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Precautionary saving in the presence of other risks: further comment

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Abstract: In a recent paper Courbage and Rey (Econ Theory 32:417–424 2007) provide conditions for precautionary saving motives under specific hypotheses concerning the relation between income risk and a background risk. Menegatti (Econ Theory 39:473-476 2009) has corrected a part of their conclusions. This comment shows that there are still other features in the proofs and propositions of the original paper that are incorrect and how they need to be reformulated.

Keywords: precautionary saving; background risk

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1 Introduction

Courbage and Rey (2007) provide conditions for income risk to have a precautionary saving effect in the presence of a background risk, under specific sets of assumptions about the dependence between the income risk and the background risk. Menegatti (2009) corrects their conclusions for the case of Bernouilli distributed variables, but does not examine their proof. It however appears that this proof is also partly incorrect. Several aspects of this proof have to be reformulated. Menegatti (2009) also corrects the conclusions of Courbage and Rey (2007) for the case of first degree stochastic correlation. Their definition of this concept is however incorrect and also needs to be reformulated. Section 2 re-examines the proof of the results for Bernouilli distributed variables, while section 3 re-examines the definition of first degree stochastic correlation.

2 The proof of the results obtained with Bernouilli distributed risks

The starting point of our reasoning is equation 7 of Courbage and Rey (2007, p. 420):

$$\hat{F}_1(\bar{s},H)=(1+r)\begin{pmatrix} E_{\tilde{\varepsilon}}v_1(\bar{s}(1+r)+E\tilde{y}_1,H+\tilde{\varepsilon}) \\ -E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}}v_1(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon})-(k-1)pq\Delta v_1 \end{pmatrix} \quad (1)$$

where $\bar{s}$ is the solution of the equation

$$u_1(y_0-\bar{s},H)=(1+r)\left(E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}}v_1(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon})+(k-1)pq\Delta v_1\right) \quad (2)$$

and where $E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}}(v_1\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon})$ is the expectation that would prevail[1] if $\tilde{y}_1$ and $\tilde{\varepsilon}$ were independent (k=1):

$$E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}}v_1(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon})=E_{\tilde{y}_1,\tilde{\varepsilon}}v_1(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon})-(k-1)pq\Delta v_1. \quad (3)$$

[1] To avoid ambiguities we have preferred to use here a specific notation $E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}}$, which is the expectation $E_{\tilde{y}_1,\tilde{\varepsilon}}$ restricted to the particular case of independence, while Courbage and Rey (2007) write their equation 7 using a general notation $E_{\tilde{y}_1,\tilde{\varepsilon}}$, but precise « with the term $E_{\tilde{y}_1,\tilde{\varepsilon}}$ … being defined for independent risks“ (p. 420).

In the above equations, the different elements are defined by:

$$\begin{aligned}
&E_{\tilde{y}_1,\tilde{\varepsilon}} v_1\left(\bar{s}(1+r)+\tilde{y}_1, H+\tilde{\varepsilon}\right)=kpqv_1\left(\bar{s}(1+r)+y_1+x_2, H+\varepsilon_2\right)\\
&+q(1-kp)v_1\left(\bar{s}(1+r)+y_1+x_1, H+\varepsilon_2\right)\\
&+p(1-kq)v_1\left(\bar{s}(1+r)+y_1+x_2, H+\varepsilon_1\right)\\
&+(1-p-q+kpq)v_1\left(\bar{s}(1+r)+y_1+x_1, H+\varepsilon_1\right)
\end{aligned}\quad, \quad (4a)$$

$$\begin{aligned}
&E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}} v_1\bar{s}(1+r)+\tilde{y}_1, H+\tilde{\varepsilon} = pqv_1\left(\bar{s}(1+r)+y_1+x_2, H+\varepsilon_2\right)\\
&+q(1-p)v_1\left(\bar{s}(1+r)+y_1+x_1, H+\varepsilon_2\right)\\
&+p(1-q)v_1\left(\bar{s}(1+r)+y_1+x_2, H+\varepsilon_1\right)\\
&+(1-p)(1-q)v_1\left(\bar{s}(1+r)+y_1+x_1, H+\varepsilon_1\right)
\end{aligned}\quad, \quad (4b)$$

$$\begin{aligned}
\Delta v_1 &= \left(v_1\left(\bar{s}(1+r)+y_1+x_2, H+\varepsilon_2\right)-v_1\left(\bar{s}(1+r)+y_1+x_1, H+\varepsilon_2\right)\right)\\
&-\left(v_1\left(\bar{s}(1+r)+y_1+x_2, H+\varepsilon_1\right)-\left(v_1\bar{s}(1+r)+y_1+x_1, H+\varepsilon_1\right)\right)
\end{aligned}. \quad (4c)$$

