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How Optimism Leads to Price Discovery and Efficiency in a Dynamic Matching Market*

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Abstract

We study a market search equilibrium with aggregate uncertainty, private information and heterogeneus beiefs. Traders initially start out optimistic and then update their beliefs based on their matching experience in the market, using the Bayes rule. It is shown that all separating equilibria converge to perfect competition in the limit as the time between matches tends to 0. We also establish existence of a separating equilibrium.

KEYWORDS: Markets with search frictions, aggregate uncertainty, heterogeneous beliefs, optimism, bargaining, foundations of Walrasian equilibrium

1 Introduction

Stories about how market prices reveal information and implement efficient allocation of resources are of general interest to economists. Textbook discussions often invoke a Walrasian auctioneer in support of the price taking paradigm, yet he or she is notably absent in most real markets, which are often decentralized and involve search frictions.

Models in which traders search, meet pairwise and bargain have proven very useful for our understanding how many decentralized markets operate.[1] It is

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[1] For example, such models have been used to study labor markets, housing markets and over-the-counter financial markets; see Nobel Prize Report (2010) for many illuminating examples of markets with search frictions.

well known that the frictions of search and private information make an efficient allocation in general unattainable even when there are many buyers and sellers in the market. However, an efficient allocation may arise in the limit as frictions vanish, implying that markets with small frictions are approximately Walrasian and efficient.

This paper proposes a new way of looking at price formation in decentralized markets with *aggregate uncertainty*, an important feature of many decentralized markets.[2] In this paper, we consider the simplest form of aggregate uncertainty: even though traders know their own willingness to pay for the good (have *private values*), initially they don't know the aggregate market demand and supply. As a consequence, the traders need to learn at what prices to trade through their market experiences.

We present a tractable model of a bilateral search market with rich two-sided private information and aggregate uncertainty, in which learning occurs through *being not matched* and efficiency obtains in the limit as frictions vanish. Our model is based on a private-information replica of Mortensen and Wright (2002) using a notion of search market equilibrium with private information introduced in Satterthwaite and Shneyerov (2007, 2008).[3]

Continua (or oceans) of buyers and sellers arrive to the market each period $..., -1, 0, 1, ...$ The period length is a small $\tau > 0$. Each trader can trade a single unit of an indivisible good, and there are search frictions due to time discounting and participation cost. These frictions are proportionate to τ and vanish as $\tau \to 0$. In the beginning of each period, the traders who are in the market are matched in pairs and make take it or leave it offers to each other with equal probability.[4] The bargaining transpires under two-sided *private information.* If the bargaining results in trade, both traders leave the market, else the current match is dissolved and they remain.

We assume that the aggregate uncertainty of the market is represented by a state $\mu \in \{L, H\}$, where $0 < L < H$. This corresponds to the popular press notions of a buyer or seller markets. The H and L values of μ reflect a high or low value of the Walrasian price $p_W(\mu)$, with $p_W(L) < p_W(H)$. Formally, the distributions of valuations v and costs c, $G_B(\cdot|\mu)$ and $G_S(\cdot|\mu)$, depend on μ and the Walrasian price $p_W(\mu)$ is determined through the intersection of the corresponding demand and supply functions:

$$G_S(p_W|\mu) = 1 - G_B(p_W|\mu).$$

In our model, while traders are rational and Bayesian, their beliefs are *not* derived from a common prior.[5] Regardless of the true state μ, the entering

[2] For example, see Rogerson, Shimer, and Wright (2005) for a discussion of its relevance in labor markets.

[3] Such a model has recently been investigated by Shneyerov and Wong (2010a,b).

[4] This is the random-proposer protocol of Rubinstein and Wolinsky (1985). Several papers in the literature have considered other bargaining protocols, notably the k-double auction (k-DA) with grid-restricted price offers. Some references are given below. However, for a k-DA with unrestricted price offers, Shneyerov and Wong (2010b) show existence of non-convergent equilibria even without aggregate uncertainty as here.

[5] With a common prior concerning μ, the analysis of dynamic matching and bargaining

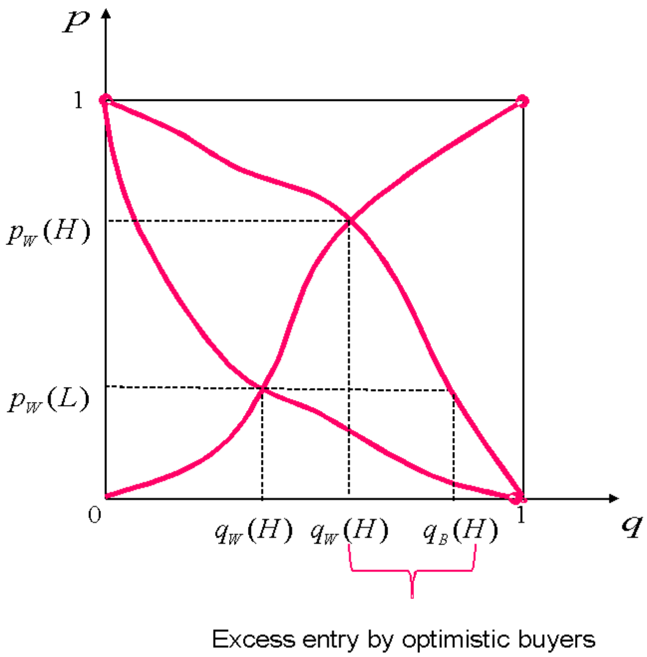

Figure 1: Fundamental market imbalance caused by excess entry of optimistic buyers when $\mu = H$.

buyers believe that they are in the buyer market, $\mu = L$, while entering sellers believe they are in the seller market, $\mu = H$. In other words, similar to Yildiz (2003), traders on both sides of the market start out with *optimistic* prior beliefs, and therefore initially agree to disagree about the state of the market. As they continue in the market, they may conclude that their original beliefs are wrong and switch to *pessimistic* beliefs: the buyers to believing that $\mu = H$, while the sellers, to believing $\mu = L$.

How do the traders discover the true state? The crucial element of our learning process is a simple matching technology whereby each period the shorter side of the market is fully matched, while the longer side is matched with probability less than 1. Our belief updating mechanism then implies that there is excessive entry by the traders who believe in the wrong state. This in turn will lead to an unbalanced market, with wrong believers on the long side. Once not matched, the wrong believers will update their optimistic beliefs, become pessimistic and trade.

Thus, in addition to the *negative* same-side search externality, in our model there is also a *positive* same-side externality because, say, additional buyers speed up the learning process of all buyers. See Figure 1 that graphically describes this basic story when τ is very small, the true state is $\mu = H$ and it is therefore the optimistic buyers who hold the wrong beliefs.

Our analysis focuses on what we call *separating equilibria*. In these equilibria, only traders who share the same (true) belief about the state μ, can trade. When

games with aggregate uncertainty is difficult due to a complicated nature of Bayes beliefs. For example, traders would need to update beliefs not only about the true state, but also about the beliefs of other traders etc. There is no parsimonious notion of a state variable describing beliefs. However, the common prior assumption is neither necessary nor always desirable.

the beliefs diverge, the traders are unable to reach mutually acceptable price and are unable to trade. Consequently, optimistic buyers can only trade with pessimistic sellers, and vice versa. When τ is small, there are incentives in place to support such an equilibrium. For example, buyers who are optimistic will prefer to wait for a pessimistic seller rather than trade with an optimistic seller at a higher price. This is because the cost of waiting will be small relative to the benefit of a lower price. Similar logic applies to the optimistic sellers. On the other hand, the pessimistic traders will have known the state and will not wait for a deal that they are sure does not exist in the market. The only delay they might have comes from private information in bargaining.

With small frictions, we show that traders who have a correct belief μ will propose or accept a price close to $p_W(\mu)$. This means two things. First, the former optimists on the long side with valuations far below (or costs far above) $p_W(\mu)$ will exit. Second, the pessimistic traders on the long side now share the same beliefs with optimistic traders on the short side and will trade with them. Provided that the state discovery by optimists is quick, their stock in the market is small, and price discovery happens quickly followed by trade (almost) at the right price $p_W(\mu)$. Thus we prove that, as $\tau \to 0$, all separating steady-state equilibria converge to the Walrasian outcome in state μ. The traders' utilities converge to their Walrasian counterparts, as if they knew the true state from the beginning. In the limit, there is both full information revelation and efficiency.

In addition to this convergence result, we also show that when both the discount rate and τ are sufficiently small, there exists a unique separating equilibrium with a *full trade* property: every meeting between the traders who share the same belief about the state results in trade.

We are unaware of any published papers that have obtained convergence results under aggregate uncertainty with private values as here. Recently, Lauermann, Merzyn, and Virag (2010) have also considered a model with aggregate uncertainty. As in Satterthwaite and Shneyerov (2008), in their model the sellers conduct auctions among the buyers they are matched with. However, their model is different in that (i) buyers (sellers) are assumed to be homogeneous in their valuations (costs), and (ii) sellers are assumed to be non-strategic. The buyers do not know the state of the market, and then learn through unsuccessful bids. Over time, the buyers become more pessimistic and bid more aggressively. It is shown that the equilibrium allocation converges to the competitive allocation as friction vanish.

Several authors have considered steady-state models with *common value* uncertainty, and double auction bargaining with a grid restricted set of price offers. In such a model, Wolinsky (1988) assumes two-sided incomplete information and obtains a negative convergence result, while Serrano and Yosha (1993) assume one-sided incomplete information and show existence of a convergent equilibrium. In addition, Blouin and Serrano (2001) consider a market with one-time entry of agents and obtain strong negative results concerning convergence.[6] In

[6]However, recently, Gottardi and Serrano (2005) revisit the issue and obtain some positive results in a somewhat different model.

addition, the seminal contributions of Reny and Perry (2006) and Pesendorfer and Swinkels (1997, 2000) provide foundations for a rational expectations equilibria in *static* models of *centralized* double-auction trade with interdependent values.

Most other papers have adopted a private values paradigm with no aggregate uncertainty; a non-exhaustive list includes Butters (1979), Gale (1986), Gale (1987), Gale (2000), Rubinstein and Wolinsky (1985),Wolinsky (1988), Wolinsky (1990), Rubinstein and Wolinsky (1990), McLennan and Sonnenschein (1991),Dagan, Serrano, and Volij (1998), Dagan, Serrano, and Volij (2000), De Fraja and Sakovics (2001), Moreno and Wooders (2002), Serrano (2002), Mortensen and Wright (2002), Satterthwaite and Shneyerov (2007), Satterthwaite and Shneyerov (2008), Atakan (2009), Lauermann (2009), and Shneyerov and Wong (2010a,b).

The structure of the paper is as follows. Section 2 introduces the model. Section 3 presents the convergence result within a more tractable class of full trade equilibria, and also establishes the existence of equilibria for small τ. Section 4 extends the convergence result to all separating equilibria.

Finally, to lend further support to extreme optimistic beliefs, in the concluding Section 5 we show that the model can be extended to include a prior choice stage based on the "change of paradigm" framework recently proposed in Ortoleva (2010). In this extended model's equilibrium, we argue that traders will *choose* their priors to be optimistic.

2 Model and Theorem

We study the steady state of a market with two-sided incomplete information and an infinite horizon. In it heterogeneous buyers and sellers meet once per period ($t = \ldots, -1, 0, 1, \ldots$) and trade an indivisible, homogeneous good. The length of each period is τ. At the beginning of each period measure τ of sellers and buyers is born and the newborn traders contemplate entering the market.

The agents in our model are potential buyers and sellers of a homogeneous, indivisible good. Each buyer has a unit demand for the good, while each seller has unit supply. All traders are risk neutral. Potential buyers are heterogeneous in their valuations (or types) v of the good. Potential sellers are also heterogeneous in their costs (or types) c of providing the good. For simplicity, we assume $v, c \in [0, 1]$.

We now introduce the main element of our model, the state of the market μ. The sate μ can take two values, high (H) and low (L) and is drawn by nature once and for all. The H and L values of μ reflect a high or low value of the Walrasian price $p_W(\mu)$, $p_W(L) < p_W(H)$. Formally, the distributions of valuations v and costs c, $G_B(\cdot|\mu)$ and $G_S(\cdot|\mu)$, depend on μ and the Walrasian price $p_W(\mu)$ is determined through the intersection of the corresponding demand and supply functions:

$$G_S(p_W|\mu) = 1 - G_B(p_W|\mu).$$

Each trader *privately* knows his valuation v if he is a buyer or cost c if he is a seller. However, the traders do not observe μ. The prior distributions of (v, c, μ) are different for buyers and sellers. Specifically, we assume that the prior distributions of μ put all the weight on $\mu = L$ for the buyers and $\mu = H$ for the sellers, while the conditional distributions $G_B(\cdot|\mu)$ and $G_S(\cdot|\mu)$ are the same. We also assume that these distributions have densities $g_B(\cdot|\mu)$ and $g_S(\cdot|\mu)$ that are supported on $[0, 1]$ and uniformly bounded from below there,

$$\inf_{v\in[0,1]} g_B(v|\mu) \equiv \underline{g}_B > 0, \quad \inf_{c\in[0,1]} g_S(c|\mu) \equiv \underline{g}_S > 0.$$

The instantaneous discount rate is $r \geq 0$, and the corresponding discount factor is $R_\tau = e^{-r\tau}$. Each period consists of the following stages.

1. The mass τ of potential buyers and sellers are born. Conditional on the true state of the market $\mu \in \{H, L\}$, the new-born buyers draw their valuations v i.i.d. from $G_B(\cdot|\mu)$ and the newborn sellers draw their costs c i.i.d. from $G_S(\cdot|\mu)$.

2. Entry (or participation, or being active): The new-born potential buyers and sellers decide whether to enter the market. Those who enter together with the current pools of traders in the market compose the set of active traders.

