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Abstract

Supplementing a finite state-space static securities market with options obtains market completeness. This study concludes that options maintain the same spanning power in the space of bounded payoff topologized by its duality with the space of the state price densities.

Keywords: Spanning; Options; Market Completeness; Efficiency

JEL classification: C0, D61, G10, G12, G19

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1 Introduction

In a seminal contribution Ross (1976) showed that a static finite state-space market can be completed by supplementing the primitive securities with ordinary call and put options written on an injective claim in the same way that adding Arrow securities would in an incomplete Arrow–Debreu economy.[1] This finding supports the view that the market structure necessary to span all contingent claims needs not to involve a complex set of securities but rather a large number of ordinary call or put options.[2] Options maintain the same spanning power in L_p-spaces for $1 < p < \infty$ that are defined over a separable measure algebra of the state-space (Galvani, 2008, Theorem 1). A similar result holds with respect to notion of approximation offered by the pointwise convergence of sequences for spaces of measurable functions (Galvani, 2008, Corollary 7). In addition, underlyers for which options bring about market completeness are shown to be dense in these spaces of contingent claims (Galvani, 2008, Corollaries 6 and 7).[3] This work analyzes the spanning power of options in spaces of bounded random variables.

Previous literature on the spanning power of options has emphasized the existence of underlyers for which ordinary options make redundant more complex derivatives on a given set of assets. Nachman (1989) proved that

[1] Baptista (2003, 2005) discusses the multi-period model.

[2] Remarkably, it might be the case that options are not replicated by any portfolio of primitive securities (Aliprantis and Tourky, 2002; Baptista, 2007).

[3] Galvani (2005 and 2007, a and b) discuss the generalization of Ross' spanning proposition for continuos underlying asset in the space of continuos payoffs and in the L_p-spaces.

two layers of options span the market completion of an at most countable collection of primitive securities N. The market completion of N is the space of contingent claims that are measurable with respect to the σ-algebra $\sigma(N)$ generated by the elements of N. In particular, options span all the derivatives that are written on the option underlyer (Nachman, 1989). Green and Jarrow (1987) obtained similar results but for the required number of option layers. From the perspective of market completeness analysis, these spanning propositions prove that if the σ-algebra modeling the market's information structure is generated by the option underlying asset, then portfolios of options span any contingent claim. In this sense options are proved to complete the market by endogenizing the market's information.

In contrast, this study obtains a generalization of Ross' spanning proposition for securities markets in which the information structure is taken as given. This departure from the previous literature is motivated by the fact that in the standard framework of empirical investigations the relevant information structure is identified with a given σ-algebra, often the state-space Borel σ-algebra.

In this work, the space of contingent claims is identified with the space of bounded measurable functions over a probability space and equipped with the weak-star topology defined by its duality with the space of state-price densities $L_1(P)$, as discussed in Jarrow et al. (1999). Options are said to complete the space of contingent claims $L_\infty(P)$ as long as finite-component portfolios of plain call options form a dense subspace of the L_∞-space. In this framework, we show that there is essentially only one L_∞-space for which an attempt to generalize Ross' spanning proposition is not futile, namely $L_\infty[0,1]$. The uniqueness of $L_\infty[0,1]$ is proved by demonstrating that spaces of bounded claims that can be complete by options are equivalent from a vectorial, topological, and latticial perspective to $L_\infty[0,1]$.

This work also shows that options on a single payoff complete a separable L_∞-space. Moreover, we also prove that underlyers for which options complete such space of bounded claims are pervasive in the sense that they form a dense subset of the space of contingent claims.

When the state-space is a completely separable metric space equipped with the completion of its Borel σ-algebra and measured by an atomless probability, we prove that options on a claim that is a.s. equal to an injective function (i.e., that is a.s. injective) complete the L_∞-space. This amounts to a direct generalization of Ross' finite-dimensional spanning result to a class of spaces of bounded claims that are extremely common in the extant literature. Also in this case, underlyers for which options obtain market completeness are shown to form a dense subset of the L_∞-space.

The structure of the paper is the following. The next section provides some background. Section 3 discusses the notion of uniqueness for the state-space $L_\infty[0,1]$ as the only L_∞-space that can be completed by options. Our main spanning proposition can be found in Section 4. Last, Section 5 offers some concluding remarks.

