DEPARTMENT OF ECONOMICS
UNIVERSITY OF ALBERTA

Working Paper No. 2009-05

Spanning with Zero-Price Investment Assets

V. Galvani
University of Alberta

A. Plourde
University of Alberta

January, 2009

Spanning with Zero-Price Investment Assets

Valentina Galvani and André Plourde*

*Both Authors are from the Department of Economics, University of Alberta, Edmonton, AB, Canada, T6G 2H4. Valentina Galvani: Tel: (780) 492-1477, Fax: (780) 492-3300. E-mail: vgalvani@ualberta.ca. André Plourde: Tel: (780) 492-3306, Fax: (780) 492-3300. E-mail: andre.plourde@ualberta.ca. Galvani thanks the Killam Research Fund for financial support.

ABSTRACT

Regression-based testing techniques has long been used to quantify whether the efficient frontier of a set of assets spans the frontier of a larger collection of investments. This work derives regression-based spanning tests for the case in which the investment possibilities set contains, or is constituted by, zero-investment assets. An empirical example illustrates that ignoring the zero-cost qualification of these assets might lead to wrong spanning propositions.

Keywords: Mean-Variance Spanning; Diversification Benefits; Portfolio Choice; Futures Markets.

JEL classification: G10, G11, G32, M21,

1 Introduction

The evaluation of the diversification benefits associated with broadening the investment opportunities set has been the subject of a sizeable literature. Studies in this field have quantified the advantages of expanding the set of tradable assets from the domestic to the international equity market (among others, Bekaert and Urias, 1996; Errunza et al., 1999; and De Roon et al., 2001, Kai et al., 2003). Other works have investigated the gains from portfolio diversification across different classes or sets of assets (e.g., Eun, et al., forthcoming).

In the mean-variance framework (Markowitz, 1952), diversification benefits can be gauged by the difference between the efficient frontiers associated with the investments that are already represented in agents' portfolios and an expanded collection of securities. If the benchmark and augmented frontiers are not significantly different, then the benchmark assets are said to span the additional (test) investment opportunities. When there is such spanning, traders do not benefit from diversifying a portfolio of the benchmark investments by broadening their holdings to include the considered test assets.

Moving closer to this note's objective, we point out that the markets examined in extant literature on the mean-variance tests for spanning are represented by assets whose purchase require a positive monetary outlay. In contrast, this work derives regression-based spanning tests for the case in which the investment possibilities set contains, or it is constituted by, zero-investment assets.[1]

As a general rule, we can think of zero-cost investments as "bets" entailing an uncertain payoff but no immediate monetary outlay. A significant example of zero-price investments are futures contracts. Traders can open a position in futures contracts without incurring any significant outlay, but for transaction costs and refundable margins. As argued by De Roon et al. (2000), investors might thus regard futures as zero-cost assets whenever trading costs are ignored. Another example of zero-price investment assets are excess returns. While these payoffs are not directly traded as zero-cost investments in actual markets, much of the theory of asset pricing is nowadays formulated in terms of excess returns.[2]

In a seminal paper, Huberman and Kandel (1987) developed a test for mean-variance spanning in the framework of multivariate regression analysis. Working within the same framework, this

[1] In this paper the terms zero-price assets, zero-cost investments and zero-investment assets are used interchangeably.

[2] An obvious difference between futures contracts and excess returns is that the latter are generated by a portfolio of price-one returns. For this reason excess returns are perhaps less representative zero-investment assets than futures contracts.

note identifies the conditions that characterize spanning when zero-investment assets are added to a selection of benchmark securities which can be purchased at a (positive) price. In addition, we propose regression-based tests for spanning for markets on which portfolios can be acquired without an immediate monetary outlay. This is the case, for example, when comparing investments within or across futures markets.

Huberman and Kandel (1987) test of diversification benefits is derived in the framework of a static portfolio selection procedure. Therefore this test fully evaluates the gains of broadening the investment opportunities set as long as it is assumed that the first and second moments of the distribution of the asset returns are time-invariant. When the expectation and volatility of returns are allowed to respond to some key information indicators, mean-variance efficient portfolios should be described within a conditional framework.

A way to incorporate conditioning variables in the mean-variance paradigm is to augment the investment opportunities set of the unconditional case with scaled returns. A scaled return is the payoff of a portfolio that is managed on the basis of the realizations of the selected information indicators (e.g., Cochrane, 2005). Regression-based testing techniques for conditional spanning that rely on managed portfolios have been first proposed by Bekaert and Urias (1996).[3] This note proposes an extension of their conditional tests for spanning for markets that include zero-price investment assets.

The structure of the paper is the following. The next section provides some background and notation. In Section 3 we present the unconditional test for spanning for zero-investment assets. The formulation of conditional tests for spanning is then reviewed in Section 4. An empirical example completes this paper.