3. The active buyers and sellers incur participation costs $\tau\kappa$.

4. The active buyers and sellers are randomly matched in pairs. The shorter side of the market is matched completely, while the longer side is appropriately rationed. The mass of the matches is given by $\min\{B(\mu), S(\mu)\}$, where $B(\mu)$ and $S(\mu)$ are the steady-state masses of active buyers and active sellers currently in the market. The probability that a buyer is matched is $\min\left\{1, \frac{S(\mu)}{B(\mu)}\right\}$, and he is equally likely to meet any active seller. Symmetrically, the seller's matching probability is $\min\left\{1, \frac{B(\mu)}{S(\mu)}\right\}$, and she is equally likely to meet any active buyer. The matching is anonymous.

5. If a type v buyer and a type c seller trade at a price p, then they leave the market with payoff $v - p$, and $p - c$ respectively. If bargaining between the matched pair breaks down, both traders can either stay in the market waiting for another match as if they were never matched, or simply exit and never come back.

6. Bargaining: Once a pair of buyer and seller is matched, they bargain without observing the type of their partner. The bargaining protocol is *random-proposal take-it-or-leave-it offer*: with probability 1/2, the seller makes a take-it-or-leave-it offer to the buyer, then the buyer chooses either to accept or reject. And with probability 1/2, the buyer proposes and the seller responds. We also assume the market is anonymous, so that the

bargainers do not know their partners' market history, e.g. how long they have been in the market, what they proposed previously, and what offers they rejected previously.

Our notion of market search equilibrium parallels that of Satterthwaite and Shneyerov (2007) and Shneyerov and Wong (2010a,b). In a market equilibrium, traders take search (continuation) values W_B, W_S of traders and market distributions of their types, Φ, Γ, as given, i.e. unaffected by their own actions. The search values determine the participation (entry) strategies of buyers and sellers, and their responding strategies $\tilde{v}, \tilde{c}$ in any given meeting. The responding strategies and the market type distributions Φ, Γ together determine the best-response price offer strategies of buyers and sellers. Our exposition in this section focuses on the differences pertinent to our new model.

Heterogeneous beliefs play a key role in our model. Let $\mu_B, \mu_S \in \{L, H\}$ be the belief of a buyer or seller about μ. In the equilibrium we are describing, μ_B, μ_S will only take two values L or H. Denote the stock of active buyers who hold a belief μ_B when the true state is μ as $B(\mu_B|\mu)$. Likewise, the stock of sellers with a belief μ_S when the true state is μ is denoted as $S(\mu_S|\mu)$. Note that $B(\mu) = B(L|\mu) + B(H|\mu)$ and $S(\mu) = S(L|\mu) + S(H|\mu)$. Obviously, there will be a distribution of the types (valuations for buyers and costs for sellers) in the above mentioned stocks of traders in the steady state. Whenever the corresponding stocks are positive we denote the distribution of active buyer and seller types as $\Phi(\cdot|\mu_B, \mu)$ and $\Gamma(\cdot|\mu_S, \mu)$.[7]

Belief Updating Mechanism. The newborn traders are assumed to start out *optimistically*, $\mu_B = L$ for the buyers and $\mu_S = H$ for the sellers. The traders will only change their beliefs if they did not succeed in meeting a partner in the previous period. If that happened, the traders will switch to pessimistic beliefs, i.e. buyers will have $\mu_B = H$, while the sellers will have $\mu_S = L$. These beliefs are fully Bayesian and consistent, provided the equilibrium satisfies the following fundamental imbalance condition.

[7]In our equilibrium, trader's strategies are functions of their first-order beliefs about the state (and of their own types). They do not include explicitly higher-order beliefs about μ. In our market context, the agents are non-atomic and therefore face 0 probability of meeting with each other in the market again. So in our model, the relevant higher-order beliefs would be necessarily about the state of the market, i.e. the buyers' beliefs about the market distribution of sellers beliefs' about μ, the buyers' beliefs about the sellers' beliefs about the distribution of buyers' beliefs about μ in the market, and so on. *We assume that these beliefs are true, i.e. that they coincide with the true equilibrium market distributions.* For example, consider optimistic buyers, $\mu_B = L$. Their (second-order) beliefs about the sellers' beliefs are assumed to be derived from the steady-state equilibrium seller stocks $S(H|L)$ and $S(L|L)$. The buyers' third-order beliefs are assumed to be equal to $B(L|H)$ and $B(H|H)$ for the stock of sellers with belief $\mu_S = H$ and to $B(L|L)$ and $B(H|L)$ for the stock of sellers with belief $\mu_S = L$.

Condition 1 (Fundamental Imbalance Condition) *The optimists with wrong beliefs about the state are on the long side of the market:*

$$B(H) > S(H), \quad S(L) > B(L).$$

At the end of this section, we verify that our equilibrium in fact satisfies Condition 1. Under this condition, no meeting in any given period is a 0 prior probability event for any optimistic buyer or seller. If such event occurs, the Bayes rule does not apply and the trader's updated belief about μ can in principle be anything. *We assume that the trader will update to a pessimistic belief.*

Denote as $W_B(v|\mu_B)$ and $W_S(c|\mu_S)$ the beginning-of-period *subjective* market utilities of type v buyer and type c seller. These market utilities are defined for all traders and for all types $v, c \in [0,1]$, even for those who in equilibrium are not active. The strategy of participating in the market will depend on μ_i. In our equilibrium, the sets of participating buyer and seller types,

$$A_B(\mu_B) = \{v : W_B(v|\mu_B) \geq 0\}, \quad A_S(\mu_S) = \{c : W_S(c|\mu_S) \geq 0\} \tag{1}$$

are intervals,

$$A_B(\mu_B) = [\underline{v}(\mu_B), 1], \quad A_S(\mu_S) = [0, \bar{c}(\mu_S)].$$

The types $\underline{v}(\mu_B)$ and $\bar{c}(\mu_S)$ are called marginal participating types of buyers and sellers.

As in Satterthwaite and Shneyerov (2007) and Shneyerov and Wong (2010a,b), the market utilities are taken as exogenous to the stage bargaining game, and are determined in the market equilibrium. If we normalize the no trade outcome as yielding 0 utilities to the traders, the maximal price and the minimal price that the buyer and seller are willing to accept respectively, or in other words their dynamic types (similar to Satterthwaite and Shneyerov (2007)), will be

$$\tilde{v}(v|\mu_B) = v - R_\tau W_B(v|\mu_B), \quad \tilde{c}(c|\mu_S) = c + R_\tau W_S(c|\mu_S). \tag{2}$$

This is because say a buyer with valuation v will, in a Perfect Bayesian equilibrium and given his belief μ_B, accept any price p such that $v - p \geq R_\tau W_B(v|\mu_B)$. The market distributions of $\tilde{c}(c|\mu_S)$ and $\tilde{v}(v|\mu_B)$ in state μ are

$$\tilde{\Gamma}(c|\mu_S, \mu) \equiv \int_{\{x:\tilde{c}(x|\mu_S)\leq c\}} d\Gamma(x|\mu_S, \mu), \quad \tilde{\Phi}(v|\mu_B, \mu) \equiv \int_{\{x:\tilde{v}(x|\mu_B)\leq v\}} d\Phi(x|\mu_B, \mu).$$

We require that $\tilde{c}(\cdot|\mu_S)$ and $\tilde{v}(\cdot|\mu_B)$ are non-decreasing functions, and only consider *separating* equilibria, i.e. those in which even the most enthusiastic optimist buyers cannot hope to trade even with the least enthusiastic optimist sellers. In other words, in our equilibrium traders only trade with partners who hold the same belief about the state of the market; i.e. optimistic buyers only trade with pessimistic sellers and vice versa. Formally, we require the following separation property to hold.

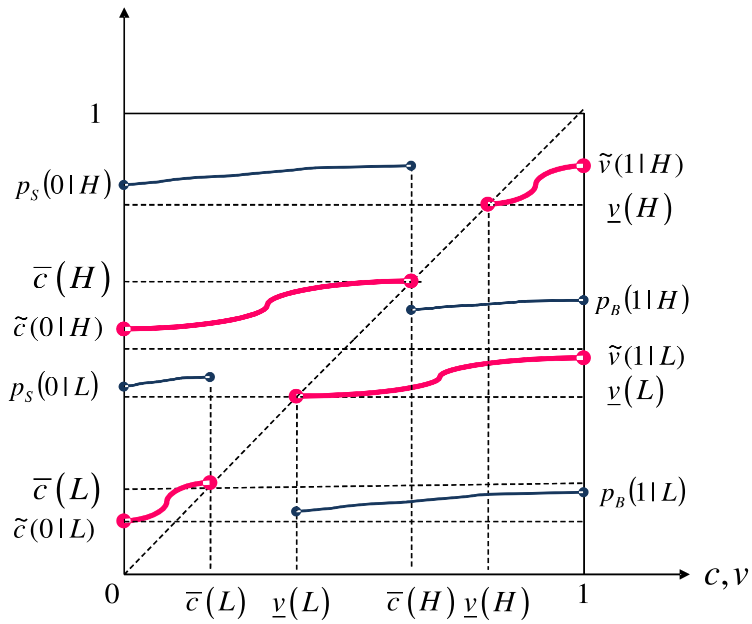

Figure 2: Separating equilibrium.

Assumption 1 (Separation Property) *In a separating equilibrium, we have*

$$\tilde{v}\,(1|L) < \tilde{c}\,(0|H)\,. \tag{3}$$

The separation property implies

$$\begin{aligned} \underline{v}(L) &< \tilde{v}(1|L) < \tilde{c}(0|H) < \underline{v}(H), \\ \bar{c}(L) &< \tilde{v}(1|L) < \tilde{c}(0|H) < \bar{c}(H). \end{aligned}$$

Figure 2 depicts a separating equilibrium of our model.

Each trader optimally chooses his or her proposed price within the support of the distribution of the dynamic types of the partners who share with the trader the same belief about the state,

$$p_B\,(v|\mu_B) \in \arg\max_{p\in[0,1]} \frac{S\,(\mu_B|\mu_B)}{S(\mu_B)}\,(\tilde{v}\,(v|\mu_B) - p)\,\tilde{\Gamma}\,(p|\mu_B,\mu_B) \tag{4}$$

$$p_S\,(c|\mu_S) \in \arg\max_{p\in[0,1]} \frac{B\,(\mu_S|\mu_S)}{B(\mu_S)}(p - \tilde{c}\,(c|\mu_S))[1 - \tilde{\Phi}\,(p|\mu_S,\mu_S)]. \tag{5}$$

Notice that in equilibrium, the buyers choose prices below their dynamic types, while the sellers choose prices above their dynamic types

$$p_B\,(v|\mu_B) \leq \tilde{v}\,(v|\mu_B)\,, \quad p_S\,(c|\mu_S) \geq \tilde{c}\,(c|\mu_S)\,.$$

Let $U_B(v|\mu_B)$ and $U_S(c|\mu_S)$ be the expected utilities in the bargaining game, over and above the market values:

$$U_B(v|\mu_B) = \frac{1}{2}\frac{S(\mu_B|\mu_B)}{S(\mu_B)} \cdot \{(\tilde{v}(v|\mu_B) - p_B(v|\mu_B))\tilde{\Gamma}(p_B(v|\mu_B)|\mu_B,\mu_B) + \int_{\{c: p_S(c|\mu_B)\le \tilde{v}(v|\mu_B)\}} (\tilde{v}(v|\mu_B) - p_S(c|\mu_B)) d\Gamma(c|\mu_B,\mu_B)\} \tag{6}$$

$$U_S(c|\mu_S) = \frac{1}{2}\frac{B(\mu_S|\mu_S)}{B(\mu_S)} \cdot \{(p_S(c|\mu_S) - \tilde{c}(c|\mu_S))(1 - \tilde{\Phi}(p_S(c|\mu_S)|\mu_S,\mu_S)) + \int_{\{v: p_B(v|\mu_S)\ge \tilde{c}(c|\mu_S)\}} (p_B(v|\mu_S) - \tilde{c}(c|\mu_S)) d\Phi(v|\mu_S,\mu_S)\} \tag{7}$$

We also call them the *interim utilities from trading.*

The intuition for these equations is as follows. For example, consider an optimistic buyer ($\mu_B = L$) who only trades with a pessimistic seller. The market proportion of pessimistic sellers is, from the buyer's point of view, $\frac{S(\mu_B|\mu_B)}{S(\mu_B)}$. With probability 1/2, the buyer is the proposer with price $p_B(v|\mu_B)$. The buyer will make a surplus of $\tilde{v}(v|\mu_B) - p_B(v|\mu_B)$ over and above the market continuation value if the offer is accepted by the seller, which happens with probability $\tilde{\Gamma}(p_B(v|\mu_B)|\mu_B,\mu_B)$ (again from the buyer's point of view). Alternatively, with probability 1/2, it is the pessimistic seller who is the proposer, with price $p_S(c|\mu_B)$. Such a price is accepted whenever $p_S(c|\mu_B) \le \tilde{v}(v|\mu_B)$.

With these in hand, we now write the *subjective* Bellman equations for $W_B(v|\mu_B)$ and $W_S(c|\mu_S)$.

$$W_B(v|\mu_B) = \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} U_B(v|\mu_B) + R_\tau \max\{W_B(v|\mu_B), 0\} - \tau\kappa, \tag{8}$$

$$W_S(c|\mu_S) = \min\left\{1, \frac{B(\mu_S)}{S(\mu_S)}\right\} U_S(c|\mu_S) + R_\tau \max\{W_S(c|\mu_S), 0\} - \tau\kappa. \tag{9}$$

The sets of participating types $A_B(\mu_B)$ and $A_S(\mu_S)$ are determined according to (1). On the equilibrium path, $W_B(v|\mu_B) \ge 0$. Off the equilibrium path, $W_B(v|\mu_B) < 0$ and a buyer who entered will have his search cost sunk for one period and will exit by the end of the period. Likewise for the sellers.

