of reaching $h$ from $h'$ if play is according to $\sigma$) and let $\mathbb{Q}_{h',\sigma}(I(h)) = \sum\limits_{x \in I(h)} \mathbb{Q}_{h',\sigma}(x)$ be the probability of reaching, from $h'$, the information set to which $h$ belongs if play is according to $\sigma$. An assessment $(\sigma, \mu)$ is *weakly consistent* if for every history $h$, whenever the probability of reaching - *from the null history* (that is, from the root of the tree) - the information set to which $h$ belongs is positive (that is, if $\mathbb{Q}_{\emptyset,\sigma}(I(h)) > 0$) then $\mu(h)$ is obtained from $\mathbb{Q}_{\emptyset,\sigma}(\cdot)$ by using Bayes' rule, that is,

$$\text{if } \mathbb{Q}_{\emptyset,\sigma}(I(h)) > 0 \text{ then } \mu(h) = \frac{\mathbb{Q}_{\emptyset,\sigma}(h)}{\mathbb{Q}_{\emptyset,\sigma}(I(h))}. \tag{WC}$$

The following result is well-known (see, for example, [11] p. 329).

**Lemma 19** *Fix an extensive-form game $G$ and an assessment $(\sigma, \mu)$. If $(\sigma, \mu)$ is weakly consistent and sequentially rational, then $\sigma$ is a Nash equilibrium.*

**Proof of Proposition 11 .** Fix an extensive-form game $G$ and let $(\sigma, \mu)$ be a perfect Bayesian equilibrium. Then $(\sigma, \mu)$ is Bayesian AGM-consistent.

Let $\precsim$ be a plausibility order on the set of histories $H$ that rationalizes $(\sigma, \mu)$ (see Definition 5) and relative to which $\precsim$ is Bayesian (see Definition 8). Let $G'$ be an arbitrary subgame of $G$ and let $(\sigma', \mu')$ be the restriction of $(\sigma, \mu)$ to $G'$. We need to show that $\sigma'$ is a Nash equilibrium of $G'$. By definition of perfect Bayesian equilibrium, $(\sigma, \mu)$ is sequentially rational in $G$ and thus $(\sigma', \mu')$ is sequentially rational in $G'$. Hence, by Lemma 19, it is sufficient to show that $(\sigma', \mu')$ is weakly consistent in $G'$, that is, if $h_0$ is the root of $G'$, then for every $h' \in H'$,[18]

$$\text{if } \mathbb{Q}_{h_0,\sigma}(I(h')) > 0 \text{ then } \mu(h') = \frac{\mathbb{Q}_{h_0,\sigma}(h')}{\mathbb{Q}_{h_0,\sigma}(I(h'))}. \tag{$WC'$}$$

Fix an arbitrary $h' \in H'$ and suppose that $\mathbb{Q}_{h_0,\sigma}(I(h')) > 0$, that is, the information set to which $h'$ belongs is reached with positive probability if $\sigma$ is played *starting from* $h_0$. Then there is an $h \in I(h')$ such that $h = h_0 a_1 ... a_m$ for some actions $a_1, ..., a_m$ and $\sigma(a_j) > 0$ for all $j = 1, ..., m$.[19] Thus, by Property $P1$ of Definition 5 every action $a_j$ is plausibility preserving (see Remark 4) and hence, by transitivity of $\precsim$,

$$h_0 \sim h. \tag{5}$$

Since every history in $H'$ has $h_0$ as a prefix, by Remark 4 $h_0 \precsim x$ for every $x \in H'$. It follows from this and (5) that $h \in Min_{\precsim} I(h')$. Let $E$ be the $\precsim$-equivalence class to which $h$ belongs (thus $E$ contains $Min_{\precsim} I(h')$). By (5) $h_0 \in E$ and by Definition 8 there is a probability measure $\nu_E$ on $H$ (whose support coincides with $E$) such that $\nu_E(I(h')) > 0$ and[20]

$$\mu(h) = \frac{\nu_E(h)}{\nu_E(I(h'))} = \frac{\nu_E(h_0) \times (\sigma(a_1) \times ... \times \ \sigma(a_m))}{\nu_E(I(h'))} = \frac{\nu_E(h_0) \times \mathbb{Q}_{h_0,\sigma}(h)}{\sum\limits_{x \in I(h')} \nu_E(x)}. \tag{6}$$

