ISTANBUL TECHNICAL UNIVERSITY ⋆ INSTITUTE OF SCIENCE AND TECHNOLOGY

DUALITY IN NONCOMMUTATIVE GAUGE THEORIES
A PARENT ACTION APPROACH

Ph.D. Thesis by

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CONTENT

ABBREVIATIONS

SYM	:	Supersymmetric Yang-Mills
BPS	:	Bogomol'nyi-Prasad-Sommerfield
UV	:	Ultraviolet
IR	:	Infrared
DBI	:	Dirac-Born-Infeld
NC	:	Noncommutative
WZ	:	Wess-Zumino

LIST OF SYMBOLS

$\mu, \nu, \rho, \sigma, ...$	:	Spacetime indices
$\alpha, \beta, \gamma, ...$	:	Supersymmetry indices
$i, j, k, ...$	:	Spatial indices
$A, \lambda, F, ...$	:	Ordinary fields
$\hat{A}, \hat{\lambda}, \hat{F}, ...$	:	Noncommutative fields
α'	:	String tension parameter
$\theta^{\mu\nu}, \Theta^{\mu\nu}$	:	Noncommutativity parameter
$Q^{\alpha}, \bar{Q}_{\dot{\alpha}}$	:	Supersymmetry generators
$\theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}$	:	Supersymmetry parameters

DUALITY IN NONCOMMUTATIVE FIELD THEORIES
A PARENT ACTION APPROACH

SUMMARY

Noncommutative field theories are defined as decoupling limit of the string theory. Hence, they enable us to study the stringy properties by using the field theory language. Duality is a powerful tool in physics to investigate the different properties of a model. S duality is especially important in the noncommutative gauge theories since it produces results peculiar to the noncommutative case: if one has a space/space noncommutative theory, then S duality leads to a space/time noncommutative theory. In such a theory although it is not possible to define hamiltonian by using the usual quantization procedure because of the noncommuting time variable, it is shown that one can define hamiltonian starting from a parent action. This hamiltonian can be used to define the worldvolume theory of $D3-$brane and hence its BPS states can be studied.

Parent action formalism is an appropriate tool for studying dual theories. One can define the partition functions of dual theories by using the path integral formulation of parent action without any other machinery. Although, on the contrary of ordinary theory it is not possible to define an explicit transformation between the partition functions in noncommutative case, it is shown that their partition functions are equivalent.

On the other hand, parent action formalism can be used to study duality in the supersymmetric generalization of the noncommutative $U(1)$ gauge theory. For this aim, supersymmetric generalization of the Seiberg-Witten map must be defined. Definition of parent action is not unique and this leads to different dual theories.

KOMÜTATİF OLMAYAN ALAN TEORİLERİNDE DUALİTE PARENT EYLEM YAKLAŞIMI

ÖZET

Komütatif olmayan (noncommutative) ayar kuramları sicim teorisinin ayrışma (decoupling) limitini tanımlarlar. Böylece, sicim kuramının özelliklerinin alan kuramı diliyle çalışılmasına imkan verirler. Dualite bir modelin farklı özelliklerinin anlaşılması için güçlü bir araçtır. Komütatif olmayan ayar kuramlarında S dualitenin çalışılması bu tip teorilere özgü yeni sonuçlar vermeleri nedeniyle ayrıca önemlidir. Uzay koordinatları arasında komütatif olmama özelliğinin tanımlı bulunduğu bir kuramda, S dualite zaman ve uzay koordinatları arasında komütatiflik bulunmayan bir kurama yol açar. Bu tip bir ayar kuramında Hamilton fonksiyonun tanımlanabilmesi bilinen kuantizasyon yöntemleriyle mümkün değilken, bir parent eylemden başlanarak bunun yapılabileceği gösterilebilir. Bu yolla $D3-$zarlarının yaşam hacim (worldvolume) kuramlarının ve bunların BPS durumlarının çalışılması mümkün olmaktadır.

Dual kuramların çalışılması için parent eylem yöntemi uygun bir araçtır. Başka bir araca gerek kalmaksızın parent eylemin path integral formalizminden hareketle dual kuramların bölüşüm fonksiyonları hesaplanabilmiştir. Her ne kadar komütatif olmayan durumda dual kuramların bölüşüm fonksiyonları arasında komütatif durumdakine benzer şekilde açık bir dönüşüm tanımlanamasa da bunların bölüşüm fonksiyonlarının eşdeğer oldukları gösterilmiştir.

Komütatif olmayan $U(1)$ ayar kuramının süpersimetrik duruma genelleştirilmesi halinde dualitenin kurulabilmesi amacıyla benzer şekilde parent eylemden yararlanılabilir. Bunun için Seiberg-Witten gönderiminin süpersimetrik duruma bir genelleştirilmesi tanımlanmalıdır. Parent eylemin farklı tanımlamalarının mümkün olmasına bağlı olarak değişik sonuçlara ulaşılır.

1 INTRODUCTION

The notion of the space has undergone some radical changes during the improvement of the physical theories. After the development of non-Euclidean geometry another example of such a great change in thinking the notion of space may be the noncommutative geometry. Actually until to come to the pioneering paper of Alain Connes [1, 2] there has already been a great amount of examples for such spaces both in physics and mathematics. For example Penrose tiling, noncommutative tori, leaf spaces of foliations, Adela class space and the duals of nonabelian groups.

At the level of topology, this issue is a part of the algebraic topology. A topological space may be completely characterized by the algebra of continuous complex-valued functions defined on it: given the continuity requirement of all functions on the manifold one may reconstruct the topology. By knowing the associative, commutative algebra $\mathcal{A}$ of the complex-valued functions one could still reconstruct the manifold M. That is due to Gel′fand $-$ Naimark theorem which makes possible to construct formally a topological space M for which $\mathcal{A}$ is naturally isomorphic to the space of functions [1, 2]. At this stage it is natural to ask what happens if one chooses an associative but noncommutative algebra $\mathcal{A}'$ for example the algebra of $N \times N$ complex-valued matrices. Noncommutative spaces result from this question. The definition of the Gel′fand transform, which is used to reconstruct a space from an algebra, becomes ambiguous for noncommutative algebras, and it is not possible to formally reconstruct the space M in this case. This ambiguity can be resolved by the Morita equivalence. The Morita equivalent spaces share many common geometrical characteristics, for example they have the same $K-$theory and cyclic homology but gauge theories or more precisely vector bundles defined over them can be very different.

Historically the first example of these spaces appear in the field theory context. At the beginning stage of constructing quantum field theory one of the most difficult problem was the divergences problem. To overcome this difficulty before developing a systematic renormalization procedure, it was suggested that replacing the ordinary spacetime coordinates x^μ with the hermitean operators $\hat{x}^\mu$ which satisfy the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{1.1}$$

this would cure the problem [3] [1]. Here $\theta^{\mu\nu}$ in general can be a constant, a function of coordinates or a function of both coordinates and momenta. In the first case theory satisfies a canonic relation and this type of theories emerge from the quantization of string theories in a background field. In the latter case noncommutativity parameter is in form of $\theta^{\mu\nu} = C^{\mu\nu}_\rho x^\rho$. It defines a Lie-algebra and this type of theories are typical in some quantum gravitational models. For instance, in the case of $C^{0i}_j = 1/\kappa$ and $C^{ij}_k = 0$ spacetime is called as $\kappa-$Minkowski spacetime and is related to a quantum deformation of Poincaré group. The commutation relations (1.1) defined between the coordinates lead to a spacetime uncertainty relation which resembles the Heisenberg uncertainty relation

$$\Delta\hat{x}^\mu \Delta\hat{x}^\nu \geq \frac{1}{2} \mid \theta^{\mu\nu} \mid \tag{1.2}$$

As a result of this uncertainty relation the notion of point on the space is no longer meaningful and a Planck cell is defined instead of the point. By this approach to define the physical processes at the $\theta-$scale one should remove the notion of the point and should work with the element of the noncommutative algebra defined above. Hence the ultraviolet divergences in the quantum field theory can be regularized by putting an ultraviolet cutoff Λ in the momentum space integrals. This cutoff in the momentum space is a direct consequence of defining a fundamental length scale which is given by Λ^{-1}. Below this length scale all the events can be neglected.

Another assumption that the structure of the spacetime at small length scale should be deformed arises from the point of view of the general relativity. The energy at which gravity and quantum effects become of comparable strength is given by the Planck energy. At the length scales corresponding to the Planck energy, the quantum gravitational fluctuations become dominant and cannot be ignored [4]. As a consequence of this, the spacetime becomes "fuzzy" at the very short distances and ordinary geometry notions fails to define the spacetime structure [5, 6].

[1] Actually the original idea is referred to the name of Heisenberg

Thus, one needs to modify the classical geometry notions. These modifications can be related to the uncertainty relations defined in (1.2). Such a minimal length scale predictions arise in the different quantum gravity models [7, 8, 9, 10, 11] and they coincide with the fundamental postulate of the noncommutative geometry which is nonlocalization property of spacetime.

So far we saw that some intuitive approaches on the problems which we come across in the different context of physics leads to an unfamiliar geometry definition. But it is still an unforeseen argument which is forced to the theory from outside. If we think that nocommutativity defines the nature at small length scales we should find it somewhere in more fundamental theories. String theory is a candidate for such a fundamental theory and it is natural to expect that noncommutativity is included in it. There also exist other physical theories in which above deformed spacetime structure arise naturally. M-theory is one of them. The known five perturbative superstring theories can be obtained from a single $11-$dimensional theory which is called M-theory. String theories correspond to the low energy limit of this single theory. Matrix-model is known a formulation of M-theory and according to this conjecture each momentum sector of the discrete light cone quantization of M-theory is described by a maximally supersymmetric Matrix-model (or Supersymmetric Yang-Mills theory) with the light cone momentum identified with the rank of gauge group. It is believed that to form a formulation of M-theory, when Matrix model compactified on a circle it must yield the string theory. Different compactifications of the Matrix model on different manifolds are possible. A class of toroidal compactifications were constructed in early stages of Matrix model development, which relied on a certain commutative subalgebras of matrices [12, 13]. Connes, Douglas and Schwarz introduced the noncommutative spaces as possible compactification manifolds. By this noncommutative space compactifications one obtain some different physical consequences. It can be summarized that it corresponds to adding a constant $3-$form background in the 11-dimensional supergravity and a major result is the Supersymmetric Yang-Mills (SYM) theory of commutative torus compactification now becomes a deformed SYM theory [14, 15]. Later it was proved that this deformed SYM theory and therefore indirectly the noncommutative torus compactifications can be realized as certain D-brane configurations in string theory

[14]. Subsequently, compactifications on more complicated spaces were studied [16] and various properties of the deformed SYM theory and their relation to string theory were considered [17, 18, 19].

Duality appears in several different context of physics. Dual theories provide to construct two different but equivalent description of the same model in the two different interaction regimes by using in general different fields. The relation between the fields is in general not known explicitly and in the most of the cases it contains nonlinear terms. An exception of this situation appears in the two-dimensional quantum field theory models. The solitons in the Sine-Gordon theory $S(\phi)$ correspond to the fermions of the massive Thirring model $S(\psi)$, where $\phi \sim \bar{\psi}\psi$ [20, 21, 22]. The possibility of writing fermions in terms of bosons (bosonization) has been a powerful method for obtaining nonperturbative information. Some of the other important dualities are Hodge duality, electric-magnetic duality, Montonen-Olive duality and string theory dualities (S,T and U duality). The importance of the S dual theories is that they enable us to work of the weak and strong coupling properties of any theory. Thus knowing the explicit relation between the fields allows perturbative calculations in the variables of the original theory both in the strong and weak coupling regimes. It should be noted that hereafter whenever we mention the duality it must be understood as S-duality unless it is stated explicitly.

Electric-magnetic duality exchanges the electric degrees of freedom of theory with the magnetic degrees of freedom. It also exchanges the electric charge quanta with the magnetic charge quanta. Electric charge quanta at the same time related to coupling constant of theory. Such a transformation, if it can be constructed, will map the strongly coupled electric degrees of freedom of theory to weakly coupled magnetic degrees of freedom of it. Hence different phases of the gauge theories can be investigated. This is especially important in the non-Abelian gauge theories, for example in QCD. Superconductivity is explained by condensation of electric charges in which magnetic fields confine, i.e, when two magnetic monopoles (for example ends of a long magnet) inserted in it potential between monopoles become linear. The dual picture of this event explains the quark confinement problem: if magnetic monopoles condense instead of electric charges, then magnetic currents are superconducting while electric charges are confined. Recently

Seiberg and Witten showed that breaking of $N = 2$ supersymmetric Yang-Mills theory down to $N = 1$ gives a semi-realistic theory of electric confinement by using a kind of electric-magnetic duality [24].

The plan of the thesis as follows: In chapter-2 we give a brief summary of the noncommutativity. Here we also give the cornerstones of the Seiberg-Witten map. This map defines a field redefinition of the noncommutative gauge field and gauge parameter in terms of the commutative ones such that both of theories constructed from that gauge fields and gauge parameters define the same physics in terms of different fields. We discuss in what conditions such an equivalence can be constructed and what are the relations between the fields.

In chapter-3 we present hamiltonian formulations of noncommutative $U(1)$ gauge theory and its dual. Dual theory has a time/space noncommutativity whereas the original theory has noncommutativity among the spatial coordinates [82]. In quantum mechanics time is the evolution parameter of the system. Contrary to the coordinate and momentum variables of the particle, time is not an operator and therefore it is not obvious what one means by the noncommutativity of time. Moreover, in such a case it is not apparent how quantization procedure can be defined. Nevertheless, examples which produce such a time/space noncommutativity arise in different cases and in string theory context it is unavoidable in a manner. For instance, the noncommutativity between the space coordinates occurs when a D-brane considered in a constant background B field which has nonvanishing components along the space directions. When background field has nonzero B_{0i} components, in other words when D-brane is put in an electric background field such a time/space noncommutativity emerges [61]. Actually an uncertainty relation between time and space can be derived from string uncertainty principles even when no electric background is present [27]. All of that leads to a better understanding of the notion of time in string theory. We will propose an alternative way to construct the hamiltonian of the space/time noncommutative theory [28]. The parent action will be the starting point of our approach. Bypassing the usual quantization procedure we were enable to obtain the hamiltonian. We show that although the time coordinate is noncommuting with the spatial coordinates it works effectively as if commuting. Under the light of these results we worked the BPS states of the noncommutative $D3-$brane.

In this chapter we will also discuss how electric-magnetic duality transformation is defined for Lagrange and Hamilton densities of noncommutative $U(1)$ gauge theory [46].

Chapter-4 contains the partition function analysis of the dual theories. Here we will focus on the construction of the partition functions starting from the parent action which yields respective partition function of both dual and original theory with respect to phase space integrations. First of all we develop formalism for ordinary $U(1)$ gauge theory. Later it is going to be extended to the noncommutative case. The results of ordinary case are compatible with the previous one which is obtained by a canonical transformation [62]. It is shown that the partition function of noncommutative $U(1)$ gauge theory and its dual are equivalent [46].

Generalization of the Seiberg-Witten map to the supersymmetric theory can be formulated in some different ways. One of them is to generalize the definition of the map by using the superfields [94]. In another one generalization is defined by using the solution of ordinary Seiberg-Witten map [95]. Duality for ordinary supersymmetric $U(1)$ is defined in terms of superfields [24]. For noncommutative supersymmetric theory duality is investigated via parent action method by using the component fields in [87]. Parent action construction is not unique. Therefore it makes possible to define different parent actions. Chapter-5 is devoted to this discussion. We will give two different parent action constructions of the duality. Although they yield the same dual theories in the ordinary case they differ for supersymmetric noncommutative theory in some ways. Somehow dual symmetry breaks and dual theories do not possess the ordinary properties. This point will be discussed and clarified. Another parent action will be proposed in order to reconstruct the symmetry and some related problems are considered.

The last chapter include the results and conclusions.

2 NONCOMMUTATIVITY IN PHYSICS

Beside the other fields of physics, general relativity provides powerful evidences that spacetime coordinates at small length scales can not be thought classically no longer in the sense that the usual notion of point is meaningless. Relativity defines the gravitation as geometry of spacetime and at the very short distances the notion of point of ordinary geometry is effected by the quantum fluctuations and hence is lost its meaning. This process deform the spacetime structure at this scale and points of space become “fuzzy”. At the most fundamental level let us consider to localize a particle to a space region of Planck size $\lambda_P \sim 10^{-33}$cm. This requires an amount of energy equivalent to Planck mass $m_P \sim 10^{19}$Gev/c^2. Black hole radius is given by $R = \frac{Gm_{\text{eff}}}{2c^2}$ where $m_{\text{eff}} = \frac{E}{c^2}$. If we consider particle's energy is localized in a space region of size a then the energy density $E = \hbar c/a$ is well defined for big a. At the limit $R = a$

$$a = \sqrt{\frac{G\hbar}{c^3}} \cong 10^{-33}\text{cm} \tag{2.1}$$

we obtain the Planck length. This implies that there exist an effective minimum length scale which form a physical bound to quantization of space. The physical events below this limit cannot be determined. One can avoid this paradox by introducing the spacetime uncertainty relation

$$\sum \Delta x^i \Delta x^j \geq \lambda_P^2 \tag{2.2}$$

Such an uncertainty relation which we familiar from quantum mechanics is a natural consequence of the commutation relation (1.2) between the coordinates. Thus we conclude from this simple example that quantization of general relativity may bring the spacetime noncommutativity.

String theory is one of the most promising candidate to being a theory of everything. In this context it also provides a consistent picture of quantum theory of gravitation. It is sensible to expect the emergence of above results in string picture. Strings have an intrinsic length l_s and using them as probes in investigation of spacetime will not give the information under this intrinsic length. String scattering amplitudes at ultra-high energies lead to string modified Heisenberg uncertainty relations

$$\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p} + l_s^2 \Delta p\right) \tag{2.3}$$

At the point particle limit, i.e. when $l_s \to 0$, this relation reduce to the standard phase space relation. Minimizing the equation (2.3) with respect to Δp will give a lower bound $\Delta x \geq \Delta x_{min}$ on the measurability of distances in the spacetime which is, as can be guessed, the length of the string

$$\Delta x_{min} = l_s \tag{2.4}$$

It is possible to realize such length scales using as probes not strings themselves but rather certain nonperturbative open string degrees of freedom known as D-branes. In fact using these objects allow one probe even shorter, sub-Planckian distance scales in string theory and they enable microscopic derivations of fairly generalized uncertainty relations which include those described above as a subset. They are therefore the natural degrees of freedom which capture phenomena related to quantum gravitational fluctuations of the spacetime and hence will be important in investigation of the spacetime noncommutativity.

2.1 Landau Problem

Landau problem [29] in a sense provides a prototype example of the noncommutativity in string framework which we are going to deal with in the next section. There exist phenomenological resemblance between two cases at least at first sight. Landau problem deals with the dynamics of the particles which are constrained to move in a two-dimensional plane with an external magnetic field $\mathbf{B}$ perpendicular to the plane. A cyclotronic motion in plane results from the interaction of magnetic field with particles due to the particles constrained in the plane. In certain limits the configuration space in which physical observables of the system take place shows interesting properties. Noncommutativity of space coordinates is one of them.

Noncommutativity in string framework comes from the quantization of open strings which are attached to a Dp-brane in presence of a background field $B_{\mu\nu}$. A Dp-brane is defined with p spatial dimensions and string end points can move on the brane worldvolume freely in absence of a background field. This background

Kalb-Ramond field form string analogue of the magnetic field in the Landau problem and couples to a string charge which corresponds to the electrical charge in the Landau problem. There is also a Maxwell ($U(1)$) gauge field lives on the D-brane and couples to the string end points. Quantization of this configuration leads to noncommutativity which will be considered in detail later. In this chapter we will review the former case. For a detailed discussion of these issues see [26]. Position of particles will be considered in the $xy-$plane

$$\mathbf{r}_i = (x_i, y_i), \qquad i = 1, 2, \cdots, N \tag{2.5}$$

and the gauge which produces the external constant magnetic field $\mathbf{B}$ is chosen as

$$\mathbf{A}(\mathbf{r_i}) = (0, Bx_i) \tag{2.6}$$

Lagrangian of the system is

$$L = \sum_{i=1}^{N} (\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \frac{e}{c} \dot{\mathbf{r}}_i \mathbf{A}(\mathbf{r}_i) - V(\mathbf{r}_i)) \tag{2.7}$$

where m_i is the particles mass and V is the electron self-energy. To quantize the system define canonical momentum

$$\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i + \frac{e}{c} \mathbf{A} \tag{2.8}$$

Canonical hamiltonian is

$$H_c = \sum_i \mathbf{p}_i \dot{\mathbf{q}}_i - L = \sum_{i=1}^{N} \frac{1}{2m_i} \pi_i^2 + V(\mathbf{r}_i) \tag{2.9}$$

here $\pi_i = m_i \dot{\mathbf{r}}_i$ shows noncanonical kinematical momentum which is related to canonical momentum by $\pi_i = \mathbf{p}_i - \frac{e}{c}\mathbf{A}$. Canonical commutation relations can be defined as

$$\begin{aligned} [x_i, p_j^x] &= i\hbar\delta_{ij} = [y_i, p_j^y], \\ [r_i, r_j] &= 0 \\ [p_i, p_j] &= 0 \end{aligned} \tag{2.10}$$

It should be noted that while the canonical momentum is not gauge invariant the kinematical momentum preserves the gauge invariance so kinematical momenta π

must be considered as physical objects. They satisfy an intriguing commutation relation

$$[\pi^i, \pi^j] = i\hbar \frac{eB}{c} \epsilon^{ij} \tag{2.11}$$

Thus the physical momenta are defined in a noncommutative space in presence of a background field. In the absence of interactions the hamiltonian can be written in terms of creation and annihilation operators

$$E = \sum_{i=1}^{N} \frac{\hbar\omega_c}{2}(n_i + 1), \qquad n_i = 0, 1, 2, \cdots \tag{2.12}$$

where

$$\omega_c = \frac{eB}{m_i c} \tag{2.13}$$

shows the classical cyclotronic frequency of the electron and these energy eigenvalues are known as the Landau levels. The gap between Landau levels is the constant

$$\Delta = \frac{1}{2}\hbar\omega_c \tag{2.14}$$

Now we consider strong field limit, $B \gg m_e$. In that case Lagrangian (2.7) reduce to

$$L' = \sum_{i=1}^{N} (\frac{eB}{c} x_i \dot{y}_i - V(\mathbf{r}_i)) \tag{2.15}$$

which is of the form $p\dot{q} - H(p, q)$. One can identify the canonical pairs as $(\frac{eB}{c} x_i, y_i)$ and they enjoy

$$[x_i^a, x_j^b] = i\delta_{ij}\theta^{ab} \tag{2.16}$$

where the noncommutativity parameter is given by

$$\theta^{ab} = \frac{\hbar c}{eB} \epsilon^{ab} \tag{2.17}$$

with ϵ^{ab} the antisymmetric tensor. The letters $a, b = 1, 2$ denotes the plane coordinates. Physically at the strong field limit, i.e. $B \to \infty$ or equivalently $m \to 0$, the gap between Landau levels diverges and the lowest level decouples from others. This forces the system to lie in the lowest level. At the same time this process degenerate the phase space into a kind of configuration space. This can be seen from constrained system analysis. The strong field limit turns the hamiltonian into a topological one

$$H' = \sum_{i=1}^{N} V(\mathbf{r}_i) \tag{2.18}$$

There is not any propagating degrees of freedom and kinematical momenta become a constraint of the system: $\pi_i = 0$. Because of the commutation relation (2.11) these are second class. This requires that normal Poisson brackets must be replaced with Dirac brackets. The result is noncommutation of coordinates.
In the following chapter we will interested in how noncommutativity arise in string states with D-branes.

2.2 Noncommutativity in String Theory

A $Dp-$brane is an extended object with p spatial dimensions defined by the property that strings can end on them. The letter D stands for Dirichlet condition. In the presence of a D-brane, the endpoints of open strings must lie on the brane. Among the quantum states of open strings attached to a D-brane we found photon states with polarizations and momentum along the D-brane directions. Thus one can deduce that a Maxwell field lives on the worldvolume of a D-brane. The existence of this Maxwell field was in fact necessary to preserve the gauge invariance of the term that couples the Kalb-Ramond field to the string in the presence of a D-brane. We also know that the endpoints of open strings carry Maxwell charge. Since any D-brane has a Maxwell field, it is physically reasonable to expect that background electromagnetic fields can exist: there may be electric or magnetic fields that permeate the D-brane. Hence the string endpoints couple to the Maxwell potential A_m in the same way as a charged particle does. In the case of N coincident Dp-brane a $U(N)$ gauge field lives on the worldvolume of the brane and this defines a $U(N)$ Yang-Mills theory.
Now consider an open string with its ends on a D-brane such that there exist a constant non-zero, static and uniform background field $B_{\mu\nu}$ in the bulk. Action of this configuration is given by the coupling of the string to this background field in addition to the standard open string action.

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma[\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu g^{ab} + \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu] \tag{2.19}$$

where a, b denote string world sheet coordinates and g^{ab} string world sheet metric whereas $\eta^{\mu\nu}$ shows the spacetime metric. Since $B_{\mu\nu}$ is a constant the second part of action can be written as a surface integral. To get the equations of motion we apply the standard variational principle to the action

$$\delta S=-\int_{\Sigma}d^2\sigma\{\eta_{\mu\nu}g^{ab}\partial_a\partial_b X^\nu\delta X^\mu\}+\int_{\partial\Sigma}d\tau\{-\eta_{\mu\nu}\delta X^\mu\partial_\sigma X^\nu+B_{\mu\nu}\delta X^\mu\partial_\tau X^\nu\}|_{\sigma=0}^{\sigma=\pi} \tag{2.20}$$

Invariance of the action under the variation leads to the equations of motion and the boundary conditions

$$(\partial_\tau^2-\partial_\sigma^2)X^\mu=0, \tag{2.21}$$

and

$$\partial_\sigma X^\mu-B_{\mu\nu}\partial_\tau X^\nu=0,\quad \sigma=0,\pi \tag{2.22}$$

Thus one can see that inclusion of a static background field to the action does not change the equations of motion but does the boundary conditions. This boundary conditions are neither Neumann nor Drichlet but a linear combination of them. The string positions X^μ can be expanded into the mode expansion by taking care of the boundary conditions as follows

$$X^\mu=x_0^\mu+p^\mu\tau+B^{\mu\nu}p_\nu\sigma+\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(ia_n^\mu\cos n\sigma+B_\nu^\mu a_n^\nu\sin n\sigma) \tag{2.23}$$

One can check that this really satisfies the boundary conditions (2.22)

$$\partial_\sigma X^\mu=B_\nu^\mu p^\nu+\sum_{n\neq 0}e^{-in\tau}(-ia_n^\mu\sin n\sigma+B_\nu^\mu a_n^\nu\cos n\sigma) \tag{2.24}$$

and

$$\partial_\tau X^\mu=p^\mu+\sum_{n\neq 0}e^{-in\tau}(a_n^\mu\cos n\sigma-iB_\nu^\mu a_n^\nu\sin n\sigma) \tag{2.25}$$

The momentum terms and the cosine terms cancel each other, leaving the sine terms. Since the boundary condition equations only hold at $\sigma=0$ and $\sigma=\pi$ these terms vanish and equations are satisfied.