The *subjective* Bellman equations require some explanation. The first thing to notice is that the traders have *point* belief. So a buyer who is optimistic believes with probability one that the true state is L. Given her current belief she believes that in the next period, the state will be the same as her believed state with probability 1. Therefore, in the subjective continuation pay-off, the posterior belief is the *same* as the prior. Thus we have a very simple Markovian structure where the "state variable" is the traders' belief about the state of the market.

To close the description of our equilibrium, we provide the steady-state equations for the distributions of trader types in the market. To state these equations, we need the true trading probabilities $q_B(v|\mu_B,\mu)$ and $q_S(c|\mu_S,\mu)$ in state μ for buyers and sellers with beliefs μ_B and μ_S respectively. Recall that in our equilibrium, traders only trade with partners who share the same, correct belief about the state. The trading probabilities are determined in parallel to (6) and (7),

$$q_B(v|\mu_B,\mu) = \frac{1}{2}\frac{S(\mu_B|\mu)}{S(\mu)} \cdot \left\{\tilde{\Gamma}\left(p_B\left(v|\mu_B\right)|\mu_B,\mu\right) + \int_{\{c:p_S(c,\mu_B)\leq\tilde{v}(v|\mu_B)\}} d\Gamma\left(c|\mu_B,\mu\right)\right\} \tag{10}$$

$$q_S(c|\mu_S,\mu) = \frac{1}{2}\frac{B(\mu_S|\mu)}{B(\mu)} \cdot \left\{\left(1-\tilde{\Phi}\left(p_S\left(c|\mu_S\right)|\mu_S,\mu\right)\right) + \int_{\{v:p_B(v,\mu_S)\geq\tilde{c}(c|\mu_S)\}} d\Phi\left(v|\mu_S,\mu\right)\right\} \tag{11}$$

The steady state equations take the following form. Consider the buyers first. For the optimistic buyers ($\mu_B = L$),

$$\tau \cdot dG_B(v|\mu) = \left[\min\left\{1, \frac{S(\mu)}{B(\mu)}\right\} q_B(v|L,\mu) + \left(1 - \min\left\{1, \frac{S(\mu)}{B(\mu)}\right\}\right)\right] \cdot d\Phi(v|L,\mu)B(L|\mu) \tag{12}$$

For the pessimistic buyers ($\mu_B = H$),

$$\left[1 - \min\left\{1, \frac{S(\mu)}{B(\mu)}\right\}\right] B(L,\mu)d\Phi(v|L,\mu) \tag{13}$$

$$= \begin{cases} \min\left\{1, \frac{S(\mu)}{B(\mu)}\right\} q_B(v|H,\mu)d\Phi(v|H,\mu)B(H|\mu), & \text{if} \quad v \in [\underline{v}(H), 1] \\ d\Phi(v|H,\mu)B(H|\mu), & \text{if} \quad v \in [\underline{v}(L), \underline{v}(H)) \end{cases}$$

Let us now explain the above two equations in detail. For the first equation (12) the left-hand side is the per-period mass of buyer types v who enter the market when the true state is μ. The term $B(L|\mu)$ is the stock of optimistic buyers who are in the market in state μ. These are the buyers who initially hold the optimistic belief. The r.h.s. consists of two parts–the mass of optimistic buyer types v who are matched and trade successfully this period and the mass of optimistic buyer types v who are not matched and become pessimistic. The term $q_B(v|L,\mu)$ denotes the probability that a buyer of type v with belief L, successfully trades in the market. Recall that $\min\left\{1, \frac{B(\mu)}{S(\mu)}\right\}$ is the probability of a buyer being matched in state μ. The r.h.s. of equation (12) therefore, reflects the outflow of buyers from the mass of optimistic buyers in each period.

For the second equation (13) the l.h.s. is the mass of buyers per-period who have turned pessimistic. Some of these buyers will exit the market which happens if the valuation of a buyer $v \in [\underline{v}(L), \underline{v}(H))$. This explains the second term on the r.h.s. If, on the other hand, a buyer's valuation $v \in [\underline{v}(H), 1]$, the buyer stays in the market and then such a buyer will only exit through trade. This explains the first term on the r.h.s.

We now state the parallel equations for the sellers. For the optimistic sellers ($\mu_S = H$),

$$\tau \cdot dG_S(c|\mu) = \left[\min\left\{1, \frac{B(\mu)}{S(\mu)}\right\} q_S(c|H,\mu) + \left(1 - \min\left\{1, \frac{B(\mu)}{S(\mu)}\right\}\right)\right] \cdot d\Gamma(c|H,\mu)S(H|\mu) \tag{14}$$

For the pessimistic sellers ($\mu_S = L$),

$$\left[1 - \min\left\{1, \frac{B(\mu)}{S(\mu)}\right\}\right] S(H|\mu)d\Gamma(c|H,\mu) \tag{15}$$

$$= \begin{cases} \min\left\{1, \frac{B(\mu)}{S(\mu)}\right\} q_S(c|L,\mu)d\Gamma(c|L,\mu)S(L|\mu), & \text{if} \quad c \in [0, \bar{c}(L)] \\ d\Gamma(c|L,\mu)S(L|\mu), & \text{if} \quad c \in [\bar{c}(L), \bar{c}(H)) \end{cases}$$

Finally, the Fundamental Imbalance Condition follows from the following simple observation. Suppose the true state is $\mu = H$. The Separation Property implies that optimistic sellers trade with optimistic buyers (who have beliefs $\mu = H$). In a steady-state equilibrium with trade, therefore, there must be a non-empty stock of pessimistic buyers. Since such buyers can only arise if the optimistic buyers have meeting probability less than 1, this implies that $S(H) < B(H)$. Similar logic shows that $S(L) > B(L)$ when $\mu = L$..

2.1 What does convergence to perfect competition mean? Statement of the main theorem

The targets for convergence are the *Walrasian utilities* of the traders in state μ:

$$W_B^*(v|\mu) \equiv \max\{v - p_W(\mu), 0\}, \qquad W_S^*(c|\mu) \equiv \max\{p_W(\mu) - c, 0\}.$$

In a frictional market ($\tau > 0$), there is an unavoidable utility loss due to costly search and discounting, and buyers and sellers will realize smaller utilities. Denote the *true market utility* of a buyer (seller) with belief μ_B (μ_S) in state μ as $w_B(v|\mu_B,\mu)$ ($w_S(c|\mu_S,\mu)$). As in e.g. Satterthwaite and Shneyerov (2007), convergence to efficiency means that the utilities of the traders in the entering cohorts converge to their Walrasian levels in state μ.

Because the traders in the entering cohorts are optimistic, we only need to demonstrate convergence of the utilities of the optimistic traders. The optimistic

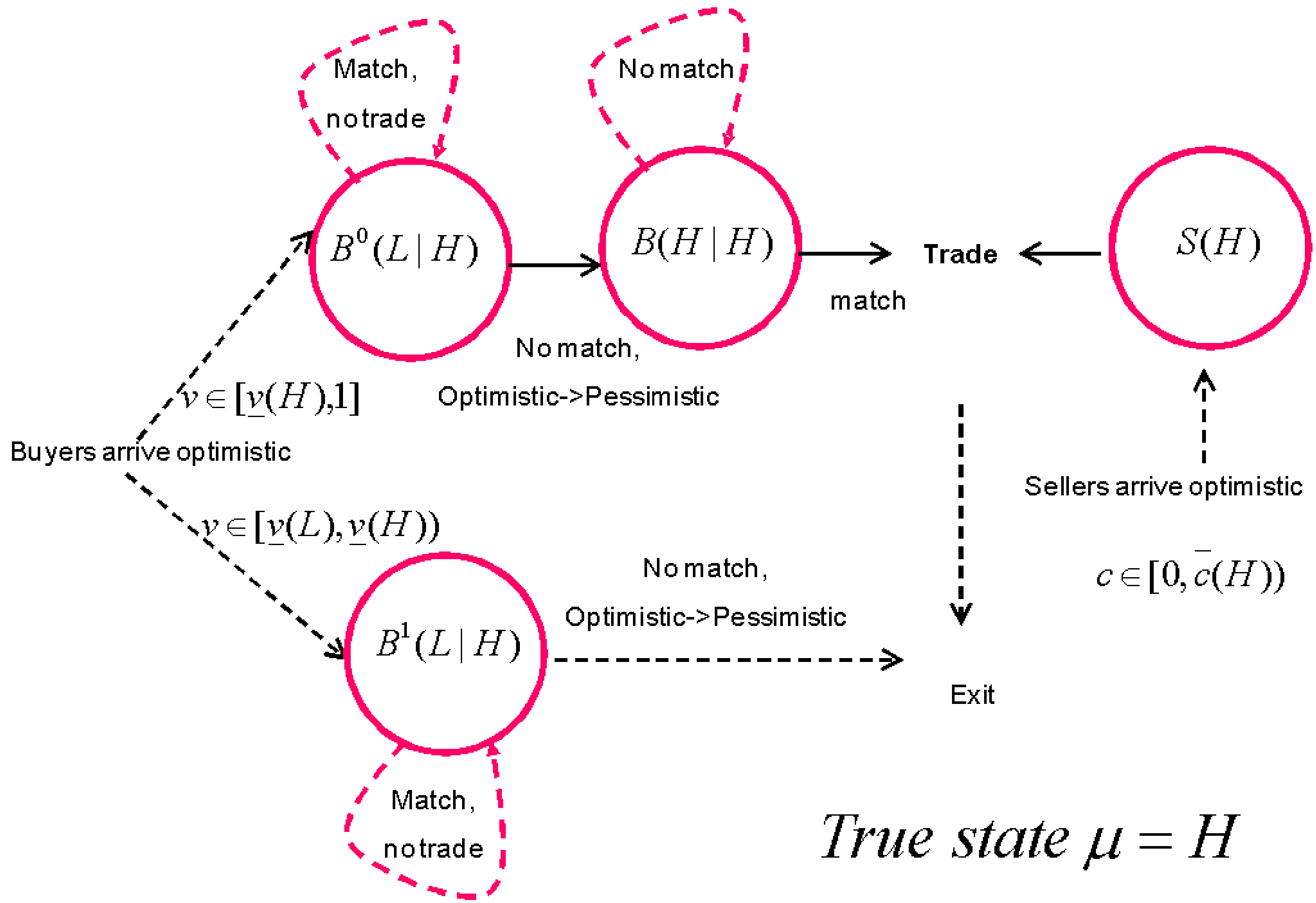

Figure 3: Steady-state flows

buyers or sellers with correct beliefs about the true state will have their market utilities equal to the subjective utilities,

$$\begin{aligned} w_B(v|L,L) &= \max\{W_B(v|L),0\}, \\ w_S(c|H,H) &= \max\left\{W_S\left(c|H\right),0\right\}, \end{aligned}$$

while the ones with wrong beliefs have their true utilities determined from the recursive equations

$$\begin{aligned} w_B\left(v|L,H\right) = R_\tau\left(1-\frac{S\left(H\right)}{B\left(H\right)}\right)\max\left\{W_B\left(v|H\right),0\right\} \\ + R_\tau\frac{S\left(H\right)}{B\left(H\right)}w_B\left(v|L,H\right)-\kappa\tau, \end{aligned} \tag{16}$$

$$\begin{aligned} w_S\left(c|H,L\right) = R_\tau\left(1-\frac{B\left(L\right)}{S\left(L\right)}\right)\max\left\{W_S\left(c|L\right),0\right\} \\ + R_\tau\frac{B\left(L\right)}{S\left(L\right)}w_S\left(c|H,L\right)-\kappa\tau. \end{aligned} \tag{17}$$

(Recall that in our equilibrium, the traders with wrong beliefs are on the long side of the market, so $\frac{S(H)}{B(H)}$, $\frac{B(L)}{S(L)} < 1$.) The intuition for (16) and (17) is that, first, optimistic traders with wrong beliefs do not trade in any meeting, and second, they learn the true state μ when they do not meet a partner in the present period. The latter event occurs with with probability $1-\frac{S(H)}{B(H)}$ for buyers when $\mu = H$ and with probability $1-\frac{B(L)}{S(L)}$ for sellers when $\mu = L$, and then the true market utilities coincide with the believed ones.

The following lemma provides additional equilibrium properties that are also useful for understanding our main Theorem 1 . Refer to Figure 2.

Lemma 1 *In any separating equilibrium,*

$$\tilde{c}\left(0|\mu_B\right) < \underline{v}\left(\mu_B\right), \quad \bar{c}\left(\mu_S\right) < \tilde{v}\left(1|\mu_S\right). \tag{18}$$

The expected mechanism payoffs for the marginal buyers and sellers are just sufficient to cover their expected search costs until the next meeting:

$$U_B\left(\underline{v}\left(\mu_B\right)\right) = \frac{\tau\kappa}{\min\left\{1,\frac{S(\mu_B)}{B(\mu_B)}\right\}}, \tag{19}$$

$$U_S\left(\bar{c}\left(\mu_S\right)\right) = \frac{\tau\kappa}{\min\left\{1,\frac{B(\mu_S)}{S(\mu_S)}\right\}}. \tag{20}$$

The Walrasian price $p_W(\mu)$ must be in between the marginal types:

$$p_W\left(\mu\right) \in \left[\min\left\{\bar{c}\left(\mu\right),\underline{v}\left(\mu\right)\right\},\max\left\{\bar{c}\left(\mu\right),\underline{v}\left(\mu\right)\right\}\right]. \tag{21}$$

The proposing strategies $p_B(\cdot|\mu)$ and $p_S(\cdot|\mu)$ are non-decreasing on $A_B(\mu_B)$ and $A_S(\mu_S)$ respectively. Moreover, $p_B(v|\mu_B) < \tilde{v}(v|\mu)$, $p_S(c|\mu) > \tilde{c}(c|\mu)$ and

$$p_B(v|\mu_B) \in [\tilde{c}(0|\mu_B), \bar{c}(\mu_B)], \quad p_S(c|\mu_S) \in [\underline{v}(\mu_S), \tilde{v}(1|\mu_S)].$$

The trading probability $q_B(v|\mu_B,\mu)$ is strictly positive and non-decreasing in v on $A_B(\mu_B)$, while $q_S(c|\mu_S,\mu)$ is strictly positive and non-increasing in c on $A_S(\mu_S)$.