2 Background

Throughout this paper the state-space Ω is assumed to be an uncountable set of states of nature. The σ-algebra modeling the market information structure is denoted by Σ, while P designates the completion of a nonatomic probability measure on Σ. The measure algebra associated with Σ and P is indicated by Σ_P and is considered a metric space under the metric induced by the L_1-norm. The space of random variables on the probability space (Ω, Σ, P) that are bounded in the essential supremum norm, with respect to P, is denoted by $L_\infty(P)$. As usual, functions in the L_∞-spaces are defined up to P-almost sure equivalence (see for example Aliprantis and Border, 2006, henceforth AB, Section 13.1). In this work, claims are identified with elements of the space $L_\infty(P)$. The space $L_\infty(P)$ is therefore called the space of contingent claims.

The space of contingent claims $L_\infty(P)$ is henceforth equipped with the weak-star topology w^* associated with the dual system $\langle L_1(P), L_\infty(P)\rangle$ defined by the duality

$$\langle f, g\rangle = \int_\Omega gf dP, \tag{1}$$

for each f in $L_1(P)$ and g in $L_\infty(P)$. The topological dual of $L_\infty(P)$ equipped with the w^*-topology is $L_1(P)$ endowed with the topology generated by the L_1-norm. We choose this topology to maintain the equivalence between market completeness and the uniqueness of a strictly positive state price density, under suitable no-arbitrage conditions, for the space of contingent claims $L_\infty(P)$.[4]

The interval $[0,1]$ is equipped with the topology induced by the Euclidean norm. If the state-space $[0,1]$ is measured by the Lebesgue measure λ, then the space of contingent claims is denoted by $L_\infty[0,1]$ and its dual by $L_1[0,1]$. The state-independent claim $\mathbf{1}_\Omega$ is defined by $\mathbf{1}_\Omega(\omega) = 1$ for each ω in Ω and is interpreted as the payoff of the riskfree bond or as the payoff of the numeraire. Whenever the domain is clearly identified by the context, the claim $\mathbf{1}_\Omega$ is denoted by $\mathbf{1}$. If k is a real number, then k stands for $k\mathbf{1}$ when appropriate.

In a static framework the payoff of a call option written on a claim x with strike price k is $(x-k)^+$, where $(x-k)^+(\omega)$ equals $\sup\{x(\omega)-k, 0\}$ for all ω in Ω. Likewise, the payoff of a put option on x with strike price k is $(k-x)^+$. If the underlyer is a positive claim, strike prices might be limited to nonnegative values.

In an effort to capture the finite nature of actual portfolio management, portfolios are from now on restricted to have finitely many nonzero weights.[5] Hence, the space $Span\{x_j\}_{j\in J}$ of linear combinations of the collection of claims $\{x_j\}_{j\in J}$ represents the space of the payoffs generated by portfolios of the claims $\{x_j\}_{j\in J}$ where the index-set J is either finite or countable. In particular, the riskfree asset and call options on $\{x_j\}_{j\in J}$ define the space of

[4] See the discussion of Artzner and Heath's paradox in Jarrow et al. (1999).

[5] Nachman (1987, 1989) allows for portfolios with infinitely many components under a mild boundness condition.

payoffs:

$$\mathcal{O}_J = Span\left\{(x_j - k)^+ : j \in J, k \in \mathbb{R}\right\}, \tag{2}$$

which is called the option space of $\{x_j\}_{j\in J}$.[6] Similarly, the riskfree asset and call options on a single claim x define the space of payoffs $\mathcal{O}_x$ by

$$\mathcal{O}_x = Span\left\{(x - k)^+ : k \in \mathbb{R}\right\}. \tag{3}$$

Options written on the collection of claims $\{x_j\}_{j\in J}$ are said to complete the space of contingent claims $L_\infty(P)$ if the space $\mathcal{O}_J$ is weak-star (w^*) dense in $L_\infty(P)$.

The reminder of this section deals with some book-keeping results we will utilize in the ensuing discussion.

Lemma 1 The space $L_\infty(P)$ is w^*-separable if and only if the state-space measure algebra Σ_P is separable.