Courbage and Rey (2007, p. 421) claim that equation (1) above, which is equation 7 in their paper, can be rewritten as:

$$\hat{F}_1(\bar{s}, H)=\hat{F}_1\left(s^*, H\right)-(1+r)(k-1)pq\Delta v \quad (5)$$

which is their equation 8, where s^* solves the equation

$$u_1\left(y_0-s^*, H\right)=E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}} v_1\left(s^*(1+r)+\tilde{y}_1, H+\tilde{\varepsilon}\right). \quad (6)$$

which corresponds to their equation 2 on p. 419. The original equation (8) of their paper wrongly uses an undefined concept s_1^*, but it is obvious from the context that the correct version meant by the authors is as above. Given the definition of $\hat{F}_1$ provided by equation 5 of Courbage and Rey (2007, p.419), it is clear that

$$\hat{F}_1\left(s^*, H\right)=(1+r)\left(E_{\tilde{\varepsilon}} v_1\left(s^*(1+r)+E\tilde{y}_1, H+\tilde{\varepsilon}\right)-E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}} v_1\left(s^*(1+r)+\tilde{y}_1, H+\tilde{\varepsilon}\right)\right) \quad (7)$$

which is generally different from $(1+r)\left(E_{\tilde{\varepsilon}} v_1\left(\bar{s}(1+r)+E\tilde{y}_1, H+\tilde{\varepsilon}\right)-E^{ind}_{\tilde{y}_1,\tilde{\varepsilon}} v_1\left(\bar{s}(1+r)+\tilde{y}_1, H+\tilde{\varepsilon}\right)\right)$, since it is only under independence (k=1) that $\bar{s}=s^*$. Therefore equation 8 of Courbage and Rey (2007, p. 421), which is equation (5) above, is incorrect. A correct formulation would have been:

$$\hat{F}_1(\bar{s}, H)=\hat{F}_1^{ind}(\bar{s}, H)-(1+r)(k-1)pq\Delta v \quad (8)$$

with

$$\hat{F}_1^{ind}(\bar{s},H)=(1+r)\left(E_{\tilde{\varepsilon}}v_1\left(\bar{s}(1+r)+E_{\tilde{y}_1}\tilde{y}_1,H+\tilde{\varepsilon}\right)-E_{\tilde{y}_1,\tilde{\varepsilon}}^{ind}v_1\left(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon}\right)\right). \quad (9)$$

Then the rest of the proof must be formulated the following way. It is only under independence[2] that $E_{\tilde{y}_1|\tilde{\varepsilon}}\tilde{y}_1=E_{\tilde{y}_1}\tilde{y}_1$. Therefore it may be written that $E_{\tilde{y}_1}\tilde{y}_1=E_{\tilde{y}_1|\tilde{\varepsilon}}^{ind}\tilde{y}_1$ where $E_{\tilde{y}_1|\tilde{\varepsilon}}^{ind}\tilde{y}_1$ is the conditional expectation computed in the particular case where risks are independent (k=1):

$$\begin{aligned}\hat{F}_1^{ind}(\bar{s},H)&=(1+r)\begin{pmatrix}E_{\tilde{\varepsilon}}v_1\left(\bar{s}(1+r)+E_{\tilde{y}_1|\tilde{\varepsilon}}{}^{ind}\tilde{y}_1,H+\tilde{\varepsilon}\right)\\-E_{\tilde{y}_1,\tilde{\varepsilon}}^{ind}v_1\left(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon}\right)\end{pmatrix}\\&=(1+r)E_{\tilde{\varepsilon}}\begin{pmatrix}v_1\left(\bar{s}(1+r)+E_{\tilde{y}_1|\tilde{\varepsilon}}{}^{ind}\tilde{y}_1,H+\tilde{\varepsilon}\right)\\-E_{\tilde{y}_1|\tilde{\varepsilon}}^{ind}v_1\left(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon}\right)\end{pmatrix}\end{aligned} \quad (10)$$