Canonical momenta are

$$P_\mu(\tau,\sigma)=\frac{\delta S}{\delta(\partial^\tau X^\mu(\tau,\sigma))}=\frac{1}{2\pi\alpha'}(\partial_\tau X_\mu-B_{\mu\nu}\partial_\sigma X^\nu) \tag{2.26}$$

More precisely

$$2\pi\alpha' P^\mu=\mathcal{M}_\nu^\mu p^\nu+\sum_{n\neq 0}e^{-in\tau}\mathcal{M}_\nu^\mu a_n^\nu\cos n\sigma \tag{2.27}$$

with

$$\mathcal{M}_\nu^\mu\equiv\delta_\nu^\mu-B^{\mu\rho}B_{\rho\nu} \tag{2.28}$$

which is symmetric. To find the commutation relations of the expansion coefficients of X^μ we impose the natural commutation relations on the conjugate pairs

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{2.29}$$

and also

$$[P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = 0 \tag{2.30}$$

From (2.29) using (2.23) and (2.27)

$$\begin{aligned}&[x_0^\mu + p^\mu\tau + B^\mu_\rho p^\rho\sigma + \sum_{n\neq 0}\frac{e^{-in\tau}}{n}(ia_n^\mu \cos n\sigma + B^\mu_\rho a_n^\rho), \\ &\mathcal{M}^\nu_\kappa p^\kappa + \sum_{n\neq 0} e^{-in\tau}\mathcal{M}^\nu_\kappa a_n^\kappa \cos n\acute{\sigma}] = 2\pi i\alpha'\eta^{\mu\nu}\delta(\sigma-\sigma')\end{aligned} \tag{2.31}$$

one can find that

$$[x_0^\mu + p^\mu\tau + \frac{\pi}{2}B^\mu_\rho p^\rho, p^\nu] = 2i\alpha'(\mathcal{M}^{-1})^{\mu\nu} \tag{2.32}$$

and

$$[\frac{1}{n}(e^{-in\tau}a_n^\mu - e^{in\tau}a_{-n}^\mu), (e^{-im\tau}a_m^\nu + e^{im\tau}a_{-m}^\nu)] = 4\alpha'(\mathcal{M}^{-1})^{\mu\nu} \tag{2.33}$$

should be satisfied. Since (2.32) should hold for any value of τ

$$[p^\mu, p^\nu] = 0 \tag{2.34}$$

and

$$[x_0^\mu, p^\nu] = 2i\alpha'(\mathcal{M}^{-1})^{\mu\nu} \tag{2.35}$$

From the equation (2.33) by the same way

$$[a_n^\mu, a_{-m}^\nu] - [a_{-n}^\mu, a_m^\nu] = 4n\alpha'\delta_{nm}(\mathcal{M}^{-1})^{\mu\nu} \tag{2.36}$$

Now using $[P^\mu, P^\nu] = 0$ will give

$$[a_n^\mu, a_{-m}^\nu] + [a_{-n}^\mu, a_m^\nu] = 0 \tag{2.37}$$

Thus we obtain

$$[a_n^\mu, a_m^\nu] = 2n\alpha'\delta_{n,-m}(\mathcal{M}^{-1})^{\mu\nu} \tag{2.38}$$

and also

$$[p^\mu, a_m^\nu] = [x_0^\mu, a_m^\nu] = 0 \tag{2.39}$$

Now let's examine $[X^\mu, X^\nu]$ commutation relations

$$\begin{aligned}[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] \quad &= [x_0^\mu, x_0^\nu] + [x_0^\mu, p^\nu\tau + B_\kappa^\nu p^\kappa \sigma'] + [p^\mu\tau + B_\rho^\mu p^\rho \sigma, x_0^\nu] + \\ &2\alpha' \sum_{n\neq 0} \frac{1}{n}\{-(\mathcal{M}^{-1})^{\mu\nu} \cos n\sigma \cos n\sigma' + i(B\mathcal{M}^{-1})^{\mu\nu} \sin n\sigma \cos n\sigma' \\ &+ i(\mathcal{M}^{-1}B)^{\mu\nu} \cos n\sigma \sin n\sigma' + (B\mathcal{M}^{-1}B)^{\mu\nu} \sin n\sigma \sin n\sigma'\}\end{aligned} \tag{2.40}$$

Utilizing the properties that $(\mathcal{M}^{-1}B)^{\mu\nu}$ is antisymmetric and $B\mathcal{M}^{-1} = \mathcal{M}^{-1}B$ in the above equation one can obtain

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [x_0^\mu, x_0^\nu] + 2i\alpha'(\mathcal{M}^{-1}B)^{\mu\nu}\{\sigma + \sigma' + \sum_{n\neq 0} \frac{1}{n} \sin n(\sigma + \sigma')\} \tag{2.41}$$

The function (on the right hand side in the curly brackets) has the values

$$\sigma + \sigma' + \sum_{n\neq 0} \frac{1}{n} \sin n(\sigma + \sigma') = \begin{cases} 0 & \sigma = \sigma' = 0 \\ 2\pi & \sigma = \sigma' = \pi \\ \pi & \text{otherwise} \end{cases} \tag{2.42}$$

and hence

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = \begin{cases} [x_0^\mu, x_0^\nu] & \sigma = \sigma' = 0 \\ [x_0^\mu, x_0^\nu] + 4\pi i\alpha'(\mathcal{M}^{-1}B)^{\mu\nu} & \sigma = \sigma' = \pi \\ [x_0^\mu, x_0^\nu] + 2\pi i\alpha'(\mathcal{M}^{-1}B)^{\mu\nu} & \text{otherwise} \end{cases} \tag{2.43}$$

There is no information on the x_0^μ commutation but even if it is chosen as 0 we have noncommutativity somewhere. One can conclude that noncommutativity arise in quantization of open strings attached to a D-brane in presence of a background field. This arbitrariness in the x_0^μ commutation has been tried to fix by some different approaches. In the [41] a time averaged symplectic form was proposed and found that it satisfies

$$[x_0^\mu, x_0^\nu] = -2\pi i\alpha'(\mathcal{M}^{-1}B)^{\mu\nu} \tag{2.44}$$

Plugging this result into the (2.43) leads to

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = \begin{cases} -2\pi i\alpha'(\mathcal{M}^{-1}B)^{\mu\nu} & \sigma = \sigma' = 0 \\ 2\pi i\alpha'(\mathcal{M}^{-1}B)^{\mu\nu} & \sigma = \sigma' = \pi \\ 0 & \text{otherwise} \end{cases} \tag{2.45}$$

This shows that open string coordinates are noncommuting at the where they are attached to the brane and hence worldvolume of D-brane also becomes noncommutative. In [42] this result arise from Dirac quantization of the string coordinates which treat the boundary conditions as constraints of system .

2.3 Gauge Theories on the Noncommutative Spaces

In this section the main interest will be the gauge theories on the noncommutative spaces. As it is noted before, open strings attached to D-branes contain in their spectrum a massless vector field. One can find the amplitude of the corresponding vertex operators. In the field theory limit, i.e., in the limit of string tension parameter $\alpha' \to 0$, these amplitudes coincide with those of an ordinary $U(1)$ gauge field theory. If, however, a constant $B_{\mu\nu}$ field with nonzero components only in the space directions parallel to the D-brane is switched on the amplitudes have changed. They are not the amplitudes of an ordinary gauge field theory, rather they correspond to the amplitudes of a noncommutative field theory, in which the noncommutativity parameter is precisely related to the value of the B field. The significance of noncommutative gauge theories can be classified mainly in the two approaches. First since these type of gauge theories are originated from string theory and D-brane worldvolume theories, the results obtained from them can be used to shed light upon the new properties of the string and D-brane theories. Secondly, by going in the reverse direction of the first, some properties of string theories can be understood in the field theory language and noncommutative gauge theories provide a natural framework to this aim. It should be stressed that although the noncommutative gauge theories are the effective theories of the dynamics of the strings there exist an equivalent description in the commutative world in terms of the ordinary fields. But in some cases the noncommutative description provide more powerful and suitable technical tools: for example $T-$duality [30, 31, 32, 33], instantons [34] and soliton solutions [35, 36]. Some other aspects are easier in the context of commutative description; in particular, in $3+1$ dimensions, electric-magnetic duality rotations.

Noncommutative theories enjoy an interesting property under the translations of the gauge fields along the noncommuting directions. Such a translation is equivalent to a gauge transformation [37]. A similar thing just appears in the general relativity where local transformations associated to general coordinate transformations. But when passed to the commutative side by Seiberg-Witten map this equivalence is lost. Instead of that another aspect that can be thought related to gravity emerges. Noncommutative field theories can be interpreted as ordinary theories immersed in a gravitational background generated by the gauge field. The θ dependent terms can be interpreted as a gravitational background which depends on the gauge field [38].

Another important property of these type theories is that they carry some stringy properties into the field theory side. One of them is the nonlocal nature of string theory. As a result of this nonlocality, noncommutative gauge theories share an intriguing property which is called UV/IR mixing.[2] In short it can be explained that if any Feynman diagram requires an ultraviolet cutoff Λ regularizing the graph, this naturally leads to an effective infrared cutoff $\Lambda_\theta = \frac{1}{\Lambda\theta}$. In the renormalization procedure two types of the diagram occur: planar and nonplanar diagrams. Nonplanar diagrams are UV finite. This is the beneficial effect of the UV/IR mixing and arise from the expected effect of the noncommutativity parameter in the high momentum region. However this advantage is compensated by an increasing singularity pattern in the IR sector[3]. This feature is actually due to the lack of the decoupling of the low energy effective field theories from the high energy dynamics. Physically this means that the quanta in noncommutative field theory include extended rigid objects whose length grows with its center of mass momentum. These quanta are responsible for many of the stringy effects that noncommutative field theories exhibit. The dipoles interact by joining at their ends and this gives a simple picture of the nonlocal nature of the interactions in noncommutative quantum field theory.

Noncommutativity can be realized mainly in two different ways. In the operator formalism coordinates and the fields which are functions of these coordinates are considered as operators (infinite size matrices). Coordinate operators satisfy the commutation relation (1.1). This approach based on the Weyl quantization idea [40]. Secondly noncommutativity of coordinates is realized by replacing the ordinary product of fields with a star product. There is an equivalence called Weyl transformation between these two formalisms. To show the equivalence we consider the Weyl quantization procedure. Weyl introduced an elegant prescription

[2]Actually all of the noncommutative field theories share the same property. For example an explanation for the issue in scalar theory see [39]

[3]It turns out that, one of the beginning motivation of the noncommutativity seems to be half achieved.As far as ultraviolet divergences are concerned, it works but in case of the infrared divergences, the problem keeps on surviving

for associating a quantum operator to a classical function of the phase space variables. One can define a noncommutative space by replacing the local coordinates x^i of $\mathbb{R}^d$ by Hermitian operators $\hat{x}^i$ obeying the commutation relation (1.1). The $\hat{x}^i$ then generate a noncommutative algebra of operators. Weyl quantization provides a one-to-one correspondence between the algebra of fields on $\mathbb{R}^d$ and this ring of operators, and it may be thought of as an analogue of the operator-state correspondence of local quantum field theory. Given the function $f(x)$ one can define its Fourier transformation[4] as

$$f(x) = \int dp\, e^{-ipx} f(p) \tag{2.46}$$

Weyl operators $\hat{f}(\hat{x})$ are defined by relating them to ordinary function $f(x)$ of ordinary variables

$$\hat{f}(\hat{x}) = \int dx\, f(x)\hat{\Delta}(x) \tag{2.47}$$

where

$$\hat{\Delta}(x) = \int dk\, e^{-ik\hat{x}} e^{ikx} \tag{2.48}$$

is a hermitian operator $\hat{\Delta}(x)^\dagger = \hat{\Delta}(x)$ and describes a mixed basis for operators and fields on spacetime. Definition of Weyl operator is invertible with the definition

$$f(x) = Tr(\hat{f}(\hat{x}).\hat{\Delta}(x)) \tag{2.49}$$

The associated function obtained from a quantum operator is known as Wigner distribution function. Hence one can form a one-to-one correspondence between the Wigner function and Weyl operator. Here the operator trace Tr is equivalent to integration over the noncommuting coordinates $\hat{x}^i$

$$Tr\hat{f}(\hat{x}) = \int dx\, f(x) \tag{2.50}$$

From (2.47) one can write the Weyl symbol as

$$\hat{f}(\hat{x}) = \int dp\, e^{-ip\hat{x}} f(p) \tag{2.51}$$

Multiplying these operator fields produce

$$\begin{aligned} \hat{f}(\hat{x})\hat{g}(\hat{x}) &\equiv (\widehat{f \circ g})(\hat{x}) \\ &= \int dp\, e^{-ip\hat{x}} f(p) \int dk\, e^{-ik\hat{x}} g(k) \end{aligned} \tag{2.52}$$

[4] we suppressed the factor $(2\pi)'$s

Two exponential do not commute, so using Baker-Campbell-Hausdorff formula

$$e^{A}\,e^{B} = e^{A+B+\frac{1}{2}[A,B]+\cdots} \tag{2.53}$$

and the commutation relation of Hermitian coordinate operators (1.1) will give

$$\hat{f}\hat{g} = \int dpdk\, e^{-i(p+k)\hat{x}} \exp[-\frac{i}{2}p_{\mu}\theta^{\mu\nu}k_{\nu}]f(p)g(k) \tag{2.54}$$

We would like to relate the operator fields to the ordinary fields, so using the inverse Fourier transformation one can manipulate

$$\begin{aligned} \hat{f}\hat{g} &= \int dpdkdxdy\; e^{-i(p+k)\hat{x}}e^{ipx}e^{iky}\exp[\frac{i}{2}\partial^{x}_{\mu}\theta^{\mu\nu}\partial^{y}_{\nu}]f(x)g(y) \\ &= \int dq\; e^{-iq\hat{x}} \int dx\; e^{iqx}\exp[\frac{i}{2}\partial^{x}_{\mu}\theta^{\mu\nu}\partial^{y}_{\nu}]f(x)g(y) \end{aligned} \tag{2.55}$$

in the last line a variable exchange and some integrations have been performed. We end up with

$$\widehat{f\circ g} \rightarrow \exp[\frac{i}{2}\partial^{x}_{\mu}\theta^{\mu\nu}\partial^{y}_{\nu}]f(x)g(y)|_{x=y} \equiv (f*g) \tag{2.56}$$

the multiplication of operator fields is equivalent to the multiplication of ordinary fields with an unusual multiplication rule.

We shall use the star product formalism through the work.

$$f*g(x) = \exp(\frac{i}{2}\frac{\partial}{\partial x^{\mu}}\theta^{\mu\nu}\frac{\partial}{\partial y^{\nu}})f(x)g(y)|_{x=y} \tag{2.57}$$

This represantation is also known as the Weyl-Moyal product. The star product is an associative but noncommuting product rule between fields.

$$f*(g*h) = (f*g)*h \tag{2.58}$$

Under the definition (2.57) it can be seen that the ordinary coordinate components satisfy the relation

$$[x^{\mu},x^{\nu}]_{*} = i\theta^{\mu\nu} \tag{2.59}$$

where commutator is defined as

$$[A,B]_{*} = A*B-B*A \tag{2.60}$$

As can be seen that the multiplication rule (2.57) is a deformation of the ordinary product and at the limit $\theta \to 0$ it gives the ordinary case. At the same time under the integral sign it satisfies

$$\begin{aligned} \int dx\, f(x) * g(x) &= \int dx\, g(x) * f(x) = \int dx\, f(x) g(x) \\ \int dx\, f * g * h &= \int dx\, (f * g) h = \int dx\, f (g * h) \end{aligned} \tag{2.61}$$

Gauge theories on noncommutative spaces are defined deformations of the ordinary gauge theories by replacing the ordinary multiplication rule with the Moyal product. Assume that there exist a noncommutative connection $\hat{A}_\mu$ with curvature

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_* \tag{2.62}$$

Gauge transformations act on this noncommutative vector gauge field as

$$\hat{A}_\mu \to U * \hat{A}_\mu * U^{-1} + iU * \partial_\mu U^{-1}, \tag{2.63}$$

where U is any gauge group. It satisfies the star unitary relation

$$U * U^{-1} = \mathbf{1} \tag{2.64}$$

Then it is easy to check that

$$\hat{F}_{\mu\nu} \to U * \hat{F}_{\mu\nu} * U^{-1} \tag{2.65}$$

and this ensures the gauge invariance of the action

$$S = -\frac{4\pi}{g^2} \int d^4x\, Tr(\hat{F}^{\mu\nu} \hat{F}_{\mu\nu}) \tag{2.66}$$

where trace is defined on the gauge indices. It should be noted that even in the $U(1)$ case we have a nontrivial deformation of ordinary case which can be seen from (2.63) that it looks like a non-Abelian theory in a sense. One would like to find the equations of motion and calculate physically interesting quantities. However, local quantities in noncommutative gauge theories are not gauge invariant. Nonlocal expressions can be gauge invariant but we deal with local quantities in the ordinary gauge theories. Hence it is not possible to compare the results obtained from both sides. There is a way of to get over this difficulty: Seiberg-Witten map.

2.4 Seiberg-Witten Map

Here we present an outline of Seiberg-Witten map [75] which stimulated a great amount of work on the noncommutative gauge theories [76, 77, 78, 79, 80, 81]. As it is stated before the effective physics on the D-branes in presence of a background field can be described both by a commutative gauge theory and by a noncommutative one. Seiberg-Witten proved that these two different descriptions arise from the same two dimensional field theory with different regularizations. Pauli-Villars regularization leads to an effective action which depends on background field B and F only in the combination $F + B$. This effective lagrangian $\mathcal{L}(F+B)$ has an ordinary gauge symmetry given by $A \to A+\Lambda$ and $B \to B-d\Lambda$ for any one-form Λ. On the other hand point splitting regularization[5] yield a noncommutative theory $\hat{\mathcal{L}}(\hat{F})$ which has noncommutative gauge symmetry and a different $B-$dependence. Since the physics does not depend on the regularization, theories obtained with different regularizations can be related to each other by coupling constant redefinition. Worldsheet lagrangians have spacetime field dependent coupling constants, therefore relating these two descriptions requires a field redefinition. Seiberg-Witten map achieves this task by mapping the standard Yang-Mills theory gauge invariance to the gauge invariance of noncommutative Yang-Mills theory. In the $\alpha' \to 0$ limit, the effective action for slowly varying fields is given by the Dirac-Born-Infeld lagrangian

$$\mathcal{L}_{DBI}(F+B) = \frac{1}{g_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{\det(\mathrm{g}+2\pi\alpha'(\mathrm{B}+\mathrm{F}))} \tag{2.67}$$

when the effective action is expressed in terms of noncommutative gauge field

$$\hat{\mathcal{L}}_{DBI}(\hat{F}) = \frac{1}{G_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}\sqrt{\det(\mathrm{G}+2\pi\alpha'\hat{\mathrm{F}})} \tag{2.68}$$

where G_s is the effective open string coupling and g_s is that of closed string. Comparing the $\mathcal{L}_{DBI}(F=0)$ and $\hat{\mathcal{L}}_{DBI}(\hat{F}=0)$ will give

$$G_s = g_s\left(\frac{\det\mathrm{G}}{\det(\mathrm{g}+2\pi\alpha'\mathrm{B})}\right)^{1/2} \tag{2.69}$$

Now one can define such a field redefinition of noncommutative gauge field $\hat{A}$ and gauge parameter $\hat{\lambda}$ in terms of ordinary ones A, λ that under this definition the

[5] different operators are taken at different points

effective actions come from two different regularizations are related as

$$\mathcal{L}_{DBI} = \hat{\mathcal{L}}_{DBI} + \text{total derivative} + \mathcal{O}(\partial \text{F}) \tag{2.70}$$

The difference in total derivative arises from the fact that the action is derived in string theory by using the equations of motions which are not sensitive to such total derivatives. The $\mathcal{O}(\partial F)$ term is possible because these two lagrangians are derived in string theory in the approximation of slowly varying fields,i.e neglecting the ∂F terms. But at this point one should be careful that this transformation is not simply a field redefinition of the gauge fields $\hat{A} = \hat{A}(A, \partial A, \partial^2 A, \cdots; \theta)$ and a simultaneous reparametrization of the gauge parameter $\hat{\lambda} = \hat{\lambda}(\lambda, \partial\lambda, \partial^2\lambda, \cdots; \theta)$. Since such a redefinition causes an isomorphy relation between the gauge groups of ordinary and noncommutative theories, this is not the case happen here. To realize this, it is enough to look at the rank one theory. The ordinary gauge group, which acts by

$$\delta A_\mu = \partial_\mu \lambda \tag{2.71}$$

is abelian, while the noncommutative gauge invariance, which acts by

$$\delta A_\mu = \partial_\mu \lambda + i\lambda * A_\mu - iA_\mu * \lambda \tag{2.72}$$

is nonabelian. So no redefinition of the gauge parameter can map the ordinary gauge parameter to noncommutative one while intertwining the gauge symmetry. Seiberg-Witten map constructs a relation between the gauge equivalence classes of ordinary and noncommutative gauge theories instead of gauge groups. In short

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}} \hat{A}(A) = \hat{A}(A + \delta_\lambda A) \tag{2.73}$$

with infinitesimal λ and $\hat{\lambda}$. This can be achieved by taking the noncommuting gauge parameter a function of both ordinary gauge parameter and gauge field while the noncommutative gauge field is a function of ordinary gauge field, i.e., $\hat{A}(A) = A + A'(A)$ and $\hat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A)$ where prime denotes the components of orders of θ . Gauge transformation for an ordinary Yang-Mills theory is given by

$$\begin{aligned} \delta_\lambda A_\mu &= \partial_\mu \lambda + i[\lambda, A_\mu], \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \\ \delta_\lambda F_{\mu\nu} &= i[\lambda, F_{\mu\nu}]. \end{aligned} \tag{2.74}$$

For noncommutative theory gauge transformations is given by the same formulas except that multiplication rule exchanged with a Moyal star product. Thus

$$\begin{aligned}\hat{\delta}_{\hat{\lambda}}\hat{A}_\mu &= \partial_\mu\hat{\lambda} + i\hat{\lambda} * \hat{A}_\mu - i\hat{A}_\mu * \hat{\lambda}, \qquad (2.75)\\ \hat{F}_{\mu\nu} &= \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu,\\ \hat{\delta}_{\hat{\lambda}}\hat{F}_{\mu\nu} &= i\hat{\lambda} * \hat{F}_{\mu\nu} - i\hat{F}_{\mu\nu} * \hat{\lambda}.\end{aligned}$$

Expanding the equation (2.73) with respect to the orders of θ and using the definition of star product (2.57) one can obtain the noncommuting fields in terms of the ordinary ones

$$\begin{aligned}\hat{A}_\mu(A) &= A_\mu - \frac{1}{4}\theta^{\nu\rho}\{A_\nu, \partial_\rho A_\mu + F_{\rho\mu}\} + \mathcal{O}(\theta^2)\\ \hat{\lambda}(\lambda, A) &= \lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu\lambda, A_\nu\} + \mathcal{O}(\theta^2) \qquad (2.76)\end{aligned}$$

From the definition (2.62) it follows that

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma}(F_{\mu\rho}F_{\nu\sigma} - A_\rho\partial_\sigma F_{\mu\nu}) + \mathcal{O}(\theta^2) \qquad (2.77)$$

3 HAMILTONIAN OF DUAL NCU(1) AS A CONSTRAINT SYSTEM

Seiberg-Witten's work displays that the noncommutative and ordinary gauge theory description of $D-$branes in a constant background $B-$field are equivalent perturbatively in the noncommutativity parameter. It is natural to ask whether this equivalence is valid nonperturbatively. Some evidence has been found in the context of noncommutative $D3-$brane BIon and dyon solutions [44]. Noncommutative $D3-$brane BIon configuration is attained when open string metric satisfies $G_{MN} = \text{diag}(-1, 1, \cdots, 1)$ where $M, N = 0, 1, \cdots, 9$. This geometry is accomplished allowing a background B-field on D3-brane worldvolume, producing a noncommutativity parameter $\theta^{01} \neq 0$ and $\theta^{02} = \theta^{03} = \theta^{ij} = 0$, where $i, j = 1, 2, 3$. At the lowest order in the string slope parameter α' and for slowly varying fields ($\partial F \sim 0$) noncommutative $D3-$brane is described in terms of noncommutative U(1) gauge theory. Although it is possible to obtain an energy density which is derived from the invariance of the theory under translations, hamiltonian description of the theory is obscure because of the noncommuting time variable.

When time is noncommuting with the spatial coordinates the usual hamiltonian method is not applicable. Some different approaches are possible. One of them is to introduce a spurious time like variable [45]. In this case the energy is the same as the one derived from Lagrangian path integral formalism of the original theory. Another approach [28] is based on the fact that the theories with noncommuting time variable are S duals of the ones with commuting time variable [61]. Similarly, in [82] noncommutative U(1) gauge theory with the noncommutativity parameter $\theta^{0i} \neq 0, \theta^{ij} = 0$ is established as the dual theory of the one whose noncommutativity parameter satisfies $\theta^{0i} = 0, \theta^{ij} \neq 0$. Dual theory can be obtained via a parent action [43] which is defined from the original theory by a Legendre transformation. Constrained system analysis [63] of the shifted action will lead to the hamiltonian formulation of both dual and initial theories without referring their Lagrangian. This bypass procedure seems interesting and useful.