2 Background

Denote by r_{Nt} and r_{Kt} the time t *net* returns of N test and K benchmark assets respectively. Huberman and Kandel (1987) proved that tests for spanning can be performed by evaluating the linear restrictions

$$\alpha = 0_N,\, \beta 1_K = 1_N, \tag{1}$$

[3] Alternative testing techniques for conditional spanning are reviewed in de Roon and Nijman (2001).

in the multivariate regression of r_{Nt} over r_{Kt} (plus the constant):

$$r_{Nt} = \alpha + \beta r_{Kt} + \varepsilon_t, \text{ for } t = 1, 2\ldots, T, \tag{2}$$

where $E_t[\varepsilon_t] = 0_N$, β is a $N \times K$ matrix, 0_N and α are vectors of a N-dimensional space and 1_K is a K-component vector of ones. It is assumed that $K \geq 1$ while N is unconstrained.[4] For future reference, the expected values of the benchmark and test net returns are denoted by μ_K and μ_N respectively.

The conditions displayed in (1) imply that the efficient frontier generated by the benchmark securities does not significantly change when the investment possibilities' set is augmented to add the test assets. In this case portfolios diversification does not deliver significant gains to mean-variance investors who are currently holding portfolios in the benchmark securities, in other words, there is spanning. An alternative interpretation of the same restrictions is that each test asset is replicated, up to a zero-mean error term, by a portfolio of the benchmark securities that sells at the same price.

Regression-based testing techniques for conditional spanning that rely on managed portfolios have been first proposed by BU. Extending in an obvious way our notation, the benchmark and test returns in the conditional model are defined by the Kronecker products $r^Z_{Kt} \equiv z_t \otimes \left(r^K_t + 1_K\right)$ and $r^Z_{Nt} \equiv z_t \otimes \left(r^N_t + 1_N\right)$ respectively, where $r^i_t + 1_i$ for $i = K, N$ are gross returns. For these products, z_t is a L-dimensional vector of the form $z_t \equiv (1, z'_t)$ in which, with a hopefully negligible abuse of notation, z'_t are also the $L - 1$ variables representing investors' information structure at time t.

The variables r^Z_{Kt} and r^Z_{Nt} are payoffs–named scaled returns of managed portfolios in which the amount invested in each asset depends on the realization of the information variables z_t. For each scaled return that is associated with a given information variable z_{lt}, for some l in $\{1, 2, \ldots, L - 1\}$, portfolio rebalancing occurs with the same frequency with which z_{lt} varies.

BU showed that tests for spanning with conditional information reduce to the evaluation of some linear restrictions on the regression coefficients of the linear model

$$r^Z_{Nt} = \alpha^Z + \beta^Z r^Z_{Kt} + \varepsilon^Z_t, \tag{3}$$

[4] For $K = 1$, Jobson and Korkie (1982) showed that the evaluation of the diversification gains entailed by an expansion of the investment opportunities set amounts to a test of the mean variance efficiency of the benchmark return.

where $E_t\left[\varepsilon_t^Z\right] = 0_N$, α^Z is a vector with NL components, β^Z is a $(NL)\times(KL)$ matrix and $cov(r_{Kt}^Z, \varepsilon_t^Z) = 0_{(NL)\times(KL)}$. They proved that the linear restrictions testing for spanning are

$$\alpha^Z = 0_{NL},\, \beta^Z q_K = q_N, \tag{4}$$

where q_K is a KL-dimensional vector defined by $q_{Kt} = E\left[1_K \otimes z_t\right]$ and q_N is the NL-dimensional vector $E\left[1_N \otimes z_t\right]$. The vectors q_i for $i = K, N$ are the expected prices of the scaled returns of the test and benchmark assets. The interpretation of these conditions follows closely that of the unconditional case.

The next sections discuss the mean-variance test for spanning, in its conditional and unconditional form, when the considered markets are represented by zero-price assets. The point of recognizing that some forms of investment can be assimilated to zero-cost assets is that the gross returns of these cannot be properly defined. This observation lays at the core of the pricing equations we rely upon to derive our tests for spanning, and it is therefore worth illustrating.

The time-t net return r_t of an investment is the relative variation of its market value V_t from $t-1$ to t, that is $\frac{V_t - V_{t-1}}{V_{t-1}}$. As long as this asset can be purchased by paying some positive amount of wealth, then its gross return is traditionally defined by the sum $1 + r_t$. In this case, the law of one price implies the following fundamental pricing equation

$$E_t\left[m_t\left(1 + r_t\right)\right] = E_t\left[m_t\right] + E_t\left[m_t r_t\right] = 1,$$

where m_t is a stochastic discount factor. Put differently, the gross return of an asset is the payoff of a long position costing one unit of wealth. In terms of net returns, the above equality amounts to

$$E_t\left[m_t r_t\right] = 1 - E_t\left[m_t\right].$$

When zero-price assets are considered, the cost of obtaining the uncertain payoff $V_t - V_{t-1}$ is of course zero. Consistently, the law of one price assigns zero price to the net return of a zero investment asset:

$$E_t\left[m_t r_t\right] = \frac{1}{V_t} E_t\left[m_t\left(V_t - V_{t-1}\right)\right] = 0. \tag{5}$$

The above pricing equation implies that the gross return of a zero-investment asset cannot be defined by simply adding one to its net return. Briefly stated:

$$E_t\left[m_t\left(1+r_t\right)\right]=E_t\left[m_t\right]+E_t\left[m_t r_t\right]=E_t\left[m_t\right],$$

which contradicts the definition of gross return as the payoff of a position costing one unit of wealth because the expected value of the stochastic discount factor need not be one.