Proof. The measure algebra Σ_P is separable if and only if the space $L_1(P)$ is separable (e.g., AB, Lemma 13.14). The Banach space $L_1(P)$ is weakly compactly generated (Fabian et al., 2001, henceforth F, Definition 11.1). Hence by the Amir and Lindenstrauss Theorem (F, Theorem 11.3), the density character of $L_1(P)$ and the weak-star density character of $L_\infty(P)$ coincide.[7]

[6] By the put-call parity relationship, the option spaces might have been equivalently defined in terms of put options or by a mixture of put and call options. Brown and Ross (1991) outlined an immediate proof of the parity relationship relying on elementary latticial properties.

[7] The density character of a Banach space X is the minimum cardinality of a dense

Therefore $L_\infty(P)$ is w^*-separable if and only if $L_1(P)$ is separable and therefore if and only if the space Σ_P is separable (AB, Lemma 13.13). ■

Linear operators between vector lattices that are onto and injective and that preserve the latticial operations are called lattice isomorphisms (AB, Definition 9.16). Such mappings are called lattice homeomorphisms when they also are topological homeomorphisms between topological spaces. Two topological vector lattices are lattice homeomorphic if there is an onto lattice homeomorphism between them. Lattice homeomorphic spaces share the same vectorial, topological, and latticial properties.

Lemma 2 The measure algebra Σ_P is separable if and only if there exists an onto lattice homeomorphisms H from $L_\infty[0,1]$ to $L_\infty(P)$ that satisfies $H\mathbf{1}_{[0,1]} = \mathbf{1}_\Omega$.

Proof. The measure algebra Σ_P is separable if and only if there exists an onto linear homeomorphism H from $L_\infty[0,1]$ to $L_\infty(P)$ that is also a lattice isomorphism and satisfies $H\mathbf{1}_{[0,1]} = \mathbf{1}_\Omega$. Only one implication needs to be proved. Assume that Σ_P is separable. Then there exists a lattice isometry Φ from the measure algebra Σ_P to the measure algebra Σ_λ on $[0,1]$ that is defined by the Lebesgue measure λ (Royden, 1988, henceforth R, Theorem 4, p. 399). The set function Φ is also surjective because P is nonatomic.[8] Thus there exists an onto lattice isometry T from $L_1(P)$ to $L_1[0,1]$ for which

subset of X. The weak-star density character of the dual Banach space is the minimum cardinality of a weak-star dense subset of X (F, Definition 11.2).

[8]In general the isomorphism Φ is not a point mapping from Ω to $[0,1]$ (R, p. 400).

$T\chi_A = \chi_{\Phi(A)}$ for each A in Σ_P (R, Exercise 7, p. 394). Denote by H the adjoint operator of T, i.e., the operator from $L_\infty[0,1]$ to $L_\infty(P)$ for which $\langle f, Hg\rangle$ equals $\langle Tf, g\rangle$, for each f in $L_1(P)$ and g in $L_\infty[0,1]$. The operator H is a surjective linear homeomorphism. Since T and its inverse are positive operators, also H and its inverse are positive operators. Thus H is also a lattice isomorphism (AB, Theorem 9.17). Because H maps the positive cone of $L_\infty(P)$ into the positive cone of $L_\infty[0,1]$, then $H\mathbf{1}_{[0,1]}$ is positive. The map Φ is measure-preserving and thus for each A in Σ_P it is

$$\langle T\chi_A, \mathbf{1}_{[0,1]}\rangle = \int_{[0,1]} \chi_{\Phi(A)}\mathbf{1}_{[0,1]} d\lambda = \lambda(\Phi(A)) = P(A).$$

Passing to the adjoint operator,

$$\langle \chi_A, H\mathbf{1}_{[0,1]}\rangle = P(A).$$

Thus

$$\langle \chi_A, \left(\mathbf{1}_\Omega - H\mathbf{1}_{[0,1]}\right)\rangle = \int_A \left(\mathbf{1}_\Omega - H\mathbf{1}_{[0,1]}\right) dP = 0,$$

which implies that $H\mathbf{1}_{[0,1]}$ is P-a.s. equal to $\mathbf{1}_\Omega$. Hence H carries $\mathbf{1}_{[0,1]}$ in $\mathbf{1}_\Omega$. By the same token, the inverse of H maps $\mathbf{1}_\Omega$ in $\mathbf{1}_{[0,1]}$. ■

In this work a claim in $L_\infty(P)$ is called a.s. injective if it is a.s. equal to an injective measurable and bounded function defined on the state-space. When the state space is a separable metric space equipped with the completion of its Borel σ-algebra, then the lattice homeomorphisms H defined in Lemma

2 maps a.s. injective claims in a.s. injective claims in $L_\infty[0,1]$.