The main result of our paper is the following theorem. To emphasize the dependence of equilibrium objects on τ, we will often index them by τ, e.g. $p_{B\tau}, p_{S\tau}$ etc.

Theorem 1 *As $\tau \to 0$, all separating equilibria converge to perfect competition: (a) the marginal participating types of buyers and sellers converge to the Walrasian prices that correspond to their beliefs,*

$$\underline{v}_\tau(\mu_B) \to p_W(\mu_B), \qquad \bar{c}_\tau(\mu_S) \to p_W(\mu_S),$$

(b) the prices $p_{B\tau}(v|\mu_B)$ and $p_{S\tau}(c|\mu_S)$ offered by buyers and sellers also converge to the Walrasian prices,

$$\sup_{v \in A_{B\tau}(\mu_B)} |p_{B\tau}(v|\mu_B) - p_W(\mu_B)| \to 0,$$
$$\sup_{c \in A_{S\tau}(\mu_S)} |p_{S\tau}(c|\mu_S) - p_W(\mu_S)| \to 0,$$

and (c) the market utilities of the entering optimistic traders $w_{B\tau}(v|\mu_B,\mu)$ and $w_{S\tau}(c|\mu_S,\mu)$ converge to the utilities that traders would realize under perfect competition,

$$\sup_{v \in [0,1]} |w_{B\tau}(v|\mu_B,\mu) - W_B^*(v|\mu)| \to 0,$$
$$\sup_{c \in [0,1]} |w_{S\tau}(c|\mu_S,\mu) - W_S^*(c|\mu)| \to 0.$$

Recursive equations (16) and (17) can be solved for the true market utilities to yield (indexing the objects by τ to emphasize their dependence on the length of the time period)

$$\begin{aligned} w_{B\tau}(v|L,H) &= \frac{R_\tau\left(1 - \frac{S_\tau(H)}{B_\tau(H)}\right)\max\{W_{B\tau}(v|H), 0\} - \kappa\tau}{1 - R_\tau \frac{S_\tau(H)}{B_\tau(H)}}, \\ w_{S\tau}(c|H,L) &= \frac{R_\tau\left(1 - \frac{B_\tau(L)}{S_\tau(L)}\right)\max\{W_S(c|L), 0\} - \kappa\tau}{1 - R_\tau \frac{B_\tau(L)}{S_\tau(L)}}. \end{aligned}$$

Consider $\mu = H$ and focus for concreteness on a buyer. As $\tau \to 0$, $R_\tau \to 1$, so the above equation for $w_{B\tau}$ implies that it converges to the Walrasian level,

$$w_{B\tau}(v|L,H) \quad \to \quad W_B^*(v|H),$$

if:

1. Conditional on meeting a seller, the terms of trade are (approximately) fixed as Walrasian, i.e. the price is fixed at $p_W(\mu)$;

2. For a pessimistic buyer, the probability of meeting a seller, $\frac{S_\tau(H)}{B_\tau(H)}$, in each period stays bounded away from 0 as $\tau \to 0$. This assures that the expected time until such meeting tends to 0; and

3. For an optimistic buyer, the probability of learning the true state in each period, $1-\frac{S_\tau(H)}{B_\tau(H)}$, does not vanish as $\tau \to 0$. This assures that the expected time until learning the true state tends to 0.

The proof of Theorem 1, contained in the following sections of the paper, essentially consists of the verification of the above conditions for convergence. Specifically, note that (a) and (b) in the theorem imply that, asymptotically as $\tau \to 0$, trade occurs at the right (Walrasian) price, as needed for (1) above. Also note that the conditions 2 and 3 above pull in opposite directions. That is, in order to quickly meet a seller, it is necessary that the stock of optimistic buyers in the market is not too large, for these buyers clog the market and make it difficult for the pessimistic buyers to match. But, if there are too few optimistic buyers in the market, then it becomes difficult for them to learn the true state (from the no meeting event), for they will be meeting sellers too frequently.

In proving that the the trade occurs at approximately the right price when τ is small, we will make frequent use of the following lemma that derives the slopes of the responding strategies. [8]

Lemma 2 *The responding strategies $\tilde{v}(\cdot|\mu_B)$ and $\tilde{c}(\cdot|\mu_S)$ are absolutely continuous functions and have slopes equal a.e.$v \in A_B(\mu_B)$ and $c \in A_S(\mu_S)$*

$$\tilde{v}'(v|\mu_B) = \frac{1-R_\tau}{1-R_\tau+R_\tau \min\left\{1,\frac{S(\mu_B)}{B(\mu_B)}\right\} q_B(v|\mu_B,\mu_B)}, \tag{22}$$

$$\tilde{c}'(c|\mu_S) = \frac{1-R_\tau}{1-R_\tau+R_\tau \min\left\{1,\frac{B(\mu_S)}{S(\mu_S)}\right\} q_S(c|\mu_S,\mu_S)}. \tag{23}$$

Since $R_\tau \to 1$ as $\tau \to 0$, Lemma 2 implies that the responding strategies will become flat in the limit as $\tau \to 0$, provided the probabilities of meeting and trading in each period are not vanishing. In fact, market balance conditions imply that they will converge to the Walrasian price.

[8] This lemma parallels Lemma 1 in Shneyerov and Wong (2009).

The proof of Theorem 1 for all separating equilibria is given in Section 4. But in order to understand exactly how the difficulties mentioned above get resolved, we first present a proof for a class of equilibria, the so called full trade equilibria. These equilibria have the virtue of allowing for (almost) explicit solutions, which is also useful for establishing the existence of separating equilibria.

3 Convergence of Full Trade Equilibria and Existence

In this section, we construct an equilibrium with a *full trade* property: each meeting between a buyer and the seller who share the same belief about μ results in trade, and show that all such equilibria converge to efficiency. When there is no aggregate uncertainty (i.e. μ is known to all traders), a full trade equilibrium is fully described in Shneyerov and Wong (2010a). Here we recall the main properties of such equilibria, with an emphasis on differences pertaining to our current setting.

A full trade equilibrium (see Figure 4) is characterized by the property that the traders can do no better than propose at the level of the marginal partner's type

$$p_B\left(v|\mu_B\right) = \bar{c}\left(\mu_B\right) \quad \left(v \in A_B\left(\mu_B\right)\right), \tag{24}$$

$$p_S\left(c|\mu_S\right) = \underline{v}\left(\mu_S\right) \quad \left(c \in A_S\left(\mu_S\right)\right). \tag{25}$$

This implies

$$\underline{v}\left(\mu\right) > \bar{c}\left(\mu\right).$$

In the seller's market, $\mu = H$, the stock of pessimistic sellers is 0. There are three relevant stocks of buyers.

1. $B^0\left(L|H\right)$, the stock of optimistic buyers with $v \in [\underline{v}\left(L\right), \underline{v}\left(H\right)]$, who will exit voluntarily after not being matched

2. $B^1\left(L|H\right)$, the stock of optimistic buyers with $v \in [\underline{v}\left(H\right), 1]$, who will only exit through trade, once again after becoming pessimistic

3. $B\left(H|H\right)$, the stock of pessimistic buyers, who exit the market only by trading

Equations (12) - (15) in the preceding section imply the following mass balance conditions for the stocks of buyers in a steady state equilibrium, which we only

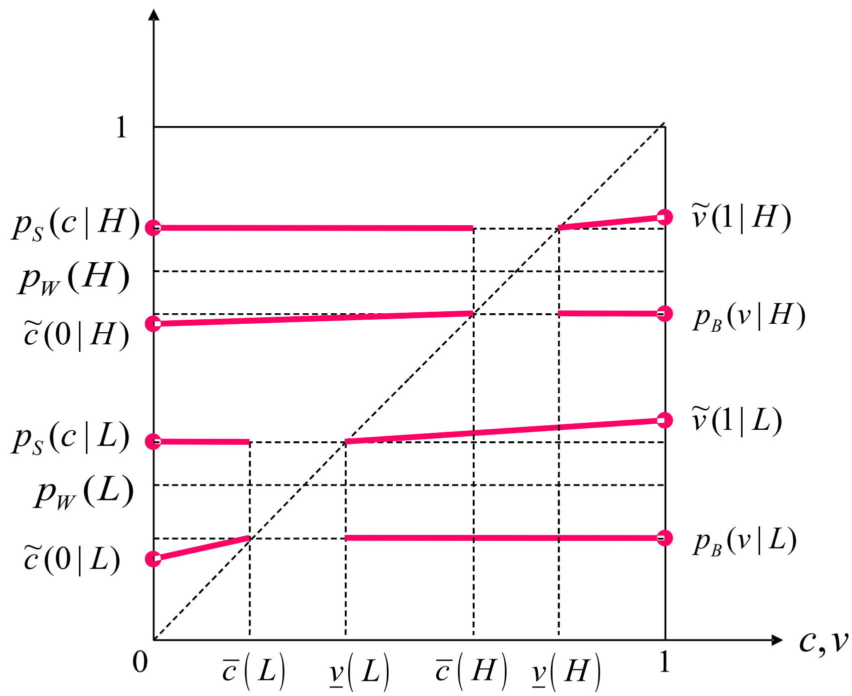

Figure 4: Full trade equilibrium.

state for $\mu = H$.

$$\tau \cdot [1 - G_B(\underline{v}(H)|H)] = (1 - \frac{S(H)}{B(H)})B^1(L|H), \tag{26}$$

$$\tau \cdot [G_B(\underline{v}(H)|H) - G_B(\underline{v}(L)|H)] = \left(1 - \frac{S(H)}{B(H)}\right)B^0(L|H), \tag{27}$$

$$\left(1 - \frac{S(H)}{B(H)}\right)B^1(L|H) = \frac{S(H)}{B(H)}\bar{q}_B(H)\,B(H|H), \tag{28}$$

and

$$B(H|H) + B^1(L|H) + B^0(L|H) = B(H), \tag{29}$$

where $\bar{q}_B(H)$ is the average trading probability of the pessimistic buyers conditional on being matched. In a full trade equilibrium,

$$\bar{q}_B(H) = 1.$$

Refer to Figure 3. Equation (26) above states that the inflowing mass of buyers with $v \in [\underline{v}(H), 1]$ in a given period is equal to the outflowing mass of optimistic buyers with $v \in [\underline{v}(H), 1]$ in the market who change their beliefs to pessimistic ones upon not meeting a seller, which happens with probability $1 - \frac{S(H)}{B(H)}$. Equation (27) is a parallel statement for the inflowing mass of buyers with $v \in [\underline{v}(L), \underline{v}(H)]$, which is equal to the outflowing mass of buyers with $v \in [\underline{v}(L), \underline{v}(H)]$ who have chosen to exit the market immediately once unmatched. Equation (28) states that the inflowing mass of pessimistic buyers is equal to the mass of buyers that leaves the market through trading, which happens with probability $\frac{S(H)}{B(H)}\bar{q}_B(H)$. Equation (29) simply re-iterates the fact that the total steady-state stock of buyers $B(H)$ is comprised of $B^0(L|H)$, $B^1(L|H)$ and $B(H|H)$.

Also, in any separating equilibrium, since , the following mass balance equations must hold:

$$1 - G_B(\underline{v}(H)|H) = G_S(\bar{c}(H)|H), \tag{30}$$

This is because the traders on the shorter side of the market can only exit through trade, and only traders who share the same belief about μ, trade. For example, when the state is $\mu = H$, the outgoing flow of sellers is $G_S(\bar{c}(H)|H)$ each period. Since traders leave in matched pairs, it is equal to the outgoing flow of pessimistic buyers. But the incoming flow into the stock of pessimistic buyers is equal to $1 - G_B(\underline{v}(H)|H)$. The steady-state requirement implies (30).

We now turn to the *indifference conditions* of the marginal participating types of buyers and sellers. The marginal trader types $\underline{v}(\mu_B)$ and $\bar{c}(\mu_S)$ must be indifferent between entering or not entering. Since the buyers only trade (with optimistic sellers) when they become pessimistic, we have from (19)

$$\frac{S(H)}{B(H)}\frac{1}{2}(\underline{v}(H) - \bar{c}(H)) = \tau\kappa. \tag{31}$$

In other words, the marginal buyers meet sellers with probability $S(H)/B(H)$ and with probability $1/2$ offer $\bar{c}(H)$, which is accepted by any active seller they meet. Their expected profit from a meeting is just sufficient to cover their participation cost $\tau\kappa$ incurred over a period.