3 Unconditional Spanning

This section proposes unconditional tests for spanning that evaluate the benefits of portfolio diversification when a collection of zero-investment assets is added to a benchmark market. We start with proposing a regression-based test for spanning for the case in which some zero-price investments are added to a collection of assets that can be purchased for a (positive) price. A direct application of standard optimization techniques, reviewed in Section 3.1, proves that there is spanning whenever the following linear restrictions on the coefficients of the baseline model (2) are satisfied:

$$H_{01}:\alpha=0_N \text{ and } \beta 1_K=0_N. \tag{6}$$

Section 3.2 shows that the null hypotheses in (6) can also be derived by exploiting the duality between the efficient frontier and the volatility bounds of investors' marginal rate of substitution discussed by Hansen and Jagannathan (1991). This approach sets the ground for the ensuing discussion of the tests for spanning when all the considered assets zero-price investments.

Before we get started with our proofs, we quickly show that the linear restrictions on the coefficients of the baseline model that are displayed in (6) can be also evaluated for the regression

$$r_{Nt}=\alpha_1+\beta\left(r_{Kt}+1_K\right)+\varepsilon_t, \text{ for } t=1,2.\ldots,T$$

in which the net returns of the zero-cost assets are regressed on the gross returns of the benchmark assets, when these are defined. To see it, notice that, of course, the slope coefficients are the same regardless of whether the benchmark market is summarized by its gross or its net returns. Moreover, the vector of constant coefficients α_1 satisfies

$$\alpha_1=\mu_N-\beta\left(1_K+\mu_K\right)=\alpha-\beta 1_K.$$

By inspection, the set of conditions

$$\alpha_1 = 0_N \text{ and } \beta 1_K = 0_N$$

are equivalent to the linear restrictions displayed in (6).

3.1 The Lagrangian Approach

With an obvious notation, the investor's problem is:

$$\min_{w} \; w'Vw,$$

$$s.t. \; w'\overline{\mu} = \mu \text{ and } w\delta_{K+N} = 1,$$

where w is the $(K+N)$-dimensional vector of portfolio weights, V is the $(K+N) \times (K+N)$-covariance matrix of the asset returns $\left[r_t^K, r_t^N\right]$, $\overline{\mu} = [\mu_K, \mu_N]$ is the $(K+N)$-dimensional vector listing the expected net returns of all assets, the scalar μ is the targeted level of expected return and δ_{K+N} is a constant $(K+N)$-dimensional vector assigning value 1 to the first K components and zero to the remaining N components. The first-order conditions of the above constrained optimization imply that:

$$w = V^{-1}\left(\lambda\overline{\mu} + \nu\delta_{K+N}\right), \tag{7}$$

where λ and ν are the Lagrangian multipliers of the constraints. Plugging (7) in the constraints, we obtain the following two equations:

$$\begin{aligned} \overline{\mu}V^{-1}\left(\lambda\overline{\mu} + \nu\delta_{K+N}\right) &= \mu, \\ \delta'_{K+N}V^{-1}\left(\lambda\overline{\mu} + \nu\delta_{K+N}\right) &= 1. \end{aligned}$$

Solving for λ and ν yields

$$\lambda = \frac{C\mu - B}{D}, \; \nu = \frac{A - B\mu}{D}$$

where

$$A = \overline{\mu}'V^{-1}\overline{\mu}, \; B = \overline{\mu}'V^{-1}\delta_{K+N}, \; C = \delta'_{K+N}V^{-1}\delta_{K+N},$$

and, $D = AC - B^2$. After substituting in (7), the weights of the minimum-variance portfolio given the targeted return μ are:

$$w = V^{-1}\frac{\overline{\mu}\,(C\mu - B) + \delta_{K+N}\,(A - B\mu)}{D}.$$

Taking derivatives of the variance of an efficient return with expected value equal to μ it can be shown that there are two efficient returns whose weights are

$$w_1 = V^{-1}\frac{\overline{\mu}}{B} \text{ and } w_2 = V^{-1}\frac{\delta_{K+N}}{C},$$