3.1 Duality in Ordinary U(1) Gauge Theory

First of all let us recall some basic facts. The symmetry of the vacuum Maxwell equations

$$\partial_\nu F^{\mu\nu} = 0 \tag{3.1}$$

$$\partial_\nu {}^\star F^{\mu\nu} = 0 \tag{3.2}$$

under exchange of the fields E and B is a well known property. Exchange of fields is defined by Hodge star operation in terms of field strength

$$F_{\mu\nu} \rightarrow {}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{3.3}$$

If one would like to extend this symmetry into the case of charges and currents are present, it is necessary that magnetic charges and currents are included as well.

$$\partial_\nu F^{\mu\nu} = -j^\mu \tag{3.4}$$

$$\partial_\nu {}^\star F^{\mu\nu} = -k^\mu \tag{3.5}$$

where $j^\mu = \{\rho, \overrightarrow{j}\}$ is electric four current and $k^\mu = \{\sigma, \overrightarrow{k}\}$ is its magnetic analogue. Now symmetry is valid under the transformation of currents among themselves beside the Hodge star operation.

$$F \rightarrow {}^\star F \quad , \quad {}^\star F \rightarrow -F \, , \tag{3.6}$$

$$j^\mu \rightarrow k^\mu \quad , \quad k^\mu \rightarrow -j^\mu . \tag{3.7}$$

This dualization procedure triggered the study of magnetic monopoles and has important consequences. In nature although electric charges can be observed any magnetic monopole has not been detected yet. In fact, this violation can be seen from a different point of view. When we investigate duality in quantum world, we should define a quantization which is based on the canonic variables.In terms of this canonical variable $F_{\mu\nu}$ is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This definition requires Bianchi identity (3.2) vanishes. In spite of this an electromagnetic potential can be constructed by putting a singularity in it. This is known as the Dirac monopole. Hence a magnetic monopole results from a topologically nontrivial configuration of the potential.

Actually there exist field theories in which these objects arise naturally. These are certain Yang-Mills-Higgs systems and they admit static finite energy field configurations. The Higgs mechanism break the gauge symmetry and a convenient choice of vacuum leads to a perturbative spectrum which contains a Higgs boson, a photon and two massive charged vector bosons and a solitonic solution. Magnetic monopoles can be associated with this solitonic solution by relating their charges to soliton numbers. An explicit construction of such a monopole was given by $'$t Hooft – Polyakov ansatz [64, 65]. $'$t Hooft – Polyakov monopole carries one unit magnetic charge and no electric charge. These models at the same time admit solutions which carry both magnetic as well as electric charges, Julia – Zee dyons [66]. For weak coupling regime of these theories the electric and magnetic charges appear in completely different characters. Electric charges appear as elementary quanta obtained by quantizing fields, by contrast magnetic monopoles arise as collective excitations of the elementary particles which are solitonic solutions and there is a quantization rule which is known as the Dirac quantization condition $q_i g_j = 2\pi\hbar n_{ij}$ for any electric charge q_i and the magnetic charge g_j. This is one of the important consequence and it says that if there exist a magnetic monopole then electrical charges are quantized. All of that imply that there exist fundamental differences between electricity and magnetism.

But this is not all of the story. Montonen and Olive bring a new insight. They explored a surprising symmetry between electricity and magnetism in the classical limit of above $4-$dimensional field theories. They saw that in these models the mass of any particle of electric charge q and magnetic charge g was given by a symmetric formula $M = \upsilon\sqrt{q^2 + g^2}$ which is invariant under the exchange of q and g. At the same time quantum of electric charge is exchanged with a multiple of the quantum of magnetic charge [67]. In short if we have a theory with weak coupling in which electric charges are elementary quanta and magnetic charges are some collective excitations we can have an equivalent picture in strong coupling regime with magnetic charges are elementary quanta and electric charges are in solitonic character.

Another natural extension of electric-magnetic duality of Abelian gauge theory is to search it in the non-Abelian case. Here the usual interchange between electric and magnetic degrees of freedom does not relate Yang-Mills theories with inverted couplings. In [68] this is examined by the loop space formulation of gauge theory. They showed that the dual theory is of Freedman-Townsend type [69]. The same result is obtained in [62] with a canonical transformation and also in [70]. Finally in supersymmetric theories the idea of electric-magnetic duality has gained its modern explanation. Here the CP violating term θ-parameter plays a crucial role and monopoles are in dyonic character [71]. These properties embed the electric-magnetic duality into a larger symmetry group $SL(2,Z)$, the modular group. For more information see [72, 73, 74].

In this chapter we will define a parent action construction of electric-magnetic duality in $U(1)$ gauge theory without source terms. Then we will exhibit the hamiltonian formulation of the theory from an extended, let us say a parent hamiltonian, by constraint analysis. Abelian gauge theory action is

$$S = -\frac{1}{4g^2}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{3.8}$$

where $F = dA$. Now we want to perform a Legendre transformation with respect to the initial variable F. At this stage F is no longer field strength of a potential and to implement the Bianchi identity we introduce a dual gauge field A_D as a Lagrange multiplier.

$$S_P = \int d^4x(-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}A^{\mu}_{D}\partial^{\nu}F^{\rho\sigma}) \tag{3.9}$$

Performing path integral over F or equivalently solving the field equations for F and replacing it in the action (3.9) leads to dual action

$$S_D = -\frac{g^2}{4}\int d^4x F^{\mu\nu}_{D}F_{D\mu\nu} \tag{3.10}$$

where $F_D = dA_D$. On the other hand repeating the same process with respect to A_D instead of F leads to Bianchi identity $\partial_\mu {}^\star F^{\mu\nu} = 0$ whose unique solution is $F = dA$ and hence the initial theory (3.8) is recovered. Canonical formulation of S_P starts with definition of canonical momenta. Here independent variables are F and A_D, so canonical momenta are

$$P_{\mu\nu} = \frac{\delta S_P}{\delta(\partial^0 F^{\mu\nu})}; \qquad P_{D\mu} = \frac{\delta S_P}{\delta(\partial^0 A^{\mu}_{D})} \tag{3.11}$$

and associated primary constraints are

$$\Phi^1_{\mu\nu} \equiv P_{\mu\nu} \approx 0, \tag{3.12}$$

$$\xi^1 \equiv P_{D0} \approx 0, \tag{3.13}$$

$$\chi^2_i \equiv P_{Di} + \frac{1}{2}\epsilon_{ijk}F^{jk} \approx 0 \tag{3.14}$$

where $i, j, k = 1, 2, 3$ and "$\approx$" denotes that constraints are weakly vanishing , i.e., they may have nonvanishing Poisson brackets with some canonical variables. The related canonical hamiltonian is

$$H_C = \int d^3x[\frac{1}{2g^2}F^{0i}F_{0i} + \frac{1}{4g^2}F^{ij}F_{ij} - \frac{1}{2}\epsilon_{ijk}\partial^i A^0_D F^{jk} + \epsilon_{ijk}\partial^i A^j_D F^{0k}] \tag{3.15}$$

Denote that we use the definition

$$\frac{\partial F_{\mu\nu}}{\partial F_{\rho\sigma}} = \frac{1}{2}(\delta_\mu{}^\rho \delta_\nu{}^\sigma - \delta_\mu{}^\sigma \delta_\nu{}^\rho). \tag{3.16}$$

By adding the primary constraints (3.12)-(3.14) to the canonical hamiltonian with some Lagrange multipliers $\alpha_i, \beta, \lambda_{ij}, \kappa_i$ one obtains the extended hamiltonian.

$$H_E = H_C + \int d^3x[\alpha_i P_{0i} + \beta P_{D0} + \lambda_{ij}P_{ij} + \kappa_i \chi^2_i] \tag{3.17}$$

Consistency of primary constraints in time may lead to some new constraints or may impose conditions on some Lagrange multipliers. Constraints which arise in this way are called secondary constraints. Hence we are left with two secondary constraints

$$\Phi^3 \equiv \{P_{D0}, H_E\} = \epsilon_{ijk}\partial^i F^{jk} \approx 0 \tag{3.18}$$

$$\chi^4_i \equiv \{P_{0i}, H_E\} = \frac{1}{g^2}F_{0i} + \epsilon_{ijk}\partial^j A^k_D \approx 0 \tag{3.19}$$

and two equations related with the multipliers κ and λ:

$$\{P_{ij}, H_E\} \equiv \kappa_i - \partial^i A^0_D + \frac{1}{g^2}\epsilon_{ijk}F^{jk} \approx 0 \tag{3.20}$$

$$\{\chi^2_i, H_E\} \equiv \lambda_{ij} + \partial_i F_{0j} \approx 0 \tag{3.21}$$

Where equal time Poisson brackets are defined as:

$$\{P^\mu_D(\mathbf{x}), A_{D\nu}(\mathbf{y})\}_{P.B.} = -\delta^\mu_\nu \delta^3(\mathbf{x} - \mathbf{y}) \tag{3.22}$$

$$\{P^{\mu\nu}(\mathbf{x}), F_{\rho\sigma}(\mathbf{y})\}_{P.B.} = -\frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.23}$$

Consistency of the secondary constraints also should be checked. This will not give further constraints and they terminate here. When constraints (3.18) and (3.14) are considered together one can obtain another constraint which is;

$$\partial_i P_D^i \approx 0 \tag{3.24}$$

Now full constraint structure of the system is obtained and they can be classified as first class or second class by following the Dirac's definition [63]. A constraint is called first class, if its Poisson bracket with all other constraints vanish. On the other hand if a constraint has a nonvanishing Poisson bracket with at least one of the other constraints it is called second class. In case of the second class constraints dynamics of any function of phase space variables is given by a modified bracket structure, Dirac bracket

$$\{A, B\}_{D.B.} = \{A, B\}_{P.B.} - \{A, \chi_i\}_{P.B.}(C^{-1})^{ij}\{\chi_j, B\}_{P.B.} \tag{3.25}$$

where χ_i stands for the second class constraints and C_{ij} for the matrix formed by the Poisson brackets of the second class constraints

$$C_{ij} \equiv \{\chi_i, \chi_j\}_{P.B.}$$

Dirac brackets satisfy the same algebraic relations of the Poisson bracket and Dirac bracket of any function with all χ vanishes. So, using the Dirac brackets instead of Poisson bracket, the weak equations may be written as strong equalities. This span a reduced phase space and quantization on this reduced space with the canonical commutators is equivalent to the Dirac quantization on the constrained phase space. The constraints (3.13) and (3.24) are first class and the rest (3.12), (3.14), (3.18), (3.19) are second class. In the reduced phase space, obtained by setting all the second class constraints equal to zero strongly and solving F, P in terms of F_D, P_D the canonical hamiltonian (3.15) becomes

$$H_D = \int d^3x[\frac{1}{2g^2}P_{Di}P_D^i + \frac{g^2}{4}F_{Dij}F_D^{ij}] \tag{3.26}$$

Moreover, there are first class constraints

$$P_{D0} \approx 0, \qquad \partial_i P_D^i \approx 0 \tag{3.27}$$

Obviously this is the same with the constraint hamiltonian formalism of the dual theory. Therefore we demonstrated that one can obtain constrained hamiltonian formulation of the dual theory beginning from the shifted action (3.9) bypassing the dual Lagrangian (3.10).

3.2 Noncommutative U(1) Gauge Theory

Noncommutative U(1) gauge theory is given by the action

$$\tilde{S} = -\frac{1}{4g^2}\int d^4x \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \tag{3.28}$$

where field strength is defined as

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu$$

By using Seiberg-Witten map (2.77) at the first order in $\theta^{\mu\nu}$ one obtains

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma}F_{\mu\rho}F_{\sigma\nu} - \theta^{\rho\sigma}A_\rho\partial_\sigma F_{\mu\nu}$$

Thus the action (3.28) can be written at the first order in $\theta^{\mu\nu}$ as

$$\tilde{S} = -\frac{1}{4g^2}\int d^4x(F_{\mu\nu}F^{\mu\nu} + 2\theta^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} - \frac{1}{2}\theta^{\mu\nu}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \tag{3.29}$$

Now we have the noncommutative action in terms of ordinary fields and in the light of the previous section we can define noncommutative parent action as

$$\tilde{S}_P = \tilde{S} + \frac{1}{2}\int d^4x A_D^\mu \epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} \tag{3.30}$$

where $F \neq dA$. As in the commutative case dual action can be found by solving the field equation for F in terms of $F_D = dA_D$ and plugging it in the action (3.30)

$$\tilde{S}_D = -\frac{g^2}{4}\int d^4x(F_D^{\mu\nu}F_{D\mu\nu} + 2\tilde{\theta}^{\mu\nu}F_{D\nu\rho}F_D^{\rho\sigma}F_{D\sigma\mu} - \frac{1}{2}\tilde{\theta}^{\mu\nu}F_{D\mu\nu}F_{D\rho\sigma}F_D^{\rho\sigma}) \tag{3.31}$$

where $\tilde{\theta}^{\mu\nu} = g^2\epsilon^{\mu\nu\rho\sigma}\theta_{\rho\sigma}$. At the first order in $\tilde{\theta}$ it can be derived from the action

$$\tilde{S}_D = -\frac{g^2}{4}\int d^4x \hat{F}_{D\mu\nu}\tilde{*}\hat{F}_D^{\mu\nu} \tag{3.32}$$

where $\tilde{*}$ is given by (2.57) by replacing θ with $\tilde{\theta}$. For $\theta^{0i} = 0$ and $\theta^{ij} \neq 0$ the dual theory is a gauge theory whose time variable is noncommuting in terms of the Moyal bracket with $\tilde{*}$, because $\tilde{\theta}^{0i} \neq 0$, $\tilde{\theta}^{ij} = 0$. For a noncommuting time canonical formalism is obscure. Thus we would like to bypass the dual action

(3.32) to obtain a phase space formulation of the dual theory using the method illustrated in the previous section.

Let $\theta^{ij} \neq 0$ and $\theta^{0i} = 0$ in the action (3.30). Definition of canonical momenta

$$\tilde{P}_{\mu\nu} = \frac{\delta \tilde{S}}{\delta(\partial^0 F^{\mu\nu})}, \tag{3.33}$$

$$\tilde{P}_{D\mu} = \frac{\delta \tilde{S}}{\delta(\partial^0 A_D^\mu)} \tag{3.34}$$

Primary constraints do not differ from the commutative case

$$\tilde{\Phi}^1_{\mu\nu} \equiv P_{\mu\nu} \approx 0, \tag{3.35}$$

$$\tilde{\xi}^1 \equiv P_{D0} \approx 0, \tag{3.36}$$

$$\tilde{\chi}^2_i \equiv P_{Di} + \frac{1}{2}\epsilon_{ijk} F_{jk} \approx 0 \tag{3.37}$$

and the canonical hamiltonian is

$$\begin{aligned} \tilde{H}_C \;=\; & \int d^3x [-\frac{1}{2}\epsilon_{ijk}\partial^i A^0_D F^{jk} + \epsilon_{ijk}\partial^i A^j_D F^{0k} + \frac{1}{2g^2} F_{0i}F^{0i} \\ & + \frac{1}{4g^2} F_{ij}F^{ij} + \frac{1}{g^2} F^{0i} F_{ij}\theta^{jk} F_{k0} + \frac{1}{2g^2} F^{ij} F_{jk} \theta^{kl} F_{li} \\ & - \frac{1}{4g^2}\theta^{ij} F_{ij} F_{0k} F^{0k} - \frac{1}{8g^2}\theta^{ij} F_{ij} F_{kl} F^{kl}] \end{aligned} \tag{3.38}$$

By choosing the Lagrange multipliers as $\tilde{\alpha}_i, \tilde{\beta}, \tilde{\lambda}_{ij}$ and $\tilde{\kappa}_i$ one can write extended hamiltonian as

$$\tilde{H}_E = \tilde{H}_C + \int d^3x [\tilde{\alpha}_i P_{0i} + \tilde{\beta} P_{D0} + \tilde{\lambda}^{ij} P_{ij} + \tilde{\kappa}_i \tilde{\chi}^2_i] \tag{3.39}$$

Preserving the primary constraints in time leads to secondary constraints

$$\tilde{\Phi}^3 \equiv \{P_{D0}, \tilde{H}_E\} = \epsilon_{ijk}\partial^i F^{jk} \approx 0 \tag{3.40}$$

and

$$\begin{aligned} \tilde{\chi}^4_i \equiv \{P_{0i}, \tilde{H}_E\} \;=\; & F^{0i} - F_{ij}\theta^{jk} F_{k0} - F^{0j} F_{jk}\theta^{ki} \\ - \; & \frac{1}{2}\theta^{jk} F_{kj} F_{0i} - g^2 \epsilon_{ijk}\partial^j A^k_D \approx 0 \end{aligned} \tag{3.41}$$

The other primary constraints will fix the multipliers $\tilde{\kappa}$, $\tilde{\lambda}$ as before. In our calculations these multipliers play no role, therefore we do not need determine them explicitly. Of course we also have

$$\tilde{\xi}^2 \equiv \partial^i P_{Di} \approx 0 \tag{3.42}$$

from (3.37) and (3.40). Consistency of second class constraints will not yield new constraints. One can check that (3.36) and (3.42) are first class and the rest are second class. In the reduced phase space where second class constraints strongly vanish, the canonical hamiltonian becomes

$$\begin{aligned}\tilde{H}_D &= \int d^3x\,[\frac{g^2}{4}F_{Dij}^2+\frac{1}{2g^2}P_{Di}^2-\frac{1}{2g^2}\epsilon_{ijk}\theta^{ij}P_D^kP_{Dl}^2\\ &\quad -\frac{g^2}{4}\epsilon_{ijk}\theta^{ij}P_D^kF_{Dlm}^2-g^2F_{Dij}P_D^j\theta^{ik}\epsilon_{klm}F_D^{lm}] \qquad (3.43)\end{aligned}$$

if we solve F, P in terms of F_D and P_D. Moreover, there are still the constraints

$$\partial^iP_{Di}\approx 0, \qquad P_{D0}\approx 0 \qquad (3.44)$$

which are first class. This hamiltonian can be written in terms of $\tilde{\theta}^{0i}=g^2\epsilon^{ijk}\theta_{jk}$ as

$$\begin{aligned}\tilde{H}_D &= \int d^3x\,[\frac{g^2}{4}F_{Dij}^2+\frac{1}{2g^2}P_{Di}^2+\frac{1}{2g^4}\tilde{\theta}_{0i}P_D^iP_{Dj}^2\\ &\quad +\frac{1}{4}\tilde{\theta}_{0i}P_D^iF_{Djk}^2+\tilde{\theta}^{0i}F_{Dji}F_{Djk}P_D^k] \qquad (3.45)\end{aligned}$$

On the other hand, although the dual action (3.31) possess a noncommuting time variable in terms of the Moyal bracket (2.57) given by $\tilde{*}$, it is originated from the action (3.30) whose time coordinate is commuting. We wonder what would be the phase space structure if we treat time coordinate as commuting in the action (3.31) written in components as

$$\begin{aligned}\tilde{S}_D &= g^2\int d^4x\,[\frac{1}{2}F_{0i}F_{0i}-\frac{1}{4}F_{ij}F_{ij}-\frac{1}{2}\tilde{\theta}^{0i}F_{i0}F_{0j}F_{0j}\\ &\quad -\tilde{\theta}^{0i}F_{ij}F_{jk}F_{k0}+\frac{1}{4}\tilde{\theta}^{0i}F_{i0}F_{jk}F_{kj}] \qquad (3.46)\end{aligned}$$

Definition of the spatial components of momentum

$$\begin{aligned}P_D^i=\frac{\delta\tilde{S}}{\delta(\partial_0A_{Di})} &= g^2[F_D^{i0}+\frac{1}{2}\tilde{\theta}^{0i}F_{D0j}F_{D0j}-\tilde{\theta}^{0j}F_{Dj0}F_D^{i0} \qquad (3.47)\\ &+\ \tilde{\theta}^{0k}F_{Dkj}F_D^{ji}-\frac{1}{4}\tilde{\theta}^{0i}F_{Djk}F_{Djk}]\end{aligned}$$

can be solved to find ∂_0A_{Di}. They lead to the same hamiltonian (3.45) which was obtained using the action (3.31). Moreover, there are same constraints (3.44).

We conclude that at the first order in $\tilde{\theta}$ whatever the method used we obtain the same hamiltonian and the constraints. However, the method of obtaining hamiltonian from the shifted action (3.30) seems easier: when the higher orders in $\tilde{\theta}$ are considered the unique change will be in the constraint (3.41), the other constraints (3.35)-(3.37), (3.40) will remain intact. Thus, finding hamiltonian of the dual theory is reduced to find solution of a constraint.

3.3 Relations Between the Electric-Magnetic Duality and the Dual Actions of Noncommutative U(1) Theory

Although electric-magnetic duality transformation is an invariance of Maxwell equations in vacuum, it is known that it maps the lagrangian density to itself up to an overall minus sign and keeps intact the hamiltonian density of U(1) gauge theory. Electric-magnetic duality transformation of the equations of motion of noncommutative U(1) theory is studied in [83]. Discussion of relations of the electric-magnetic duality with the dual description of the noncommutative gauge theory utilizing the lagrangian and the hamiltonian densities was made in [46]. Let us write the action (3.29) and (3.31) in terms of the electric and magnetic fields: when the magnetic field vector

$$B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk} \tag{3.48}$$

and the electric field vector $E_i = F_{0i}$ are employed, the original action becomes [86]

$$\tilde{S} = \int d^4x[\frac{1}{2g^2}(\mathbf{E}^2 - \mathbf{B}^2)(1 - \theta\cdot\mathbf{B}) + \frac{1}{g^2}\theta\cdot\mathbf{E}\mathbf{E}\cdot\mathbf{B}] \tag{3.49}$$

where the vector θ is defined by $\theta^{ij} = \epsilon^{ijk}\theta_k$. For the dual case we adopt the same notation: $E_i = F_{D0i}$ and

$$B_i = -\frac{1}{2}\epsilon_{ijk}F_D^{jk}. \tag{3.50}$$

Hence, the dual action can be written as

$$\tilde{S}_D = \int d^4x[\frac{g^2}{2}(\mathbf{E}^2 - \mathbf{B}^2)(1 + \tilde{\theta}\cdot\mathbf{E}) + g^2\tilde{\theta}\cdot\mathbf{B}\mathbf{E}\cdot\mathbf{B}] \tag{3.51}$$

where $\tilde{\theta}$ vector is defined as $\tilde{\theta}^i \equiv \tilde{\theta}^{0i}$. One can observe that under the transformation

$$\mathbf{E} \to g^2\mathbf{B}, \quad \mathbf{B} \to -g^2\mathbf{E}, \tag{3.52}$$

(3.49) is mapped into the dual action (3.51) up to an overall minus sign. This is a well known property of abelian gauge theory action. Thus, it persists in the noncommutative theory.

We also would like to obtain relations between electric–magnetic duality and the (S) duality transformation of noncommutative $U(1)$ theory in hamiltonian formalism. Canonical hamiltonian associated with (3.29) can be derived as

$$\begin{aligned}\tilde{H} &= \int d^3x \Big[\frac{g^2}{2}P_i^2 + \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{2g^2}\theta^{ij}F_{jk}F^{kl}F_{li} - \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl} \\ &\quad + g^2\theta^{ij}P_jP^kF_{ki} - \frac{g^2}{4}\theta^{ij}F_{ji}P_k^2\Big], \qquad (3.53)\end{aligned}$$

where we choose the subsidiary condition $A_0 = 0$ which corresponds to the constraint $P_0 = 0$. Furthermore, there is the constraint $\partial_i P^i = 0$. Hamiltonian of the dual noncommutative $U(1)$ gauge theory (3.45) is obtained in the previous subsection by two different approaches as

$$\begin{aligned}\tilde{H}_D &= \int d^3x \Big[\frac{1}{2g^2}P_{Di}^2 + \frac{g^2}{4}F_{Dij}F_D^{ij} + \frac{1}{2g^4}\tilde{\theta}_{0i}P_D^iP_{Dj}^2 + \frac{1}{4}\tilde{\theta}_{0i}P_D^iF_{Djk}F_D^{jk} \\ &\quad + \tilde{\theta}_{0i}F_D^{ij}F_{Djk}P_D^k\Big] \qquad (3.54)\end{aligned}$$

with the constrained $\partial_i P_D^i = 0$ after setting $P_{D0} = 0,\ A_{D0} = 0$.

Let us introduce the vector field $P_i = g^{-2}D_i$ and the magnetic fields as before (3.48). Hence, we write the hamiltonian (3.53) as

$$\tilde{H} = \int d^3x \left[\frac{1}{2g^2}(\mathbf{D}^2 + \mathbf{B}^2) - \frac{1}{2g^2}\theta \cdot \mathbf{B}(\mathbf{B}^2 - \mathbf{D}^2) - \frac{1}{g^2}\theta \cdot \mathbf{D}\mathbf{B} \cdot \mathbf{D}\right]. \qquad (3.55)$$

Similarly, let us introduce $P_{Di} = g^2 D_i$ and the magnetic field as in (3.50). Then, the hamiltonian (3.45) becomes

$$\tilde{H}_D = \int d^3x \left[\frac{g^2}{2}(\mathbf{D}^2 + \mathbf{B}^2) - \frac{g^2}{2}\tilde{\theta} \cdot \mathbf{D}(\mathbf{D}^2 - \mathbf{B}^2) - g^2\tilde{\theta} \cdot \mathbf{B}\mathbf{B} \cdot \mathbf{D}\right]. \qquad (3.56)$$

One can show that under the map

$$\mathbf{D} \to -g^2\mathbf{B}, \quad \mathbf{B} \to g^2\mathbf{D} \qquad (3.57)$$

the hamiltonian (3.55) transforms into the dual hamiltonian (3.56). Thus, the noncommutative electric-magnetic duality transformation in the hamiltonian formulation is given by (3.57). Observe that the lagrangian and the hamiltonian description of electric–magnetic duality transformations, (3.52) and (3.57), seem to be “inverted”.