Fix an arbitrary $x \in I(h')$. If $x \notin E$, then $\nu_E(x) = 0$; furthermore, $h \prec x$ so that, by (5), $h_0 \prec x$. It follows from this that $x = h_0 b_1 ... b_r$ for some actions $b_1, ..., b_r$ at least one of which - say $b_j$ - is not plausibility preserving. Hence, by Property $P1$ of Definition 5, $\sigma(b_j) = 0$ and thus $\mathbb{Q}_{h_0,\sigma}(x) = 0$. Hence if $x \notin E$ then $\nu_E(x) = \mathbb{Q}_{h_0,\sigma}(x) = 0$. Now consider the case where $x \in E$, that is, $x \sim h$ so that, by (5), $x \sim h_0$. Hence, $x = h_0 b_1 ... b_s$ where, for every $k = 1, ..., s$, $b_k$ is a plausibility preserving action so that, by Property $P1$ of Definition 5, $\sigma(b_k) > 0$, for every $k = 1, ..., s$. Thus, by Property (ii) of Definition 8, $\nu_E(x) = \nu_E(h_0) \times (\sigma(b_1) \times ... \times \ \sigma(b_s)) = \nu_E(h_0) \times \mathbb{Q}_{h_0,\sigma}(x)$. It follows from this and (6) that

$$\mu(h) = \frac{\nu_E(h_0) \times \mathbb{Q}_{h_0,\sigma}(h)}{\nu_E(h_0) \times \sum\limits_{x \in I(h')} \mathbb{Q}_{h_0,\sigma}(x)} = \frac{\mathbb{Q}_{h_0,\sigma}(h)}{\sum\limits_{x \in I(h')} \mathbb{Q}_{h_0,\sigma}(x)}. \tag{7}$$

[18] Note that - since $\sigma'$ is the restriction of $\sigma$ to $G'$ - for every $h' \in H'$, $\mathbb{Q}_{h_0,\sigma}(h') = \mathbb{Q}_{h_0,\sigma'}(h')$.

[19] Since $\sigma'$ is the restriction of $\sigma$ to the subgame $G'$, $\sigma'(a_j) = \sigma(a_j)$ for all $j = 1, ..., m$.

[20] The probability of reaching $h$ from $h_0$, if play is according to $\sigma$, is $\mathbb{Q}_{h_0,\sigma}(h) = \sigma(a_1) \times ... \times \ \sigma(a_m)$.

Hence the restriction of $(\sigma, \mu)$ to the subgame $G'$ with root $h_0$ is weakly consistent in $G'$ and thus, by Lemma 19, $\sigma'$ is a Nash equilibrium of $G'$. ■

In order to prove Proposition 12 we need some preliminary definitions and lemmas.

**Definition 20** *Fix an extensive-form game. Let $(\sigma, \mu)$ be an assessment and $\precsim$ a plausibility order on the set of histories $H$ that rationalizes $(\sigma, \mu)$. Let $E \subseteq H$ be an $\precsim$-equivalence class. A* path *in $E$ is a sequence $\langle h_1, ..., h_m \rangle$ in $E$ such that, for every $i = 1, ..., m-1$, either $h_i$ is a prefix of $h_{i+1}$ or $h_{i+1}$ is a prefix of $h_i$ or $h_{i+1} \in I(h_i)$. We say that $h, h' \in E$ are* linked *if there exists a path $\langle h_1, ..., h_m \rangle$ in $E$ with $h_1 = h$ and $h_m = h'$. A subset $F \subseteq E$ is* connected *if every pair in $F$ is linked. A subset $F \subseteq E$ is* maximally connected *if it is connected and, for every connected $G \subseteq E$, if $F \subseteq G$ then $F = G$.*

For example, consider the (partial) extensive form of Figure 6 and a plausibility order $\precsim$ for which the following is an equivalence class: $E = \{a, b, c, d, em, en, bf, cg\}$. Then histories $a$ and $d$ are linked by the path $\langle a, b, bf, cg, c, d \rangle$ ($b \in I(a)$, $b$ is a prefix of $bf$, $cg \in I(bf)$, $c$ is a prefix of $cg$ and $d \in I(c)$), while histories $a$ and $em$ are not linked; furthemore, the maximally connected subsets of $E$ are $F_1 = \{a, b, c, d, bf, cg\}$ and $F_2 = \{em, en\}$.