The vector of weights w_1 corresponds to the tangency portfolio when the riskfree asset is a zero net return asset. The second vector of weights defines the Global Minimum Variance (GMV) portfolio. Proceeding as in Kan and Zhou (2001), the weights of these portfolios on the N test assets, denoted by w_1^N and w_2^N, are:

$$w_1^N = \Sigma^{-1}\frac{\alpha}{B} \text{ and } w_2^N = \frac{\left[\Sigma^{-1}\beta, \Sigma^{-1}\right]\delta_{K+N}}{C},$$

where Σ is the $(N \times N)$ lower right block of the inverse of V, the $(N \times K)$ matrix β is defined by $\beta \equiv \Sigma_{NK}\Sigma_K^{-1}$ and the vector α is the function of β defined by $\alpha \equiv \mu^N - \beta\mu^K$. The way δ_{K+N} is defined we also have:

$$\left[\Sigma^{-1}\beta, \Sigma^{-1}\right]\delta_{K+N} = \Sigma^{-1}\beta 1_K + \Sigma^{-1}0_N = \Sigma^{-1}\beta 1_K.$$

It is known that we can obtain any other efficient return by forming portfolios of any given pair of linearly independent efficient returns (Merton, 1972). Hence if the linear restrictions in (6) are satisfied, then all the efficient returns of the augmented frontier have zero weights in the test assets. In this case the augmented and benchmark mean-variance frontiers coincide, and we have spanning.

3.2 The Stochastic Discount Factor Approach

Hansen and Jagannathan (1991) showed that among the stochastic discount factors with a given expected value that correctly price a collection of assets, the one which bears the lowest level of variability is linear in the priced asset returns. In particular, the stochastic discount factor m^ν with expected value ν that displays the lowest variance, and that prices the benchmark returns, is an affine transformation of the returns of the benchmark assets, that is:

$$m_t^{\nu} = v + (r_{Kt} - \mu_K)' \varphi. \tag{8}$$

The vector φ can be identified by imposing that m_t^{ν} correctly prices the benchmark returns, that is by solving the system of equations:

$$E\left[m_t^{\nu}(r_{Kt} + 1_K)\right] = 1_K. \tag{9}$$

Plugging (8) in the above pricing formula and solving for φ we obtain

$$\varphi' = \Sigma_K^{-1}\left(1_K - \nu\left(\mu_K + 1_K\right)\right), \tag{10}$$

where Σ_K is the covariance matrix of the benchmark returns. Substituting φ in (8) we obtain:

$$m_t^{\nu} = v + (r_{Kt} - \mu_K)' \Sigma_K^{-1}\left(1_K - \nu\left(\mu_K + 1_K\right)\right). \tag{11}$$

For future reference we emphasize that the same discount factor can be identified when the pricing restriction involves the net returns of the benchmark assets. From (9), the minimum-variance stochastic discount factor m_t^{ν} correctly prices the benchmark net returns whenever:

$$\begin{aligned} E\left[m_t^{\nu} r_{Kt}\right] &= (1-\nu)\,1_K = E\left[\left(v + (r_{Kt} - \mu_K)'\varphi\right) r_{Kt}\right] \\ &= \nu\left(\mu_K\right) + E\left[\varphi'\left(r_{Kt}\right)'\left(r_{Kt} - \mu_K\right)\right] \\ &= \nu\mu_K + \Sigma_K\varphi', \end{aligned}$$

The above pricing equation deliver the expression of φ displayed in (10).

As noted in BU, the hypothesis of mean-variance spanning can be reformulated in terms of conditions on the minimum-variance stochastic discount factor. In particular, there is spanning whenever the stochastic discount factor m_t^{ν} that is defined by the benchmark securities as in (11) prices correctly the test assets for each value of ν. The BU approach can be interpreted in view of the fact that the stochastic discount factor is the inter-temporal marginal rate of substitution of an expected utility-maximizing investor calculated at the optimal consumption bundle. In fact, if the stochastic discount factor that is implied by the benchmark securities does not respond to the expansion of the investment opportunities set, then from an investor' perspective the collection of assets available for trade remains unchanged after the inclusion of the test assets.

The zero-investment assets payoffs r_{Nt} are correctly priced by the stochastic discount factor m_t^ν whenever $E\left[r_{Nt} m_t^\nu\right] = 0_N$. Relying on (11) we obtain the following chain of equalities:

$$
\begin{aligned}
E\left[r_{Nt} m_t^\nu\right] &= E\left[r_{Nt}\nu + r_{Nt}\left(r_{Kt} - \mu_K\right)' \Sigma_K^{-1}\left(1_K - \nu\left(1_K + \mu_K\right)\right)\right] \\
&= \nu\mu_N - \nu E\left[r_{Nt}\left(r_{Kt} - \mu_K\right)'\left(\Sigma_K^{-1}\right)\left(1_K + \mu_K\right)\right] \\
&\quad + E\left[r_{Nt}\left(r_{Kt} - \mu_K\right)' \Sigma_K^{-1} 1_K\right] \\
&= \nu\left(\mu_N - \Sigma_{NK}\Sigma_K^{-1}\left(1_K + \mu_K\right)\right) + \left(\Sigma_{NK}\Sigma_K^{-1} 1_K\right) \\
&= \nu\alpha - (1 - \nu)\,\beta 1_K .
\end{aligned}
$$

By inspection, the spanning conditions for zero-beta assets displayed in (6) are satisfied if and only if the stochastic discount factor m_t^ν correctly prices the test investment opportunities for each ν.