Definition of the canonical momenta P_i following from (3.29) can be used to express P_i in terms of the electric field $E_i = F_{0i}$. Then, one can express the hamiltonian (3.53) as [86]

$$\tilde{H} = \int d^3x \left[\frac{1}{2g^2}(\mathbf{E}^2 + \mathbf{B}^2)(1 - \theta \cdot \mathbf{B}) + \frac{1}{g^2}\theta \cdot \mathbf{E}\mathbf{E} \cdot \mathbf{B} \right]. \tag{3.58}$$

Analogously, the canonical momenta P_{Di} derived from (3.31) can be expressed in terms of the electric field $E_i = F_{D0i}$. Making use of it in the hamiltonian (3.45) one obtains

$$\tilde{H}_D = \int d^3x \left[\frac{g^2}{2}(\mathbf{E}^2 + \mathbf{B}^2) + g^2\tilde{\theta} \cdot \mathbf{E}\mathbf{E}^2 \right]. \tag{3.59}$$

(3.58) and (3.59) are not related with a transformation resembling the electric–magnetic duality transformation (3.57).
Electric–magnetic duality transformation of the noncommutative hamiltonians cannot be given in terms of $\mathbf{E}$, $\mathbf{B}$ fields but using $\mathbf{D}$, $\mathbf{B}$. This is an expected result: Hamiltonians should be written in momenta P_i or P_{Di} not by using the "velocities" F_{0i} or F_{D0i}. In the commuting case this difference does not appear due to the fact that $\mathbf{P} = \mathbf{E}$.

3.4 BPS States Of Noncommutative D3-Brane

The notion of BPS states plays a fundamental role in discussion of nonperturbative duality symmetries. Massive BPS states appear in theories with extended supersymmetry. It just so happens that supersymmetry representations are sometimes shorter than usual. This is due to some of the supersymmetry operators being "null", so that they cannot create new states. The vanishing of some supercharges depends on the relation between the mass of a multiplet and some central charges appearing in the supersymmetry algebra. These central charges depend on electric and magnetic charges of the theory as well as on expectation values of scalars. In 1978 Witten and Olive noted that in supersymmetric theories with solitons the topological charges play the role of the central charges of the super Poincaré algebra. In a sector with given charges, the BPS states are the lowest lying states and they saturate the so called BPS bound. BPS states behave in a very special way: they are absolutely stable. The reason is the dependence of their mass on conserved charges. For a detailed discussion see [47].

In the zero slope limit, $\alpha' \to 0$, and considering slowly varying fields noncommutative DBI action becomes noncommutative gauge theory (3.31) up to constant terms [75]. Noncommutative $D3-$brane worldvolume action can be extracted from 10 dimensional noncommutative gauge theory in the static gauge where the first three spatial coordinates are taken equal to brane worldvolume coordinates and the rest of the coordinates as scalar field on the brane. We consider only one scalar field. $D3-$brane worldvolume Hamiltonian density resulting from (3.45) when $\tilde{\theta}^{0i} \neq 0,\ \tilde{\theta}^{ij} = 0$, is

$$\begin{aligned} H &= \frac{1}{2}P_i^2 + \frac{1}{4}F_{ij}^2 - \frac{1}{2}\tilde{\theta}^{0i}P_iP_j^2 - \frac{1}{4}\tilde{\theta}^{0i}P_iF_{jk}^2 + \tilde{\theta}^{0i}F_{ji}F_{jk}P^k \\ &+\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i\phi)^2 - \frac{1}{2}\tilde{\theta}^{0i}P_i\pi^2 + \tilde{\theta}^{0i}\pi F_{ij}\partial_j\phi \\ &-\tilde{\theta}^{0i}P_j\partial_i\phi\partial_j\phi + \frac{1}{2}\tilde{\theta}^{0i}P_i(\partial_j\phi)^2 \end{aligned} \tag{3.60}$$

The scalar field and the corresponding canonical momentum denoted as ϕ and π. Moreover we renamed the dual variables $F_D,\ P_D$ as F, P. We choose $\pi = 0$ to deal with the static case.
To discuss bounds on the value of the Hamiltonian we would like to write (3.60) as

$$H = \frac{1}{2}\widehat{P}_i^2 + \frac{1}{2}\widehat{B}_i^2 + \frac{1}{2}(\widehat{\partial_i\phi})^2 \tag{3.61}$$

with the restrictions

$$\widehat{P}_i|_{P=0} = 0,\ \widehat{B}_i|_{F=0} = 0,\ \widehat{\partial_i\phi}|_{\phi=0} = 0,$$

These are fulfilled by

$$\widehat{P}_i = P_i - a_1\tilde{\theta}^{oi}P_j^2 - a_2\tilde{\theta}^{oj}P_jP_i, \tag{3.62}$$

$$\widehat{B}_i = \frac{1}{2}\epsilon_{ijk}(F_{jk} - \frac{1}{2}\tilde{\theta}^{0l}P_lF_{jk} + b_1\tilde{\theta}^{0l}P_kF_{jl} + b_2\tilde{\theta}^{0k}P_lF_{jl}), \tag{3.63}$$

$$\widehat{\partial_i\phi} = \partial_i\phi + \frac{1}{2}\tilde{\theta}^{0j}P_j\partial_i\phi - c_1\tilde{\theta}^{0i}\partial_j\phi P_j - c_2\tilde{\theta}^{0j}\partial_j\phi P_i, \tag{3.64}$$

where $a_{1,2}$ $b_{1,2}$ $c_{1,2}$ are constants which should satisfy

$$a_1 + a_2 = \frac{1}{2},\ b_1 + b_2 = -2,\ c_1 + c_2 = 1, \tag{3.65}$$

otherwise arbitrary. These do not correspond to the Seiberg–Witten map (2.77). There the fields of commutative and noncommutative gauge theories are mapped into each other by changing the gauge group from commutative $U(1)$ to noncommutative one such that (2.73) is satisfied. In our case gauge group is always $U(1)$. Although we write the Hamiltonian (3.61) in terms of $\tilde{\theta}^{0i}$ dependent fields we still have the constraint $\partial_i P_i = 0$, indicating $U(1)$ gauge group. Seiberg–Witten map in phase space is studied in [48, 52].

Now, in terms of an arbitrary angle α the Hamiltonian density (3.60) can be put into the form

$$\begin{aligned} H &= \frac{1}{2}(\widehat{P}_i - \sin\alpha\ \widehat{\partial_i\phi})^2 + \frac{1}{2}(\widehat{B}_i - \cos\alpha\ \widehat{\partial_i\phi})^2 \\ &\quad + \sin\alpha\ \widehat{P}_i\widehat{\partial_i\phi} + \cos\alpha\ \widehat{B}_i\widehat{\partial_i\phi}. \end{aligned} \tag{3.66}$$

Thus, we can write a bound on total energy E relative to the worldvolume vacuum of noncomutative $D3-$brane as

$$E \geq \sqrt{\tilde{Z}_{el}^2 + \tilde{Z}_{mag}^2}, \tag{3.67}$$

where, we introduced

$$\tilde{Z}_{el} = \int_{D3} d^3x\ \widehat{P}_i\widehat{\partial_i\phi}, \tag{3.68}$$

$$\tilde{Z}_{mag} = \int_{D3} d^3x\ \widehat{B}_i\widehat{\partial_i\phi}. \tag{3.69}$$

In the commutative case $\tilde{Z}_{el}$ and $\tilde{Z}_{mag}$ become topological charges due to the Gauss law and the Bianchi identity: $\partial_i P_i = 0,\ \partial_i B_i = 0$. In the commuting case (3.67) is known as BPS bound [53, 54]. However, in our case we do not have an integrability conditions for $\hat{P}_i,\ \hat{B}_i$. Nevertheless, it will be shown that they can be topological charges for some specific configurations.

The bound (3.67)is saturated for

$$\widehat{P}_i = \widehat{\partial_i\phi},\ \widehat{B}_i = 0,\ \sin\alpha = 1. \tag{3.70}$$

This can be accomplished at the first order in $\tilde{\theta}^{0i}$, when

$$F_{ij} = 0,\ P_i = \partial_i\phi, \tag{3.71}$$

if we fix the parameters as

$$a_1 = c_1,\ a_2 = c_2 - \frac{1}{2}, \tag{3.72}$$

which are consistent with (3.65). Because of the constraint (3.42) ϕ should satisfy

$$\partial_i^2\phi = 0. \tag{3.73}$$

For this configuration $\tilde{Z}_{mag}$ vanishes: $\tilde{Z}^{(1)}_{mag} = 0$, and $\tilde{Z}_{el}$ reads

$$\tilde{Z}^{(1)}_{el} = \int d^3x\partial_i(\phi\partial_i\phi) - \int d^3x\ \tilde{\theta}^{0i}\partial_i\phi(\partial\phi)^2. \tag{3.74}$$

For the commutative case isolated singularities of ϕ satisfying these conditions are called BIon [53]. The simplest choice satisfying (3.73) is [54]

$$\phi(r) = \frac{e}{4\pi r}, \tag{3.75}$$

where r is the radial variable. In general we cannot write $\tilde{\theta}^{0i}$ dependent part as a surface integral. However, this choice of harmonic function (3.73) renders it possible. Indeed, we can write $\tilde{Z}^{(1)}_{el}$ as an integral over a sphere of radius ϵ about the origin and find

$$\tilde{Z}^{(1)}_{el} = (e - \frac{\tilde{\theta}e^2}{20\pi\epsilon^4})\lim_{\epsilon\to 0}\phi(\epsilon), \tag{3.76}$$

where $\tilde{\theta} \equiv \sqrt{\tilde{\theta}^{0i}\tilde{\theta}^{0i}}$.

Observe that the usual BIon solution (3.75) leads to a solution for the noncommutative case (3.70). This is similar to the fact that linearized and full DBI actions lead to the same BIon solution with the same energy [55]. Here the solutions are the same but energies differ. When one sets $P_i = 0$ the terms depending on the noncommutativity parameter $\tilde{\theta}^{0i}$ disappear. This is what we expected: noncommutativity is only between time and space coordinates not between spatial coordinates. Thus, when momenta vanish noncommutativity should cease to exist. For $P_i = 0$ the bound (3.67) is saturated for

$$\frac{1}{2}\epsilon_{ijk}F_{jk} = \partial_i\phi, cos\alpha\ = 1 \tag{3.77}$$

where as before ϕ should satisfy (3.73). For this commuting configuration $\tilde{Z}_{el}$ and $\tilde{Z}_{mag}$ become $\tilde{Z}^{(2)}_{el} = 0$, and

$$\tilde{Z}^{(2)}_{mag} = \int d^3x\partial_i(\phi\partial_i\phi). \tag{3.78}$$

To satisfy (3.77) and (3.73) consider a magnetic charge at the origin

$$\phi(r) = \frac{g}{4\pi r}. \tag{3.79}$$

Let the integral be over a sphere of radius ϵ about the origin which yields

$$\tilde{Z}^{(2)}_{mag} = g \lim_{\epsilon \to 0} \phi(\epsilon). \tag{3.80}$$

There is another configuration

$$\widehat{P}_i = \sin\alpha \ \widehat{\partial_i \phi}, \ \ \widehat{B}_i = \cos\alpha \ \widehat{\partial_i \phi}, \tag{3.81}$$

which saturates the bound (3.67). The constant angle α is defined as

$$\tan\alpha = \frac{\tilde{Z}_{el}}{\tilde{Z}_{mag}}. \tag{3.82}$$

This can be realized if the commuting variables are fixed as

$$P_i = \sin\alpha \ \partial_i \phi, \ \ \frac{1}{2}\epsilon_{ijk} F_{jk} = \cos\alpha \ \partial_i \phi \tag{3.83}$$

and the free parameters in (3.62)–(3.64) satisfy (3.72) and

$$c_1 = b_1/2, \ \ c_2 = 1 - b_1/2. \tag{3.84}$$

These are consistent with (3.65). Thus, in the hatted quantities (3.62)–(3.64) now, there is only one free constant parameter. For this configuration $\tilde{Z}_{el}$ and $\tilde{Z}_{mag}$ are given by

$$\tilde{Z}^{(3)}_{el} = \int d^3x \sin\alpha \ \partial_i(\phi \partial_i \phi) - \int d^3x \ \tilde{\theta}^{0i} \sin^2\alpha \ \partial_i \phi (\partial\phi)^2, \tag{3.85}$$

$$\tilde{Z}^{(3)}_{mag} = \int d^3x \cos\alpha \ \partial_i(\phi \partial_i \phi), - \int d^3x \ \tilde{\theta}^{0i} \cos^2\alpha \ \partial_i \phi (\partial\phi)^2. \tag{3.86}$$

Similar to the other configurations, ϕ should satisfy (3.73) and we consider the simplest choice

$$\phi(r) = \frac{g}{4\pi \cos\alpha \ r}. \tag{3.87}$$

For this choice of the harmonic function (3.87) the integrals in (3.85) and (3.86) can be performed over a sphere of radius ϵ about the origin. Therefore, the energy can be calculated as

$$E = \left[(e - \frac{\tilde{\theta} e^2}{20\pi\epsilon^4})^2 + (g - \frac{\tilde{\theta} g^2}{20\pi\epsilon^4})^2 \right]^{1/2} \lim_{\epsilon \to 0} \phi(\epsilon), \tag{3.88}$$

where $e/g = \tan\alpha$. Similar to the above mentioned configurations ordinary $D3-$brane dyon solution (3.87), provide a solution of the noncommutative condition (3.81).

Hamiltonian formulation of noncommutative $D3-$brane when Moyal bracket of time coordinate with spatial coordinates is nonvanishing, i.e. $\tilde{\theta}^{0i} \neq 0$ is obtained without introducing any new machinery. This follows from the fact that its action can be obtained from an action where time is as usual, commuting. The result which we obtained is only at the first order in noncommutativity parameter, however it can be generalized to the higher orders. Obviously, one of the method is to solve $\partial_0 A_D$ in terms of $P_D,\ F_D$ from the generalization of (3.47). However, it is highly non-linear. On the other hand using the shifted action as it is illustrated here seems more manageable. It is an encouraging property that one should only solve a constraint similar to (3.41). The other constraints (3.35)-(3.37),(3.40) remain intact.

Noncommuting $D3-$brane formulation which we deal with is somehow different from the one considered in [44, 56, 57, 58, 59, 60]. There, gauge group is noncommutative $U(1)$, in our case although Hamiltonian depend on the noncommutativity parameter, gauge group is still $U(1)$. This seems to be the basic reason that the BPS solutions of ordinary case [53, 54] provide solutions of the noncommutative case as it happens between linearized and full DBI action [55].

4 PARTITION FUNCTIONS OF DUAL THEORIES

In four dimensional gauge theories with complexified coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ S-duality emerges as a symmetry of the theory. This is known as the modular transformations of the τ and under this transformations partition function transforms as a modular form with a weight proportional to the Euler characteristic of 4-manifold. This analysis for $U(1)$ gauge theory without charges could be done explicitly at the level of path integral and it was shown that there is an analogy to the transformation law for the dilaton under T duality in non-linear sigma model [23]. In supersymmetric theories, especially in $N = 2$ and $N = 1$ supersymmetric Yang-Mills theories, S-duality has been also studied [24, 25].

In the previous chapter we studied the parent action construction of S-dual theories of noncommutative gauge theory. We would like to obtain partition functions of these dual theories by using the same machinery. Our treatment will be a minimalistic approach to the problem: we will neglect the θ–term in the τ and work nonsupersymmetric theory. When parent action is employed in the path integral if one integrates over dual field A_D the partition function of the ordinary $U(1)$ theory results. Instead of A_D one can integrate antisymmetric second rank tensor $F_{\mu\nu}$ which yields the partition function of the dual $U(1)$ theory. Thus one can easily show equivalence of partition function for the $U(1)$ and its dual theory, up to a normalization constant. On the other hand hamiltonian description of these theories are shown to be connected by a canonical transformation and as a consequence it followed that the partition functions in their phase spaces are the same [62]. This equivalence can also be demonstrated directly in terms of the hamiltonian formulation of the parent action [46]. In the light of this strategy partition function of S-dual theories in noncommutative spacetime can be obtained.

For $U(1)$ gauge theory the parent action can be used in path integrals to derive relations between the original and the dual theories. But, for noncommutative theory one should employ equations of motion to obtain the initial or dual noncommutative $U(1)$ theory and relation between their partition functions is unknown. We will show that partition functions of noncommutative $U(1)$ theory with spatial noncommutativity and its dual whose time coordinate is effectively noncommuting with spatial coordinates, are equivalent in appropriate phase spaces. To achieve this we will follow the approach presented for the commutative gauge theory.

4.1 Partition Functions of $U(1)$ Gauge Theory and Its Dual

The parent action which gives $U(1)$ gauge theory and its dual and constraint structure of this action was obtained in the chapter-3. When one inserts this parent action into path integral it contains all of the degrees of the freedom and hence it is highly redundant. Because of that it requires a careful analysis of constraints. Let us find out the number of physical phase space fields: the constraint (3.13) is obviously first class. Besides it, the linear combination

$$\xi^2 \equiv \partial_i \chi_i^2 - \frac{1}{2}\Phi^3 = \partial_i P_{Di} \approx 0, \tag{4.1}$$

is also a first class constraint. A vector can be completely described by giving its divergence and rotation (up to a boundary condition). (4.1) is derived taking divergence of χ_i^2, so that, there are still two linearly independent second class constraints following from the curl of χ_i^2. Obviously, the constraints Φ^1, Φ^3, χ_i^4 are all second class and linearly independent. Therefore, the number of physical phase space fields is four.

To deal with path integrals, we choose the gauge fixing (subsidiary) conditions

$$\Lambda^1 = A_{D0} \approx 0, \;\; \Lambda^2 = \partial_i A_{Di} \approx 0 \tag{4.2}$$

for the first class constraints (3.13) and (4.1). The linearly independent second class constraints resulting from the curl of χ_i^2 can be taken as

$$\Phi_n^2 \equiv C_n^i \chi_i^2 \equiv K_n^i \epsilon_{ijk} \partial_j \chi_k^2 \approx 0, \tag{4.3}$$

where $n = 1, 2$, and K_n^i are some constants which should be chosen in accordance with solutions of the other constraints when they vanish strongly. For the later convenience, instead of dealing with χ_i^4 we introduce another set of linearly independent second class constraints:

$$\Phi_n^4 \equiv M_n^i \chi_i^4 \equiv L_n^i \epsilon_{ijk}\partial_j \chi_k^4 \approx 0, \quad \Phi_3^4 \equiv \partial^i F_{0i} \approx 0. \tag{4.4}$$

L_n^i are some constants. As we will see, explicit forms of K_n^i and L_n^i play no role in our calculations.

Partition function associated with the hamiltonian (3.15) in the total phase space is

$$Z = \int DA_D DF DP_D DP \,\Delta \exp\left\{ i \int d^4x \left[P_{D\mu}\dot{A}_D^\mu + P_{\mu\nu}\dot{F}^{\mu\nu} - H_C \right] \right\}. \tag{4.5}$$

We suppressed the indices of the integration variables and the measure Δ is defined as[84], [85]

$$\Delta = \det\{\xi^\alpha, \Lambda^\beta\} \det{}^{1/2}\{\Phi^a, \Phi^b\} \prod_{\sigma=1}^{2} \delta(\xi^\sigma)\delta(\Lambda^\sigma) \prod_{c=1}^{4} \delta(\Phi^c). \tag{4.6}$$

The determinant related to first class constraints and their subsidiary conditions is

$$\det\{\xi^\alpha, \Lambda^\beta\} = \det \partial_i \partial^i \equiv \det(\partial^2)$$

The determinant due to the second class constraints can be calculated as

$$\det{}^{1/2}\{\Phi^a, \Phi^b\} = \det(g^4)\det(\partial^2)\det\left(\epsilon_{ijk}\partial^i C_1^j C_2^k\right)\det\left(\epsilon_{ijk}\partial^i M_1^j M_2^k\right), \tag{4.7}$$

where C_n^i and M_n^i are defined in (4.3) and (4.4). Here the determinants of these linear operators should be interpreted as multiplication of their eigenvalues. Explicit form of these determinants and calculations can be found in the appendix. Performing functional integrations over the variables $F^{\mu\nu}$,$P_{\mu\nu}$ and A_D^0, P_D^0 we obtain the partition function of the dual theory in hamiltonian formalism

$$\begin{aligned} Z \;&=\; \int D\mathbf{A}_D D\mathbf{P}_D \delta(\partial\cdot\mathbf{P}_D)\delta(\partial\cdot\mathbf{A}_D)\det(\partial^2) \\ &\quad \exp\left\{ i\int d^3x \left[P_{Di}\dot{A}_D^i - \frac{1}{2g^2}P_{Di}P_D^i - \frac{g^2}{4}F_D^{ij}F_{Dij} \right] \right\}. \end{aligned} \tag{4.8}$$

Here, the factor $\det^{1/2}\{\Phi^a, \Phi^b\}$ is canceled with the determinant arising from the Dirac delta functions $\delta(\Phi^a)$ when we use them to express $F_{\mu\nu}$ in terms of the "physical" fields $\mathbf{A}_D$, $\mathbf{P}_D$. Although here this can be observed by direct calculation[6], it is true in general when one gets rid of second class constraints by imposing them strongly and deal with the reduced phase space path integrals [85].

Now, in (4.5) we would like to perform integrations over the dual fields $A_{D\mu}$, $P_{D\mu}$ and the momenta $P_{\mu\nu}$. Vanishing of the constraint (3.18) strongly, i.e. $\Phi^3 = 0$, dictates that

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{4.9}$$

Being a second class constraint $\Phi^3 = 0$ should eliminate one phase space variable. However, the number of independent components of F_{ij} and A_i are the same. So that, solving $\Phi^3 = 0$ as (4.9) and dealing with A_i instead of F_{ij}, has to be accompanied with a condition on A_i. The constraint χ_i^2 involves only curl of A_i, therefore, $\Phi_n^2 = 0$ give information only about the two components of A_i. In order to describe A_i completely one needs to furnish its divergence. Thus, we choose as the missing condition

$$\partial_i A^i = 0. \tag{4.10}$$

After performing the $A_{D\mu}, P_{D\mu}$ and $P_{\mu\nu}$ integrations in (4.5) we obtain

$$\begin{aligned} Z &= detg^{-4} \int D\mathbf{A} DF_{0j} \det(\partial^2) \delta(\partial^{\mathrm{l}} \mathrm{F}_{0\mathrm{l}}) \delta(\partial \cdot \mathbf{A}) \\ & \exp \left\{ i \int d^3x [-\frac{1}{g^2} F_{0i} \dot{A}^i + \frac{1}{2g^2} F^{0i} F_{0i} - \frac{1}{4g^2} F^{ij} F_{ij}] \right\}. \end{aligned} \tag{4.11}$$

We used the fact that expressing A_{Di} and P_{Di} in terms of the "physical" fields A_i, F_{0i}, using the Dirac delta functions $\delta(\Phi^a)$, $\delta(\partial \cdot \mathbf{P}_D)$, $\delta(\partial \cdot \mathbf{A}_D)$, contributes to the measure as

$$\left[\det(\mathrm{g}^4) \det(\partial^2) \det \left(\epsilon_{\mathrm{ijk}} \partial^{\mathrm{i}} \mathrm{C}_1^{\mathrm{j}} \mathrm{C}_2^{\mathrm{k}} \right) \det \left(\epsilon_{\mathrm{ijk}} \partial^{\mathrm{i}} \mathrm{M}_1^{\mathrm{j}} \mathrm{M}_2^{\mathrm{k}} \right) \right]^{-1} \tag{4.12}$$

See the appendix-A for details. Moreover, here F_{ij} is given by (4.9) and we performed the change of variables $F_{ij} \rightarrow A_i$. We choose domains of the integrals such that in (4.5) we can perform the replacement

$$\begin{aligned} & DF_{ij} \delta(\epsilon^{klm} \partial_k F_{lm}) \delta \left(C_n^i \left(P_D^i + \frac{1}{2} \epsilon_{ijk} F^{jk} \right) \right) \rightarrow \\ & det(\partial^2) DA_i \delta(\partial_j A^j) \delta \left(C_n^i (P_{Di} + \epsilon_{ijk} \partial^j A^k) \right). \end{aligned} \tag{4.13}$$

[6] To obtain (4.8) we do not need to deal with the set (4.4). It is easier to employ (3.19) with an appropriate determinant.

One can observe that $\det(\partial^2)$ should be included in the measure when one deals with the gauge fields A_i satisfying the condition (4.10), considering this change of variables from the beginning with an appropriate change of the momenta $P_{ij} \rightarrow P_i$, where the later are canonical momenta of A_i.
Observe that in (4.11) the variables F_{0i} can be renamed as

$$F_{0i} = -g^2 P_i, \tag{4.14}$$

where P_i are the canonical momenta associated to A_i. Thus, (4.11) becomes

$$\begin{aligned} Z &= \det g^{-4} \int \mathrm{D}\mathbf{A}\mathrm{D}\mathbf{P}\det(\partial^2)\delta(\partial \cdot \mathbf{P})\delta(\partial \cdot \mathbf{A}) \\ &\quad \exp\left\{ i \int d^4x [P_i \dot{A}^i - \frac{g^2}{2} P_i P^i - \frac{1}{4g^2} F^{ij} F_{ij}] \right\}. \end{aligned} \tag{4.15}$$

Although (4.14) is resulted after performing functional integrals in (4.5), we could derive it from the constraint structure using Dirac brackets:

$$\begin{aligned} \{F_{0i}(x), P_{Dj}(y)\}_{\text{Dirac}} &= \{F_{0i}, P_{0k}\}\{P_{0k}, \Phi^4_l\}^{-1}\{\Phi^4_l, P_{Dj}\} \\ &= g^2 \epsilon_{ikj} \frac{\partial \delta^3(x-y)}{\partial x_k}. \end{aligned} \tag{4.16}$$

On the other hand making use of (4.9) in $\chi_i^2 = 0$ yields

$$P_{Di} = -\epsilon_{ijk} \partial^j A^k \tag{4.17}$$

Plugging (4.17) into the left hand side of the Dirac bracket (4.16), leads to

$$-\epsilon_{jkl} \frac{\partial}{\partial y_k} \{F_{0i}(x), A_l(y)\}_{\text{Dirac}} = g^2 \epsilon_{ijk} \frac{\partial \delta^3(x-y)}{\partial x_k}, \tag{4.18}$$

which is solved as

$$\{F_{0i}(x), A_j(y)\}_{\text{Dirac}} = g^2 \delta_{ij} \delta^3(x-y). \tag{4.19}$$

Thus, (4.14) follows.
We choose the normalization such that partition function for Maxwell theory in hamiltonian formalism is given by

$$\begin{aligned} Z_H \equiv Z_N(g) &= \det g^{-2} \int \mathrm{D}\mathbf{A}\mathrm{D}\mathbf{P}\delta(\partial \cdot \mathbf{A})\delta(\partial \cdot \mathbf{P}) \\ &\quad \exp\left\{ i \int d^4x \left[P_i \dot{A}^i - \frac{g^2}{2} P_i P^i - \frac{1}{4g^2} F^{ij} F_{ij} \right] \right\}. \end{aligned} \tag{4.20}$$

We denoted the normalized partition function as $Z_N(g)$. The normalized partition function of the dual theory in phase space is

$$Z_{HD} \equiv Z_N(g^{-1}) = \det g^2 \int \mathrm{D}\mathbf{A}\mathrm{D}\mathbf{P}\delta(\partial\cdot\mathbf{A})\delta(\partial\cdot\mathbf{P}) \tag{4.21}$$

$$\exp\left\{i\int d^4x\left[P_i\dot{A}^i - \frac{1}{2g^2}P_iP^i - \frac{g^2}{4}F^{ij}F_{ij}\right]\right\}, \tag{4.22}$$

where we renamed A^i_D, P^i_D as A^i, P^i. By comparing Z obtained in (4.8) and (4.15) we conclude that in hamiltonian formalism partition functions for Maxwell theory and its dual are the same

$$Z_H = Z_{HD}, \tag{4.23}$$

which can equivalently be written in terms of the normalized partition function as

$$Z_N(g) = Z_N(g^{-1}). \tag{4.24}$$

This result was obtained in [62] in terms of canonical transformations without gauge fixing factor and with another normalization.