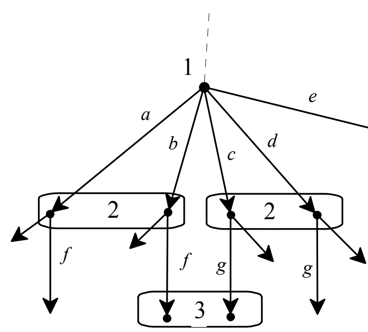



Illustration of Definition 20
Figure 6

**Lemma 21** *Let $\precsim$ be a plausibility order on $H$ and $E$ an $\precsim$-equivalence class. If $F$ and $G$ are two maximally connected subsets of $E$ then either $F = G$ or $F \cap G = \varnothing$.*

**Proof.** Suppose that $F$ and $G$ are two maximally connected subsets of $E$ with $F \neq G$ and $F \cap G \neq \varnothing$. Fix an arbitrary $h \in F \cap G$ and arbitrary $h_1 \in F$ and $h_2 \in G$. Then there is a path from $h_1$ to $h$ and there is a path from $h$ to $h_2$. Joining these two paths we get that $h_1$ is linked to $h_2$. Thus every two histories in $F \cup G$ are linked and therefore $F \cup G$ is connected. Since $F \neq G$, $F \cup G$ is a proper superset of either $F$ or $G$, contradicting the hypothesis that $F$ and $G$ are maximally connected. ■

**Corollary 22** *Let $\precsim$ be a plausibility order on $H$ and $E$ an $\precsim$-equivalence class. Then $E$ can be partitioned into a collection of maximally connected subsets .*

Corollary 22 is an immediate consequence of Lemma 21 and finiteness of $H$.

**Lemma 23** *Fix an extensive-form game. Let $(\sigma, \mu)$ be an assessment and $\precsim$ a plausibility order on the set of histories $H$ that rationalizes $(\sigma, \mu)$. Let $E \subseteq H$ be an $\precsim$-equivalence class and $F$ a maximally connected subset of $E$ (Definition 20). Let $\nu : H \to [0, 1]$ be a probability measure such that*

*(i) Supp$(\nu) \cap F \neq \varnothing$,*

*(ii) if $h, h' \in F$ and $h' = ha_1...a_r$ (that is, $h'$ is a prefix of $h$) then $\nu(h') = \nu(h) \times \sigma(a_1) \times ... \times \sigma(a_r)$,*

*(iii) for every $h \in F$ if $\nu(I(h)) > 0$ then $\mu(h) = \frac{\nu(h)}{\nu(I(h))}$.*

*Then, for all $h \in F$, $\nu(h) > 0$, $h \in Min_{\precsim} I(h)$ and $\mu(h) = \frac{\nu(h)}{\nu(I(h))} > 0$.*