3.3 The Case of Zero-Price Benchmark Assets

Assume that the benchmark market is represented by K zero-cost investments. We aim to determine conditions that evaluate the gains associated with broadening the investment opportunities set by adding a given collection of zero-price test assets. This would be the case, for example, when evaluating the diversification benefits yielded by an expansion of agents' portfolio holdings within a market of futures contracts.

Because the constraint on the wealth level in the consumer problem is missing, the direct approach that has been reviewed in Section 3.1 does not yield conditions for spanning that are similar to those displayed in (6). However, test for spanning can be obtained by imposing that the stochastic discount factor implied by the benchmark investment opportunities need not be modified to account for the pricing information summarized by the test returns.

Following the logic we used to derive (11), the stochastic discount factor that is determined by the K zero-cost benchmark assets is

$$
m_t^\nu = v - \left(r_{Kt} - \mu_K\right)' \Sigma_K^{-1} \nu \mu_K . \tag{12}
$$

Relying on (12) we obtain:

$$
\begin{aligned}
E\left[r_{Nt} m_t^{\nu}\right] &= E\left[r_{Nt}\nu - r_{Nt}\left(r_{Kt}-\mu_K\right)' \Sigma_K^{-1}\left(\nu\mu_K\right)\right] \\
&= \nu\mu_N - \nu E\left[r_{Nt}\left(r_{Kt}-\mu_K\right)'\left(\Sigma_K^{-1}\right)\left(\mu_K\right)\right] \\
&= \nu\left(\mu_N - \Sigma_{NK}\Sigma_K^{-1}\mu_K\right) \\
&= \nu\left(\mu_N - \Sigma_{NK}\Sigma_K^{-1}\mu_K\right) \\
&= \nu\alpha.
\end{aligned}
\tag{13}
$$

By imposing that m_t^{ν} correctly prices the zero-price test assets for each ν we conclude that there is spanning whenever the constant is statistically insignificant in the baseline model. The intuition is that the examined zero-price investments fail to offer significant diversification benefits if they can be replicated—modulo some measurement errors—by a portfolio of the benchmark assets. Because these latter are zero-cost investments, no wealth constraint has to be imposed on the portfolio weights, i.e. on the slopes of the baseline model.

The linear restrictions on the constant of the baseline model that are displayed in (13) mirror a known performance measure gauging the variations in the risk-bearing compensation associated with an expansion of the investment opportunity set. In the mean-variance framework, the risk compensation of a portfolio is summarized by its Sharpe ratio, (i.e., the ratio between its excess return and standard deviation). Jobson and Korkie (1982, 1984, 1989) illustrated that the increase of the Sharpe ratio entailed by an expansion of collection of assets that are available for trade can be measured by the constant coefficient in a regression of the excess returns of the test assets over those of the benchmark securities, plus a constant.

Jobson and Korkie's measure is defined with respect to the hyperbolic efficient frontier associated with the gross returns from which the excess returns are constructed. Therefore it can be interpreted within the framework of the familiar mean-variance investors' portfolio selection procedure. In contrast, the spanning condition in (13) is not associated with the shifts of an efficient frontier but rather with the changes of the Hansen-Jagannathan bounds that are implied by the expansion of the set of tradable assets.

To conclude this section we examine the case in which the benchmark market is represented by zero-price assets, while the test investment opportunities can be purchased at a positive price. A reasonable conjecture is that in this case the benchmark assets fail to span the test securities.

In fact, if there was spanning, a portfolio of zero-cost assets would replicate–up to a zero-mean disturbance–the payoff of an asset whose purchase implies some non-zero expense. As long as the preferences of the individuals are consistent with the mean-variance paradigm, the price system entailing this trading opportunities would not be sustainable. Nicely enough, this intuition finds some corroboration.

Following the same logic we used to derive the linear restrictions in (13), there is spanning whenever for each expected value ν of the stochastic discount factor m_t^ν that prices the benchmark zero-investment assets also correctly prices the gross returns that summarize the test market. Relying on the stochastic discount factor that is implied by a collection of zero-price benchmark assets, this observation implies the following chain of inequalities:

$$\begin{aligned}
1_N &= E\left[(r_{Nt}+1_N)\,\nu - (r_{Nt}+1_N)\,(r_{Kt}-\mu_K)'\,\Sigma_K^{-1}\,(\nu\mu_K)\right] \\
&= \nu\,(\mu_{Nt}+1_N) - \nu E\left[(r_{Nt}+1_N)\,(r_{Kt}-\mu_K)'\left(\Sigma_K^{-1}\right)\mu_K\right] \\
&= \nu\left[(\mu_{Nt}+1_N) - \Sigma_{NK}\Sigma_K^{-1}\mu_K\right] \\
&= \nu\,(\alpha+1_N)\,.
\end{aligned}$$

Of course, the above equality might be violated by infinitely many values of ν and therefore there is no spanning.