Corollary 1 Let Ω be a complete and separable metric space equipped with its Borel σ-algebra, then the lattice homeomorphisms H maps a.s. injective claims in a.s. injective claims in $L_\infty[0,1]$.

Proof. It suffices to note that, under the hypotheses, the set mapping Φ defined in the proof of Lemma 2 is an injective point mapping from Ω onto $[0,1]$ (R, Proposition 12, p. 407). ■

In the interest of clarity, we recall that the σ-algebra $\sigma(x)$ induced by a claim x is the smallest σ-algebra with respect to which x is measurable and is defined by the counter images of the Lebesgue sets of the real line. The measure algebra obtained from $\sigma(x)$ is called the measure algebra associated with x. The next lemma shows that the measure algebra associated with an a.s. injective claim on $[0,1]$ is the Lebesgue measure algebra.

Lemma 3 The measure algebra associated with an a.s. injective claim in $L_\infty[0,1]$ coincides with the Lebesgue measure algebra Σ_λ on $[0,1]$.

Proof. Let x be an a.s. injective element of $L_\infty[0,1]$. For simplicity, we also denote with x an injective representative of the equivalent class of functions on $[0,1]$ that are a.s. equal to x. Let W be an element of the Borel σ-algebra $\mathcal{B}$ of $[0,1]$. Since x is one-to-one, then W coincides with $x^{-1}x(W)$. However $x(W)$ is an element of $\mathcal{B}_{\mathbb{R}}$ because the state-space is a Polish space (AB, Theorem 12.29). Therefore each Borel set of $[0,1]$ is the counter-image via x

of a Borel set of the real line. Since x is Lebesgue measurable, and thus also Borel measurable, this shows that $\sigma(x)$ coincides with the Borel σ-algebra of $[0,1]$ and thus, up to zero-measure sets, with the σ-algebra of the Lebesgue sets. Hence the measure algebra associated with $\sigma(x)$ coincides with Σ_λ. ■

We conclude this section with the observation that a.s. injective claims are pervasive in the space $L_\infty[0,1]$.

Lemma 4 The collection of claims that are a.s. injective are w^*-dense in $L_\infty[0,1]$.

Proof. The step functions are dense with respect to the norm of the essential supremum in $L_\infty[0,1]$ (e.g., AB, Theorem 13.8). Adding an appropriate multiple of $x(t) = t$ for t in $[0,1]$ to a given step function transforms this latter in an injective function (e.g., Galvani, 2008, proof of Lemma 2). Therefore a.s. injective claims are dense in $L_\infty[0,1]$ with respect to the norm of the essential supremum. Density with respect to this norm implies density in the w^*-topology induced by $L_1[0,1]$. Therefore injective claims are w^*-dense in $L_\infty[0,1]$. ■

3 Uniqueness

The next result indicates that only w^*-separable L_∞-spaces can be completed by options. Put differently, options do not complete a L_∞-space that is non-separable in the weak-star topology.

Lemma 5 If options on a collection of at most countably many claims $\{x_j\}_{j \in J}$ complete $L_\infty(P)$, then $L_\infty(P)$ is w^*-separable. In particular, countably many options suffice.