**Proof.** Fix an $h_1 \in F$ such that $\nu(h_1) > 0$ (it exists by hypothesis (i)); then, since $h_1 \in I(h_1)$, $\nu(I(h_1)) > 0$ so that, by hypothesis (iii), $\mu(h_1) = \frac{\nu(h_1)}{\nu(I(h_1))}$. Hence $\mu(h_1) > 0$ and therefore, by Definition 5 (since $(\sigma, \mu)$ is rationalized by $\precsim$), $h_1 \in Min_{\precsim} I(h_1)$. Fix an arbitrary $h \in F$. Then there is a path $\langle h_1, h_2, ..., h_m \rangle$ in $F$ with $h_m = h$. First we show that $\nu(h_2) > 0$ and $h_2 \in Min_{\precsim} I(h_2)$. If $h_1$ is a prefix of $h_2$ then $h_2 = h_1 a_1 ... a_r$ for some actions $a_1, ..., a_r$. By hypothesis $h_1$ and $h_2$ belong to the same equivalence class, that is, $h_1 \sim h_2$ and therefore every action $a_j$ $(j = 1, ..., r)$ is plausibility preserving (see Remark 4) so that, by Definition 5, $\sigma(a_j) > 0$, for all $j = 1, ..., r$. By hypothesis (ii), $\nu(h_2) = \nu(h_1) \times \sigma(a_1) \times ... \times \ \sigma(a_r)$. Thus $\nu(h_2) > 0$ and, therefore, $\nu(I(h_2)) > 0$. Hence, by hypothesis (iii), $\mu(h_2) = \frac{\nu(h_2)}{\nu((I(h_2))}$. Thus $\mu(h_2) > 0$ and, therefore, by Definition 5, $h_2 \in Min_{\precsim} I(h_2)$. The proof for the case where $h_2$ is a prefix of $h_1$ is similar. If $h_2 \in I(h_1)$ then, since $h_1 \in Min_{\precsim} I(h_1)$ and $h_2 \sim h_1$, it follows that $h_2 \in Min_{\precsim} I(h_1) = Min_{\precsim} I(h_2)$ (since $I(h_1) = I(h_2)$) and thus, by Definition 5, $\mu(h_2) > 0$. Since $\nu(I(h_1)) > 0$ (and $I(h_1) = I(h_2)$), by hypothesis (iii), $\mu(h_2) = \frac{\nu(h_2)}{\nu(I(h_2))}$. Hence (since $\mu(h_2) > 0$) $\frac{\nu(h_2)}{\nu(I(h_2))} > 0$ and thus $\nu(h_2) > 0$. The proof can now be completed by induction by showing (for $k = 2, ..., m-1$, using the same argument) that if $h_k$ is such that $\nu(h_k) > 0$ then $\nu(h_{k+1}) > 0$ and $h_{k+1} \in Min_{\precsim} I(h_{k+1})$. ■

**Proof of Proposition 12.** Fix an extensive-form game and let $(\sigma, \mu)$ be a sequential equilibrium, that is, $(\sigma, \mu)$ is sequentially rational and KW-consistent. By Proposition 7, KW-consistency implies AGM-consistency. Thus we only need to show that $(\sigma, \mu)$ is Bayesian relative to some plausibility order that rationalizes $\precsim$. Let $\precsim$ be the plausibility order described in the proof of Proposition 7 (obtained from an $A$-weighting, whose existence is guaranteed by (a) of Lemma 18). Fix an arbitrary history $\tilde{h} \in H$ and let $Min_{\precsim} I(\tilde{h})$ be the set of most plausible histories in $I(\tilde{h})$ (thus, for every $h \in I(\tilde{h})$, $\mu(h) > 0$ if $h \in Min_{\precsim} I(\tilde{h})$ and $\mu(h) = 0$ otherwise). Let $E$ be the $\precsim$-equivalence class that contains $Min_{\precsim} I(\tilde{h})$. Thus if $h, h' \in E$ are such that $h' = ha_1...a_m$ then $\sigma(a_j) > 0$ for all $j = 1, ..., m$.

We need to show that there is a probability measure $\nu_E$ on $H$ that satisfies the three properties of Definition 8. By definition of KW-consistency, there is a sequence $\langle \sigma^1, ..., \sigma^m, ... \rangle$ of completely mixed behavior-strategy profiles such that (1) $\lim_{m\to\infty} \sigma^m = \sigma$ and (2) for every $h \in H$, $\lim_{m\to\infty} \mu_m(h) = \mu(h)$ where $\mu_m(h)$ is the conditional probability of $h$ given $I(h)$ obtained by Bayes' rule from $\sigma^m$, that is, $\mu_m(h) = \frac{\mathbb{Q}_{\emptyset,\sigma^m}(h)}{\sum\limits_{x\in I(h)} \mathbb{Q}_{\emptyset,\sigma^m}(x)}$. To simplify the notation denote $\mathbb{Q}_{\emptyset,\sigma^m}(h)$ by $\nu_m(h)$. Let $F$ be the maximally connected subset of $E$ that contains $Min_{\precsim} I(\tilde{h})$ (see Definition 20). Define $\nu_F : H \to [0,1]$ as follows:[21]

$$\nu_F(h) = \begin{cases} 0 & \text{if } h \notin F \\ \lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)} & \text{if } h \in F \end{cases} \tag{8}$$