4 Conditional Spanning

As it has been reviewed in Section 2, the test for conditional spanning compares the investment possibilities that are offered by information-based portfolios of the benchmark assets with those associated with an expanded set of portfolios that respond to the same information indicators. This test must be modified from BU's original formulation to accommodate the fact that the payoff of any managed portfolio of zero-investment assets can be purchased at zero cost.

We start our discussion by proposing a test for conditional spanning for the case in which N zero-price assets are added to a market that is represented by K investments that can be purchased at a (positive) price. It is now shown that there is conditional spanning whenever the linear restrictions

$$H_{03} : \alpha^Z = 0_{NL},\, \beta^Z q_K = 0_{NL} \tag{14}$$

are satisfied. The coefficients α^Z and β^Z are obtained from the regression displayed in (3), where $r_{Kt}^Z \equiv z_t \otimes \left(r_t^K + 1_K\right)$ as in BU's model, while the scaled returns r_{Nt}^Z are obtained by interacting the selected information variables with the net returns of the test assets, i.e., $r_{Nt}^Z \equiv z_t \otimes r_t^N$. The vector q_K is defined by $q_K \equiv z_t \otimes 1_K$ and it represents the cost of the managed portfolios of the benchmark securities. Of course, the above spanning conditions are not equivalent to the linear restrictions proposed by BU, but for the trivial case in which the instruments are constant.

Following the notation of Section 2, the scaled returns of the benchmark and test assets, and associated variables, are identified by a z superscript. Hence, the expected values of the scaled returns r_{it}^Z are μ_i^Z for $i = K, N$, while Σ_K^Z and Σ_{NK}^Z denote the covariance of r_{Kt}^Z and the covariance between r_{Nt}^Z and r_{Kt}^Z, respectively. Relying on an argument in Hansen and Jagannathan (1991), the minimum-variance stochastic discount factor $m_t^{Z\nu}$ that prices the scaled benchmark returns is:

$$m_t^{Z\nu} = \nu + \left(r_{Kt}^Z - \mu_K^Z\right)' \gamma, \tag{15}$$

where:

$$\gamma = \left(\Sigma_K^Z\right)^{-1} \left(q_K - \nu \mu_K^Z\right).$$

From the argument presented Section 3, it is straightforward to conclude that there is conditional spanning whenever for each expected value ν the discount factor $m_t^{Z\nu}$ assigns the correct value to the test assets' scaled returns. Because trading positions on the test assets can be obtained at no cost, there is conditional spanning whenever the expected value $E\left[m_t^{Z\nu} r_{Nt}^Z\right]$ vanishes for each ν. Relying on the expression of the stochastic discount factor displayed in (15), we obtain the following chain of equalities:

$$\begin{aligned}
E\left[m_t^{Z\nu} r_{Nt}^Z\right] &= E\left[r_{Nt}^Z \nu + r_{Nt}^Z \left(r_{Kt}^Z - \mu_K^Z\right)' \left(\Sigma_K^Z\right)^{-1} \left(q_K - \nu \mu_K^Z\right)\right] \\
&= \nu \mu_N^Z - \nu E\left[r_{Nt}^Z \left(r_{Kt}^Z - \mu_K^Z\right)' \left(\Sigma_K^Z\right)^{-1} \mu_K^Z\right] \\
&\quad + E\left[r_{Nt}^Z \left(r_{Kt}^Z - \mu_K^Z\right)' \left(\Sigma_K^Z\right)^{-1} q_K\right] \\
&= \nu \left(\mu_N^Z - \Sigma_{NK}^Z \left(\Sigma_K^Z\right)^{-1} \mu_K^Z\right) + \left(\Sigma_{NK} \left(\Sigma_K^Z\right)^{-1} q_K\right) \\
&= \nu \left(\mu_N^Z - \beta^Z \mu_K^Z\right) + \left(\beta^Z q_K\right) \\
&= \nu \alpha^Z + \beta^Z q_K.
\end{aligned}$$

Of course, the linear conditions displayed in (14) are implied by requiring that the NL linear equations displayed above are verified for each value of the parameter ν.

An argument virtually identical, but for the use of scaled returns rather than net returns, to that delivering the restrictions for unconditional spanning with zero-investment assets displayed in (13) shows that the linear restrictions that characterize conditional spanning when all considered assets can be acquired at zero cost are: $\alpha^Z = 0_N$, where α^Z is the NL-dimensional vector of coefficients in the regression:

$$r_{Nt}^Z = \alpha^Z + \beta^Z r_{Kt}^Z + \varepsilon_t^Z,$$

in which $r_{Nt}^Z = z_t \otimes r_t^i$ for $i = K, N$.