which implies that $\hat{F}_1^{ind}(\bar{s},H)>(=,<)0$ if $v_{111}<(=,>)0$ since for any value of $\tilde{\varepsilon}$, using Jensen inequality, $v_1\left(\bar{s}(1+r)+E_{\tilde{y}_1|\tilde{\varepsilon}}{}^{ind}\tilde{y}_1,H+\tilde{\varepsilon}\right)>(=,<)E_{\tilde{y}_1|\tilde{\varepsilon}}^{ind}v_1\left(\bar{s}(1+r)+\tilde{y}_1,H+\tilde{\varepsilon}\right)$ if $v_{111}<(=,>)0$. The term Δv_1 expresses how an increase of $\tilde{\varepsilon}$ (from ε_1 to ε_2) affects the reaction of v_1 to an increase of $\tilde{y}_1$ (from y_1+x_1 to y_1+x_2). Since the reaction of v_1 to $\tilde{y}_1$ is governed by v_{11}, the variation of this reaction due to $\tilde{\varepsilon}$ is driven by v_{112}. This is why $\Delta v_1>(=,<)0$ is equivalent to $v_{112}>(=,<)0$ as pointed out by Courbage and Rey (2007).

Menegatti (2009) draws the correct conclusions of all these results. However this paper keeps the formulation of Courbage and Rey (2007) in proposition 2 according to which the found conditions are "*necessary and sufficient for any introduction of a non-financial Bernouillan risk to have a precautionary motive*". It seems to be a misleading interpretation of the results. Indeed these papers find conditions under which the introduction of an income risk increases saving, in the presence of a non financial background risk, as compared to a situation where income is certain.

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[2] It is true for any distribution and thus for a Bernouilli distribution. $E_{\tilde{y}_1}\tilde{y}_1=y_1+(1-p)x_1+px_2$ but $E_{\tilde{y}_1|\tilde{\varepsilon}=\varepsilon_1}\tilde{y}_1=y_1+\left(\dfrac{1-p-q+kpq}{1-q}\right)x_1+px_2$ and $E_{\tilde{y}_1|\tilde{\varepsilon}=\varepsilon_2}\tilde{y}_1=y_1+(1-kp)x_1+kpx_2$. It is for k=1 that $E_{\tilde{y}_1}\tilde{y}_1=E_{\tilde{y}_1|\tilde{\varepsilon}=\varepsilon_1}\tilde{y}_1=E_{\tilde{y}_1|\tilde{\varepsilon}=\varepsilon_2}\tilde{y}_1$.

3 The definition of first-degree stochastic correlation

Courbage and Rey (2007) propose the following definition of first-order stochastic correlation:

Definition 1 *Consider a pair of random variables ($\tilde{x}$, $\tilde{\varepsilon}$) with marginal cdf G for $\tilde{\varepsilon}$ and cdf F for $\tilde{x}$ conditional to ε. We say that there is a positive* (*negative*) *FSC correlation between $\tilde{x}$ and $\tilde{\varepsilon}$ if F is non increasing* (*decreasing*) *in x for all ε.* (Courbage and Rey 2007, p. 421)

This definition is incorrect. A cumulative distribution function, whether marginal or conditional, is always non decreasing in its argument. Here $F(x|\varepsilon) = \Pr(\tilde{x} < x|\tilde{\varepsilon} = \varepsilon)$. By definition it never decreases when x increases, whatever the value of ε. It is thus always non decreasing in x, for any value of ε. The intuitive meaning of positive first-degree stochastic correlation is that, when the realization ε of $\tilde{\varepsilon}$ increases, $\Pr(\tilde{x} < x|\tilde{\varepsilon} = \varepsilon)$ decreases or remains constant for any value of x. In addition it is useless to define the marginal cdf of $\tilde{\varepsilon}$ to define this concept of FSC. This definition should thus be reformulated as follows:

Definition 1 *Consider a pair of random variables ($\tilde{x}$, $\tilde{\varepsilon}$) with cdf $F(x|\varepsilon)$ for $\tilde{x}$ conditional to ε. We say that there is a positive* (*negative*) *FSC correlation between $\tilde{x}$ and $\tilde{\varepsilon}$ if $F(x|\varepsilon)$ is non increasing* (*decreasing*) in *ε for all x.*

This definition corresponds to the concept[3] of positive (negative) regression dependence of Tukey (1958). It implies that $\mathrm{cov}(\tilde{x},\tilde{\varepsilon})$ is positive (negative). However a positive (negative) covariance is less restrictive than positive (negative) regression dependence or FSC. This concept is re-examined by Lehmann (1966) who provides an easy interpretation: $\tilde{x}$ is positively (negatively) regression dependent on $\tilde{\varepsilon}$ means that knowledge of ε being large increases (decreases) the probability of x being large.

[3] The distribution of x given y shows complete negative (positive) regression dependence on y if $F_{\tilde{x}|\tilde{y}}(x|y_2) \leq F_{\tilde{x}|\tilde{y}}(x|y_1)$ for $y_2 \leq y_1$.

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