First we show that $\nu_F$ satisfies the properties of Lemma 23. Property (i) is clearly satisfied, since the support of $\nu$ is a subset of $F$. Fix arbitrary $h, h' \in F$ such that $h' = ha_1...a_r$. Then $\nu_m(h') = \nu_m(h) \times \sigma^m(a_1) \times ... \times \ \sigma^m(a_r)$ and thus $\nu_F(h') = \lim_{m\to\infty} \frac{\nu_m(h')}{\sum\limits_{y\in F} \nu_m(y)} = \lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)} (\sigma^m(a_1) \times ... \times \ \sigma^m(a_r)) =$ $\lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)} \times \lim_{m\to\infty} (\sigma^m(a_1) \times ... \times \ \sigma^m(a_r)) = \nu_F(h) \times \sigma(a_1) \times ... \times \sigma(a_r)$, so that Property (ii) is also satisfied. Fix an arbitrary $h \in F$ such that $\sum\limits_{x\in I(h)} \nu_F(x) = \nu_F(I(h)) > 0$. Then

$$\begin{aligned} \frac{\nu_F(h)}{\sum\limits_{x\in I(h)} \nu_F(x)} &= \frac{\lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)}}{\sum\limits_{x\in I(h)} \lim_{m\to\infty} \frac{\nu_m(x)}{\sum\limits_{y\in F} \nu_m(y)}} = \frac{\lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)}}{\lim_{m\to\infty} \sum\limits_{x\in I(h)} \frac{\nu_m(x)}{\sum\limits_{y\in F} \nu_m(y)}} \\ &= \lim_{m\to\infty} \frac{\frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)}}{\sum\limits_{x\in I(h)} \frac{\nu_m(x)}{\sum\limits_{y\in F} \nu_m(y)}} = \lim_{m\to\infty} \frac{\frac{1}{\sum\limits_{y\in F} \nu_m(y)} \nu_m(h)}{\frac{1}{\sum\limits_{y\in F} \nu_m(y)} \sum\limits_{x\in I(h)} \nu_m(x)} \\ &= \lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{x\in I(h)} \nu_m(x)} = \lim_{m\to\infty} \mu_m(h) = \mu(h). \end{aligned} \tag{9}$$

[21] Note that, since $\sigma^m$ is completely mixed, $0 < \nu_m(h) < 1$ for every $h \in H$ and thus, for every $F \subseteq H$ and every $h \in F$, $0 < \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)} < 1$. Hence, by the Bolzano–Weierstrass theorem, the sequence $\left\{ \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)} \right\}_{m=1,2,...}$ has a convergent subsequence. Thus, if necessary, we can take a convergent subsequence to define $\lim_{m\to\infty} \frac{\nu_m(h)}{\sum\limits_{y\in F} \nu_m(y)}$.

Thus Property (iii) of Lemma 23 is also satisfied. Hence, by Lemma 23, for every $h \in F$, $\nu_F(h) > 0$ and $\mu(h) = \frac{\nu(h)}{\nu(I(h))} > 0$.[22] If $F = E$ then the proof of Proposition 12 is complete. If $F$ is a proper subset of $E$, then, by Corollary 22, $E$ can be partitioned into a collection of maximally connected subsets $\{G_1, ..., G_m\}$ one of which is $F$. For every $G_j$ ($j = 1, ..., m$) repeat the same argument as for $F$ and obtain a probability measure $v_{G_j}$ on $G_j$ such that $v_{G_j}(h) > 0$ for all $h \in G_j$ and that satifies the properties of Lemma 23. Let $\alpha_1, ..., \alpha_m$ be arbitrary numbers such that $0 < a_j < 1$, for all $j = 1, ..., m$, and $\sum_{j=1}^{m} a_j = 1$ and let $\nu_E : H \to [0, 1]$ be defined by $\nu_E(h) = \sum_{j=1}^{m} a_j v_{G_j}(h)$. Then $\nu_E$ satisfies the properties of Definition 8 and thus $(\sigma, \mu)$ is Bayesian relative to $\precsim$. ■

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[22] Note that if $I$ is an information set such that $Min_{\precsim} I \subseteq F$, $h' \in I$ and $h' \notin F$ then, by (8), $\nu_F = 0$ and by Definition 5 $\mu(h') = 0$. Thus $\mu(h') = \frac{\nu(h')}{\nu(I)} = 0$.

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