Since in our equilibrium $S(H) < B(H)$ (to be verified later), the sellers always meet a buyer; but they only trade if they meet a pessimistic buyer. Their indifference condition (from (20)) is

$$\frac{B(H|H)}{B(H)}\frac{1}{2}(\underline{v}(H) - \bar{c}(H)) = \tau\kappa \tag{32}$$

These two equations, together with four steady state conditions (26) − (29) for trader stocks and the mass balance equation (30) give us a system of 7 characterizing equations. Parallel equations can be written when the state of the market is $\mu = L$, where the definitions of the seller stocks $S^0(L|L), S^1(H|L), S(H|L)$ mirror those of $B^0(H|H), B^1(L|H), B(L|H)$, and the counterpart of (30) is

$$1 - G_B(\underline{v}(L)|L) \quad = \quad G_S(\bar{c}(L)|L). \tag{33}$$

(We omit the equations for state $\mu = L$ to save on notation.) In total, this gives us a system of $2 \cdot 7 = 14$ equations for 14 unknowns

$$\begin{aligned}
&B(H), S(H), B^0(H|H), B^1(L|H), B(L|H), \underline{v}(H), \bar{c}(H), \\
&B(L), S(L), S^0(L|L), S^1(H|L), S(H|L), \underline{v}(L), \bar{c}(L).
\end{aligned}$$

Proposition 1 below invokes The Implicit Function theorem to show the existence of a unique solution to this system. Recall the steady state balance conditions (30) and (33). They allow us to define a function

$$\phi(z|\mu) \equiv G_S^{-1}[1 - G_B(z|\mu)|\mu]$$

that gives the marginal seller's type given the marginal buyer's type z in a steady state of the market. In particular, the equilibrium marginal types are related as

$$\bar{c}(\mu) = \phi(\underline{v}(\mu)|\mu).$$

Definition 1 *For any two real-valued functions $x_\tau : \mathbb{R}_+ \to \mathbb{R}_+$, we say that x_τ is* asymptotically proportionate *to τ, and write $x_\tau \asymp \tau$ if, as $\tau \to 0$, the ratio x_τ/τ is bounded away from both 0 and infinity, i.e. $lim_{\tau\to 0}\, inf\, x_\tau/\tau > 0$ and $lim_{\tau\to 0}\, sup\, x_\tau/\tau < \infty$.*

Proposition 1 *The system of equations characterizing a full trade equilibrium reduces to*

$$\frac{1}{2}\left(\underline{v}_\tau(H) - \phi\left(\underline{v}_\tau(H)|H\right)\right) = \tau \cdot \kappa \left(1 + \sqrt{\frac{1 - G_B\left(\underline{v}_\tau(L)|H\right)}{1 - G_B\left(\underline{v}_\tau(H)|H\right)}}\right), \tag{34}$$

$$\frac{1}{2}\left(\underline{v}_\tau(L) - \phi\left(\underline{v}_\tau(L)|L\right)\right) = \tau \cdot \kappa \left(1 + \sqrt{\frac{G_S\left(\phi\left(\underline{v}_\tau(H)|H\right)|L\right)}{G_S\left(\phi\left(\underline{v}_\tau(L)|L\right)|L\right)}}\right). \tag{35}$$

There exists a unique solution $(\underline{v}_\tau(H), \underline{v}_\tau(L))$ for all sufficiently small $\tau \geq 0$. Moreover, as $\tau \to 0$, for $\mu \in \{H, L\}$,

$$\begin{aligned}
\underline{v}_\tau(\mu) &= p_W(\mu) + O(\tau), \\
\bar{c}_\tau(\mu) &= p_W(\mu) - O(\tau),
\end{aligned}$$

and all trader stocks are asymptotically proportionate to τ:

$$\begin{aligned}
B_\tau(H), S_\tau(H), B_\tau^0(H|H), B_\tau^1(H|H), B_\tau(H|H) &\asymp \tau, \\
B_\tau(L), S_\tau(L), S_\tau^0(L|L), S_\tau^1(H|L), S_\tau(H|L) &\asymp \tau.
\end{aligned}$$

In general, this solution may or may not be a separating equilibrium, because the nonlinear system does not impose all equilibrium conditions. First, we need to verify the Fundamental Imbalance Condition (Assumption 1). This is immediate because, in state $\mu = H$, dividing the buyer's entry indifference condition (31) by the seller's (32), we obtain

$$\begin{aligned} S(H) &= B(H|H) \\ &< B(H), \end{aligned}$$

and in parallel, when the state is $\mu = L$, $B(L) < S(L)$.

In addition, we need to show that (i) the marginal participating types do not have an incentive to deviate to making offers that would be accepted with probability less than 1; and (ii) that the separation property holds. The remaining Lemmas 3 and 4 establish these properties for sufficiently small τ and r. For these results, we impose the following standard assumption that will assure the quasi-concavity of the expected profit functions in the proofs.

Assumption 2 *The Myerson virtual type functions*

$$J_B(\cdot|\mu) \equiv v - \frac{1 - G_B(\cdot|\mu)}{g_B(\cdot|\mu)}, \quad J_S(\cdot|\mu) \equiv c + \frac{G_S(\cdot|\mu)}{g_S(\cdot|\mu)}$$

are non-decreasing.

Lemma 3 *If $\tau < 1$, then the marginal types $\underline{v}(H)$ and $\bar{c}(H)$ do not have an incentive to deviate if*

$$r < \log\left(1 + 2\kappa \min\left\{\underline{g}_B, \underline{g}_S\right\}\right). \tag{36}$$

Lemma 4 *There exist $\bar{\tau}, \bar{r} > 0$ such that the separation property is satisfied for all (r, τ) such that both $r \in [0, \bar{r}]$ and $\tau \in (0, \bar{\tau}]$.*

To gain intuition, let $\mu = H$, and recall that according to Lemma 2, in a full trade equilibrium, the responding strategies, e.g buyers', are

$$\begin{aligned} \tilde{v}'(v|\mu_B) &= \frac{1 - R_\tau}{1 - R_\tau + R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} q_B(v|\mu_B, \mu_B)} \\ &= \frac{1 - R_\tau}{1 - R_\tau + R_\tau \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\}}, \end{aligned}$$

Because all trader stocks are asymptotically proportionate to τ according to Proposition 1, the perceived meeting probability

$$\min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} = \begin{cases} \frac{S_\tau(H)}{B_\tau(H)}, & \mu_B = H \\ 1, & \mu_B = L \end{cases}$$

remains bounded away from 0 as $\tau \to 0$. Therefore, the responding strategies become progressively flatter and in the limit become exactly horizontal. In addition, the same Proposition 1 implies that the gaps between the marginal participating types converges to 0. Together, these observations imply the Separation Property.

Also, it implies that the buyers with marginal types face diminishing incentives to reduce their price offers, because the probability of these offers rejected increases. Similarly, the sellers face diminishing incentives to increase their price offers. However, the "no deviation" result in Lemma 3 is not immediate since the profit at stake for the marginal types, equal to the difference between the seller's and buyer's marginal type, is proportionate to τ and therefore also becomes small. The proof of Lemma 3 (in the Appendix) shows that, still, the first effect dominates.

We now show how Proposition 1 and Lemmas 3 and 4 together immediately imply Theorem (1). For concreteness, assume $\mu = H$. Recall our discussion in the previous section: we need to show that (i) the trades occur at prices that converge to the Walrasian price in the limit; (ii) the pessimistic buyer's probability to meet a seller, $\frac{S_\tau(H)}{B_\tau(H)}$, is bounded away from 0 as $\tau \to 0$; and (iii) the optimistic buyer's probability of learning the true state in each period, $1 - \frac{S_\tau(H)}{B_\tau(H)}$, is bounded away from 0. But all are immediate given the result in Proposition 1. (i) follows directly from the fact that price offers converge to the Walrasian price. To see (ii) and (iii), note that since all trader stocks are asymptotically proportionate to τ, the ratio $\frac{S_\tau(H)}{B_\tau(H)}$ stays bounded away from both 0 and 1 as $\tau \to 0$.

4 All Separating Equilibria Converge to Perfect Competition

In this section, we generalize our convergence result to the class of *all* separating equilibria, thus providing a complete proof of our main result, Theorem 1. The proof of this result is split into several lemmas. Recall that only the meetings where the traders share the same beliefs about the state can lead to trade in a separating equilibrium. Let's call these "serious" meetings, and let

$$\ell_B^*(\mu) = \min\left\{\frac{S(\mu|\mu)}{S(\mu)}, \frac{S(\mu|\mu)}{B(\mu)}\right\}, \quad \ell_S^*(\mu) = \min\left\{\frac{B(\mu|\mu)}{B(\mu)}, \frac{B(\mu|\mu)}{S(\mu)}\right\}$$

be the "serious" meeting probabilities for buyers and sellers. The next lemma that establishes a lower bound on either ℓ_B^* or ℓ_S^* and is crucial for our results.

Lemma 5 *There exists a constant $\underline{\ell} > 0$ that doesn't depend on τ such that*

$$\max\{\ell_B^*(\mu), \ell_S^*(\mu)\} \geq \underline{\ell}. \tag{37}$$

We next prove the following lemma that establishes a bound on the entry gap (if it exists) in terms of ℓ_B^* and ℓ_S^* (refer to Figure 2).

Lemma 6 *We have*

$$\max\{0, \underline{v}_\tau(\mu) - \bar{c}_\tau(\mu)\} \le \frac{2\tau\kappa}{\max\{\ell_B^*(\mu), \ell_S^*(\mu)\}}. \tag{38}$$

As a corollary of Lemmas 5 and 6, we show that the entry gap (if there is any) converges to 0. This proves part (a) of Theorem 1.

Corollary 1 *The entry gap converges to* 0*:*

$$\lim_{\tau\to 0} \max\{0, \underline{v}_\tau(\mu) - \bar{c}_\tau(\mu)\} = 0.$$

The following lemma establishes an upper bound for $\tilde{v}_\tau(1|\mu) - \tilde{c}_\tau(0|\mu)$ (refer to Figure 2).

Lemma 7 *We have*

$$\tilde{v}_\tau(1|\mu) - \tilde{c}_\tau(0|\mu) \le \tau \cdot 2\frac{r+\kappa}{\kappa}(4r+\kappa)\frac{1}{\underline{\ell}}. \tag{39}$$

Lemmas 5 and 7 imply the following important corollary, which proves part (b) of Theorem 1.

Corollary 2 *We have*

$$\tilde{v}_\tau(1|\mu) - \tilde{c}_\tau(0|\mu) = O(\tau) \text{ as } \tau \to 0.$$

This corollary allows us to show that traders' search values converge to their Walrasian counterparts *if their beliefs about the state are true*. For $v \in [\underline{v}(\mu), 1]$, $\max\{W_{B\tau}(v|\mu), 0\} = W_{B\tau}(v|\mu)$ and from the definition $\tilde{v}(v|\mu) = v - R_\tau W_{B\tau}(v|\mu)$ we have

$$\begin{aligned} W_B^*(v|\mu) - R_\tau W_{B\tau}(v|\mu) &= \tilde{v}(v|\mu) - \underline{v}(\mu) + \underline{v}(\mu) - p_W(\mu) \\ &= O(\tau) \end{aligned}$$

where the last equality follows from Lemma 7. For $v \in [0, \underline{v}(\mu)]$, $\max\{W_{B\tau}(v|\mu), 0\} = 0$, and $W_B^*(v|\mu) > 0$ only if $v \in [p_W(\mu), \underline{v}(\mu)]$, where $W_B^*(v|\mu) = O(\tau)$ since $\underline{v}(\mu) - p_W(\mu) = O(\tau)$ again by Lemma 7. Therefore

$$\max\{W_{B\tau}(v|\mu), 0\} - W_B^*(v|\mu) = O(\tau) \qquad (v \in [0,1]) \tag{40}$$

and a parallel argument shows

$$\max\{W_{S\tau}(c|\mu), 0\} - W_S^*(c|\mu) = O(\tau) \qquad (c \in [0,1]). \tag{41}$$

Remark 1 *In fact, the convergence above is uniform over* $[0,1]$ *because the functions* $\max\{W_{B\tau}(\cdot|\mu), 0\}$ *and* $\max\{W_{S\tau}(\cdot|\mu), 0\}$ *are uniformly Lipschitz by Lemma 1, with slopes within* $[0,1]$*,and pointwise convergence of a sequence of uniformly Lipschitz functions implies uniform convergence.*

Finally, we now prove part (c) of Theorem 1. That is, the true search values of optimistic buyers and sellers with wrong beliefs, $w_{B\tau}(v|L,H)$ and $w_{S\tau}(c|H,L)$, also converge to $W_B^*(v|\mu)$ and $W_S^*(c|\mu)$ respectively. We will need the following result establishing upper bounds on the probabilities of meeting a serious trading partner.

Lemma 8 *There exists $\bar{\ell} \in (0,1)$ such that*

$$\ell_B^*(H) \leq \bar{\ell}, \qquad \ell_S^*(L) \leq \bar{\ell}.$$

Recalling $\ell_B^*(H) = \frac{S(H)}{B(H)}$, the recursive equation (16) for $w_{B\tau}(v|H)$ implies

$$w_{B\tau}(v|L,H) = \frac{R_\tau (1-\ell_B^*(H)) \max\{W_{B\tau}(v|H), 0\} - \kappa\tau}{1 - R_\tau \ell_B^*(H)},$$

$$\begin{aligned} w_{B\tau}(v|L,H) - \max\{W_{B\tau}(v|H), 0\} = &-\frac{1-R_\tau}{1-R_\tau \ell_B^*(H)} \max\{W_{B\tau}(v|H), 0\} \\ &- \frac{\kappa\tau}{1-R_\tau \ell_B^*(H)}. \end{aligned}$$

Similarly,

$$w_{S\tau}(c|H,L) = \frac{R_\tau (1-\ell_S^*(L)) \max\{W_{S\tau}(c|L), 0\} - \kappa\tau}{1 - R_\tau \ell_S^*(L)},$$

$$\begin{aligned} w_{S\tau}(c|H,L) - \max\{W_{S\tau}(c|L), 0\} = &-\frac{1-R_\tau}{1-R_\tau \ell_S^*(L)} \max\{W_{S\tau}(c|L), 0\} \\ &- \frac{\kappa\tau}{1-R_\tau \ell_S^*(L)}. \end{aligned}$$

Since $R_\tau \to 1$ as $\tau \to 0$, if the probabilities $\ell_B^*(H)$ and $\ell_S^*(L)$ stay bounded away from 1 as $\tau \to 0$ by Lemma 8, we have

$$\sup_{v\in[0,1]} |w_{B\tau}(v|L,H) - W_{B\tau}(v|H)| \to 0, \tag{42}$$

$$\sup_{c\in[0,1]} |w_{S\tau}(c|H,L) - W_{S\tau}(c|L)| \to 0, \tag{43}$$

Since $w_{B\tau}(v|L) = W_{B\tau}(v|L)$ and $w_{S\tau}(c|H) = W_{S\tau}(c|H)$, (42) and (43) together with (40) and (41) imply part (c) of Theorem 1.