5 An Empirical Example

The purpose of this section is to illustrate the results discussed in this note by means of an empirical exercise. We rely on a data set of daily observations covering from the beginning of January 1990 to the end of February 2008. We examine the diversification benefits, as measured by the shifts of the efficient frontier, offered by futures contracts on light sweet crude oil (WTI) issued by the New York Mercantile Exchange (NYMEX) over a collection of energy stocks that are traded on the New York Stock Exchange.[5] The baseline model (2) is a regression of the net returns of the WTI futures contracts over the net returns of the selected stocks. In addition, we evaluate the diversification benefits that the WTI futures contracts offer over futures on unleaded gasoline, natural gas and Brent crude. Approximately $4,740$ daily returns are available per asset. All data are from Datastream International.

The continuous time series of daily futures returns are defined on a moving source, as for example in de Roon et al. (2000). A position is taken in the nearest-to-maturity contract until the delivery month, at which time the position changes to the following nearest-to-maturity contract. The US-based companies that have been selected to summarize the energy sector are divided in three groups: integrated, refiners, and producers. The integrated firms are represented by Amarada-Hess, ExxonMobil, Chevron, ConocoPhillips, and Marathon Oil. The selected refiners are Valero Energy, Sunoco, Tesoro Petroleum, Holly, and Frontier Oil, while the crude oil producers are Occidental Oil & Gas, Kerr-McGee, Apache, Plains Exploration & Production, and Devon Energy.

The Wald test statistic of the linear restrictions for unconditional spanning, as displayed in

[5] Galvani and Plourde (2008) investigate the diversification benefits offered by futures contracts on selected energy commodities to investors who are holding a portfolio of energy stocks.

(6), is a whopping 221.88.[6] The associated P-value can be obtained from a chi-square distribution with 2 degree of freedom (e.g., Huberman and Kandel, 1987) and is zero. In this example, the unconditional test for spanning indicates that the WTI futures offer sizeable diversification benefits with respect to efficient portfolios of the selected energy stocks.

Tests for conditional spanning compare the payoffs of dynamic trading strategies that are determined on the basis of variables that summarize relevant pricing information. In our example we rely on a variable that measures the order imbalances carried by large hedgers on the WTI futures contracts. We argue that this indicator accounts for relevant information on the ground that the difference between buy and sell orders has been shown to reveals market pressure on prices (e.g., Chordia et al. 2001). Also, there is empirical evidence that returns of futures contracts are correlated with the net positions of large hedgers (Bessembinder, 1992 and de Roon et al. 2000).

Positions of large traders in North American futures markets are reported to the U.S. Commodity Futures Trading Commission (CFTC). The CFTC reports are usually released to the public at the end of each trading week. Following de Roon et al. (2000) we construct a hedging pressure variable for each futures contract traded on NYMEX based on the positions of large hedgers as reported to the CFTC:

$$q = \frac{\text{number of short hedge positions -number of long hedge positions}}{\text{total number of hedge positions}}. \tag{16}$$

To eliminate biases due to the benefits of hindsight, the managed portfolios associated with the variable q are rebalanced at the beginning of the trading week *following* the release of the CFTC reports.[7]

The Wald test statistic of the conditional spanning test is 7.37 with a P-value of 0.11 obtained from a χ^2-distribution with 4 degrees of freedom (e.g. BU, page 843). At the 5% significance level the P-value indicates that the WTI futures contracts fail to improve the investment opportunities offered by energy stocks once investors observe the positions held by large hedgers.

Our empirical exercise continues with the evaluation of the diversification benefits offered by the WTI futures over futures contracts on unleaded gasoline and natural gas as traded on NYMEX. In addition we also include futures on the Brent crude which are traded on the former International Petroleum Exchange (IPE). Because the in-sample coefficient of linear correlation of the futures

[6] The statistics reported in this note are corrected for heteroskedasticity and autocorrelation of the error terms.

[7] Until October 1992, the CFTC reports were published only every two weeks. Hence, for the first 719 observations, the managed portfolios have been rebalanced every two weeks.

contracts on WTI with those on the Brent blend is 0.8 and with those on unleaded gasoline is 0.7 we expect the null hypotheses of spanning to be rejected. In fact, the Wald test of the linear restrictions displayed in (13) is 0.12 with a P-value of 0.73.