4.2 Partition Functions of Noncommutative $U(1)$ Theory and Its Dual

Here we will make a similar discussion for the noncommutative theory. We know from the previous chapter that except the constraint (3.41) the other constraints are the same as in the commuting case. Hence the constraint (3.36) and the linear combination of the (3.37) and (3.40)

$$\tilde{\xi}^2 \equiv \partial_i\tilde{\chi}^2_i - \frac{1}{2}\tilde{\Phi}^3 = \partial_i P^i_D \approx 0 \tag{4.25}$$

are first class constraints. Curl of χ^2_i leads to two linearly independent second class constraints:

$$\tilde{\Phi}^2_n \equiv C^i_n\tilde{\chi}^2_i \equiv K^i_n\epsilon_{ijk}\partial_j\tilde{\chi}^2_k \approx 0, \tag{4.26}$$

where $n = 1, 2$. Analogous to the commuting case, instead of $\tilde{\chi}^4_i$ we deal with the following set of second class constraints

$$\tilde{\Phi}^4_n \equiv M^i_n\tilde{\chi}^4_i \equiv L^i_n\epsilon_{ijk}\partial_j\tilde{\chi}^4_k \approx 0, \tag{4.27}$$

$$\tilde{\Phi}^4_3 \equiv \partial_i\left(F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \tfrac{1}{2}\theta^{jk}F_{kj}F_{0i}\right) \approx 0. \tag{4.28}$$

K_n^i and L_n^i are some constants which should be determined by taking into account the other constraints when they vanish strongly. The constraints (3.35) and (3.40) are also second class. Structure of the constraints is similar to commuting case discussed in the previous chapter. In fact, the number of physical phase space fields is four.

In phase space, partition function associated with the parent action for noncommutative $U(1)$ theory (3.30) is defined as

$$\tilde{Z} = \int DPDP_D DFDA_D \; \tilde{\Delta} \exp\left\{ i \int d^4x \left[P_D^\mu \dot{A}_{D\mu} + P_{\mu\nu}\dot{F}^{\mu\nu} - \tilde{H}_C \right] \right\}. \quad (4.29)$$

Indices of integration variables are suppressed. We have adopted the gauge fixing conditions

$$\tilde{\Lambda}^1 = A_{D0} \approx 0, \qquad \tilde{\Lambda}^2 = \partial_i A_{Di} \approx 0. \quad (4.30)$$

Therefore, the measure $\tilde{\Delta}$ is

$$\tilde{\Delta} = \det\{\tilde{\xi}^\alpha, \tilde{\Lambda}^\beta\} \det^{\frac{1}{2}}\{\tilde{\Phi}^a, \tilde{\Phi}^b\} \prod_{\sigma=1}^{2} \delta(\tilde{\xi}^\sigma)\delta(\tilde{\Lambda}^\sigma) \prod_{c=1}^{4} \delta(\tilde{\Phi}^c). \quad (4.31)$$

Contribution of the first class constraints $\tilde{\xi}^\alpha$ and their subsidiary conditions $\tilde{\Lambda}^\alpha$ to the measure is

$$\det\{\tilde{\xi}^\alpha, \tilde{\Lambda}^\beta\} = \det(\partial^2). \quad (4.32)$$

The second class constraints $\tilde{\Phi}^a$ contribute to the measure as

$$\begin{aligned} \det^{\frac{1}{2}}\{\tilde{\Phi}^a, \tilde{\Phi}^b\} = \quad & g^4 \det(\partial^2) \det\left(\epsilon_{ijk}\partial^i M_1^j M_2^k\right) \\ & \det\left(\epsilon_{ijk}\partial^i C_1^j C_2^k\right) \det\left(-1 + \frac{1}{2}\theta^{ij}F_{ji}\right). \end{aligned} \quad (4.33)$$

$\epsilon_{ijk}\partial^i M_1^j M_2^k$ and $\epsilon_{ijk}\partial^i C_1^j C_2^k$ denote multiplication of the three linear differential operators and as usual, determinants of them are defined as multiplication of the eigenvalues of the linear operators. The last term in (4.33) is to be interpreted as multiplication of the value of $(-1 + \frac{1}{2}\theta^{ij}F_{ji})$ over all spacetime. The determinant should be regularized, however as we will show, our results are independent of their regularizations.

Performing the functional integrations over $F^{\mu\nu}$ and $P_{\mu\nu}$ in (4.29) we obtain

$$\begin{aligned} \tilde{Z} = \quad & \int D\mathbf{A}_D D\mathbf{P}_D \delta(\partial \cdot \mathbf{P}_D)\delta(\partial \cdot \mathbf{A}_D)\det(\partial^2) \\ & \exp\left\{ i \int d^3x \left[P_{Di}\dot{A}_D^i - \frac{1}{2g^2}P_{Di}P_D^i - \frac{g^2}{4}F_{Dij}F_D^{ij} \right.\right. \\ & \left.\left. + \frac{1}{2g^4}\tilde{\theta}^{0i}P_{Di}P_D^2 + \tilde{\theta}^{0i}F_{Dij}F_D^{jk}P_{Dk} + \frac{1}{4}\tilde{\theta}^{0i}P_{Di}F_D^2 \right] \right\}. \end{aligned} \quad (4.34)$$

The determinant (4.33) is cancelled[7] when we used $\delta(\tilde{\Phi}^a)$ to express the "redundant" fields $F^{\mu\nu},\ P_{\mu\nu}$ in terms of the "physical" fields $A^i_D,\ P^i_D$. Obviously, there are other solutions of (4.27) and (4.28) which would be useful to express another set of fields in terms of the remaining ones. We take the solution yielding the partition function which we desire. We observe that in (4.34) the exponential term is the first order of the dual noncommutative $U(1)$ theory whose hamiltonian is (3.54).

Like the commuting case discussed in the previous subsection, when $\tilde{\Phi}^3 = 0$ is used to write

$$F_{ij} = \partial_i A_j - \partial_j A_i,$$

we demand that the constraint

$$\partial_i A^i \approx 0$$

should be fulfilled. Moreover, when we change the variables $F_{ij} \rightarrow A_i$ we choose the normalization and domains of integrations in (4.29) such that (4.13) is satisfied. Equipped with these, we perform the integrations over the fields $A_{D\mu},\ P_{D\mu},\ P_{\mu\nu}$ in (4.29) which yields

$$\begin{aligned}
\tilde{Z} \;=\; & \mathrm{det g}^{-4} \int \mathrm{D}\mathbf{A}\mathrm{D}\mathrm{F}^{0\mathrm{i}} \delta(\partial \cdot \mathbf{A}) \det(\partial^2) \det\left(-1 + \frac{1}{2}\theta^{\mathrm{ij}}\mathrm{F}_{\mathrm{ji}}\right) \\
& \delta\left(\partial_i (F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i})\right) \\
& \exp\Big\{ i \int d^3x \Big[\frac{1}{g^2}\dot{A}^i (F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i}) \\
& + \frac{1}{2g^2}F_{0i}F^{0i} - \frac{1}{4g^2}F_{ij}F^{ij} + \frac{1}{g^2}F^{0i}F^{0j}F_{jk}\theta^{ki} - \frac{1}{4g^2}\theta^{jk}F_{jk}F_{0i}F^{0i} \\
& + \frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl} \Big] \Big\}.
\end{aligned} \tag{4.35}$$

We made use of the fact that employing $\delta(\tilde{\Phi}^a)$, $\delta(\partial \cdot \mathbf{P}_D)$, $\delta(\partial \cdot \mathbf{A}_D)$ to express P^i_D, A^i_D in terms of F_{0i} and A_i gives the following contribution to the measure

$$\left[\det(\mathrm{g}^4)\det\left(\epsilon_{\mathrm{ijk}}\partial^{\mathrm{i}}\mathrm{C}^{\mathrm{j}}_1\mathrm{C}^{\mathrm{k}}_2\right)\det\left(\epsilon_{\mathrm{ijk}}\partial^{\mathrm{i}}\mathrm{M}^{\mathrm{j}}_1\mathrm{M}^{\mathrm{k}}_2\right)\right]^{-1} \tag{4.36}$$

To deal with P_i which are the canonical momenta of A_i, let us adopt the change of variables

$$g^2 P^i = F^{0i} - F_{ij}\theta^{jk}F_{k0} - F^{0j}F_{jk}\theta^{ki} - \frac{1}{2}\theta^{jk}F_{kj}F_{0i}, \tag{4.37}$$

[7] Obviously, to obtain (4.34) one does not need to separate $\tilde{\chi}^4_i$ as (4.27)– (4.28).

by inspecting the terms multiplying $\dot{A}^i$. Thus, the partition function (4.35) can be written as

$$\begin{aligned}\tilde{Z} &= \det(\mathrm{g}^{-4})\int \mathrm{D}\mathbf{A}\mathrm{D}\mathbf{P}\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\det(\partial^2)\\ &\quad \exp\Big\{i\int d^3x\Big[\dot{A}^iP_i-\frac{g^2}{2}P_iP^i-\frac{1}{4g^2}F^{ij}F_{ij}-g^2\theta^{ij}P_iP^kF_{jk}\\ &\quad +\frac{g^2}{4}\theta^{ij}F_{ji}P^2+\frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl}\Big]\Big\}. \end{aligned} \tag{4.38}$$

In the exponential factor of (4.38) we recognize the hamiltonian of the noncommutative $U(1)$ theory (3.53).

It could be possible to show that the canonical momenta P_i are given as in (4.37) using Dirac brackets:

$$\begin{aligned}\{F_{0i}(x),P_{Dj}(y)\}_{\mathrm{Dirac}} &= \{F_{0i},P_{0k}\}\{P_{0k},\tilde{\Phi}^4_l\}^{-1}\{\tilde{\Phi}^4_l,P_{Dj}\}=g^2\epsilon_{jkl}[\delta^k_i+F^{km}\theta_{mi}\\ &\quad +F_{im}\theta^{mk}+\frac{1}{2}\delta^k_i\theta^{mn}F_{nm}]\partial^l_x\delta^3(x-y). \end{aligned} \tag{4.39}$$

Vanishing of (3.37) and (3.40) strongly the left hand side of (4.39) can equivalently be written as

$$\{F_{0i}(x),P_{Dj}(y)\}_{\mathrm{Dirac}}=-\epsilon_{jkl}\partial^k_y\{F_{0i}(x),A^l(y)\}_{\mathrm{Dirac}}. \tag{4.40}$$

By comparing the right hand sides of (4.39) and (4.40) we observe that they are compatible when

$$F_{0i}=-g^2(P_i+F_{ij}\theta^{jk}P_k+F^{jk}\theta_{ki}P_j-\frac{1}{2}F_{jk}\theta^{kj}P_i). \tag{4.41}$$

Solving this equation for P_i at the first order in θ_{ij}, gives rise to (4.37).

We adopt the normalization consistent with the ordinary case to write partition function of the noncommutative $U(1)$ theory in phase space as

$$\begin{aligned}\tilde{Z} &= \det(\mathrm{g}^{-2})\int \mathrm{D}\mathbf{A}\mathrm{D}\mathbf{P}\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\det(\partial^2)\\ &\quad \exp\Big\{i\int d^3x\Big[\dot{A}^iP_i-\frac{g^2}{2}P_iP^i-\frac{1}{4g^2}F^{ij}F_{ij}-g^2\theta^{ij}P_iP^kF_{jk}\\ &\quad +\frac{g^2}{4}\theta^{ij}F_{ji}P^2+\frac{1}{8g^2}\theta^{ij}F_{ij}F_{kl}F^{kl}\Big]\Big\} \end{aligned} \tag{4.42}$$

Accordingly, the dual partition function is given by

$$\begin{aligned}\tilde{Z}_D &= \det(\mathrm{g}^{2})\int \mathrm{D}\mathbf{A}\mathrm{D}\mathbf{P}\delta(\partial\cdot\mathbf{P})\delta(\partial\cdot\mathbf{A})\det(\partial^2)\\ &\quad \exp\Big\{i\int d^3x\Big[\dot{A}^iP_i-\frac{1}{2g^2}P_iP^i-\frac{g^2}{4}F^{ij}F_{ij}+\frac{1}{2g^4}\tilde{\theta}^{0i}P_iP^2\\ &\quad +\tilde{\theta}^{0i}F_{ij}F^{jk}P_k+\frac{1}{4}\tilde{\theta}^{0i}P_iF^2\Big]\Big\} \end{aligned} \tag{4.43}$$

where we renamed A_{Di}, P_{Di} as A_i, P_i.
We conclude that in phase space, partition functions for the noncommutative $U(1)$ theory and its dual are the same

$$\tilde{Z} = \tilde{Z}_D. \tag{4.44}$$

This results demonstrates that strong-weak duality transformation is helpful to make calculations in weak coupling regions to extract information about physical quantities in the strong coupling regions. Contrary to the usual $U(1)$ theory, momentum integrals in $\tilde{Z}$ and $\tilde{Z}_D$ are not easily computable. Because of this we cannot derive any relation between the partition functions in configuration space. Nevertheless, the result obtained (4.44) demonstrates that strong–weak duality can be helpful to calculate physical quantities in weak coupling regions to extract information about the strong coupling regions.

5 SUPERSYMMETRIC NONCOMMUTATIVE U(1) GAUGE THEORY

Supersymmetry is a graded Lie algebra which is the only one can be added consistently to the S-matrix symmetries. This is a symmetry between bosons and fermions and hence in a manner supersymmetric theories are the attempts of unifying the matter and interactions. Historically in the context of string theory the first examples of these kind of theories were introduced by Neveu, Schwarz and Ramond [88]. Especially an important property of supersymmetric theories is that radiative corrections tend to be less important in them due to cancelations between fermion loops and boson loops. As a result certain quantities that are small or vanish classically will remain so once radiative corrections are taken into account.

According to the Coleman-Mandula theorem [89] the internal symmetries such as spin, electric charge, hypercharge, etc. do not mix with space-time symmetries. This means that the symmetry generators associated with internal quantum degrees must be translationally and rotationally invariant. In Coleman-Mandula theorem this internal symmetries defined by a Lie group with real parameters and the charge operators associated with such Lie groups obey commutation relations with each other. More precisely the particle states which are related with each other by an internal symmetry transformation must have the same mass and spin.

Haag-Lopuszanski-Sohnius proved that by relaxing one assumption of the Coleman-Mandula, space-time symmetries can be related with internal symmetries [90]. In that case symmetry operators are fermionic and obey an anticommutation relation. Hence bosons and fermions appear in the same representation which is called multiplet and have the same mass. The symmetry operations will transform different members of a multiplet into each other. These multiplets contain the same number of fermions and bosons. The minimal supersymmetry have one supersymmetry generators and called $N = 1$ supersymmetry. The number of supersymmetry generators is constrained with consistency condition. This is four for supersymmetry and eight for supergravity. Theories with more then one generators are called extended supersymmetries.

Here we of course can not give a complete discussion of superymmetry but only the part of that we will deal with, $N = 1$ supersymmetry. There is a great amount of material but we will especialy refer to the [91] and[92]. Conventional details and spinor multiplication rules are defined in the appendix-B. So in the first part we will take a look at $N = 1$ supersymmetry and then deal with the supersymmetric gauge theory in the context of noncommutative space. Mainly we will exhibit how duality can be defined in noncommutative supersymmetric $U(1)$ gauge theory by using our previous approach.

Supersymmetry is defined by the algebra of the supersymmetry transformation generators in addition to Poincaré algebra. That is

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \tag{5.1}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0 \tag{5.2}$$

$$\{P_\mu, Q^A_\alpha\} = \{P_\mu, \bar{Q}_{\dot{\alpha}A}\} = 0 \tag{5.3}$$

where $\alpha, \beta, \dot{\alpha}, \dot{\beta} = 1, 2$ denote components of Weyl spinors, μ, ν are Lorentz indices and take values from 0 to 3, and A, B refer to an internal space degree which is in our case equal to 1. This supersymmetry algebra can be defined in terms of commutators by introducing anticommuting parameters $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ which satisfy

$$\{\theta^\alpha, \theta^\beta\} = \{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} = \cdots = [P_\mu, \theta^\alpha] = 0 \tag{5.4}$$

Here it should be noted that we made a change of parametrization of variables. Here after we will use the parameter θ for supersymmetry parameter and Θ for noncommutativity paramater. Hence the algebra (5.1)-(5.3) become

$$\begin{aligned} [\theta Q, \bar{\theta}\bar{Q}] &= 2\theta\sigma^\mu\bar{\theta}P_\mu \qquad (5.5) \\ [\theta Q, \theta Q] &= [\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}] = 0 \\ [P_\mu, \theta Q] &= [P_\mu, \bar{\theta}\bar{Q}] = 0 \end{aligned}$$

Realization of this algebra on fields is especially important. For example being the most simplest supersymmetric theory Wess-Zumino model contains a chiral and an anti-chiral multiplet. Chiral multiplet is formed by two complex scalars and a chiral Weyl spinor

$$\Phi = (\phi, \psi_\alpha, F). \tag{5.6}$$

Multiplet constructions can be found in [91] in detail. Beginning from a ground state, we use the term ground state to denote the element of multiplet with the smallest spin number from which other elements of multiplet can be obtained, and acting the generators on this state in a consistent way with the supersymmetry algebra give the entire multiplet. At some stage of algebra it requires to impose some constraints. For chiral multiplet the constraint is $[\phi, \bar{Q}_{\dot{\alpha}}] = 0$ for ground state. Component fields in the multiplet transform under the supersymmetry as

$$\delta\phi = \sqrt{2}\theta\psi \tag{5.7}$$

$$\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\phi + \sqrt{2}\theta_\alpha F \tag{5.8}$$

$$\delta F = i\sqrt{2}\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi \tag{5.9}$$

where

$$\delta A = [\theta Q + \bar{\theta}\bar{Q}, A]. \tag{5.10}$$

Anti-chiral multiplet can be obtained from chiral multiplet by hermitian conjugation;

$$\Phi^\dagger = (\phi^\dagger, \bar{\psi}_{\dot{\alpha}}, F^\dagger). \tag{5.11}$$

Constraint equation for anti-chiral multiplet is $[\phi^\dagger, Q] = 0$. By these definitions (anti)chiral multiplet forms a linear representation of the algebra. The action

$$\begin{aligned} S_{WZ} = & \int d^4x\{-\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}\partial\psi + F^\dagger F \\ & + [m(\phi F - \frac{1}{2}\psi\psi) + g(\phi\phi F - \psi\psi\phi) + h.c.]\} \end{aligned} \tag{5.12}$$

is invariant under above supersymmetry variations.

The superspace formalism is useful for calculations in supersymmetric theories especially in $N = 1$. Fields in superspace are defined as functions of the superspace coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$; where x^μ are usual space-time coordinates and $\theta, \bar{\theta}$ are independent spinorial coordinates. Naturally, because of the anticommutation property of θ's and $\bar{\theta}$'s any superfield does not contain terms bigger than θ^2 and $\bar{\theta}^2$. Hence any superfield can always be expanded as

$$\begin{aligned} F(x,\theta,\bar{\theta}) &= f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta}n(x) \\ &+ \theta\sigma^{\mu}\bar{\theta}v_{\mu} + \theta\theta\bar{\theta}\lambda(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned} \tag{5.13}$$

This definition is identical to express the components of multiplet as power series expansion together with certain constraints. Supersymmety generators are realized as differential operators in superspace.

$$\begin{aligned} Q &= \frac{\partial}{\partial\theta^{\alpha}} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} \\ \bar{Q} &= \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}\partial_{\mu} \end{aligned} \tag{5.14}$$

Above differential operators generate a motion in the parameter space $(x,\theta,\bar{\theta})$ and obey the same anticommutation relation (5.1). Definition (5.13) contains all possible terms with respect to the powers of $\theta, \bar{\theta}$ and by this form they form reducible representations of supersymmetry. Covariant derivatives are defined as

$$\begin{aligned} D_{\alpha} &= \frac{\partial}{\partial\theta^{\alpha}} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} \end{aligned} \tag{5.15}$$

Covariant derivative operators form a different realization of the super $-$ Poincaré group and yield an inverted motion with respect to the operators Q and $\bar{Q}$. They satisfy the following anticommutation relations

$$\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} \tag{5.16}$$

$$\{D, D\} = \{\bar{D}, \bar{D}\} = 0 \tag{5.17}$$

and they anticommute with Q's

$$\{D,Q\} = \{\bar{D},Q\} = \{D,\bar{Q}\} = \{\bar{D},\bar{Q}\} = 0 \tag{5.18}$$

Now a chiral superfield is defined by putting the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{5.19}$$

on superfield and an anti-chiral one by

$$D_{\alpha}\bar{\Phi} = 0 \tag{5.20}$$

These constraints are more tractable in the new coordinate system $(y^\mu, \theta, \bar{\theta})$ where

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}. \tag{5.21}$$

Covariant derivatives become

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu} \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \end{aligned} \tag{5.22}$$

The most general solution of (5.19) can be given as

$$\begin{aligned} \Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\ &= \phi(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) \\ &+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta\theta F(x) \end{aligned} \tag{5.23}$$

and anti-chiral superfield can be obtained easily from (5.23) by hermitian conjugation instead of solving the constraint (5.20),

$$\Phi^\dagger = \phi^*(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^*(y^\dagger) \tag{5.24}$$

where $y^{\dagger\mu} = x^\mu - i\theta\sigma^\mu\bar{\theta}$ and in this coordinates operators are

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^{\dagger\mu}} \end{aligned} \tag{5.25}$$

As can be seen easily, field content of above superfields are consistent with the definition of chiral and anti-chiral multiplets (5.6),(5.11). Products of chiral superfields are again a chiral superfield and it is also so for anti-chiral ones.
Vector superfield is defined by reality condition

$$V = V^\dagger \tag{5.26}$$

The corresponding superfield which satisfy above condition is given by

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\ &+ \frac{i}{2}\theta\theta(M(x) + iN(x)) - \frac{i}{2}\bar{\theta}\bar{\theta}(M(x) - iN(x)) \\ &- \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)] \\ &- i\bar{\theta}\bar{\theta}\theta[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(x) + \frac{1}{2}\Box C(x)] \end{aligned} \tag{5.27}$$

Here C, M, N and D are real scalars,υ_μ is a real vector field and λ, χ are Weyl spinors. Addition of chiral and anti-chiral superfields (5.23),(5.24) gives a real superfield and one can observe that combination of this addition with vector superfield yield the supersymmetric generalization of gauge transformation.

$$V \to V + \Phi + \Phi^\dagger \tag{5.28}$$

Under this transformation component fields transform as

$$\begin{aligned} C &\to C + \phi + \phi^\dagger \\ \chi &\to \chi - i\sqrt{2}\psi \\ M + iN &\to M + iN - 2iF \\ \upsilon_\mu &\to \upsilon_\mu - i\partial_\mu(\phi - \phi^\dagger) \\ \lambda &\to \lambda \\ D &\to D \end{aligned} \tag{5.29}$$

As can be seen transformation of vector component resembles an ordinary gauge transformation, while the fields λ and D are gauge invariants. This gauge freedom provides a special gauge in which C, χ, M and N are all zero. Thus the vector superfield has a more simple form.

$$V_{WZ} = -\theta\sigma^\mu\bar{\theta}\upsilon_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{5.30}$$

Powers of V satisfy the properties

$$V^2 = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\upsilon_\mu\upsilon^\mu \tag{5.31}$$

$$V^3 = 0 \tag{5.32}$$

This gauge is known as Wess-Zumino gauge. Gauge choice breaks the supersymmetry but fermionic and bosonic degrees of freedoms still equal to each other. Thus superfield V can be viewed as the supersymmetric generalization of the Yang-Mills potential. Corresponding supersymmetric field strengths are

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \tag{5.33}$$

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \tag{5.34}$$

which are chiral and anti-chiral superfield respectively, i.e,

$$\bar{D}_{\dot{\alpha}} W_\beta = 0 \tag{5.35}$$

$$D_\alpha \bar{W}_{\dot{\beta}} = 0 \tag{5.36}$$

Component expansion in WZ gauge and in $(y^\mu, \theta, \bar{\theta})$ coordinates is

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - i\sigma^{\mu\nu}_\alpha{}^\beta \theta_\beta(\partial_\mu \upsilon_\nu(y) - \partial_\nu \upsilon_\mu(y)) + \theta\theta\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \bar{\psi}^{\dot{\alpha}}(y) \tag{5.37}$$

and

$$\begin{aligned} \bar{W}^{\dot{\alpha}} &= i\bar{\lambda}^{\dot{\alpha}}(y^\dagger) + \bar{\theta}^{\dot{\alpha}} D^\dagger(y^\dagger) + i\bar{\sigma}^{\mu\nu}{}_{\dot{\beta}}^{\dot{\alpha}} \bar{\theta}^{\dot{\beta}}(\partial_\mu \upsilon_\nu(y^\dagger) - \partial_\nu \upsilon_\mu(y^\dagger)) \\ &+ \bar{\theta}\bar{\theta}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu \partial^\mu \psi_\alpha(y^\dagger) \end{aligned} \tag{5.38}$$

Field strengths derived from vector superfield satisfy the additional constraint equation,supersymmetric Bianchi identity.