5 Concluding remarks

A limitation of our analysis is that the entering cohorts of buyers and sellers are assumed to have extreme optimistic priors: the buyers believe the market is in the low state, while the sellers believe it is in the high state. Here we propose an extended model in which the optimistic priors arise endogenously.

We consider a model based on the "change of paradigm" framework recently proposed by Ortoleva (2010). Assume that buyers and sellers may have multiple priors, with the set of priors restricted to $\{\pi_L, \pi_H\}$, where the prior π_L puts full weight on state L, and π_H puts full weight on state H:

$$\pi_H = \delta_{\{H\}}, \quad \pi_L = \delta_{\{L\}}.$$

The traders start out with a *prior over priors* $\aleph$ that puts positive weight on both π_H and π_L. $\aleph$ is assumed to be common among the traders. We adopt a version of Ortoleva (2010) in which the agent first chooses his prior π, and then updates it whenever the Bayes rule applies. If, however, a zero probability event occurs and the Bayes rule does not apply, then the agent changes his/her paradigm: he/she updates the prior over priors $\aleph$ instead (to which the Bayes rule does apply), and then chooses a prior from the support of the updated $\aleph$.

To resolve the initial indeterminacy in the choice of the priors, we now augment our original model with an initial *prior choice* stage. We claim that, at least for τ sufficiently small, our game now has an equilibrium in which the buyers and sellers *will initially choose their respective optimistic priors*, and then play according to an equilibrium described in the previous sections of our paper.

Suppose, by the way of contradiction, that one trader, say a buyer, deviates from this proposed equilibrium strategy. Because the players are non-atomic and are matched anonymously, this deviation will never be detected by other players. Therefore, as before, if the optimistic prior π_L is chosen by the buyer, then the event n of not meeting a seller has zero probability. So the buyer will now first update his prior over priors $\aleph$. Since $\aleph$ puts positive weight on both π_H and π_L, the Bayes rule applies. Following Ortoleva (2010), the buyer will form a Bayesian update $\aleph_B(\pi_H|n)$ according to

$$\aleph_B(\pi_H|n) \quad = \quad \frac{\aleph(\pi_H) \cdot \pi_H(n)}{\aleph(\pi_H) \cdot \pi_H(n) + \aleph(\pi_L) \cdot \pi_L(n)} = 1$$

where the last equality follows because $\pi_L(n) = 0$. Thus, the buyer's update $\aleph_B(\cdot|n)$ will assign full probability mass to the pessimistic prior π_H. If, on the other hand, the buyer initially chooses the pessimistic prior π_H, then the probability of the event n is positive and the prior is never updated. Similar reasoning applies to the case of a seller.

As the updating rule in this extended model coincides with the one we used to have, the buyer will believe that his expected utility upon entry is $W_B(v|H)$ if he chooses the prior π_H, and $W_B(v|L)$ if he chooses the prior π_L. Our Proposition 1 implies that, as $\tau \to 0$, the expected utilities converge to their Walrasian values:

$$max\{W_B(v|\hat{\mu}_B), 0\} \quad \to \quad max\{v - p_W(\hat{\mu}_B), 0\} \qquad (\hat{\mu}_B = H, L).$$

Because $p_W(L) < p_W(H)$, we see that such a deviation is not profitable for sufficiently small τ: for all $v \in [0, 1]$, $W_B(v|L) \geq W_B(v|H)$. Similarly, we can show that a seller's deviation is not profitable, which completes the argument showing that in this extended model's equilibrium, buyers and sellers *choose* to be optimistic.

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Appendix

Proof of Lemma 1. For (18), note that otherwise say the marginal buyers $\underline{v}(\mu_B)$ would not be able to trade profitably with even the lowest cost sellers who share the same belief, because the latter would prefer to search for a better match in the market. Next, since $W_B(\underline{v}(\mu_B)|\mu_B) = W_S(\bar{c}(\mu_S)|\mu_S) = 0$, the marginal participating types are equal to the corresponding dynamic types: $\bar{c}(\mu_S) = \tilde{c}(\bar{c}(\mu_S)|\mu_S)$, $\underline{v}(\mu_B) = \tilde{v}(\underline{v}(\mu_B)|\mu_B)$. Evaluating (8) and (9) at $v = \underline{v}(\mu_B)$ and $c = \bar{c}(\mu_S)$, we obtain(19) and (20). To show (21), note that because the demand and supply functions intersect at $p = p_W(\mu)$, the mass balance equations equations (30) and (33) imply that $p_W(\mu)$ must in between $\bar{c}(\mu)$ and $\underline{v}(\mu)$. The proof of the remainder of this lemma parallels that of Lemma 2 in Shneyerov and Wong (2009) and is omitted. However, the intuition is as follows. The proposing strategies must be non-decreasing by standard single-crossing arguments. Individual rationality implies that the price offers are below reservation values for the buyers and above reservation values for the sellers. The buyer's equilibrium offer cannot be smaller than $\tilde{c}(0|\mu_B)$, since otherwise it will be surely rejected by any active seller. Also, any offer over and above $\bar{c}(\mu)$ will surely be accepted by any active seller, so in equilibrium, no buyer will choose to make an offer greater than $\bar{c}(\mu)$. Similar logic shows that $p_S(c|\mu_S) \in [\underline{v}(\mu_S), \tilde{v}(1|\mu_S)]$. *Q. E. D.*

Proof of Lemma 2. A formal proof would be parallel to the proof of Lemma 1 of Shneyerov and Wong (2009) and is omitted for the sake of brevity. To gain the intuition for e.g. (22), assume that $W_B(\cdot|\mu_B)$ is differentiable on A_B. Then the Envelope Theorem applied to taking into account the fact that prices are chosen optimally according to (4), yields for any $v \in A_B$,

$$\begin{aligned} U_B'(v|\mu_B) &= \tilde{v}'(v|\mu_B)\, q_B(v|\mu_B, \mu_B) \\ &= (1 - R_\tau W_B'(v|\mu_B))\, q_B(v|\mu_B, \mu_B). \end{aligned} \tag{44}$$

Differentiating the recursive equation (8) and substituting the slope $U_B'(v)$ from (44), we have

$$\begin{aligned} W_B'(v|\mu_B) &= \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} U_B'(v|\mu_B) + R_\tau W_B'(v|\mu_B) \\ &= \min\left\{1, \frac{S(\mu_B)}{B(\mu_B)}\right\} (1 - R_\tau W_B'(v|\mu_B))\, q_B(v|\mu_B, \mu_B) + R_\tau W_B'(v|\mu_B) \end{aligned}$$

for $v \in A_B(\mu_B)$. Solving the above equation for $W_B'(v|\mu_B)$ yields the integrand that appears in (22).

Proof of Proposition 1. Equations (31) and (32) imply that in a full trade equilibrium, the stock of pessimistic buyers is equal to the stock of sellers,

$$B(H|H) = S(H). \tag{45}$$

Equations (26) and (27) imply

$$\begin{aligned}B^0(L|H) &= \frac{G_B(\underline{v}(H)|H) - G_B(\underline{v}(L)|H)}{1 - G_B(\underline{v}(H), H)} B^1(L|H) \\ &= \beta \cdot B^1(L|H)\end{aligned} \tag{46}$$

where

$$\beta \equiv \frac{G_B(\underline{v}(H)|H) - G_B(\underline{v}(L)|H)}{1 - G_B(\underline{v}(H)|H)} > 0.$$

Equation (28) is equivalent to

$$\left(B^1(L|H) + B^0(L|H)\right) B^1(L|H) = B(H|H)^2,$$

which upon the substitution of (46) for $B^0(L|H)$ can be solved for $B(H|H)$,

$$B(H|H) = (1+\beta)^{1/2} B^1(L|H). \tag{47}$$

Substituting (46) and (47) into (26) gives us the solution for $B^1(L|H)$ and

$$B^1(L|H) = \tau \cdot [1 - G_B(\underline{v}(H), H)] \frac{1 + \beta + (1+\beta)^{1/2}}{1+\beta}, \tag{48}$$

and the other stocks $B^0(L|H)$, $B(H|H)$ are then determined from (46) and (47). The probability of meeting a pessimistic buyer is

$$\begin{aligned}\theta_B(H|H) &= \frac{B(H|H)}{B(H)} \\ &= \frac{(1+\beta)^{1/2}}{1+\beta+(1+\beta)^{1/2}} \\ &= \left(1 + \sqrt{\frac{1 - G_B(\underline{v}(L)|H)}{1 - G_B(\underline{v}(H)|H)}}\right)^{-1}.\end{aligned}$$

The entry equation, say (32) in $\mu = H$ is then equivalent to

$$\frac{1}{2}(\underline{v}(H) - \bar{c}(H)) = \tau \cdot \kappa \left(1 + \sqrt{\frac{1 - G_B(\underline{v}(L)|H)}{1 - G_B(\underline{v}(H)|H)}}\right). \tag{49}$$

For $\mu = L$ we obtain in parallel

$$\frac{1}{2}(\underline{v}(L) - \bar{c}(L)) = \tau \cdot \kappa \left(1 + \sqrt{\frac{G_S(\bar{c}(H)|L)}{G_S(\bar{c}(L)|L)}}\right). \tag{50}$$

The marginal types must also satisfy the mass balance conditions (30) and (33), and for $\mu \in \{H, L\}$,

$$p_W(\mu) \in [\underline{v}(\mu), \bar{c}(\mu)]. \tag{51}$$

Equations (49) and (50), together with the mass balance conditions (30) and (33), form a system of four equations for 4 unknowns, now denoted as $(\underline{v}_\tau(H), \bar{c}_\tau(H), \underline{v}_\tau(L), \bar{c}_\tau(L))$. For $\tau = 0$, these equations imply

$$\underline{v}_0(\mu) = \bar{c}_0(\mu) = p_W(\mu).$$

The Implicit Function Theorem implies that $\bar{\tau} > 0$ exists such that a solution exists for all $\tau \in [0, \bar{\tau}]$ provided the Jacobian of this system is nonzero. Moreover, as $\tau \to 0$, the marginal types converge to the corresponding Walrasian prices, $\underline{v}(H), \bar{c}(H) \to p_W(H)$ and $\underline{v}(L), \bar{c}(L) \to p_W(L)$.

To evaluate the Jacobian, it is convenient to reduce this system by eliminating $\bar{c}(\mu)$ from equations (30) and (33):

$$\begin{aligned}\bar{c}(\mu) &= G_S^{-1}(1 - G_B(\underline{v}(\mu)|\mu)|\mu) \\ &\equiv \phi(\underline{v}(\mu)|\mu),\end{aligned}$$

where the mapping $\phi(\cdot|\mu) : [p_W(\mu), 1] \to [0, p_W(\mu)]$ (smoothly extended to an open εneighborhood of $p_W(\mu)$) has the derivative at $p_W(\mu)$ equal to

$$\phi'(p_W(\mu)|\mu) = -\frac{g_B(p_W(\mu)|\mu)}{g_S(p_W(\mu)|\mu)} < 0. \tag{52}$$

Now the system of equations for $(\underline{v}(H), \underline{v}(L))$ becomes

$$\frac{1}{2}(\underline{v}(H) - \phi(\underline{v}(H)|H)) - \tau \cdot \kappa \left(1 + \sqrt{\frac{1 - G_B(\underline{v}(L)|H)}{1 - G_B(\underline{v}(H)|H)}}\right) = 0, \tag{53}$$

$$\frac{1}{2}(\underline{v}(L) - \phi(\underline{v}(L)|L)) - \tau \cdot \kappa \left(1 + \sqrt{\frac{G_S(\phi(\underline{v}(H)|H)|L)}{G_S(\phi(\underline{v}(L)|L)|L)}}\right) = 0. \tag{54}$$

The Jacobian of this system at $\tau = 0$ is

$$\begin{aligned}&\begin{vmatrix} \frac{1}{2}(1 - \phi'(p_W(H)|H)) & 0 \\ 0 & \frac{1}{2}(1 - \phi'(p_W(L)|L)) \end{vmatrix} \\ &= \frac{1}{4}(1 - \phi'(p_W(H)|H))(1 - \phi'(p_W(L)|L)) \\ &> 0,\end{aligned}$$

where the inequality in the last line follows from (52). *Q. E. D.*

For notational expedience, from now on we denote the matching probabilities as

$$\ell_B(\mu) \equiv \min\left\{1, \frac{S(\mu)}{B(\mu)}\right\}, \quad \ell_S(\mu) \equiv \min\left\{1, \frac{B(\mu)}{S(\mu)}\right\}.$$

Also define the market fractions of buyers (sellers) with belief μ_B (μ_S)

$$\theta_B(\mu_B|\mu) \equiv \frac{B(\mu_B|\mu)}{B(\mu)}, \quad \theta_S(\mu_S|\mu) \equiv \frac{S(\mu_S|\mu)}{S(\mu)}.$$

Proof of Lemma 3. Without loss of generality, let's assume that the state is $\mu = H$.

