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Characterization of solutions for bankruptcy problems $R^n \times R$

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Abstract

This paper provides an analysis of solutions to bankruptcy problems from an axiomatic point of view. In particular, we provide characterizations of certain classes of solutions involving the properties of linearity, symmetry and efficiency. Furthermore, we show that there is a unique solution satisfying the previous axioms and inessentiality.

Keywords: Bankruptcy problems, axiomatic solutions.

JEL Classification: A120, C710, C020.

Resumen

Este documento ofrece un análisis de las soluciones a los problemas de bancarrota desde el punto de vista axiomático. En particular, ofrecemos caracterizaciones de ciertos tipos de soluciones que implican las propiedades de linealidad, simetría y eficiencia. Además, mostramos que hay una única solución que satisface los axiomas anteriores e inesencialidad.

Palabras clave: problemas de bancarrota, soluciones axiomáticas.

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1 Introduction

A bankruptcy problem is a distribution problem involving the allocation of a given amount of a single (perfectly divisible) good among a group of agents, when this amount is insufficient to satisfy all their demands. This type of problem arises in many real life situations. The canonical example is that of a bankrupt firm that is to be liquidated; namely, a situation in which the creditors' entitlements exceed the worth of the firm. Another familiar example refers to the division of an estate among several heirs when the estate falls short of the deceased's commitments. A different case is that in which, in a fixed-price setting, the demand for a given commodity exceeds the available supply. The collection of a given amount of taxes in a community can also be given this form.

The available quantity of the good to be divided is usually called the estate. The agents are also referred to as creditors, whereas the term claims is meant to describe the agents' entitlements, demands or needs, depending upon the problem at hand. A solution to a bankruptcy problem is to be interpreted as the application of an allocation rule that gives a sensible distribution of the estate as a function of agents' claims. Therefore, we are interested in the analysis of axioms or properties that can be applied to all bankruptcy problems, rather than in the solvability of a particular one.

Alternative rules typically represent different ways of applying some ethical principles and some operational criteria to the resolution of bankruptcy problems. The analysis of their structural properties permits one to select a particular rule by choosing the set of these properties that this rule satisfies. This venue becomes more fruitful the closer we get to the following recommendations:

- Each property is intuitive and represents a single and clear ethical principle.
- We can identify each rule as the only one satisfying a distinctive set of properties (that is, a collection of these properties characterizes the rule); moreover all these properties are logically independent.
- This set of distinctive properties is small whereas alternative rules share most of the properties (in order to clearly identify their ethical differences).

Structural properties express invariance of the solutions with respect to changes in the parameters, and are usually motivated by particular concerns. They are intended to ensure that the solution has some desirable features or to prevent some inconveniences. Hence it is not surprising that a particular rule can be characterized by different sets of independent axioms. Each characterization provides an insight on the type of problems for which a rule is satisfactory. The reader is referred to Thomson (1998) for a discussion of the axiomatic method.

The resolution of bankruptcy-like situations is a major practical issue and has a long history as a conceptual problem (see the references provided in Rebinovitch, 1973; O'Neill, 1982; Aumann and Maschler, 1985; Young, 1994, Ch. 4). Modern economic analysis has addressed this class of problems from two main perspectives. The first one is the game theoretic approach, where a bankruptcy problem is formulated either as a TU game or as a bargaining problem (see, for instance, O'Neill, 1982; Aumann and Maschler, 1985; Curiel et al., 1988; Dagan and Volij, 1993). The second one is the axiomatic method, where alternative solutions are characterized in terms of intuitive properties that express different value judgements (e.g., Young, 1987; Dagan, 1996; Herrero et al., 1999; Chun, 1988a). The reader is referred to Thomson (2003) and Moulin (2001) for a survey of this literature.

This paper provides an analysis of solutions to bankruptcy problems from an axiomatic point of view. In particular, we provide characterizations of certain classes of solutions involving the properties of linearity, symmetry and efficiency. Furthermore, we show that there is a unique solution satisfying the previous axioms and inessentiality.

2 Preliminaries

A bankruptcy problem occurs when a company goes bankrupt owing money to some investors, but the company has only an amount E to cover debts. Investors demand quantities $d_1, d_2, \ldots, d_n$ so that the sum of these claims exceeds the amount E to be distributed. The problem can also motivate a tax problem, the d_i represents income from taxpayers and E represents what the government needs to raise.