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \tag{5.39}$$

We have defined a gauge invariant field strength which is constructed from a vector superpotential. The supersymmetric gauge invariant generalization of the Lagrangian for a vector field can be defined from this chiral superfield. For this aim observe that $\theta\theta$ component of the product $W^\alpha W_\alpha$ give a space derivative. The same is valid for $\bar{\theta}\bar{\theta}$ component of $\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$.

$$S = \frac{1}{4g^2}\int d^4x(\int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}) \tag{5.40}$$

This reduce to the action of supersymmetric U(1) gauge theory

$$S = \int d^4x \frac{1}{g^2}\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2\} \tag{5.41}$$

This is pure $N = 1$ gauge theory and component field υ_μ is a gauge boson, λ is the supersymmetric partner of the gauge boson, gaugino, and D is a real scalar, auxiliary field.

5.1 Duality in Supersymmetric $U(1)$ Gauge Theory

Parent action of "ordinary" supersymmetric $U(1)$ gauge theory was formulated by superfields [24]. In terms of component fields we will define two different parent actions which yield the same dual symmetric actions. Duality transformation of supersymmetric U(1) gauge theory can be formulated in terms of a general superfield $\tilde{W}_\alpha$ and a dual vector superfield V_D as,

$$I_P = \frac{1}{4g^2}\int d^4x(\int d^2\theta\tilde{W}^2 + \int d^2\bar{\theta}\bar{\tilde{W}}^2) + \frac{1}{2}\int d^4x d^4\theta(V_D D\tilde{W} - V_D\bar{D}\bar{\tilde{W}}) \quad (5.42)$$

where with general superfield we mean that it is not field strength of a vector superfield (5.33). The equations of motion with respect to the dual superfield V_D gives

$$D\tilde{W} - \bar{D}\bar{\tilde{W}} = 0 \quad (5.43)$$

that is the supersymmetric generalization of Bianchi identity, and solution of this restriction gives the ordinary superfield (5.33). Substituting this solution in the parent action (5.42), one gets the action of supersymmetric $U(1)$ gauge theory,

$$I = \frac{1}{4g^2}\int d^4x(\int d^2\theta W^2 + \int d^2\bar{\theta}\bar{W}^2) \quad (5.44)$$

On the other hand, when solutions of the equations of motion with respect to $\tilde{W}_\alpha$ and $\bar{\tilde{W}}^{\dot{\alpha}}$ following from I_P are plugged into (5.42), one obtains the dual action in terms of superfields

$$I_D = \frac{g^2}{4}\int d^4x(\int d^2\theta W_D^2 + \int d^2\bar{\theta}\bar{W}_D^2) \quad (5.45)$$

where W_D is the dual superfield strength $W_{D\alpha} = \frac{1}{4}\bar{D}^2 D_\alpha V_D$.

The original and the dual actions (5.44) and (5.45) are in the same form except with g^{-2} replaced with g^2. Thus, one can conclude that supersymmetric $U(1)$ gauge theory possesses (S) duality symmetry.

Instead of superfields, we would like to consider duality transformations in terms of their component fields. It is straightforward to construct a general chiral superfield $\tilde{W}_\alpha$ that does not satisfy the condition(5.39) as

$$\tilde{W}_\alpha(y) = -i\lambda_\alpha(y) + \theta_\alpha\tilde{D}(y) - i\sigma_\alpha^{\mu\nu\,\beta}\theta_\beta\tilde{F}_{\mu\nu}(y) + \theta\theta\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}}(y) \quad (5.46)$$

Here, λ and $\bar{\psi}$ are two independent Weyl spinors, $\tilde{F}_{\mu\nu}$ is a complex anti-symmetric second rank tensor field and $\tilde{D}$ is a complex scalar field. Hermitean conjugate of the chiral superfield $\tilde{W}_\alpha$ can be written as,

$$\bar{\tilde{W}}^{\dot{\alpha}}(y^\dagger) = i\bar{\lambda}^{\dot{\alpha}}(y^\dagger) + \bar{\theta}^{\dot{\alpha}}\tilde{D}^\dagger(y^\dagger) + i\bar{\sigma}^{\mu\nu\dot{\alpha}}_{\ \ \ \ \dot{\beta}}\bar{\theta}^{\dot{\beta}}\tilde{F}^\dagger_{\mu\nu}(y^\dagger) + \bar{\theta}\bar{\theta}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\partial^\mu\psi_\alpha(y^\dagger). \quad (5.47)$$

It should be stressed that these are not the same fields as (5.37) and (5.38). There Weyl spinors are hermitian conjugate of each other and $F_{\mu\nu}$ is field strength of vector potential A_μ. Moreover the auxiliary field D is real.
Plugging (5.46) and (5.47) and the real vector superfield

$$V_D = -(\theta\sigma^\mu\bar{\theta})A_{D\mu} + i\theta\theta\bar{\theta}\bar{\lambda}_D - i\bar{\theta}\bar{\theta}\theta\lambda_D + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D_D \tag{5.48}$$

into (5.42) the parent action in component fields is obtained

$$I_p = I_o[\tilde{F}, \psi, \lambda, \tilde{D}] + I_l, \tag{5.49}$$

where we defined

$$\begin{aligned} I_o &= \frac{1}{g^2}\int d^4x[-\frac{1}{8}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{i}{16}\epsilon^{\mu\nu\lambda\kappa}\tilde{F}_{\mu\nu}\tilde{F}_{\lambda\kappa} - \frac{1}{8}\tilde{F}^{\dagger\mu\nu}\tilde{F}^\dagger_{\mu\nu} + \\ &\quad \frac{i}{16}\epsilon^{\mu\nu\lambda\kappa}\tilde{F}^\dagger_{\mu\nu}\tilde{F}^\dagger_{\lambda\kappa} - \frac{i}{2}\lambda\partial\!\!\!/\bar{\psi} - \frac{i}{2}\bar{\lambda}\bar{\partial\!\!\!/}\psi + \frac{1}{4}\tilde{D}^2 + \frac{1}{4}\tilde{D}^{\dagger 2}], \end{aligned} \tag{5.50}$$

and the Legendre transformation term

$$\begin{aligned} I_l &= \frac{1}{2}\int d^4x[-i\tilde{F}^{\mu\nu}\partial_\mu A_{D\nu} + \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}\tilde{F}_{\mu\nu}\partial_\lambda A_{D\kappa} + i\tilde{F}^{\dagger\mu\nu}\partial_\mu A_{D\nu} \\ &\quad + \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}\tilde{F}^\dagger_{\mu\nu}\partial_\lambda A_{D\kappa} + \frac{1}{2}\lambda_D\partial\!\!\!/\bar{\psi} + \lambda\partial\!\!\!/\bar{\lambda}_D \\ &\quad - \frac{1}{2}\bar{\lambda}_D\bar{\partial\!\!\!/}\psi - \bar{\lambda}\bar{\partial\!\!\!/}\lambda_D + iD_D(\tilde{D} - \tilde{D}^\dagger)]. \end{aligned} \tag{5.51}$$

here we use $\partial\!\!\!/$ for $\sigma^\mu\partial_\mu$ and $\bar{\partial\!\!\!/}$ for $\bar{\sigma}^\mu\partial_\mu$.

We now proceed as before to derive supersymmetric $U(1)$ gauge theory in terms of the component fields from the parent action (5.49): the equations of motion with respect to the dual vector field $A_{D\mu}$

$$[\frac{i}{2}(\partial_\mu\tilde{F}^{\mu\kappa} - \partial_\mu\tilde{F}^{\dagger\mu\kappa}) - \frac{1}{4}\epsilon^{\mu\nu\lambda\kappa}\partial_\lambda(\tilde{F}_{\mu\nu} + \tilde{F}^\dagger_{\mu\nu})]_{\tilde{F}=F} = 0. \tag{5.52}$$

lead to $F_{\mu\nu}$ which satisfy

$$F_{\mu\nu} = F^\dagger_{\mu\nu}, \quad \epsilon^{\mu\nu\lambda\kappa}\partial_\lambda F_{\mu\nu} = 0 \tag{5.53}$$

which are solved by taking $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is the field strength of the vector field A_μ. When we also use the equation of motion with respect to the other dual fields

$$\partial\!\!\!/\bar{\psi} = \partial\!\!\!/\bar{\lambda}\ ,\ \bar{\partial\!\!\!/}\psi = \bar{\partial\!\!\!/}\lambda\ ,\ \tilde{D} - \tilde{D}^\dagger|_{\tilde{D}=D} = 0. \tag{5.54}$$

in the parent action (5.49) we obtain the supersymmetric $U(1)$ gauge theory action in terms of component fields

$$I = \frac{1}{g^2}\int d^4x[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\lambda\partial\!\!\!/\bar{\lambda} - \frac{i}{2}\bar{\lambda}\bar{\partial\!\!\!/}\lambda + \frac{1}{2}D^2] \tag{5.55}$$

Note also that when the above equations are substituted in the general superfield given in (5.46), one finds the standard chiral vector field W_α that can be obtained from the N=1 vector field V as $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V$ in the Wess-Zumino gauge. Similarly, we can obtain the dual action (5.45) in terms of the component fields using the equations of motion (5.49) with respect to the fields $\tilde{F}_{\mu\nu}, \lambda, \bar{\psi}, \tilde{D}$:

$$(\eta^{\mu\lambda}\eta^{\nu\kappa} - \eta^{\mu\kappa}\eta^{\nu\lambda} + i\epsilon^{\mu\nu\lambda\kappa})F_{\lambda\kappa} = -ig^2(\eta^{\mu\lambda}\eta^{\nu\kappa} - \eta^{\mu\kappa}\eta^{\nu\lambda} + i\epsilon^{\mu\nu\lambda\kappa})F_{D\lambda\kappa} \tag{5.56}$$

$$\partial\!\!\!/\bar{\psi} = -ig^2\partial\!\!\!/\bar{\lambda}_D, \quad \bar{\partial\!\!\!/}\lambda = -ig^2\bar{\partial\!\!\!/}\lambda_D, \quad \tilde{D} = -ig^2 D_D \tag{5.57}$$

and the equations of motion with respect to $\tilde{F}^\dagger_{\mu\nu}, \bar{\lambda}, \psi, \tilde{D}^\dagger$:

$$(\eta^{\mu\lambda}\eta^{\nu\kappa} - \eta^{\mu\kappa}\eta^{\nu\lambda} - i\epsilon^{\mu\nu\lambda\kappa})\tilde{F}^\dagger_{\lambda\kappa} = ig^2(\eta^{\mu\lambda}\eta^{\nu\kappa} - \eta^{\mu\kappa}\eta^{\nu\lambda} - i\epsilon^{\mu\nu\lambda\kappa})F_{D\lambda\kappa} \tag{5.58}$$

$$\partial\!\!\!/\bar{\lambda} = ig^2\partial\!\!\!/\bar{\lambda}_D, \quad \bar{\partial\!\!\!/}\psi = ig^2\bar{\partial\!\!\!/}\lambda_D, \quad \tilde{D}^\dagger = ig^2 D_D \tag{5.59}$$

where $F_{D\mu\nu} = \partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}$. These equations can be solved for the original fields to substitute them in the parent action (5.49) yielding the dual supersymmetric $U(1)$ gauge action

$$I_D = g^2\int d^4x[-\frac{1}{4}F_D^{\mu\nu}F_{D\mu\nu} - \frac{i}{2}\lambda_D\partial\!\!\!/\bar{\lambda}_D - \frac{i}{2}\bar{\lambda}_D\bar{\partial\!\!\!/}\lambda_D + \frac{1}{2}D_D^2] \tag{5.60}$$

Instead of the complex field $\tilde{F}_{\mu\nu}$ we can deal with the real antisymmetric tensor field $F_{R\mu\nu}$ from the beginning. We propose the following parent action for this case

$$S_p = S_o[F_R, \psi, \lambda, \tilde{D}] + S_l, \tag{5.61}$$

where

$$S_o = \frac{1}{4g^2}\int d^4x[-F_R^{\mu\nu}F_{R\mu\nu} - 2i\bar{\lambda}\sigma^\mu\partial_\mu\psi - 2i\lambda\sigma^\mu\partial_\mu\bar{\psi} + \tilde{D}^2 + \tilde{D}^{\dagger 2}] \tag{5.62}$$

and the Legendre transformation part

$$\begin{aligned} S_l \;&=\; \frac{1}{2}\int d^4x[\epsilon^{\mu\nu\rho\sigma}F_{R\mu\nu}\partial_\rho A_{D\sigma} + \lambda_D\sigma^\mu\partial_\mu\bar{\psi} + \bar{\lambda}_D\bar{\sigma}^\mu\partial_\mu\lambda \\ &\quad -\lambda_D\sigma^\mu\partial_\mu\bar{\lambda} - \bar{\lambda}_D\bar{\sigma}^\mu\partial_\mu\psi + iD_D(\tilde{D} - \tilde{D}^\dagger)]. \end{aligned} \tag{5.63}$$

Now equations of motions with respect to the dual fields, $A_D,\ \lambda_D,\ \bar{\lambda}_D,\ D_D$, yield

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{R\rho\sigma}|_{F_R=F} &= 0, && (5.64)\\
\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}} - \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} &= 0, && (5.65)\\
\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\lambda_\alpha - \bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha &= 0, && (5.66)\\
\left(\tilde{D}-\tilde{D}^\dagger\right)_{\tilde{D}=D} &= 0 && (5.67)
\end{aligned}$$

When solution of this equations with respect to $F_{\mu\nu}$ and real scalar field D used in (5.61) yields the supersymmetric $U(1)$ gauge theory (5.55).

The equations of motions with respect to the original fields $F_{R\mu\nu}, \lambda, \psi, \bar{\lambda}, \tilde{D}, \bar{\psi}, \tilde{D}^\dagger$ are

$$\begin{aligned}
-\frac{1}{g^2}F_R^{\mu\nu} + \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_{D\sigma} &= 0 && && (5.68)\\
\frac{1}{g^2}\tilde{D}^\dagger - iD_D &= 0, && \frac{1}{g^2}\tilde{D} + iD_D = 0 &&\\
\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu(-\frac{i}{g^2}\bar{\psi}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}}_D) &= 0, && \bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu(-\frac{i}{g^2}\psi_\alpha - \lambda_{D\alpha}) = 0 &&\\
\partial_\mu(-\frac{i}{g^2}\bar{\lambda}_{\dot{\alpha}} + \bar{\lambda}_{D\dot{\alpha}})\bar{\sigma}^{\mu\dot{\alpha}\alpha} &= 0, && \partial_\mu(-\frac{i}{g^2}\lambda^\alpha + \lambda^\alpha_D)\sigma^\mu_{\alpha\dot{\alpha}} = 0 &&
\end{aligned}$$

Solving these equations for the dual fields and substituting them in the parent action (5.61) yield the dual of action of $N=1$ supersymmetric $U(1)$ gauge theory (5.60). We conclude that both of the parent actions (5.49) and (5.61) generate supersymmetric $U(1)$ gauge theory and its dual.

5.2 Supersymmetric Seiberg–Witten map

Generalization of the Seiberg-Witten map to supersymmetric gauge theories can be formulated in some different ways. One of these is to generalize the definition of the map between $\hat{A}(A), \hat{\lambda}(\lambda, A)$ and A, λ to $\hat{V}(V), \hat{\Lambda}(\Lambda, V)$ and V, Λ. Here V is a vector superfield, Λ is a chiral superfield and $\hat{V}$ and $\hat{\Lambda}$ are corresponding noncommutative superfields [94]. Infinitesimal gauge transformation of the noncommutative supervector field $\hat{V}$ is defined by

$$\hat{\delta}_{\hat{\Lambda}}\hat{V} = i(\hat{\Lambda} - \hat{\bar{\Lambda}}) - \frac{i}{2}[(\hat{\Lambda} + \hat{\bar{\Lambda}}) * \hat{V} - \hat{V} * (\hat{\Lambda} + \hat{\bar{\Lambda}})] \qquad (5.69)$$

It has the properties of a nonabelian gauge transformation, although the ordinary vector field V gauge transforms as

$$\delta_\Lambda V = i(\Lambda - \bar{\Lambda}) \tag{5.70}$$

Supersymmetric Seiberg-Witten map is defined as

$$\hat{V}(V) + \hat{\delta}_{\hat{\Lambda}}\hat{V}(V) = \hat{V}(V + \delta_\Lambda V). \tag{5.71}$$

In [94] a solution of this equation is given in terms of superfields. However, it is nonlocal and do not yield the original solution

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\Theta^{kl}(A_k\partial_l A_\mu + A_k F_{l\mu}), \tag{5.72}$$

at the first order in the noncommutativity parameter $\theta^{\mu\nu}$.
On the other hand the approach suggested in [95] is to generalize the solution (5.72) to supersymmetric case as

$$\begin{aligned}
\hat{V}(V) &= V + aP^{\mu\nu}\partial_\mu\nabla_\nu V + bP^{\alpha\beta}D_\alpha V W_\beta + cP^{\alpha\beta}V D_\alpha W_\beta + c.c & (5.73)\\
\hat{\Lambda}(\Lambda, V) &= \Lambda + d\bar{D}^2(P^{\alpha\beta}D_\alpha D_\beta V) & (5.74)
\end{aligned}$$

where a,b,c,d are some constants which should be derived using (5.72). Here P and ∇ are some operators which do not depend on fields.
We would like to obtain a generalization of Seiberg-Witten map to supersymmetric $U(1)$ gauge theory in terms of the components of the superfield V. This will be performed utilizing both of the methods mentioned above. We adopt the definition (5.71) for supersymmetric Seiberg-Witten map but solve it for components of the superfield V by keeping the original solution (5.72).
The vector superfield V in Wess-Zumino gauge and chiral and anti-chiral superfields Λ and $\bar{\Lambda}$, respectively, are given as

$$\begin{aligned}
V &= -(\theta\sigma^\mu\bar{\theta})A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D, & (5.75)\\
\Lambda &= \beta + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\beta + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\beta + \sqrt{2}\theta\kappa - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\kappa\sigma^\mu\bar{\theta} + \theta\theta f, & (5.76)\\
\bar{\Lambda} &= \beta^* - i(\theta\sigma^\mu\bar{\theta})\partial_\mu\beta^* + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial^2\beta^* + \sqrt{2}\bar{\theta}\bar{\kappa} + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\kappa} + \bar{\theta}\bar{\theta}f^* & (5.77)
\end{aligned}$$

Noncommuting superfields $\hat{V}, \hat{\Lambda}, \hat{\bar{\Lambda}}$ can be written in the same form in terms of their components. At the first order in $\theta^{\mu\nu}$ let us denote the noncommutative fields as

$$\hat{V}(V) = V + V_{(1)} \tag{5.78}$$

$$\hat{\Lambda}(V,\Lambda) = \Lambda + \Lambda_{(1)}(V_i,\Lambda_i) \tag{5.79}$$

$$\hat{\bar{\Lambda}}(V,\bar{\Lambda}) = \bar{\Lambda} + \bar{\Lambda}_{(1)}(V_i,\bar{\Lambda}_i) \tag{5.80}$$

and plug them into the definition (5.71)

$$\begin{aligned}
&\theta\sigma^\mu\bar{\theta}[A_{(1)\mu}(V_i+\delta V_i) - A_{(1)\mu}(V_i) - \partial_\mu(\beta_{(1)}(V_i,\Lambda_i) + \beta^*_{(1)}(V_i,\Lambda_i))] \\
&-i\theta\theta\bar{\theta}[\bar{\lambda}_{(1)}(V_i+\delta V_i) - \bar{\lambda}_{(1)}(V_i) - \frac{i}{\sqrt{2}}\bar{\sigma}^\mu\partial_\mu\kappa_{(1)}(V_i,\Lambda_i)] \\
&+i\bar{\theta}\bar{\theta}\theta[\lambda_{(1)}(V_i+\delta V_i) - \lambda_{(1)}(V_i) + \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar{\kappa}_{(1)}(V_i,\bar{\Lambda}_i)] \\
&-\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D_{(1)}(V_i+\delta V_i) - D_{(1)}(V_i) - \frac{i}{2}\partial^2(\beta_{(1)}(V_i,\Lambda_i) - \beta^*_{(1)}(V_i,\Lambda_i))] \\
&+\sqrt{2}i\theta\kappa_{(1)}(V_i,\Lambda_i) - \sqrt{2}i\bar{\theta}\bar{\kappa}_{(1)}(V_i,\Lambda_i) + i\theta\theta f_{(1)}(V_i,\Lambda_i) \\
&-i\bar{\theta}\bar{\theta}f^*_{(1)}(V_i,\Lambda_i) + i(\beta_{(1)}(V_i,\Lambda_i) - \beta^*_{(1)}(V_i,\Lambda_i)) \\
&= \frac{1}{2}\Theta^{\nu\rho}[-(\theta\sigma^\mu\bar{\theta})\partial_\nu A_\mu\partial_\rho(\beta+\beta^*) + i\theta\theta\bar{\theta}\partial_\nu\bar{\lambda}\partial_\rho(\beta+\beta^*) \\
&-i\bar{\theta}\bar{\theta}\theta\partial_\nu\lambda\partial_\rho(\beta+\beta^*) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\{[\partial_\nu D\partial_\rho(\beta+\beta^*) \\
&+\partial_\nu A_\mu\partial_\rho\partial_\mu(\beta-\beta^*) - \sqrt{2}i\varepsilon^{\alpha\beta}\partial_\nu\lambda_\alpha\partial_\rho\kappa_\beta \\
&+\sqrt{2}i\varepsilon_{\dot{\alpha}\dot{\beta}}\partial_\nu\bar{\lambda}^{\dot{\alpha}}\partial_\rho\bar{\kappa}^{\dot{\beta}}]\} - \sqrt{2}(\theta\sigma^\mu\bar{\theta})\partial_\nu A_\mu\partial_\rho(\theta\kappa + \bar{\theta}\bar{\kappa})]
\end{aligned} \tag{5.81}$$

Here V_i and Λ_i denote the components. The equations which component fields satisfy can be obtained by matching the same θ order terms. These will give rise to following equations

$$\beta_{(1)} - \beta^*_{(1)} = 0 \tag{5.82}$$

$$f_{(1)} = f^*_{(1)} = \kappa_{(1)} = \bar{\kappa}_{(1)} = 0. \tag{5.83}$$

Moreover, there are the equations

$$A^{(1)}_\mu(V_i+\delta V_i) - A_{(1)\mu}(V_i) - \partial_\mu\beta_{(1)} = -\Theta^{\nu\rho}\partial_\nu A_\mu\partial_\rho\beta \tag{5.84}$$

$$\lambda_{(1)}(V_i+\delta V_i) - \lambda_{(1)}(V_i) = -\Theta^{\nu\rho}\partial_\nu\lambda\partial_\rho\beta \tag{5.85}$$

$$\bar{\lambda}_{(1)}(V_i+\delta V_i) - \bar{\lambda}_{(1)}(V_i) = -\Theta^{\nu\rho}\partial_\nu\bar{\lambda}\partial_\rho\beta \tag{5.86}$$

$$D_{(1)}(V_i+\delta V_i) - D_{(1)}(V_i) = -\Theta^{\nu\rho}\partial_\nu D\partial_\rho\beta \tag{5.87}$$

where V_i and Λ_i denote the component fields. Obviously, one can write (5.71) in terms of a general vector superfield instead of choosing the Wess-Zumino gauge

(5.75), which would have drastically changed the equations for component fields. However, we prefer to choose V as (5.75), so that, we deal with the equations (5.82)-(5.87) as defining supersymmetric Seiberg-Witten map. One can solve the above equations and get the noncommutative fields in terms of the commutative ones at the first order of noncommutativity parameter.

$$A_\mu^{(1)} = -\frac{1}{2}\Theta^{\nu\rho}(A_\nu\partial_\rho A_\mu + A_\nu F_{\rho\mu}) \tag{5.88}$$

$$\lambda_{(1)} = -\Theta^{\nu\rho}\partial_\nu\lambda A_\rho \tag{5.89}$$

$$\bar{\lambda}_{(1)} = -\Theta^{\nu\rho}\partial_\nu\bar{\lambda} A_\rho \tag{5.90}$$

$$D_{(1)} = -\Theta^{\nu\rho}\partial_\nu D A_\rho. \tag{5.91}$$

(5.88) and (5.89) are also found in [96] considering deformations of supersymmetric Yang–Mills theory while preserving supersymmetry. To define a parent action to obtain duality transformation we also need to define

$$\psi_{(1)} = -\Theta^{kl}\partial_k\psi A_l \tag{5.92}$$

$$\bar{\psi}_{(1)} = -\Theta^{kl}\partial_k\bar{\psi} A_l \tag{5.93}$$

5.3 Duals of Noncommutative Supersymmetric $U(1)$ Gauge Theory

Noncommutative generalization of supersymmetric $U(1)$ gauge theory [93] can be written in terms of the so called noncommuting component fields, although they satisfy the usual (anti)commutation relations, by the star product as

$$S_{NC} = \int d^4x[-\frac{1}{4g^2}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} - \frac{i}{2g^2}\hat{\bar{\lambda}}\bar{\sigma}^\mu\hat{D}_\mu * \hat{\lambda} - \frac{i}{2g^2}\hat{\lambda}\sigma^\mu\hat{D}_\mu * \bar{\lambda} + \frac{1}{2g^2}\hat{D}\hat{D}] \tag{5.94}$$

where $\hat{D}_\mu * \hat{\lambda} = \partial_\mu\hat{\lambda} + i(\hat{A}_\mu * \hat{\lambda} - \hat{\lambda} * \hat{A}_\mu)$. The action is invariant under the supersymmetry transformations given by the fermionic constant spinor parameter ξ as

$$\hat{\delta}_\xi\hat{A}_\mu = i\xi\sigma^\mu\hat{\bar{\lambda}} + i\bar{\xi}\bar{\sigma}^\mu\hat{\lambda}, \tag{5.95}$$

$$\hat{\delta}_\xi\hat{\lambda} = \sigma^{\mu\nu}\xi\hat{F}_{\mu\nu} + i\xi\hat{D}, \tag{5.96}$$

$$\hat{\delta}_\xi\hat{D} = \bar{\xi}\bar{\sigma}^\mu\hat{D}_\mu\lambda - \xi\sigma^\mu\hat{D}_\mu\hat{\bar{\lambda}}. \tag{5.97}$$

Making use of the generalization of Seiberg-Witten map to the supersymmetric case (5.84)-(5.87) we write, up to the first order in Θ, the action of noncommutative supersymmetric $U(1)$ gauge theory (5.94) in terms of the ordinary fields as

$$\begin{aligned} S_{NC}[F,\lambda,D,\Theta] &= \int d^4x\{-\frac{1}{4g^2}(F^{\mu\nu}F_{\mu\nu}+2\Theta^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}-\frac{1}{2}\Theta^{\mu\nu}F_{\nu\mu}F_{\rho\sigma}F^{\sigma\rho}) \\ &\quad -\frac{i}{g^2}(\frac{1}{2}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda+\Theta^{\mu\nu}[\frac{1}{4}\bar{\lambda}\bar{\sigma}^\rho\partial_\rho\lambda F_{\mu\nu}+\frac{1}{2}\bar{\lambda}\bar{\sigma}^\rho\partial_\mu\lambda F_{\nu\rho}\} \\ &\quad +\frac{1}{2}\lambda\sigma^\mu\partial_\mu\bar{\lambda}+\Theta^{\mu\nu}[\frac{1}{4}\lambda\sigma^\rho\partial_\rho\bar{\lambda}F_{\mu\nu}+\frac{1}{2}\lambda\sigma^\rho\partial_\mu\bar{\lambda}F_{\nu\rho}]) \\ &\quad +\frac{1}{2g^2}(D^2+\frac{1}{2}\Theta^{\mu\nu}D^2F_{\mu\nu})] \end{aligned} \tag{5.98}$$

When we write this action we set the surface terms to zero while performing required partial integrations. The same action was also obtained in [97] using a completely different approach.