Proof. The option space $\mathcal{O}_J$ of $\{x_j\}_{j \in J}$ is defined in (2). Assume that $\mathcal{O}_J$ is w^*-dense in $L_\infty(P)$. Define the subset $\mathcal{O}_{J\mathbb{Q}}$ of $\mathcal{O}_J$ obtained by restricting the portfolio weights and the call options' strike price to be rational numbers. Also, as a matter of notation, denote by $\overline{A}^\infty$ the closure of a subset A of $L_\infty(P)$ with respect to the norm of the essential supremum and by $\overline{A}^*$ its closure in the weak-star topology. Since the weak-star topology is weaker than the topology generated by the norm of the essential supremum, the weak-star closure of $\overline{\mathcal{O}}_J^\infty$ coincides with the weak-star closure of the option space $\mathcal{O}_J$, which, by hypothesis, is the entire space $L_\infty(P)$. Now notice that every element of $\mathcal{O}_J$ can be uniformly approximated by elements of $\mathcal{O}_{J\mathbb{Q}}$ and thus the norm-closed sets $\overline{\mathcal{O}}_J^\infty$ and $\overline{\mathcal{O}}_{J\mathbb{Q}}^\infty$ coincide. Therefore

$$L_\infty(P) = \overline{\mathcal{O}}_J^* = \overline{\left(\overline{\mathcal{O}}_J^\infty\right)}^* = \overline{\left(\overline{\mathcal{O}}_{J\mathbb{Q}}^\infty\right)}^*.$$

Because the closure in the weak-star topology of $\overline{\mathcal{O}}_{J\mathbb{Q}}^\infty$ coincides with the closure in the weak-star topology of $\mathcal{O}_{J\mathbb{Q}}$, then

$$L_\infty(P) = \overline{\left(\mathcal{O}_{J\mathbb{Q}}\right)}^*,$$

which proves that $\mathcal{O}_{J\mathbb{Q}}$ is w^*-dense in $L_\infty(P)$. By hypothesis, the index set J in $\{x_j\}_{j\in J}$ is either finite of countable. Hence $L_\infty(P)$ contains a countable dense subset, i.e. is separable. Moreover, the collection of options in $\mathcal{O}_J$ with rational strike price suffice to span $L_\infty(P)$. ■

An argument similar to the one proving Lemma 5, shows that options fail to complete the familiar space of claims $L_\infty[0,1]$ when this space is topologized by the usual norm of the essential supremum. In fact, $L_\infty[0,1]$ is non-separable with respect to this topology (F, Proposition 1.27).

In general, spaces of contingent claims are considered indistinguishable as long as they are equipped with equivalent vectorial and topological structures. However, when considering the spanning properties of options the preservation of vectorial and topological qualities must be matched by that of space ordering, bar foregoing the intrinsic mathematical qualities of option payoffs. By inspection of the claim $(x-k)^+$, a linear mapping between spaces of contingent claims carries option payoffs in themselves as long as it preserves the pointwise supremum and the constants.

As recalled in Section 2, linear operators between vector lattices that are onto and injective and that preserve the latticial operations are called lattice isomorphisms (AB, Definition 9.16). If, in addition, these operators also are topological homeomorphisms between topological spaces then they are called lattice homeomorphisms. Two topological vector lattices are lattice homeomorphic if there is an onto lattice homeomorphism between them. Lattice homeomorphic spaces share the same vectorial, topological, and latticial properties. Hence, whenever between two spaces of contingent claims there exists a lattice homeomorphism that preserves the constants, then these spaces are equivalent from the perspective of the spanning properties of options.

The next lemma proves that all the L_∞-spaces that can be completed by options are equivalent, from the perspective of the spanning power of options, to the familiar space $L_\infty[0,1]$.

Lemma 6 The space $L_\infty[0,1]$ is the unique L_∞-space that can be completed by options with respect to the weak-star topology w^*. The uniqueness is defined modulo onto lattice homeomorphisms.

Proof. Lemma 5 indicates that we can concern ourselves only with w^*-separable L_∞-spaces. Lemmas 1 and 2 show that there exists an onto linear homeomorphism H from $L_\infty[0,1]$ to $L_\infty(P)$ that is also a lattice isomorphism and satisfies $H\mathbf{1}_{[0,1]} = \mathbf{1}_\Omega$. Thus options on a collection of claims $\{x_j\}_{j\in J}$ in $L_\infty(P)$ complete the space $L_\infty(P)$ if and only if the span:

$$Span\left\{(Hx_j - k)^+ : j \in J, k \in \mathbb{R}\right\}, \tag{4}$$

is w^*-dense in $L_\infty[0,1]$. The vector space defined in (4) is the option space of the collection of claims $\{Hx_j\}_{j\in J}$. Therefore options on a set of claims $\{x_j\}_{j\in J}$ complete $L_\infty(P)$ if and only if options on the claims $\{Hx_j\}_{j\in J}$ complete $L_\infty[0,1]$. This complete the proof. ■

Lemma 6 indicates that $L_\infty[0,1]$ is essentially the only L_∞-space for which options might obtain the allocative efficiency of a complete market structure. This uniqueness is defined up to lattice homeomorphisms that preserve the constants, i.e. the riskfree asset's payoff.