Definition 1 *Let* $N = \{1,...,n\}$ *be a finite set of agents (creditors). A general bankruptucy problem is an ordered pair* (D, E)*, where* $E \in R$ *and* $D = (D_1, D_2, ..., D_n) \in R^n$ *such that* $D_i \geq 0$ *for all* $1 \leq i \leq n$ *and* $0 \leq E \leq \sum_{j \in N} D_j$.

We suppose a problem with n creditors and we interpret D_i as the amount that the i-th creditor demands, whereas E is the total amount that may be repaid.

We can think bankruptcy problems as elements $(D,E)\in R^n \times R$ such that $\sum_{j\in N} D_j \geq E$.

Given $(D,E),(D^{'},E^{'})\in R^n \times R$ and $c\in R$, we define the sum $(D,E)+(D^{'},E^{'})$ and the product $c(D,E)$, in $R^n \times R$, in the usual form, i.e.,

$$(D,E)+(D^{'},E^{'})=(D+D^{'},E+E^{'}) \text{ and } \quad c(D,E)=(cD,cE)$$

respectively. It is well known that $R^n \times R$ is a vector space with these operations, and dim $R^n \times R = n+1$.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta$ is bijective$\}$, acts on R^n and on $R^n \times R$ in the natural way; i.e., for $\theta \in S_n$:

$$\theta \cdot (x_1, x_2,..., x_n) = (x_{\theta(1)}, x_{\theta(2)},..., x_{\theta(n)})$$

and

$$\theta \cdot (D,E) = (\theta \cdot D, E)$$

Here, S_n is acting trivially on R.

Definition 2 *An allocation for a bankruptcy problem* (E,d), *is an* n*-tuple* $x=(x_1,x_2,\ldots,x_n)\in R^n$ *of real numbers satisfying* $\sum_{j\in N} x_j = E$, *where* x_i *represents the amount allocated to creditor* i, $1\leq i\leq n$.

So, we are interested in solutions on $R^n \times R$, which are just operators of the form

$$\varphi : R^n \times R \to R^n$$

where $\varphi_i(D,E)$ represents the amount that creditor i will receive from the bankruptcy problem (D,E).

Next, we define some desirable axioms which are asked solutions $\varphi : R^n \times R \to R^n$ to satisfy :

Axiom 3 (Linearity) *The solution* φ *is linear if*

$\varphi(D,E)+(D^{'},E^{'})]=\varphi(D,E)+\varphi(D^{'},E^{'})$ *and* $\varphi c(D,E)]=c\varphi(D,E)$, *for all* $(D,E),(D^{'},E^{'})\in R^n\times R$ *and* $c\in R$.

Axiom 4 (Symmetry) *The solution* φ *is said to be symmetric if and only if*

$$\varphi\theta\cdot(D,E)]=\theta\cdot\varphi(D,E)$$

for every $\theta\in S_n$ and $(D,E)\in R^n\times R$, where the problem $\theta\cdot(D,E)$ is defined as $\theta\cdot(D,E)=(\theta\cdot D,E)$.

Axiom 5 (Efficiency) *The solution* φ *is efficient if*

$$\sum_{i\in N}\varphi_i(D,E)=E$$

for all $(D,E)\in R^n\times R$.

Axiom 6 (Inessentiality) *The solution* φ *is inessential if*

$$\varphi\left(D,\sum\nolimits_{j\in N}d_j\right)=D$$

for every $\left(D,\sum\nolimits_{j\in N}d_j\right)\in R^n\times R$.

3 Characterizations

In this section we present some characterization of solutions, combining the above properties. For example, an expression for all linear and symmetric solutions:

Proposition 7 *If the solution* $\varphi:R^n\times R\to R^n$ *satisfies the linearity and symmetry axioms, then there exist unique real numbers* α,β,γ *such that*

$$\varphi_i(D,E)=\alpha D_i+\beta E+\gamma\sum_{j\in N}D_j \qquad (1)$$

Conversely, for any real numbers α,β,γ, the solution given by (1) is linear and symmetric.

Proof. Let $\varphi : R^n \times R \to R^n$ be a linear and symmetric solution. Since the bankruptcy problems $\{(e_k,0)\}_{k\in N} \cup \{(\bar{0},1)\}$ form a basis[5] of $R^n \times R$, then for every problem $(D,E)\in R^n \times R$, $(D,E) = \sum_{j\in N} D_j \cdot (e_j,0) + E\cdot(\bar{0},1)$. Set $a_{i,j} = \varphi_i(e_j,0)$ if $i \neq j$, $b = \varphi_i(e_i,0)$ and $\beta = \varphi_i(\bar{0},1)$, then

$$\varphi_i(D,E) = \sum_{j\in N} D_j \cdot \varphi_i(e_j,0) + E\cdot \varphi_i(\bar{0},1)$$

$$= \sum_{j\in N\setminus\{i\}} a_{i,j}D_j + bD_i + \beta E$$

for every $i \in N$.