Supersymmetry transformations which leave (5.98) invariant can be read from (5.95)-(5.97) as

$$\begin{aligned} \delta_\xi A_\mu &= i\xi\sigma_\mu\bar{\lambda}+i\bar{\xi}\bar{\sigma}_\mu\lambda-i\Theta^{\rho\kappa}(\xi\sigma_\rho\bar{\lambda}+\bar{\xi}\bar{\sigma}_\rho\lambda)(\frac{1}{2}F_{\kappa\mu}+\frac{1}{2}\partial_\kappa A_\mu) \\ &\quad -i\Theta^{\rho\kappa}\frac{1}{2}(\xi\sigma_\rho\partial_\mu\bar{\lambda}+\bar{\xi}\bar{\sigma}_\rho\partial_\mu\lambda)A_\kappa \end{aligned} \tag{5.99}$$

$$\begin{aligned} \delta_\xi\lambda &= \sigma^{\mu\nu}\xi F_{\mu\nu}+i\xi D+\Theta^{\rho\kappa}\partial_\rho\lambda(i\xi\sigma_\kappa\bar{\lambda}+i\bar{\xi}\bar{\sigma}_\kappa\lambda) \\ &\quad +i\Theta^{\rho\kappa}\sigma^{\mu\nu}\xi F_{\mu\rho}F_{\nu\kappa} \end{aligned} \tag{5.100}$$

$$\begin{aligned} \delta_\xi D &= \bar{\xi}\bar{\sigma}^\mu\partial_\mu\lambda-\xi\sigma^\mu\partial_\mu\bar{\lambda}-i\Theta^{\rho\kappa}(\xi\sigma_\rho\bar{\lambda}+\bar{\xi}\bar{\sigma}_\rho\lambda)\partial_\kappa D \\ &\quad +\Theta^{\rho\kappa}\xi\sigma^\mu F_{\rho\mu}\partial_\kappa\bar{\lambda}-\Theta^{\rho\kappa}\bar{\xi}\bar{\sigma}^\mu F_{\rho\mu}\partial_\kappa\lambda \end{aligned} \tag{5.101}$$

We would like to generalize the parent actions of the ordinary supersymmetric gauge theory to the noncommutative case. To this aim let us first take $\hat{F}^{\mu\nu}$ complex and deal with

$$\begin{aligned} I_{oNC} &= \frac{-1}{g^2}\int d^4x[\frac{1}{8}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}+\frac{i}{16}\epsilon^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu}\hat{F}_{\rho\sigma}+\frac{1}{8}\hat{F}^{\dagger\mu\nu}\hat{F}^\dagger_{\mu\nu}+\frac{i}{16}\epsilon^{\mu\nu\rho\sigma}\hat{F}^\dagger_{\mu\nu}\hat{F}^\dagger_{\rho\sigma} \\ &\quad +\frac{i}{2}\hat{\lambda}\sigma^\mu\hat{D}_\mu*\hat{\bar{\psi}}+\frac{i}{2}\hat{\bar{\lambda}}\bar{\sigma}^\mu\hat{D}_\mu*\hat{\psi}-\frac{1}{4}\hat{D}^2-\frac{1}{4}\hat{D}^{\dagger 2}] \end{aligned} \tag{5.102}$$

It is possible to discuss supersymmetry and gauge transformations of (5.102), however, it is not needed for the purpose of this work. Although the transformations (5.88)-(5.91) are derived for a read vector superfield, we suppose that they are also valid for complex fields. We perform the transformations (5.88)-(5.93) and their complex conjugates to write (5.102) as

$$\begin{aligned} I_{oNC} &= I_0[F,\lambda,\psi,D] - \frac{\Theta^{\mu\nu}}{g^2}\int d^4x[\frac{1}{4}F^{\rho\sigma}F^{\rho\mu}F_{\nu\sigma} + \frac{1}{16}F_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \\ &+\frac{i}{8}\epsilon^{\lambda\kappa\rho\sigma}F_{\lambda\kappa}F_{\rho\mu}F_{\nu\sigma} + \frac{i}{32}\epsilon^{\lambda\kappa\rho\sigma}F_{\mu\nu}F_{\lambda\kappa}F^{\rho\sigma}) \\ &+\frac{i}{4}\lambda\sigma^\rho\partial_\rho\bar{\psi}F_{\mu\nu} - \frac{i}{2}\lambda\sigma^\rho\partial_\nu\bar{\psi}F_{\mu\rho} - \frac{1}{4}F_{\mu\nu}D^2 + \text{c.c.}]. \end{aligned} \tag{5.103}$$

where I_o is defined in (5.50). We define the parent action

$$I_P = I_{oNC}[\tilde{F},\lambda,\psi,\tilde{D}] + I_l \tag{5.104}$$

where I_l is given in (5.51). We would like to emphasize that $\tilde{F}_{\mu\nu}$ is not a field strength but a complex, antisymmetric field. When the solutions of the equations of motion with respect to dual fields are used in the parent action, it leads to the noncommutative supersymmetric $U(1)$ gauge theory action (5.98). However, when the equations of motion with respect to the fields $\tilde{F},\lambda,\psi,\tilde{D}$ and their complex conjugates are solved and used in the parent action (5.103) one finds

$$\begin{aligned} I_{DNC} &= I_D + \frac{g^4}{4}\Theta^{\mu\nu}\int d^4x\,[\frac{1}{4}\epsilon^{\lambda\kappa\rho\sigma}F_{D\lambda\kappa}F_{D\rho\mu}F_{D\nu\sigma} \\ &+\frac{1}{16}\epsilon^{\lambda\kappa\rho\sigma}F_{D\mu\nu}F_{D\lambda\kappa}F_{D\rho\sigma})] \end{aligned} \tag{5.105}$$

where F_D is the field strength of A_D. Obviously, we cannot define any duality symmetry between (5.98) and (5.105). The latter does not possess any contribution in terms of the fields λ, D at the first order in $\Theta^{\mu\nu}$.

As the other possibility, let us take $\hat{F}_{\mu\nu}$ real and deal with

$$S_{oNC} = \int d^4x[-\frac{1}{4g^2}\hat{F}^{\mu\nu}_R\hat{F}_{R\,\mu\nu} - \frac{i}{2g^2}\hat{\bar{\lambda}}\bar{\sigma}^\mu\hat{D}_\mu*\hat{\psi} - \frac{i}{2g^2}\hat{\lambda}\sigma^\mu\hat{D}_\mu*\hat{\bar{\psi}} + \frac{1}{2g^2}\hat{D}\hat{\bar{D}}] \tag{5.106}$$

Through the supersymmetric Seiberg-Witten map (5.88)-(5.93) we write the action (5.106) as

$$\begin{aligned} S_{oNC}[F_R,\lambda,\psi,D] &= \int d^4x\{-\frac{1}{4g^2}(F^{\mu\nu}_R F_{R\mu\nu} + 2\Theta^{\mu\nu}F_{R\nu\rho}F^{\rho\sigma}_R F_{R\sigma\mu} - \frac{1}{2}\Theta^{\mu\nu}F_{R\nu\mu}F_{R\rho\sigma}F^{\sigma\rho}_R) \\ &-\frac{i}{2g^2}(\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\psi + \Theta^{\mu\nu}\bar{\lambda}\bar{\sigma}^\rho\partial_\mu\psi F_{R\nu\rho} + \frac{1}{2}\Theta^{\mu\nu}\bar{\lambda}\bar{\sigma}^\rho\partial_\rho\psi F_{R\mu\nu}) \\ &-\frac{i}{2g^2}(\lambda\sigma^\mu\partial_\mu\bar{\psi} + \Theta^{\mu\nu}\lambda\sigma^\rho\partial_\mu\bar{\psi}F_{R\nu\rho} + \frac{1}{2}\Theta^{\mu\nu}\lambda\sigma^\rho\partial_\rho\bar{\psi}F_{R\mu\nu}) \\ + &\frac{1}{4g^2}[(D^2+\bar{D}^2) + \frac{1}{2}\Theta^{\mu\nu}(D^2+\bar{D}^2)F_{R\mu\nu}]\} \end{aligned} \tag{5.107}$$

Now, we define the parent action as

$$S_P = S_{0NC}[F_R, \lambda, \psi, \tilde{D}] + S_l \tag{5.108}$$

where as before $F_{R\mu\nu}$ denotes an antisymmetric real fields and the Legendre transformation part S_l is given in (5.63).

Equations of motion with respect to the dual fields $A_D, \lambda_D, \bar{\lambda}_D, D_D$ are given as before by (5.64)-(5.67). Plugging their solution into S_{oNC} leads to the noncommutative supersymmetric $U(1)$ gauge theory (5.98). Equations of motion with respect to the other fields are

$$\begin{aligned}
&\frac{1}{g^2}F_R^{\mu\nu} + \frac{1}{g^2}\Theta^{\rho[\mu}F_R^{\nu]\sigma}F_{R\sigma\rho} + \frac{1}{2g^2}\Theta^{\rho\sigma}F_{R\sigma[\mu}F_{R\nu]\rho} - \frac{1}{4g^2}\Theta^{\mu\nu}F_{R\rho\sigma}F_R^{\rho\sigma} \\
&-\frac{1}{2g^2}\Theta^{\rho\sigma}F_{R\rho\sigma}F_R^{\mu\nu} + \frac{i}{2g^2}(\Theta^{\rho\mu}\bar{\lambda}\bar{\sigma}^\nu - \Theta^{\rho\nu}\bar{\lambda}\bar{\sigma}^\mu)\partial_\rho\psi + \frac{i}{2g^2}\Theta^{\mu\nu}(\bar{\lambda}\bar{\sigma}^\rho\partial_\rho\psi) \\
&+\frac{i}{2g^2}(\Theta^{\rho\mu}\lambda\sigma^\nu - \Theta^{\rho\nu}\lambda\sigma^\mu)\partial_\rho\bar{\psi} + \frac{i}{2g^2}\Theta^{\mu\nu}\lambda\sigma^\rho\partial_\rho\bar{\psi} \\
&-\frac{1}{4g^2}\Theta^{\mu\nu}(\tilde{D}^2 + \tilde{D}^{\dagger 2}) + \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_{D\sigma} = 0, &(5.109)\\
&\frac{i}{2g^2}\sigma^\mu\partial_\mu\bar{\psi} + \frac{i}{4g^2}\Theta^{\mu\nu}\sigma^\rho\partial_\rho\bar{\psi}F_{R\mu\nu} + \frac{i}{2g^2}\Theta^{\mu\nu}\sigma^\rho\partial_\mu\bar{\psi}F_{R\nu\rho} &(5.110)\\
&-\frac{1}{2}\sigma^\mu\partial_\mu\bar{\lambda}_D = 0, \\
&\frac{i}{2g^2}\bar{\sigma}^\mu\partial_\mu\psi + \frac{i}{4g^2}\Theta^{\mu\nu}\bar{\sigma}^\rho\partial_\rho\psi F_{R\mu\nu} + \frac{i}{2g^2}\Theta^{\mu\nu}\bar{\sigma}^\rho\partial_\mu\psi F_{R\nu\rho} &(5.111)\\
&+\frac{1}{2}\bar{\sigma}^\mu\partial_\mu\lambda_D = 0, \\
&\partial_\mu\left[\frac{i}{2g^2}\bar{\lambda}\bar{\sigma}^\mu - \frac{i}{4g^2}\Theta^{\rho\nu}\bar{\lambda}\bar{\sigma}^\mu F_{R\rho\nu} - \frac{i}{2g^2}\Theta^{\mu\nu}\bar{\lambda}\bar{\sigma}^\rho F_{R\nu\rho} - \frac{1}{2}\bar{\lambda}_D\bar{\sigma}_\mu\right] = 0, &(5.112)\\
&\partial_\mu\left[\frac{i}{2g^2}\lambda\sigma^\mu + \frac{i}{4g^2}\Theta^{\rho\nu}\lambda\sigma^\mu F_{R\rho\nu} + \frac{i}{2g^2}\Theta^{\mu\nu}\lambda\sigma^\rho F_{R\nu\rho} - \frac{1}{2}\lambda_D\sigma^\mu\right] = 0, &(5.113)\\
&\frac{1}{2g^2}\tilde{D} + \frac{1}{4g^2}\Theta^{\mu\nu}\tilde{D}F_{R\mu\nu} + \frac{i}{4}D_D = 0, &(5.114)\\
&\frac{1}{2g^2}\tilde{D}^\dagger + \frac{1}{4g^2}\Theta^{\mu\nu}\tilde{D}^\dagger F_{R\mu\nu} - \frac{i}{4}D_D = 0. &(5.115)
\end{aligned}$$

We solve these equations for $F_R, \psi, \lambda, \tilde{D}$ and plug the solutions into (5.108) to obtain the dual action

$$\begin{aligned}
S_{NCD} &= \int d^4x[-\frac{g^2}{4}(F_D^{\mu\nu}F_{D\mu\nu} + 2\tilde{\Theta}^{\mu\nu}F_{D\nu\rho}F_D^{\rho\sigma}F_{D\sigma\mu} - \frac{1}{2}\tilde{\Theta}^{\mu\nu}F_{D\nu\mu}F_{D\rho\sigma}F^{D\sigma\rho}) \\
&\quad -ig^2(\frac{1}{2}\lambda_D\sigma^\mu\partial_\mu\bar{\lambda}_D + \frac{1}{2}\bar{\lambda}_D\bar{\sigma}^\mu\partial_\mu\lambda_D + \frac{1}{4}\tilde{\Theta}^{\mu\nu}\lambda_D\sigma_\mu\partial^\rho\bar{\lambda}_D F_{D\rho\nu}) \qquad (5.116)\\
&\quad +\frac{g^2}{4}\tilde{\Theta}^{\mu\nu}\bar{\lambda}_D\bar{\sigma}_\mu\partial^\rho\lambda_D F_{D\rho\nu} + \frac{g^2}{2}(D_D^2 + \frac{g^2}{2}\tilde{\Theta}^{\mu\nu}D_D^2 F_{D\mu\nu})].
\end{aligned}$$

where

$$\tilde{\Theta}^{\mu\nu} \equiv g^2 \epsilon^{\mu\nu\rho\sigma} \Theta_{\rho\sigma}, \tag{5.117}$$

When the fermionic and auxiliary fields λ_D, D_D set equal to zero one obtains the results of [82]: there is a duality symmetry under the replacement of A^μ with A^μ_D and $\Theta^{\mu\nu}$ with $\tilde{\Theta}^{\mu\nu}$. Unfortunately, this symmetry accompanied by the replacement of λ, D with λ_D, D_D cease to exist between the noncommutative supersymmetric action (5.98) and its dual (5.117). Inspecting the terms which obstruct the duality symmetry we can find actions in terms of the component fields which possess this symmetry. Let us define the action

$$\Sigma(\Theta, F, \lambda, \bar{\lambda}, D) = S_{NC} - \frac{i}{g^2} \int d^4x \Theta^{\mu\nu} \left(\lambda \sigma_\mu \partial^\rho \bar{\lambda} + \bar{\lambda} \bar{\sigma}_\mu \partial^\rho \lambda \right) F_{\rho\nu}, \tag{5.118}$$

which can be obtained from the parent action

$$\Sigma_P = S_P - \frac{i}{2g^2} \int d^4x \Theta^{\mu\nu} \left(\psi \sigma_\mu \partial^\rho \bar{\lambda} + \bar{\psi} \bar{\sigma}_\mu \partial^\rho \lambda + \lambda \sigma_\mu \partial^\rho \bar{\psi} + \bar{\lambda} \bar{\sigma}_\mu \partial^\rho \psi \right) F_{R\rho\nu}. \tag{5.119}$$

Dual theory which follows from (5.119) can be shown to be

$$\Sigma_D = g^4 \Sigma(\tilde{\Theta}, F_D, \lambda_D, \bar{\lambda}_D, D_D). \tag{5.120}$$

Therefore we conclude that the action (5.118) possesses duality symmetry when the original fields are substituted by the dual ones and the noncommutativity parameter Θ is replaced with $\tilde{\Theta}$. However, whether the action (5.118) is supersymmetric or not is an open question. However, it is explicitly gauge invariant.

6 RESULTS AND DISCUSSION

In this thesis we provide a complete and consistent study of the electric-magnetic duality in the noncommutative $U(1)$ gauge theory. Noncommutative gauge theories emerge in the string theory context. Therefore study of these theories provide appropriate tools to understand the different properties of the string theory. Duality in noncommutative theories has interesting consequences in some ways. First of all electric-magnetic duality plays role in the study of different phases of the gauge theories. If one has a strong coupling theory in terms of its dual theory it become possible to obtain information from this weakly coupled theory, especially by using the powerful technics of the perturbatif calculations. On the other hand duality leads to another important consequence in the noncommutative theories: starting from a space/space noncommutative theory, by duality one passes to a space/time noncommutative theory. Such space/time noncommutative theories are typical examples of string theory. The better understanding of this type of theories may have consequences in the string theory side.

The first part of our work includes investigation of how hamiltonian can be defined in such a space/time noncommutative theory. Because of the noncommutation property of time the usual quantization procedure is not obvious. For this aim the parent action seems to be an appropriate tool. It is shown that it becomes possible to define hamiltonian starting from the parent action by using the Dirac's constraint system analysis. For ordinary case our results consistent with the previous ones. We extended the formalism to the noncommutative case and obtained the hamiltonian of the dual theory. It is also shown that the hamiltonian which is obtained from the parent action coincides with the one calculated from the dual action by using the usual quantization procedure and pretending as if the time is commuting. This analysis performed at the first order of $\tilde{\theta}$ parameter. However, the method of obtaining hamiltonian from the shifted action seems easier. When higher order terms are considered the unique change will be in a single constraint while other constraints remain intact. Results of this section are used in the study of the $D3-$brane worldvolume theories. The worldvolume action of noncommutative $D3-$brane is obtained from the noncommutative gauge theory in $10-$dimensions by using the static gauge. The first three spatial coordinates are taken as spatial coordinates of the brane and the rest of the coordinates as scalar fields on the brane. We considered the existence of only one scalar field on the brane. For this configuration we obtained the hamiltonian density by using the static gauge. BPS states are investigated for this configuration. Noncommutative $D3-$brane formulation which we deal with is somehow different from the one considered previously [44, 56, 57, 58, 59, 60]. The difference stems from the difference of the gauge groups. In our case although hamiltonian depends on the noncommutativity parameter, gauge group is still $U(1)$ but there gauge group is noncommutative $U(1)$. In chapter-1 we also studied the electric-magnetic duality transformation of both lagrangian and hamiltonian densities. It is well known that duality maps the lagrangian to itself up to an overall minus sign and keeps intact the hamiltonian density. In the noncommutative theory this property persists. We show that duality transformation of hamiltonian density is given by a somehow inverted one with respect to the transformation rule of lagrangian density.

In chapter-4 partition functions of these dual theories were established. We started from the path integral formulation of parent action which include the constraints as Dirac delta functions in the measure. By definition determinant of first class and second class constraints also included in the measure. This path integral definition gives partition function of both dual and original theory with respect to appropriate phase space integrations. We showed that partition functions for the noncommutative $U(1)$ theory and its dual are equivalent. This result demonstrates that strong weak duality transformations is helpful to make calculations in weak coupling regime to extract information about physical quantities in the strong coupling regions. We would like to emphasize the difference between the results obtained for the commutative case and for the noncommutative $U(1)$ theory. In $U(1)$ gauge theory, partition functions for the initial and the dual theories are equivalent and they are related with the map $g \to g^{-1}$. However, the partition function of noncommutative $U(1)$ does not yield the partition function of

its dual by only inverting the coupling constant, although they are equivalent. Application of the approach presented here to noncommutative supersymmetric $U(1)$ gauge theory may shed light on the duality symmetry of the supersymmetric noncommutative theory. We dealt with free theories, although introducing source terms into the starting path integral to gain insight about relations of the Green functions of the noncommutative $U(1)$ theory would be interesting.
In the chapter-5 we studied the supersymmetric noncommutative $U(1)$ theory. First of all we investigated that how parent action can be defined for ordinary supersymmetric $U(1)$ theory. We introduced two different parent actions which yield the same results. Then to generalize these parent actions to noncommutative case we studied the generalization of the Seiberg-Witten map to the supersymmetric case. There are two different approaches through superfields to achieve this [94, 95]. We utilized both of them to define a generalization of the Seiberg-Witten map in terms of component fields. By using the results of these approaches we proposed two different parent actions. Both of them generate noncommutative supersymmetric $U(1)$ gauge theory given by the component fields defined in commuting spacetime. However, they yield different dual actions contrary to the ordinary case. At the first order in noncommutativity parameter one of the dual actions does not have any contribution from the fermionic and the auxiliary fields. Moreover, it does not lead to the dual action of non-supersymmetric gauge theory. The other parent action generates a dual theory which embraces the results of previous works. However, this dual action is not in the same form with the noncommutative $U(1)$ gauge theory. Thus, duality symmetry of the non-supersymmetric theory given by replacing the field strength $F^{\mu\nu}$ with the dual one $F_D^{\mu\nu}$ and the noncommutativity parameter $\Theta^{\mu\nu}$ with $\tilde{\Theta}^{\mu\nu} = g^2\epsilon^{\mu\nu\rho\sigma}\Theta_{\rho\sigma}$ is not satisfied when actions are considered. We introduced a parent action for the component fields which generates actions possessing this duality symmetry. Unfortunately, it is not clear if these duality symmetric actions are supersymmetric, though they are explicitly gauge invariant.