First, we focus on the incentives of the sellers (a symmetric argument will apply for the buyers, with obvious changes). The expected profit contingent on proposing $\lambda \geq \underline{v}(\mu)$ is

$$\pi_S(\bar{c}(\mu), \lambda|\mu) = (\lambda - \bar{c}(\mu))\left(1 - \tilde{\Phi}(\lambda|\mu, \mu)\right),$$

and its slope is

$$\begin{aligned}\frac{\partial \pi_S(\bar{c}(\mu), \lambda|\mu)}{\partial \lambda} &= \left(1 - \tilde{\Phi}(\lambda|\mu, \mu)\right) - (\lambda - \bar{c}(\mu))\,\tilde{\Phi}'(\lambda|\mu, \mu) \\ &= -\tilde{\Phi}'(\lambda|\mu, \mu)\left[\tilde{J}_B(\lambda|\mu) - \bar{c}(\mu)\right]\end{aligned} \tag{55}$$

where $\tilde{J}_B(\lambda|\mu)$ is the "virtual type" that corresponds to the distribution of dynamic types $\tilde{\Phi}(\cdot|\mu, \mu)$,

$$\tilde{J}_B(\lambda|\mu) \equiv \lambda - \frac{1 - \tilde{\Phi}(\lambda|\mu, \mu)}{\tilde{\Phi}'(\lambda|\mu, \mu)}.$$

Notice that $\tilde{\Phi}(\lambda|\mu, \mu) = \Phi\left(\tilde{v}^{-1}(\lambda|\mu)|\mu, \mu\right)$. Contingent on meeting a seller, pessimistic buyers trade with probability 1 regardless of their type. Therefore, their distribution of types in the market is a truncation of the inflow distribution,

$$1 - \Phi(v|\mu, \mu) = \frac{1 - G_B(v|\mu)}{1 - G_B(\underline{v}(\mu)|\mu))} \quad (v \geq \underline{v}(\mu)).$$

From Lemma 1, the dynamic type $\tilde{v}(v|\mu)$ is a linear function with the slope

$$\tilde{v}'(v|\mu) = \frac{1 - R_\tau}{1 - R_\tau + R_\tau \ell_B^*(\mu)}$$

(recall that the probability of meeting a seller is equal to $S(H)/B(H)$, while $S(H) = B(H|H)$ from (45)). Since $\tilde{v}(\underline{v}(\mu)|\mu) = \underline{v}(\mu)$, we can explicitly solve for the responding strategy,

$$\tilde{v}(v|\mu) = \frac{(1 - R_\tau)v + R_\tau \ell_B^*(\mu)\underline{v}(\mu)}{(1 - R_\tau) + R_\tau \ell_B^*(\mu)}. \tag{56}$$

From (56), the inverse responding strategy is

$$\tilde{v}^{-1}(\lambda) = \frac{(1 - R_\tau) + R_\tau \ell_B^*}{1 - R_\tau}\lambda - \frac{R_\tau \ell_B^* \underline{v}(\mu)}{1 - R_\tau}.$$

Then

$$1 - \tilde{\Phi}(\lambda|\mu, \mu) = \frac{1 - G_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}{1 - G_B(\underline{v}(\mu)|\mu)},$$

$$\tilde{\phi}(\lambda|\mu,\mu) = \frac{d\tilde{v}^{-1}(\lambda|\mu)}{d\lambda}\frac{g_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}{1-G_B(\underline{v}(\mu)|\mu)}$$
$$= \frac{(1-R_\tau)+R_\tau\ell_B^*}{1-R_\tau}\frac{g_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}{1-G_B(\underline{v}(\mu)|\mu)},$$

and

$$\tilde{J}_B(\lambda|\mu) \equiv \lambda - \frac{1-\tilde{\Phi}(\lambda|\mu,\mu)}{\tilde{\phi}(\lambda|\mu,\mu)}$$
$$= \lambda - \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}\frac{1-G_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}{g_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}$$
$$= \lambda - \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}\tilde{v}^{-1}(\lambda|\mu)$$
$$+ \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}\left(\tilde{v}^{-1}(\lambda|\mu) - \frac{1-G_B\left(\tilde{v}^{-1}(v|\mu)|\mu\right)}{g_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right)}\right)$$
$$= \frac{R_\tau\ell_B^*\underline{v}(\mu)}{(1-R_\tau)+R_\tau\ell_B^*} + \frac{1-R_\tau}{(1-R_\tau)+R_\tau\ell_B^*}J_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right).$$

Equivalently,

$$\tilde{J}_B(\lambda|\mu) = \frac{1}{(1-R_\tau)+R_\tau\ell_B^*} \cdot \left((1-R_\tau)J_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right) + R_\tau\ell_B^*\underline{v}(\mu)\right). \tag{57}$$

Substituting (57) in the slope formula (55), we obtain

$$\frac{\partial\pi_S(\bar{c}(\mu),\lambda|\mu)}{\partial\lambda} = -\tilde{\Phi}'(\lambda|\mu,\mu)\{\frac{1}{(1-R_\tau)+R_\tau\ell_B^*} \cdot ((1-R_\tau)J_B\left(\tilde{v}^{-1}(\lambda|\mu)|\mu\right) + R_\tau\ell_B^*\underline{v}(\mu)) - \bar{c}(\mu)\}. \tag{58}$$

Clearly, a deviation to $\lambda < \underline{v}(\mu)$ is not profitable, so we only need to consider $\lambda > \underline{v}(\mu)$. A necessary condition for such a deviation to be not profitable is that $\partial\pi_S(\bar{c}(\mu),\lambda|\mu)/\partial\lambda \leq 0$ at $\lambda = \underline{v}(\mu)$, i.e. the expression in the brackets on the right-hand side of equation (58) is non-negative when $\lambda = \underline{v}(\mu)$. This is also sufficient because of the assumed monotonicity of $J_B(\cdot|\mu)$ (Assumption 2). This gives us the inequality

$$\frac{(1-R_\tau)J_B(\underline{v}(\mu)|\mu) + R_\tau\ell_B^*\underline{v}(\mu)}{(1-R_\tau)+R_\tau\ell_B^*} - \bar{c}(\mu) \geq 0.$$

We now show that this inequality is satisfied for small r. We can rewrite it as

$$\underline{v}(\mu) - \bar{c}(\mu) - \frac{(1-R_\tau)}{(1-R_\tau)+R_\tau\ell_B^*}\frac{1-G_B(\underline{v}(\mu)|H)}{g_B(\underline{v}(\mu)|H)} \geq 0. \tag{59}$$

Now $\ell_B^* = \theta_B(H|H)$ for $\mu = H$, and from either (31) or (32) we have $\underline{v}(\mu) - \bar{c}(\mu) = 2\tau\kappa/\theta_B(H|H)$. Substituting these into (59) and replacing $\frac{1-G_B(\underline{v}(\mu)|H)}{g_B(\underline{v}(\mu)|H)}$

with an upper bound $1/\underline{g}_B$, and $(1-R_\tau)+R_\tau\theta_B(H|H)$ with $R_\tau\theta_B(H|H)$, we have a stronger inequality that is sufficient for no deviation:

$$\frac{2\tau\kappa}{\theta_B(H|H)}-\frac{(1-R_\tau)}{R_\tau\theta_B(H|H)}\frac{1}{\underline{g}_B}\geq 0.$$

Alternatively,

$$\frac{1-e^{-r\tau}}{\tau e^{-r\tau}}\leq 2\kappa\underline{g}_B. \tag{60}$$

The l.h.s. of the above equation, $(e^{r\tau}-1)/\tau$, is an increasing function of τ because $e^{r\tau}-1$ is a convex, increasing function of τ taking value 0 at $\tau=0$. Therefore, if $\tau\leq 1$, it is sufficient to require

$$e^r-1\leq 2\kappa\underline{g}_B,$$

or equivalently

$$r\leq\log\left(1+2\kappa\underline{g}_B\right).$$

A similar argument applied to marginal sellers yields $r\leq\log\left(1+2\kappa\underline{g}_S\right)$, from which (36) in the statement of the Lemma follows. *Q. E. D.*

Proof of Lemma 4. First note that $\underline{v}(H)>p_W(L)$ and $\bar{c}(L)<p_W(H)$. These and the stead state mass balance equations imply

$$\begin{aligned}1-G_B(\underline{v}(H)|H)&=G_S(\bar{c}(H)|H)\geq G_S(p_W(L)|H),\\ G_S(\bar{c}(L)|L)&=1-G_B(\underline{v}(L)|L)\geq 1-G_B(p_W(H)|L).\end{aligned}$$

From (49) and (50), these bounds in turn imply the following bounds on the entry gaps for $\mu=H$ and $\mu=L$:

$$\underline{v}(H)-\bar{c}(H)\leq\tau a,\quad \underline{v}(L)-\bar{c}(L)\leq\tau a \tag{61}$$

These bounds imply that $\underline{v}(L)$ and $\bar{c}(H)$ are within $O(\tau)$ distance from the Walrasian prices for the corresponding states:

$$\underline{v}(L)\leq p_W(L)+\tau a, \tag{62}$$

$$\bar{c}(H)\geq p_W(H)-\tau a. \tag{63}$$

Now that the slopes of the responding strategies are bounded from Lemma 1, e.g. for the buyers

$$\begin{aligned}\tilde{v}'(v|\mu_B)&=\frac{1-R_\tau}{1-R_\tau+R_\tau\ell_B(\mu_B)q_B(v|\mu_B,\mu_B)}\\ &\leq\frac{1-R_\tau}{R_\tau\kappa\tau}\\ &=\frac{e^{r\tau}-1}{\tau}\frac{1}{\kappa}\\ &\leq\frac{e^r-1}{\kappa}\end{aligned}$$

where the first inequality follows from $\kappa\tau \leq \ell_B(\mu_B) U_B(v|\mu_B) \leq \ell_B(\mu_B) q_B(v|\mu_B)$, which is implied by the Bellman equation (8) and the definition of U_B in (6), while the second inequality follows from the fact that $(e^{r\tau}-1)/\tau$ is the definition of R_τ and the fact that $\tau \leq 1$. A similar bound obtains for slope of the sellers' responding strategy:

$$\tilde{c}'(c|\mu_S) \leq \frac{e^r - 1}{\kappa}.$$

These two bounds together with (62) and (63) imply

$$\tilde{v}(1|L) \leq p_W(L) + a\tau + \frac{e^r - 1}{\kappa},$$
$$\tilde{c}(0|H) \geq p_W(H) - a\tau - \frac{e^r - 1}{\kappa}.$$

Then the separating property $\tilde{c}(0|H) \geq \tilde{v}(1|L)$ holds if

$$a\tau + \frac{e^r - 1}{\kappa} \leq \frac{1}{2}(p_W(H) - p_W(L)).$$

It is sufficient to impose a stronger inequality

$$2\max\left\{a\tau, \frac{e^r - 1}{\kappa}\right\} \leq \frac{1}{2}(p_W(H) - p_W(L)),$$

or, equivalently, the conditions

$$\tau \leq \bar{\tau} \equiv \min\left\{1, \frac{1}{4a\kappa}(p_W(H) - p_W(L))\right\}, \tag{64}$$
$$r \leq \bar{r} \equiv \log\left(1 + \frac{\kappa}{2}(p_W(H) - p_W(L))\right), \tag{65}$$

where

$$a \equiv 2\kappa \max\left\{1 + \sqrt{\frac{1}{G_S(p_W(L)|H)}}, 1 + \sqrt{\frac{1}{1 - G_B(p_W(H)|L)}}\right\}$$

Q. E. D.

Proof of Lemma 5. To economize on notation, in this proof we suppress index τ. We prove the result for $\mu = H$ only; the proof for $\mu = L$ is parallel. Equation (28) implies

$$B^1(L|H) = \frac{\frac{S(H)}{B(H)}}{1 - \frac{S(H)}{B(H)}} \frac{B(H|H)}{B(H)} B(H)\bar{q}_B(H)$$
$$= \frac{\ell_B^*(H)}{1 - \ell_B^*(H)} \ell_S^*(H) B(H)\bar{q}_B(H). \tag{66}$$

Dividing (27) by (26) we have

$$\frac{B^0(L|H)}{B^1(L|H)} = \frac{G_B(\underline{v}(H)|H) - G_B(\underline{v}(L)|H)}{1 - G_B(\underline{v}(H)|H)} \equiv M(H). \tag{67}$$

Therefore, from (29)

$$(1 + M(H)) B^1(H|H) + B(H|H) = B(H),$$

which implies

$$B^1(H|H) = m(H)(B(H) - B(H|H)), \tag{68}$$

where

$$m(H) \equiv (1 + M(H))^{-1}. \tag{69}$$

Substituting (68) into (66) and dividing by $B(H) - B(H|H)$,

$$\begin{aligned} m(H) &= \frac{\ell_B^*(H)}{1 - \ell_B^*(H)} \frac{\ell_S^*(H)}{B(H) - B(H|H)} B(H)\bar{q}_B(H) \\ &= \frac{\ell_B^*(H)}{1 - \ell_B^*(H)} \frac{\ell_S^*(H)}{1 - \ell_S^*(H)} \bar{q}_B(H) \end{aligned} \tag{70}$$

Since $\bar{q}_B(H) \leq 1$,

$$\max\left\{\frac{\ell_B^*(H)}{1 - \ell_B^*(H)}, \frac{\ell_S^*(H)}{1 - \ell_S^*(H)}\right\} \geq m(H)^{1/2}.$$