4 Spanning

The next result shows that the topological separability of the L_∞-space is equivalent to the ability of options to complete the market. As illustrated by Lemma 1, the separability of the state-space measure algebra Σ_P is equivalent to the w^*-separability of $L_\infty(P)$. Therefore the separability of the state-space turns out being a sufficient and necessary conditions for options to complete $L_\infty(P)$. This finding complements the results of previous works on the spanning power of options for infinite-dimensional spaces of contingent claims in which the separability of the state-space is, instead, directly assumed (e.g., Nachman, 1987, 1989).

Theorem 7 There exists a claim for which options complete the market if and only if the space of contingent claims $L_\infty(P)$ is w^*-separable. Moreover claims for which options complete the market form a w^*-dense subset of $L_\infty(P)$.

Proof. In view of Lemma 5, only one implication needs to be proven. Also, Lemma 6 guarantees that our analysis can be confined to $L_\infty[0,1]$ with no loss of generality. Let x be an a.s. injective claim in $L_\infty[0,1]$. For each positive integer n define the function φ_n on $[0,1]$ by:

$$\varphi_n(t) = \begin{cases} 0 & if \quad x(t) < \beta \\ (x(t) - \beta)\, n & if \quad \beta \leq x(t) \leq \beta + \frac{1}{n} \\ 1 & if \quad x(t) > \beta \end{cases}$$

where β is a scalar. For future reference, note that the payoff φ_n represents a spread in call options on the claim x and therefore it belongs to the option space $\mathcal{O}_x$.[9]. In fact:

$$\varphi_n(t) = n\left[(x(t) - \beta)^+ - \left(x(t) - \left(\beta + \frac{1}{n}\right)\right)^+\right],$$

where β is a scalar. Let f be an element of $L_1[0,1]$. Notice that the sequence $\{f\varphi_n\}_n$ converges pointwisely to $f\chi_\beta$ where χ_β is the indicator function of the set of t in $[0,1]$ for which $x(t) > \beta$. Since x is bounded, the dominate convergence theorem implies that

$$\lim_{n\to\infty} \langle f, \varphi_n \rangle = \int_0^1 \chi_\beta f d\lambda. \qquad (5)$$

Suppose there exists a function f in $L_1[0,1]$ for which $\langle f, \varphi \rangle$ is zero for each element of the option space $\mathcal{O}_x$ of x defined in (3). Then $\langle f, \varphi_n \rangle$ is zero for all n because φ_n is an element of the option space of x. Thus 5 implies that

[9] This function is similar to that utilized for the proof of Theorem 2 in Nachman (1989).

for each scalar β it is:

$$\int_0^1 \chi_\beta f d\lambda = 0.$$

Varying β, the collection of the sets of $[0,1]$ for which $x(t) > \beta$ define the σ-algebra generated by the claim x. Therefore for each set A in $\sigma(x)$ it is:

$$\int_A f d\lambda = 0.$$

By Lemma **??** the measure algebra associated with $\sigma(x)$ coincides with the Lebesgue measure algebra on $[0,1]$. Therefore f is a.s. equal to the constant-zero function. Hence each function f in $L_1[0,1]$ that satisfies the equality $\langle f, \varphi \rangle = 0$ for all claims φ in the option space $\mathcal{O}_x$, must be zero. The Hahn-Banach theorem for the weak-star topology on $L_\infty(P)$ then implies that $\mathcal{O}_x$ is w^*-dense in $L_\infty[0,1]$. Lemma 4 completes the proof. ■

Ross proved that options on an injective claim complete the Euclidean space (Ross, 1976, Theorem 4). In contrast, Theorem 7 does not require that the underlying asset for which options complete a separable L_∞-space is one-to-one. In fact, if we demand to identify underlyers for which options complete the L_∞-spaces by means of a pointwise relationship, we must allow some latitude in what is taken as to be the standard state-space structure, which until now, besides separability, has been left unconstrained. The next result shows that for a large class of securities market models options on an a.s. injective claim complete the market.