Now, let $k,l \in N\setminus\{i\}$ and $\theta \in S_n$ be such that $\theta(k) = l$ and $\theta(i) = i$. Since $\theta\cdot(e_k,0) = (e_l,0)$ then, by symmetry:

$$\varphi_i(e_k,0) = \varphi_{\theta(i)}(e_k,0) = \varphi_i(\theta\cdot(e_k,0)) = \varphi_i(e_l,0)$$

Therefore, $a_{i,k} = a_{i,l}$ if $k,l \in N\setminus\{i\}$ are such that $k \neq l$. Thus, if we set $\gamma = a_{i,j}$ for each $j\in N\setminus\{i\}$ and $\alpha = b-\gamma$, we obtain

$$\varphi_i(d,E) = \gamma \sum_{j\in N\setminus\{i\}} D_j + bD_i + \beta E$$

$$= \alpha D_i + \beta E + \gamma \sum_{j\in N} D_j$$

Uniqueness: to check uniqueness it is enough to prove that if

$$0 = \alpha D_i + \beta E + \gamma \sum_{j\in N} D_j$$

for every bankruptcy problem (D,E) and for every creditor i, then the numbers α,β,γ vanish.

Thus, for given $\beta\in R$ let $i\in N$ and $(D,E) = (\bar{0},1)$. Then the above sum reduces to

$$0 = \beta$$

Similarly, given $\gamma \in R$ let $i,j\in N$ such that $i\neq j$ and $(D,E) = (e_j,0)$. In this case the sum is just

$$0 = \gamma$$

[5] $\{e_k\}_{k=1}^n$ denotes the standard basis for R^n.

And for given $\alpha \in R$ let $i \in N$ and $(D,E)=(e_i,0)$. Then we get

$$0=\alpha$$

Finally, it is straightforward to check that formula (1) defines a linear and symmetric solution for any choice of coefficients.

Once we have a global description of every linear symmetric solution, we can add other axioms and characterize such solutions, like the efficiency property:

Theorem 8 *The solution* $\varphi : R^n \times R \to R^n$ *satisfies linearity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(D,E)=\frac{E}{n}+\delta\left(D_i-\frac{1}{n}\sum_{j\in N}D_j\right) \qquad (2)$$

for any real number δ.

Moreover, such representation is unique.

Proof. By the previous Proposition,

$$\varphi_i(D,E)=\alpha D_i+\beta E+\gamma\sum_{j\in N}D_j$$

for some constants α,β,γ.

Efficiency implies:

$$0=\sum_{i\in N}\varphi_i(e_i,0)=\alpha+n\gamma$$

and

$$1=\sum_{i\in N}\varphi_i(\bar{0},1)=n\beta$$

Hence,

$$\alpha=-n\gamma \quad \text{and} \quad \beta=\frac{1}{n}$$

Therefore

$$\varphi_i(D,E)=-n\gamma D_i+\frac{E}{n}+\gamma\sum_{j\in N}D_j$$

Set $\delta=-n\gamma$, then

$$\varphi_i(D,E) = \frac{E}{n} + \delta\left(D_i - \frac{1}{n}\sum_{j\in N} D_j \right)$$

The converse is a straightforward computation, and uniqueness follows from the uniqueness part of Proposition 5.

Now, adding the inessential axiom to the class of linear symmetric solutions, we get a formula for all solutions satisfying these properties:

Theorem 9 *If the solution* $\varphi : R^n \times R \to R^n$ *satisfies the linearity, symmetry and inessential axioms, then there exist a unique real number* γ *such that*

$$\varphi_i(D,E) = D_i + \gamma\left(\sum_{j\in N} D_j - E \right) \tag{3}$$

Conversely, for any real number γ, the solution given by (3) is linear, symmetric and inessential.

Proof. First of all, it is easy to verify that the bankruptcy problems $\{(e_k,1)\}_{k\in N} \cup \{(\bar{0},1)\}$ form a basis for $R^n \times R$.