We conclude this section with an example of how the evaluation of the diversification benefits by means of the standard mean-variance spanning tests might be misleading when zero-price assets are involved. To this purpose, we contrast the foregoing empirical example with an analysis of the diversification benefits offered by the WTI futures that is conducted under the assumption that the gross returns of futures contracts are calculated by simply adding one to their net returns. Working within the framework proposed by Huberman and Kandel, the diversification benefits offered by the WTI futures can be evaluated by making inferences on the restrictions displayed in (1), for the test of unconditional spanning, and on those proposed by BU and reported in (4), for the tests of conditional spanning. The Wald test statistic of the unconditional spanning tests is 176 with a zero P-value. The Wald test statistics for conditional spanning, where the scaled returns are determined by the hedging pressure variable defined in (16), is 170.9 with again a zero P-value. Hence at any relevant significance value the conditional and the unconditional tests for spanning would indicate that future contracts on the WTI do improve the investment opportunities offered by the selected energy stocks. In contrast the analysis proposed at the beginning of our empirical exercise indicates that the WTI futures contracts can be replicated by a zero-cost portfolio of managed portfolios of energy stocks.

Similar conclusions can be obtained for the evaluation of the diversification benefits offered by the WTI futures contracts to investors who are currently holding a portfolio of futures on other energy commodities. In fact the Wald statistic of the test for spanning of Huberman and Kandel's linear restrictions is 14.12 which entails a zero P-value. This result would indicate, erroneously, that the WTI futures offer significant diversification benefits over the selected zero-investment assets.

References

[1] Bekaert, G., Urias, M.S., 1996. Diversification, integration and emerging market closed-end funds, *Journal of Finance*, 51(3), 835-869, Papers and proceedings of the fifty-sixth annual meeting of the American finance association, San Francisco, Ca.

[2] Bessembinder, H., 1992. Systematic risk, hedging pressure, and risk premiums in futures mar-

kets, *Review of Financial Studies*, 5(4), 637-667.

[3] Chordia, T., Roll, R., Subrahmanyam, A., 2001, Market liquidity and trading activity, *The Journal of Finance*, 56 (2), 501-530.

[4] Cochrane, J.H., 2005. *Asset Pricing*, Princeton University Press, Princeton, NJ.

[5] De Roon, F.A., Nijman, T.E., 2001. Testing for mean-variance spanning: a survey, *Journal of Empirical Finance*, 8(2), 111-156.

[6] De Roon, F.A., Nijman, T.E., and Veld, C., 2000. Hedging pressure effects in futures markets, *Journal of Finance*, 55(3), 1437-1456.

[7] Errunza, V., Hogan, K., Hung, M.W., 1999. Can the gains from international diversification be achieved without trading abroad?, *Journal of Finance*, 54(6), 2075-2107

[8] Eun C.S., Huang, W., Lai, S., forthcoming. International diversification with large-and small-cap stocks, *Journal of Financial and Quantitative Analysis*.

[9] Galvani, V., Plourde, A., 2008. Portfolio Diversification in Energy Markets, Manuscript, Department of Economics, University of Alberta, Edmonton, AB, Canada.

[10] Hansen, L., Jagannathan, R., 1991, Implications of Security Market Data for Models of Dynamic Economies, *The Journal of Political Economy*, 99(2), 225-262.

[11] Huberman, G., Kandel, S. 1987. Mean-variance spanning, *Journal of Finance*, 42(4), 873-888.

[12] Jobson, J.D., Korkie, B., 1982. Potential performance and tests of portfolio efficiency, *Journal of Financial Economics*, 10(4), 433-466.

[13] Jobson, J.D., Korkie, B., 1984. On the Jensen measure and marginal improvements in portfolio performance: A note, *Journal of Finance*, 39(1), 245-251.

[14] Jobson, J.D., Korkie, B., 1989. A performance interpretation of multivariate tests of asset set intersection, spanning, and mean-variance efficiency, *Journal of Financial and Quantitative Analysis*, 24(2), 185-204.

[15] Kai, L, Asani, S. and Zhenyu Wang, 2003, Diversification benefits of emerging markets subject to portfolio constraints, *Journal of Empirical Finance*, 10, 57-80.

Department of Economics, University of Alberta Working Paper Series

http://www.economics.ualberta.ca/working_papers.cfm

Recent Working Papers

2009-04 Options and Efficiency in Spaces of Bounded Claims – **Galvani**, Troitsky
2009-03 Identifying the Poorest Older Americans – Fisher, Johnson, **Marchand**
2009-02 Time-Saving Innovations, Time Allocation, and Energy Use: Evidence from Canadian Households – **Brenčič, Young**
2009-01 Trigger Happy or Gun Shy? Dissolving Common-Value Partnerships with Texas Shootouts – Brooks, **Landeo**, Spier

Department of Economics
8-14 HM Tory
University of Alberta
Edmonton, AB
T6G 2H4
Canada

Telephone (780) 492-3406
Fax (780) 492-3300
www.ualberta.ca/economics

FACULTY OF ARTS

[16] Kan, R., Zhou, G., 2001. Mean-Variance spanning tests, Working Paper, University of Washington–St. Louis.

[17] Markowitz, H., 1952. Portfolio selection, *Journal of Finance* 7(1), 77-91.

[18] Merton, R.C., 1972. An analytic derivation of the efficient portfolio frontier, *Journal of Financial and Quantitative Analysis*, 7(4), 1851-1872.