Since $x \mapsto x/(1-x)$ is an increasing on $(0,1)$ function, this in turn implies

$$\frac{\max\{\ell_B^*(H), \ell_S^*(H)\}}{1 - \max\{\ell_B^*(H), \ell_S^*(H)\}} \geq m(H)^{1/2},$$

or

$$\max\{\ell_B^*(H), \ell_S^*(H)\} \geq \frac{m(H)^{1/2}}{1 + m(H)^{1/2}} \geq \frac{1}{2} m(H)^{1/2}, \tag{71}$$

where the last inequality follows from $m(H) < 1$. Now

$$\begin{aligned} m(H) &\geq \frac{1}{1 + \frac{1}{1 - G_B(\underline{v}(H)|H)}} \\ &= \frac{1}{1 + \frac{1}{G_S(\bar{c}(H)|H)}} \\ &\geq \frac{1}{1 + \frac{1}{G_S(p_W(L)|H)}} \\ &\geq \frac{1}{2} G_S(p_W(L)|H) \end{aligned} \tag{72}$$

where the equality follows from the mass balance condition (30) and the second to last inequality follows because in a separating equilibrium $\bar{c}(H) \geq p_W(L)$. Combining (71) and (72) and using $\sqrt{2} < 2$ gives (37) with

$$\underline{\ell} = \frac{1}{4}\left(G_S\left(p_W\left(L\right)|H\right)\right)^{1/2}.$$

Q. E. D.

Proof of Lemma 6. The buyer with type $\underline{v}_\tau(\mu)$ can offer $\bar{c}_\tau(\mu)$, and this offer will be accepted by any seller with $c < \bar{c}_\tau(\mu)$. This strategy guarantees him the expected payoff $\frac{1}{2}\ell_B^*\left(\underline{v}_\tau(\mu) - \bar{c}_\tau(\mu)\right)$. The equilibrium condition (19) then implies $\kappa\tau \geq \frac{1}{2}\ell_B^*\left(\underline{v}_\tau(\mu) - \bar{c}_\tau(\mu)\right)$. Similarly, we can show that $\kappa\tau \geq \frac{1}{2}\ell_S^*\left(\underline{v}_\tau(\mu) - \bar{c}_\tau(\mu)\right)$, and therefore

$$\underline{v}_\tau(\mu) - \bar{c}_\tau(\mu) \leq \tau \cdot \min\left\{\frac{\kappa}{\frac{1}{2}\ell_B^*(\mu)}, \frac{\kappa}{\frac{1}{2}\ell_S^*(\mu)}\right\},$$

from which (38) follows. *Q. E. D.*

Proof of Lemma 7. To economize on notation, in this proof also we suppress index τ.

Step 1: We claim that

$$\tilde{v}(1|\mu) - \tilde{c}(0|\mu) \leq \frac{r+\kappa}{\kappa}\min\left\{\underline{v}(\mu) - \tilde{c}(0|\mu), \tilde{v}(1|\mu) - \bar{c}(\mu)\right\} \tag{73}$$

We only prove the first inequality,

$$\tilde{v}(1|\mu) - \tilde{c}(0|\mu) \leq \frac{r+\kappa}{\kappa}\left(\underline{v}(\mu) - \tilde{c}(0|\mu)\right); \tag{74}$$

the proof of the other inequality is parallel. First note that $p_B\left(\underline{v}(\mu)|\mu\right) \geq \tilde{c}(0|\mu)$. Since q_B is non-decreasing, (19) then implies $\ell_B q_B(v|\mu,\mu)(\underline{v}(\mu) - \tilde{c}(0|\mu)) \geq \kappa\tau$ whenever $v \in [\underline{v}(\mu), 1]$. Then for almost all $v \in [\underline{v}(\mu), 1]$,

$$\ell_B q_B(v|\mu,\mu) \geq \frac{\kappa\tau}{\underline{v}(\mu) - \tilde{c}(0|\mu)},$$

and therefore

$$\tilde{v}'(v|\mu) = \frac{1-R_\tau}{1-R_\tau+R_\tau\ell_B q_B(v|\mu,\mu)} \leq \frac{r\tau}{\ell_B q_B(v|\mu,\mu)} \leq \frac{r}{\kappa/(\underline{v}(\mu) - \tilde{c}(0|\mu))}.$$

where the first inequality follows from the concavity of the function $1-e^{-x}$. Hence

$$\tilde{v}(1|\mu) - \underline{v}(\mu) = \int_{\underline{v}(\mu)}^1 \tilde{v}'(v|\mu)dv \leq \frac{r}{\kappa/(\underline{v}(\mu) - \tilde{c}(0|\mu))},$$

$$\frac{\tilde{v}(1|\mu) - \underline{v}(\mu)}{\underline{v}(\mu) - \tilde{c}(0|\mu)} \leq \frac{r}{\kappa},$$

$$\begin{aligned}\frac{\underline{v}(\mu) - \tilde{c}(0|\mu)}{\tilde{v}(1|\mu) - \tilde{c}(0|\mu)} &= \frac{1}{1 + \frac{(\tilde{v}(1|\mu) - \underline{v}(\mu))}{\underline{v}(\mu) - \tilde{c}(0|\mu)}} \\ &\geq \frac{1}{1 + \frac{r}{\kappa}} \\ &= \frac{\kappa}{r + \kappa},\end{aligned}$$

from which (74) follows.

Step 2: We claim that

$$\text{(a):} \quad \tilde{c}'(c|\mu) \leq \tau \frac{4r(r+\kappa)}{\kappa} \frac{1}{\ell_S^*(\mu)},$$

$$\text{(b):} \quad \tilde{v}'(v|\mu) \leq \tau \frac{4r(r+\kappa)}{\kappa} \frac{1}{\ell_B^*(\mu)}.$$

Again by symmetry, we only provide a proof for (a) only, the other one is parallel. Let

$$y \equiv \min\{\bar{c}(\mu), \underline{v}(\mu)\} - \tilde{c}(0|\mu). \tag{75}$$

Consider a type c seller with $\tilde{c}(c|\mu) \leq \tilde{c}(0|\mu) + y/2$. By proposing the price $\underline{v}(\mu)$, she can guarantee the expected payoff of $\frac{1}{2}\theta_B(\mu|\mu)[\underline{v}(\mu) - \tilde{c}(c|\mu)]$, since this offer is accepted in equilibrium by any buyer with $v > \underline{v}(\mu)$ who shares the same belief μ. Therefore the equilibrium expected payoff in the bargaining game is bounded from below by $\frac{1}{2}\theta_B(\mu|\mu)[\underline{v}(\mu) - \tilde{c}(c|\mu)]$:

$$q_S(c|\mu,\mu)[\bar{p}_S(c|\mu) - \tilde{c}(c|\mu)] \geq \frac{\theta_B(\mu|\mu)}{2}[\underline{v}(\mu) - \tilde{c}(c|\mu)],$$

where $\bar{p}_S(c|\mu)$ is the expected price conditional on trading. Since $\underline{v}(\mu) - \tilde{c}(c|\mu) \geq \underline{v}(\mu) - (\tilde{c}(0|\mu) + y/2)$, and our definition of y implies that $y \leq \underline{v}(\mu) - \tilde{c}(0|\mu)$, it follows that $\underline{v}(\mu) - \tilde{c}(c|\mu) \geq (\underline{v}(\mu) - \tilde{c}(0|\mu))/2$ and therefore

$$q_S(c|\mu,\mu)[\bar{p}_S(c|\mu) - \tilde{c}(c|\mu)] \geq \frac{\theta_B(\mu|\mu)}{2} \frac{\underline{v}(\mu) - \tilde{c}(0|\mu)}{2}.$$

Since no offer above $\tilde{v}(1|\mu)$ will be accepted in equilibrium by a buyer with belief μ, $p_S(c|\mu) \leq \tilde{v}(1|\mu)$. Since $\tilde{c}(c)$ is non-decreasing by Lemma 2, we must also have $\tilde{c}(c|\mu) \geq \tilde{c}(0|\mu)$, and therefore

$$q_S(c|\mu,\mu) \geq \frac{\theta_B(\mu|\mu)}{4} \frac{\underline{v}(\mu) - \tilde{c}(0|\mu)}{\tilde{v}(1|\mu) - \tilde{c}(0|\mu)},$$

$$q_S(c|\mu,\mu) \geq \frac{\theta_B(\mu|\mu)\kappa}{4(r+\kappa)},$$

where the last inequality follows from applying the bound from Step 1,

$$\frac{\underline{v}(\mu)-\tilde{c}(0|\mu)}{\tilde{v}(1|\mu)-\tilde{c}(0|\mu)} \geq \frac{\kappa}{r+\kappa}.$$

Then from (23) in Lemma 2,

$$\begin{aligned}\tilde{c}'(c|\mu) &= \frac{1-R_\tau}{1-R_\tau+R_\tau \ell_S q_S(c|\mu,\mu)} \leq \frac{r\tau}{\ell_S q_S(c|\mu,\mu)} \\ &\leq \frac{r\tau}{\ell_S \theta_B(\mu|\mu)\frac{1}{2}\frac{\kappa}{2(r+\kappa)}} \\ &= \tau \frac{4r(r+\kappa)}{\kappa}\frac{1}{\ell_S^*(\mu)}.\end{aligned}$$

Step 3: We now combine the bound on the entry gap in Lemma with steps 1 and 2 of this proof to show (39). From (73) in step 1, we have

$$\begin{aligned}\tilde{v}(1|\mu)-\tilde{c}(0|\mu) &\leq \frac{r+\kappa}{\kappa}\min\{\underline{v}(\mu)-\tilde{c}(0|\mu),\tilde{v}(1|\mu)-\bar{c}(\mu)\} \\ &\leq \frac{r+\kappa}{\kappa}\max\{\underline{v}(\mu)-\bar{c}(\mu),0\} \\ &+\frac{r+\kappa}{\kappa}\min\{\bar{c}(\mu)-\tilde{c}(0|\mu),\tilde{v}(1|\mu)-\underline{v}(\mu)\}\end{aligned}$$

Lemma implies $\max\{\underline{v}(\mu)-\bar{c}(\mu),0\}\to 0$, while the bounds in step 2 imply

$$\begin{aligned}\tilde{v}(1|\mu)-\underline{v}(\mu) &\leq \max_{v\in A_B}\tilde{v}'(v|\mu) \\ &\leq \tau\frac{4r(r+\kappa)}{\kappa}\frac{1}{\ell_B^*(\mu)},\end{aligned}$$

$$\begin{aligned}\bar{c}(\mu)-\tilde{c}(0|\mu) &\leq \max_{c\in A_S}\tilde{c}(0|\mu) \\ &\leq \tau\frac{4r(r+\kappa)}{\kappa}\frac{1}{\ell_S^*(\mu)}.\end{aligned}$$

Therefore

$$\begin{aligned}\tilde{v}(1|\mu)-\tilde{c}(0|\mu) &\leq 2\tau\frac{4r(r+\kappa)}{\kappa}\min\left\{\frac{1}{\ell_B^*(\mu)},\frac{1}{\ell_S^*(\mu)}\right\}+\frac{r+\kappa}{\kappa}\frac{2\tau\kappa}{\max\{\ell_B^*(\mu),\ell_S^*(\mu)\}} \\ &= \tau\cdot 2\frac{r+\kappa}{\kappa}(4r+\kappa)\frac{1}{\max\{\ell_B^*(\mu),\ell_S^*(\mu)\}} \\ &\leq \tau\cdot 2\frac{r+\kappa}{\kappa}(4r+\kappa)\frac{1}{\underline{\ell}}.\end{aligned}$$

Q. E. D.

Proof of Lemma 8. Recall equation (70) in the proof of Lemma 5:

$$m(H)=\frac{\ell_B^*(H)}{1-\ell_B^*(H)}\frac{\ell_S^*(H)}{1-\ell_S^*(H)}\bar{q}_B(H),$$

and recall that $m(H) \leq 1$. Therefore

$$\begin{aligned}
\frac{\ell_B^*(H)}{1-\ell_B^*(H)} \frac{\ell_S^*(H)}{1-\ell_S^*(H)} \bar{q}_B(H) \leq 1, & \\
\frac{\ell_B^*(H)}{1-\ell_B^*(H)} &\leq \frac{1}{\frac{\ell_S^*(H)}{1-\ell_S^*(H)} \bar{q}_B(H)} \\
&\leq \frac{1}{\ell_S^*(H)\, \bar{q}_B(H)}, \\
\ell_B^*(H) &\leq \frac{1}{1+\ell_S^*(H)\, \bar{q}_B(H)}. \qquad (76)
\end{aligned}$$

We next establish lower bounds on $\ell_S^*(H)$ and $\bar{q}_B(H)$. To this end, (19) and (20) together with the definition of U_B and U_S imply

$$\begin{aligned}
\ell_B^*(H)\, q_B(\underline{v}(H)|H) &\geq \frac{\kappa\tau}{\tilde{v}_\tau(1|H) - \tilde{c}_\tau(0|H)} \qquad (77) \\
&\geq \frac{\kappa}{2\frac{r+\kappa}{\kappa}(4r+\kappa)\frac{1}{\underline{\ell}}} \\
&\geq \frac{\kappa^2 \underline{\ell}}{2(r+\kappa)(4r+\kappa)} \\
&\equiv \gamma \in (0,1),
\end{aligned}$$

where the second inequality follows from Lemma 7. Exactly the same lower bound holds for the sellers,

$$\ell_S^*(H)\, q_S(\bar{c}(H)|H) \geq \gamma. \qquad (78)$$

Since $q_B(\cdot|\mu_B)$ is a non-decreasing function, $q_B(\underline{v}(H)|H) \leq \bar{q}_B(H)$ and (77), (78) together imply

$$\bar{q}_B(H), \ell_S^*(H) \geq \gamma.$$

Substituting the above bound in (76), we get

$$\begin{aligned}
\ell_B^*(H) &\leq \frac{1}{1+\gamma^2} \\
&\equiv \bar{\ell}.
\end{aligned}$$

A parallel argument shows that exactly the same bound applies for the sellers when the state is $\mu = L$, $\ell_S^*(L) \leq \bar{\ell}$. *Q. E. D.*