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APPENDIX-A

Here we present some calculational details. When we evaluate the determinant of the second class constraints we have established the following matrix.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^1 & M_2^1 & \partial^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^2 & M_2^2 & \partial^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_1^3 & M_2^3 & \partial^3 \\
0 & 0 & 0 & 0 & 0 & 0 & -C_1^3 & -C_2^3 & -\partial^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_2^2 & \partial^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -C_1^1 & -C_2^1 & -\partial^1 & 0 & 0 & 0 \\
0 & 0 & 0 & C_1^3 & -C_1^2 & C_1^1 & 0 & 0 & 0 & k & l & 0 \\
0 & 0 & 0 & C_2^3 & -C_2^2 & C_2^1 & 0 & 0 & 0 & m & n & 0 \\
0 & 0 & 0 & \partial^3 & -\partial^2 & \partial^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M_1^1 & -M_1^2 & -M_1^3 & 0 & 0 & 0 & -k & -m & 0 & 0 & 0 & 0 \\
-M_2^1 & -M_2^2 & -M_2^3 & 0 & 0 & 0 & -l & -n & 0 & 0 & 0 & 0 \\
-\partial^1 & -\partial^2 & -\partial^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Non zero Poisson brackets are

$$\{P_{01}(x), \Phi_1^4(y)\} = \frac{1}{g^2} M_1^1 \delta(x-y) \quad , \quad \{P_{01}(x), \Phi_2^4(y)\} = \frac{1}{g^2} M_2^1 \delta(x-y), \tag{A.1}$$

$$\{P_{01}(x), \Phi_3^4(y)\} = \partial_y^1 \delta(x-y), \tag{A.2}$$

$$\{P_{02}(x), \Phi_1^4(y)\} = \frac{1}{g^2} M_1^2 \delta(x-y) \quad , \quad \{P_{02}(x), \Phi_2^4(y)\} = \frac{1}{g^2} M_2^2 \delta(x-y), \tag{A.3}$$

$$\{P_{02}(x), \Phi_3^4(y)\} = \partial_y^2 \delta(x-y), \tag{A.4}$$

$$\{P_{03}(x), \Phi_1^4(y)\} = \frac{1}{g^2} M_1^3 \delta(x-y) \quad , \quad \{P_{03}(x), \Phi_2^4(y)\} = \frac{1}{g^2} M_2^3 \delta(x-y), \tag{A.5}$$

$$\{P_{03}(x), \Phi_3^4(y)\} = \partial_y^3 \delta(x-y), \tag{A.6}$$

$$\{P_{12}(x), \Phi_1^2(y)\} = -C_1^3 \delta(x-y) \quad , \quad \{P_{12}(x), \Phi_2^2(y)\} = -C_2^3 \delta(x-y)\,, \tag{A.7}$$

$$\{P_{12}(x), \Phi^3(y)\} = -\partial^3 \delta(x-y)\,, \tag{A.8}$$

$$\{P_{13}(x), \Phi_1^2(y)\} = C_1^2 \delta(x-y) \quad , \quad \{P_{13}(x), \Phi_2^2(y)\} = C_2^2 \delta(x-y)\,, \tag{A.9}$$

$$\{P_{13}(x), \Phi^3(y)\} = \partial^2 \delta(x-y)\,, \tag{A.10}$$

$$\{P_{23}(x), \Phi_1^2(y)\} = -C_1^1 \delta(x-y) \quad , \quad \{P_{23}(x), \Phi_2^2(y)\} = -C_2^1 \delta(x-y)\,, \tag{A.11}$$

$$\{P_{23}(x), \Phi^3(y)\} = -\partial^1 \delta(x-y)\,, \tag{A.12}$$

We have used the abbreviations k, l, m, n for the following Poisson brackets

$$k \equiv \{\Phi_1^2(x), \Phi_1^4(y)\} = \epsilon_{ilm} C_1^i(x) M_1^l(y) \partial_y^m \delta(x-y), \tag{A.13}$$

$$l \equiv \{\Phi_1^2(x), \Phi_2^4(y)\} = \epsilon_{ilm} C_1^i(x) M_2^l(y) \partial_y^m \delta(x-y), \tag{A.14}$$

$$m \equiv \{\Phi_2^2(x), \Phi_1^4(y)\} = \epsilon_{ilm} C_2^i(x) M_1^l(y) \partial_y^m \delta(x-y), \tag{A.15}$$

$$n \equiv \{\Phi_2^2(x), \Phi_2^4(y)\} = \epsilon_{ilm} C_2^i(x) M_2^l(y) \partial_y^m \delta(x-y), \tag{A.16}$$

To obtain (4.8) one should solve the constraint equations for $F_{\mu\nu}$ in terms of the physical fields A_D, P_D. Delta functions contribute the determinant of the following matrix

$$\underbrace{\begin{pmatrix} M_1^1 & M_1^2 & M_1^3 & 0 & 0 & 0 \\ M_2^1 & M_2^2 & M_2^3 & 0 & 0 & 0 \\ \partial^1 & \partial^2 & \partial^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1^3 & -C_1^2 & C_1^1 \\ 0 & 0 & 0 & C_2^3 & -C_2^2 & C_2^1 \\ 0 & 0 & 0 & \partial^3 & -\partial^2 & \partial^1 \end{pmatrix}}_{S_{ij}} \cdot \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{12} \\ F_{13} \\ F_{23} \end{pmatrix} = A_{ij} \cdot \begin{pmatrix} A_{D1} \\ A_{D2} \\ A_{D3} \\ P_{D1} \\ P_{D2} \\ P_{D3} \end{pmatrix}$$

It can be easily seen that the related determinant is

$$\mathrm{detS_{ij}} = \mathrm{det}(\epsilon_{\mathrm{ijk}} \mathrm{C_1^i C_2^j} \partial^{\mathrm{k}}) \mathrm{det}(\epsilon_{\mathrm{ijk}} \mathrm{M_1^i M_2^j} \partial^{\mathrm{k}}) \tag{A.17}$$

For (4.12) one can establish the following matrix equation from the related constraints

$$\underbrace{\begin{pmatrix} C_1^1 & C_1^2 & C_1^3 & 0 & 0 & 0 \\ C_2^1 & C_2^2 & C_2^3 & 0 & 0 & 0 \\ \partial^1 & \partial^2 & \partial^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & \partial^1 & \partial^2 & \partial^3 \end{pmatrix}}_{Z_{ij}} \cdot \begin{pmatrix} P_{D1} \\ P_{D2} \\ P_{D3} \\ A_{D1} \\ A_{D2} \\ A_{D3} \end{pmatrix} = B_{ij} \cdot \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{12} \\ F_{13} \\ F_{23} \end{pmatrix}$$

where we have used the a, b, c, d, e, f, for

$$a = g^2(M_1^2\partial^3 - M_1^3\partial^2) \quad , \quad b = g^2(M_1^3\partial^1 - M_1^1\partial^3), \tag{A.18}$$

$$c = g^2(M_1^1\partial^2 - M_1^2\partial^1) \quad , \quad d = g^2(M_2^2\partial^3 - M_2^3\partial^2)$$

$$e = g^2(M_2^3\partial^1 - M_2^1\partial^3) \quad , \quad f = g^2(M_2^1\partial^2 - M_2^2\partial^1)$$

and

$$\mathrm{det(Z_{ij})} = \mathrm{det(g^4)} \mathrm{det}(\epsilon_{\mathrm{ijk}} \mathrm{C_1^i C_2^j} \partial^{\mathrm{k}}) \mathrm{det}(\epsilon_{\mathrm{ijk}} \mathrm{M_1^i M_2^j} \partial^{\mathrm{k}}) \mathrm{det}(\partial^2) \tag{A.19}$$

For noncommutative case determinant of the second class constraints is given by the following matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^1_{(\theta)1} & M^1_{(\theta)2} & \partial^1_{(\theta)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{(\theta)1} & M^2_{(\theta)2} & \partial^2_{(\theta)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^3_{(\theta)1} & M^3_{(\theta)2} & \partial^3_{(\theta)} \\
0 & 0 & 0 & 0 & 0 & 0 & -C^3_1 & -C^3_2 & -\partial^3 & a^1_{(\theta)} & a^2_{(\theta)} & a^3_{(\theta)} \\
0 & 0 & 0 & 0 & 0 & 0 & C^2_1 & C^2_2 & \partial^2 & b^1_{(\theta)} & b^2_{(\theta)} & b^3_{(\theta)} \\
0 & 0 & 0 & 0 & 0 & 0 & -C^1_1 & -C^1_2 & -\partial^1 & c^1_{(\theta)} & c^2_{(\theta)} & c^3_{(\theta)} \\
0 & 0 & 0 & C^3_1 & -C^2_1 & C^1_1 & 0 & 0 & 0 & k & l & 0 \\
0 & 0 & 0 & C^3_2 & -C^2_2 & C^1_2 & 0 & 0 & 0 & m & n & 0 \\
0 & 0 & 0 & \partial^3 & -\partial^2 & \partial^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M^1_{(\theta)1} & -M^2_{(\theta)1} & -M^3_{(\theta)1} & -a^1_{(\theta)} & -b^1_{(\theta)} & -c^1_{(\theta)} & -k & -m & 0 & 0 & 0 & 0 \\
-M^1_{(\theta)2} & -M^2_{(\theta)2} & -M^3_{(\theta)2} & -a^2_{(\theta)} & -b^2_{(\theta)} & -c^2_{(\theta)} & -l & -n & 0 & 0 & 0 & 0 \\
-\partial^1_{(\theta)} & -\partial^2_{(\theta)} & -\partial^3_{(\theta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where the subscript θ denotes that the related terms are θ dependent and the explicit form of them are displayed below

$$
\begin{aligned}
\{P_{oi}(x), \Phi^4_n(y)\} &= M^j_n(y)(-\delta_{ji} + F_{jk}\theta^{kl}\delta_{li} + \delta_{ki}F^{kl}\theta_{lj} && \text{(A.20)}\\
&\quad -\frac{1}{2}\theta^{kl}F_{lk}\delta_{ji})(y)\delta(x-y) \\
\{P_{0i}(x), \Phi^4_3(y)\} &= (-\delta_{ji} + F_{jk}\theta^{kl}\delta_{li} + \delta_{ki}F^{kl}\theta_{lj} && \text{(A.21)}\\
&\quad -\frac{1}{2}\theta^{kl}F_{lk}\delta_{ji})(y)\partial^j_y\delta(x-y) \\
\{P_{ij}(x), \Phi^2_n(y)\} &= -\epsilon_{ijk}C^k_n(y)\delta(x-y) && \text{(A.22)}\\
\{P_{ij}(x), \Phi^3(y)\} &= -\epsilon_{ijk}\partial^k_y\delta(x-y) && \text{(A.23)}\\
\{P_{ij}(x), \Phi^4_n(y)\} &= [(M^i_n\theta^{jk} - M^j_n\theta^{ik})F_{k0} + (F^{0i}\theta^{jk} - F^{0j}\theta^{ik})M^k_n && \text{(A.24)}\\
&\quad -\theta^{ij}F_{0k}M^k_n](y)\delta(x-y) \\
\{P_{ij}(x), \Phi^4_3(y)\} &= [(\partial^i_y\theta^{jk} - \partial^j_y\theta^{ik})F_{k0} + (F^{0i}\theta^{jk} - F^{0j}\theta^{ik})\partial^k_y && \text{(A.25)}\\
&\quad -\theta^{ij}F_{0k}\partial^k_y](y)\delta(x-y) \\
\{\Phi^2_s(x), \Phi^4_r(y)\} &= g^2\epsilon_{ijk}C^i_s(x)M^j_r(y)\partial^k_y\delta(x-y) && \text{(A.26)}
\end{aligned}
$$

Although it seems very confusing the calculations are performed for the first order of the θ. Hence determinant of the matrix give rise to the (4.33).

Like the commuting case solving the $F_{\mu\nu}$ and $P_{\mu\nu}$ in terms of A_D, P_D requires to evaluate the following matrix equation

$$\underbrace{\begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & 0 & 0 & 0 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & 0 & 0 & 0 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1^3 & -C_1^2 & C_1^1 \\ 0 & 0 & 0 & C_2^3 & -C_2^2 & C_2^1 \\ 0 & 0 & 0 & \partial^3 & -\partial^2 & \partial^1 \end{pmatrix}}_{\tilde{S}_{ij}} \cdot \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{12} \\ F_{13} \\ F_{23} \end{pmatrix} = \tilde{A}_{ij} \cdot \begin{pmatrix} A_{D1} \\ A_{D2} \\ A_{D3} \\ P_{D1} \\ P_{D2} \\ P_{D3} \end{pmatrix}$$

where

$$\begin{aligned} \tilde{a}_1 &= -M_1^1 + 2M_1^1(F_{12}\theta^{21} + F_{13}\theta^{31}) + M_1^2(F_{13}\theta^{32} + F_{23}\theta^{31}) \\ &\quad +M_1^3(F_{12}\theta^{23} + F_{32}\theta^{21}) - \frac{1}{2}M_1^1\theta^{jk}F_{kj} \end{aligned} \tag{A.27}$$

$$\begin{aligned} \tilde{a}_2 &= -M_1^2 + 2M_1^2(F_{12}\theta^{21} + F_{23}\theta^{32}) + M_1^1(F_{13}\theta^{32} + F_{23}\theta^{31}) \\ &\quad +M_1^3(F_{12}\theta^{31} + F_{13}\theta^{21}) - \frac{1}{2}M_1^2\theta^{jk}F_{kj} \end{aligned} \tag{A.28}$$

$$\begin{aligned} \tilde{a}_3 &= -M_1^3 + 2M_1^3(F_{31}\theta^{13} + F_{32}\theta^{23}) + M_1^1(F_{12}\theta^{23} + F_{32}\theta^{21}) \\ &\quad +M_1^2(F_{21}\theta^{13} + F_{31}\theta^{12}) - \frac{1}{2}M_1^3\theta^{jk}F_{kj} \end{aligned} \tag{A.29}$$

$$\begin{aligned} \tilde{b}_1 &= -M_2^1 + 2M_2^1(F_{12}\theta^{21} + F_{13}\theta^{31}) + M_2^2(F_{13}\theta^{32} + F_{23}\theta^{31}) \\ &\quad +M_2^3(F_{12}\theta^{23}F_{32}\theta^{21}) - \frac{1}{2}M_2^1\theta^{jk}F_{kj} \end{aligned} \tag{A.30}$$

$$\begin{aligned} \tilde{b}_2 &= -M_2^2 + 2M_2^2(F_{21}\theta^{12} + F_{23}\theta^{32}) + M_2^1(F_{13}\theta^{32} + F_{23}\theta^{31}) \\ &\quad M_1^3(F_{12}\theta^{31} + F_{13}\theta^{21}) - \frac{1}{2}M_3\theta^{jk}F_{kj} \end{aligned} \tag{A.31}$$

$$\begin{aligned} \tilde{b}_3 &= -M_2^3 + 2M_2^3(F_{31}\theta^{13} + F_{23}\theta^{32}) + M_2^1(F_{12}\theta^{23} + F_{32}\theta^{21}) \\ &\quad M_2^2(F_{21}\theta^{13} + F_{31}\theta^{12}) - \frac{1}{2}M_2^3\theta^{jk}F_{kj} \end{aligned} \tag{A.32}$$

$$\begin{aligned} \tilde{c}_1 &= -\partial^1 + 2\partial^1(F_{12}\theta^{21} + F_{13}\theta^{31}) + \partial^2(F_{13}\theta^{32} + F_{23}\theta^{31}) \\ &\quad \partial^3(F_{12}\theta^{23} + F_{32}\theta^{21}) - \frac{1}{2}\partial^1\theta^{jk}F_{kj} \end{aligned} \tag{A.33}$$

$$\begin{aligned} \tilde{c}_2 &= -\partial^2 + 2\partial^2(F_{21}\theta^{12}F_{23}\theta^{32}) + \partial^1(F_{13}\theta^{32} + F_{23}\theta^{31}) \\ &\quad +\partial^3(F_{12}\theta^{31} + F_{13}\theta^{21}) - \frac{1}{2}\partial^2\theta^{jk}F_{kj} \end{aligned} \tag{A.34}$$

$$\begin{aligned} \tilde{c}_3 &= -\partial^3 + 2\partial^3(F_{31}\theta^{13} + F_{32}\theta^{23}) + \partial^1(F_{12}\theta^{23} + F_{32}\theta^{21}) \\ &\quad +\partial^2(F_{21}\theta^{13} + F_{31}\theta^{12}) - \frac{1}{2}\partial^3\theta^{jk}F_{kj} \end{aligned} \tag{A.35}$$

By the same way we construct the following equation in order to solve the dual fields in terms of the ordinary ones.

$$
\underbrace{\begin{pmatrix} C_1^1 & C_1^2 & C_1^3 & 0 & 0 & 0 \\ C_2^1 & C_2^2 & C_2^3 & 0 & 0 & 0 \\ \partial^1 & \partial^2 & \partial^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{k}_1 & \tilde{k}_2 & \tilde{k}_3 \\ 0 & 0 & 0 & \tilde{l}_1 & \tilde{l}_2 & \tilde{l}_3 \\ 0 & 0 & 0 & \partial^1 & \partial^2 & \partial^3 \end{pmatrix}}_{\tilde{Z}_{ij}} \cdot \begin{pmatrix} P_{D1} \\ P_{D2} \\ P_{D3} \\ A_{D1} \\ A_{D2} \\ A_{D3} \end{pmatrix} = \tilde{B}_{ij} \cdot \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{12} \\ F_{13} \\ F_{23} \end{pmatrix}
$$

and

$$\tilde{k}_1 = g^2(M_1^2\partial^3 - M_1^3\partial^2) \quad (A.36)$$

$$\tilde{k}_2 = g^2(M_1^3\partial^1 - M_1^1\partial^3) \quad (A.37)$$

$$\tilde{k}_3 = g^2(M_1^1\partial^2 - M_1^2\partial^1) \quad (A.38)$$

$$\tilde{l}_1 = g^2(M_2^2\partial^3 - M_2^3\partial^2) \quad (A.39)$$

$$\tilde{l}_2 = g^2(M_2^3\partial^1 - M_2^1\partial^3) \quad (A.40)$$

$$\tilde{l}_3 = g^2(M_2^1\partial^2 - M_2^2\partial^1) \quad (A.41)$$

Determinant of this matrix produce the following result

$$\mathrm{det}\tilde{\mathrm{Z}}_{\mathrm{ij}} = \mathrm{det}(\mathrm{g}^4)\mathrm{det}(\partial^2)\mathrm{det}(\epsilon_{\mathrm{ijk}}\mathrm{C}_1^{\mathrm{i}}\mathrm{C}_2^{\mathrm{j}}\partial^{\mathrm{k}})\mathrm{det}(\epsilon_{\mathrm{ijk}}\mathrm{M}_1^{\mathrm{i}}\mathrm{M}_2^{\mathrm{j}}\partial^{\mathrm{k}}) \quad (A.42)$$

APPENDIX-B

Throughout the work we used the conventions of the Wess-Bagger [92]. Greek letters are used to denote the spinor indices while the Latin letters to denote the vector and tensor indices. Metric convention is

$$\eta_{\mu\nu} = diag(-1, 1, 1, 1) \tag{B.1}$$

Antisymmetric tensors with dotted and undotted index are

$$\epsilon^{12} = -\epsilon^{21} = \epsilon^{\dot{1}\dot{2}} = -\epsilon^{\dot{2}\dot{1}} \;\; = \;\; 1 \tag{B.2}$$

$$\epsilon_{12} = -\epsilon_{21} = \epsilon_{\dot{1}\dot{2}} = -\epsilon_{\dot{2}\dot{1}} \;\; = \;\; -1 \tag{B.3}$$

and

$$\epsilon^{0123} = -\epsilon_{0123} = 1 \tag{B.4}$$

Raising or lowering the dotted and undotted Weyl spinors, which form the $(0, 1/2)$ and $(1/2, 0)$ representations of the $SL(2, C)$ respectively, are performed with the antisymmetric tensor

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad , \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \tag{B.5}$$

$$\psi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}} \qquad , \qquad \psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}} \tag{B.6}$$

Multiplication of the spinors is

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \tag{B.7}$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \tag{B.8}$$

$$(\psi\chi)^\dagger = \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi} \tag{B.9}$$

Sigma matrices are

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.10}$$

Index structure of the sigma matrices and the operations with respect to the both spinor indices and Lorentz indices of them are given by

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\beta} \quad , \quad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}} \tag{B.11}$$

$$\bar{\sigma}^0 = \sigma^0 \quad , \quad \bar{\sigma}^i = -\sigma^i \quad i = 1, 2, 3 \tag{B.12}$$

$$Tr(\sigma^{\mu}\bar{\sigma}^{\nu}) = -2\eta^{\mu\nu} \quad , \quad \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_{\mu} = -2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{B.13}$$

$$(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})^{\beta}_{\alpha} = -2\eta^{\mu\nu}\delta^{\beta}_{\alpha} \quad , \quad (\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}_{\dot{\beta}} = -2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{B.14}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\lambda} + \sigma^{\lambda}\bar{\sigma}^{\nu}\sigma^{\mu} = 2(\eta^{\mu\lambda}\sigma^{\nu} - \eta^{\nu\lambda}\sigma^{\mu} - \eta^{\mu\nu}\sigma^{\lambda}) \tag{B.15}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\lambda} + \bar{\sigma}^{\lambda}\sigma^{\nu}\bar{\sigma}^{\mu} = 2(\eta^{\mu\lambda}\bar{\sigma}^{\nu} - \eta^{\nu\lambda}\bar{\sigma}^{\mu} - \eta^{\mu\nu}\bar{\sigma}^{\lambda}) \tag{B.16}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\lambda} - \sigma^{\lambda}\bar{\sigma}^{\nu}\sigma^{\mu} = 2i\epsilon^{\mu\nu\lambda\kappa}\sigma_{\kappa} \quad , \quad \bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\lambda} - \bar{\sigma}^{\lambda}\sigma^{\nu}\bar{\sigma}^{\mu} = -2i\epsilon^{\mu\nu\lambda\kappa}\bar{\sigma}_{\kappa} \tag{B.17}$$

$$\sigma^{\mu\nu\,\beta}_{\ \alpha} = \frac{1}{4}(\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}) \quad , \quad \bar{\sigma}^{\mu\nu\,\dot{\alpha}}_{\ \dot{\beta}} = \frac{1}{4}(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}) \tag{B.18}$$

$$\sigma^{\mu\nu\,\alpha}_{\ \alpha} = 0 = \bar{\sigma}^{\mu\nu\dot{\alpha}}_{\ \dot{\alpha}} \quad , \quad \sigma^{\mu\nu\,\beta}_{\ \alpha}\epsilon_{\beta\gamma} = \sigma^{\mu\nu\,\beta}_{\ \gamma}\epsilon_{\beta\alpha} \tag{B.19}$$

$$\epsilon^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa} = -2i\sigma^{\mu\nu} \quad , \quad \epsilon^{\mu\nu\lambda\kappa}\bar{\sigma}_{\lambda\kappa} = 2i\bar{\sigma}^{\mu\nu} \tag{B.20}$$

$$\epsilon^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa} = -2i\sigma^{\mu\nu} \quad , \quad \epsilon^{\mu\nu\lambda\kappa}\bar{\sigma}_{\lambda\kappa} = 2i\bar{\sigma}^{\mu\nu} \tag{B.21}$$

$$Tr\sigma^{\mu\nu}\bar{\sigma}^{\lambda\kappa} = -\frac{1}{2}(\eta^{\mu\lambda}\eta^{\nu\kappa} - \eta^{\mu\kappa}\eta^{\nu\kappa}) - \frac{i}{2}\epsilon^{\mu\nu\lambda\kappa} \tag{B.22}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} - \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = 2(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + 2(\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{B.23}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma^{\nu}_{\beta\dot{\beta}} + \sigma^{\nu}_{\alpha\dot{\alpha}}\sigma^{\mu}_{\beta\dot{\beta}} = -\eta^{\mu\nu} + 4(\sigma^{\lambda\mu}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{\lambda\nu})_{\dot{\alpha}\dot{\beta}} \tag{B.24}$$

$$\sigma^{\mu\nu}\sigma^{\lambda} = \frac{1}{2}(-\eta^{\lambda\nu}\sigma^{\mu} + \eta^{\lambda\mu}\sigma^{\nu} + i\epsilon^{\lambda\mu\nu\kappa}\sigma_{\kappa}) \tag{B.25}$$

$$\sigma^{\mu}\bar{\sigma}^{\nu\lambda} = \frac{1}{2}(\eta^{\mu\lambda}\sigma^{\nu} - \eta^{\mu\nu}\sigma^{\lambda} + i\epsilon^{\mu\nu\lambda\kappa}\sigma_{\kappa}) \tag{B.26}$$

$$\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\lambda} = \frac{1}{2}(-\eta^{\lambda\nu}\bar{\sigma}^{\mu} + \eta^{\lambda\mu}\bar{\sigma}^{\nu} - i\epsilon^{\lambda\mu\nu\kappa}\bar{\sigma}_{\kappa}) \tag{B.27}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu\lambda} = \frac{1}{2}(\eta^{\mu\lambda}\bar{\sigma}^{\nu} - \eta^{\mu\nu}\bar{\sigma}^{\lambda} - i\epsilon^{\mu\nu\lambda\kappa}\bar{\sigma}_{\kappa}) \tag{B.28}$$

$$\sigma^{\mu\nu}\sigma_{\nu} = \sigma_{\nu}\bar{\sigma}^{\nu\mu} = -\frac{3}{2}\sigma^{\mu} \quad , \quad \bar{\sigma}^{\mu\nu}\bar{\sigma}_{\nu} = \bar{\sigma}_{\nu}\sigma^{\nu\mu} = -\frac{3}{2}\bar{\sigma}^{\mu} \tag{B.29}$$

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\sigma_{\nu\,\gamma\dot{\gamma}} = \frac{1}{2}(\sigma^{\mu}_{\delta\dot{\gamma}}\epsilon_{\gamma\alpha}\epsilon^{\beta\delta} - \sigma^{\mu}_{\alpha\dot{\gamma}}\delta^{\beta}_{\gamma}) \tag{B.30}$$

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\bar{\sigma}_{\nu}^{\dot{\alpha}\gamma} = \frac{1}{2}(\bar{\sigma}^{\mu\,\dot{\alpha}\delta}\epsilon_{\alpha\delta}\epsilon^{\beta\gamma} + \bar{\sigma}^{\mu\,\dot{\alpha}\beta}\delta^{\gamma}_{\alpha}) \tag{B.31}$$

$$\bar{\sigma}^{\mu\nu\,\dot{\alpha}}{}_{\dot{\beta}}\bar{\sigma}_{\nu}^{\dot{\gamma}\gamma} = \frac{1}{2}(\bar{\sigma}^{\mu\,\dot{\delta}\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\delta}\dot{\beta}} - \bar{\sigma}^{\mu\,\dot{\alpha}\gamma}\delta^{\dot{\gamma}}_{\dot{\beta}}) \tag{B.32}$$

$$\bar{\sigma}^{\mu\nu\,\dot{\alpha}}{}_{\dot{\beta}}\sigma_{\nu\,\alpha\dot{\gamma}} = \frac{1}{2}(\sigma^{\mu}_{\alpha\dot{\delta}}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\beta}\dot{\gamma}} + \sigma^{\mu}_{\alpha\dot{\beta}}\delta^{\dot{\alpha}}_{\dot{\gamma}}) \tag{B.33}$$

Some useful spinor identities:

$$\theta^{\alpha}\theta^{\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \quad , \quad \theta_{\alpha}\theta_{\beta} = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \tag{B.34}$$

$$\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \quad , \quad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \tag{B.35}$$

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta) \quad , \quad (\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = -\frac{1}{2}(\bar{\phi}\bar{\psi})(\bar{\theta}\bar{\theta}) \tag{B.36}$$

$$\chi\sigma^{\mu}\bar{\psi} = -\bar{\psi}\bar{\sigma}^{\mu}\chi \quad , \quad (\chi\sigma^{\mu}\bar{\psi})^{\dagger} = \psi\sigma^{\mu}\bar{\chi} \tag{B.37}$$

$$\chi\sigma^{\mu}\bar{\sigma}^{\nu}\psi = \psi\sigma^{\nu}\bar{\sigma}^{\mu}\chi \quad , \quad (\chi\sigma^{\mu}\bar{\sigma}^{\nu}\psi)^{\dagger} = \bar{\psi}\bar{\sigma}^{\nu}\sigma^{\mu}\bar{\chi} \tag{B.38}$$

$$\theta\sigma^{\mu}\bar{\theta}\theta\sigma^{\nu}\bar{\theta} = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu}, \tag{B.39}$$

$$(\psi\phi)\bar{\chi}_{\dot{\beta}} = -\frac{1}{2}(\theta\sigma^{\mu}\bar{\chi})(\psi\sigma^{\mu})_{\dot{\beta}}. \tag{B.40}$$

Differentiation and integration of the Grassmann variables:

$$\frac{\partial\theta^{\alpha}}{\partial\theta^{\beta}} = \delta^{\alpha}_{\beta} \ , \tag{B.41}$$

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\beta}} = -\frac{\partial}{\partial\theta_{\alpha}} \quad , \quad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} \ , \tag{B.42}$$

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha}}\frac{\partial}{\partial\theta^{\beta}}\theta\theta = 4 \quad , \quad \epsilon_{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}_{\dot{\beta}}}\bar{\theta}\bar{\theta} = 4 \ , \tag{B.43}$$

$$\int d\eta = 0 \quad , \quad \int d\eta\,\eta = 1 \tag{B.44}$$

BIOGRAPHY

Barış Yapışkan was born in Tunceli in 1975. He graduated from the High School of Zeytinburnu İhsan Mermerci in 1991. He obtained his BSc. degree in 1996 and MSc. degree in 2000 from İstanbul Technical University, Department of Physics . He started PhD education at the same department in 2000. He has been working in İstanbul Technical University, Department of Physics as a research assistant since 1999.