By Propostition 21,

$$\varphi_i(D,E) = \alpha D_i + \beta E + \gamma \sum_{j\in N} D_j$$

for some constants α, β, γ.

Inessentiality implies:

$$1 = \varphi_i(e_i,1) = \alpha + \beta + \gamma$$

and

$$0 = \varphi_j(e_i,1) = \beta + \gamma$$

Therefore, $\alpha = 1$ and $\beta = -\gamma$. Then

$$\varphi_i(D,E) = D_i + \gamma\left(\sum_{j\in N} D_j - E \right)$$

The proof in the other direction is straightforward, and again the uniqueness follows from the uniqueness part of Proposition 21.

Finally, we can now state the main result of this section:

Theorem 10 *The solution* $\varphi : R^n \times R \to R^n$ *given by*

$$\varphi_i(D,E) = D_i + \frac{1}{n}\left(E - \sum_{j\in N} D_j\right) \quad (4)$$

for each $i \in N$ and each bankruptcy problem $(D,E) \in R^n \times R$; is the unique solution satisfying linearity, symmetry, efficiency and inessentiality axioms.

Proof. Recall that the collection of bankruptcy problems $\{(e_k,1)\}_{k\in N} \cup \{(\bar{0},1)\}$ constitutes a basis of $R^n \times R$.

It is easy to check that the solution given by (4) is of the form (2), so it satisfies linearity, symmetry and efficiency axioms; and it is straightforward to prove that it also satisfies the inessentiality axiom. We prove uniqueness. Let φ be a solution satisfying the four axioms. For any $(D,E) \in R^n \times R$, there exist unique real numbers $\{\lambda_k\}_{k=1}^{n+1}$ such that $(D,E) = \sum_{k\in N} \lambda_k (e_k,1) + \lambda_{n+1}(\bar{0},1)$. Then, by linearity $\varphi(D,E) = \sum_{k\in N} \lambda_k \varphi(e_k,1) + \lambda_{n+1}\varphi(\bar{0},1)$. We will show that $\varphi(D,E)$ is determined for all $(D,E) \in R^n \times R$ and by the previous discussion, it is therefore sufficient to show that $\varphi(e_k,1)$ is determined for all $k \in N$ and also to determine $\varphi(\bar{0},1)$. In this way, φ is unique for each $(e_k,1),(\bar{0},1)$ and so for (D,E).

Notice that,

By the inessentiality axiom,

$$\varphi_i(e_i,1) = 1$$

for each $i \in N$.

If $i, j, l \in N$ and $\theta \in S_n$ is such that $\theta(i) = i$ and $\theta(j) = l$, then $\theta \cdot (e_i,1) = (e_i,1)$ and by symmetry: $\varphi_l(e_i,1) = \varphi_{\theta(j)}(e_i,1) = \varphi_j(\theta \cdot (e_i,1)) = \varphi_j(e_i,1)$.

Hence, by the efficiency axiom,

$$\varphi_i(e_j,1) = 0$$

for each $i \in N \setminus \{j\}$.

If $j, l \in N$ and $\theta \in S_n$ is such that $\theta(j) = l$, then by symmetry:
$\varphi_l(\bar{0},1) = \varphi_{\theta(j)}(\bar{0},1) = \varphi_j(\theta \cdot (\bar{0},1)) = \varphi_j(\bar{0},1)$.

Thus, by the efficiency axiom,

$$\varphi_i(\bar{0},1) = \frac{1}{n}$$

for each $i \in N$.

In the decomposition of (D,E), the precise values of the numbers $\{\lambda_k\}_{k=1}^{n+1}$, are

$$\lambda_k = \left\{ ll D_k if k \in N E - \sum_{j\in N} D_j if k = n+1 \right.$$

And so,

$$\varphi_i(D,E) = \sum_{k\in N} \lambda_k \varphi_i(e_k,1) + \lambda_{n+1}\varphi_i(\bar{0},1)$$

$$= D_i + \frac{1}{n}\left(E - \sum_{j\in N} D_j \right)$$

4 Conclusions

We have provided three basic axioms (linearity, symmetry and efficiency) in order to characterize solutions to bankruptcy problems. Our Theorem 8 provides a general form characterizing every solution but it depends of one real parameter. Introduction of inessentiality axiom permit us to get a unique solution in Theorem 10. However there is a problem, some coordinates of solution can be negative. Our result is very general but we need to change last axiom or some of the others in order to characterize the most known solutions like, proportional, CEA or CEL. This will be doing as a future research.

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