Corollary 2 Let Ω be a complete and separable metric space equipped with the completion of its Borel σ-algebra, then options on an a.s. injective claim complete the space of contingent claims $L_{\infty}(P)$. Moreover, a.s. injective claims are w^{*}-dense in $L_{\infty}(P)$.

Proof. Obvious in view of Theorem 7, Lemma 6 and Corollary 1. ■

Corollary 2 indicates that options on a.s. injective claims complete the L_{∞}-spaces, provided that the state-space is a metrizable complete space equipped with the completion of its Borel σ-algebra. Examples of such spaces of states of nature include familiar probability spaces like the Euclidean spaces and their closed and bounded intervals, e.g. $[0, 1]$, endowed with the σ-algebra of the Lebesgue measurable set. The choice of the topological qualities of the underlying state-space is easily justified by the pervasiveness in the economic literature of probability spaces defined over an Euclidean space (see Nachman, 1987, p. 342 for a discussion of this point).

It is easy to show that if the state-space is completely metrizable and separable, then any a claim in $L_{\infty}(P)$ is a.s. injective *if and only if* one of its realizations is injective on a full-measure subset of the state-space. From this perspective, Corollary 2 indicates that options written on an underlyer that differentiates all but a negligible set of states of nature complete the market. In addition such payoffs are pervasive, in the sense of being dense, in the space of contingent claims.

5 Some Remarks

Adopting a terminology from Green and Jarrow (1987) and Nachman (1987, and 1989), a payoff x is efficient with respect to a collection N of at most countable many claims whenever $\sigma(x)$ and $\sigma(N)$ coincide, where $\sigma(N)$ is the σ-algebra generated by these claims. The σ-algebra $\sigma(N)$ contains all the information that is payoff relevant for payoffs of portfolios of the collection of claims N. Therefore, an efficient asset summarizes all the relevant information for all derivatives whose payoff solely depend on these securities. An obvious modification of the proof of Lemma 3 shows that, provided that the state-space is a completely metrizable probability space equipped with the completion of its Borel σ-algebra (e.g., an Euclidean space equipped with the Lebesgue σ-algebra), an a.s. injective claim is efficient with respect to the entire space of all contingent claims.

Under the assumption that the state-space measure algebra is separable, Nachman (1989, Corollary 5) proved that options on an asset that is efficient for a collection of N securities are pointwise dense in the space of $\sigma(N)$-measurable claims. In particular, when all claims are also p-integrable, then options on an efficient asset x complete the space of p-integrable and $\sigma(N)$ measurable claims with respect to the standard L_p-norm.

Among other results, this article presents an extension of Nachman's spanning propositions to the spaces of bounded claims. It is shown that the separability of the state-space is equivalent to the ability of options to complete the markets. Furthermore, the separability of the state-space is linked to the separability of the space of contingent claims as a whole. This allows categorizing the spaces that can be completed by options without directly involving the information structure underlying the securities market model. In addition, this article proves that underlyers for which options complete the market are pervasive, in the sense of being dense, among the contingent claims.

In the finite dimensional setting, Ross proved that options on an injective claim complete the market (Ross, 1976, Theorem 4). Arditti and John (1980) and John (1984) generalized this result to countable state-space models. This work shows that options on an (a.s.) injective claim span the space of bounded claims written on completely separable metric space equipped with the completion of their Borel σ-algebra. Hence this article generalizes Ross' spanning proposition for spaces of contingent claims that are commonly encountered in the extant literature.

This article also shows that when the state-space is a complete and separable metric space equipped with its Borel σ-algebra, options on an injective underlying claim complete the market, as it is the case in the finite-dimensional case (Ross, 1976, Theorem 4). Because injective claims are dense in L_∞-space defined on such state-spaces, underlyers for which options bring about market completeness are pervasive in these spaces of contingent claims.

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