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The Prenucleolus for Games with Communication Structures*

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Abstract

It is well-known that the prenucleolus on the class of TU games is characterized by singlevaluedness, covariance under strategic equivalence, anonymity, and the reduced game property. We show that the prenucleolus on the class of TU games restricted to the connected coalitions with respect to communication structures may be characterized by the same axioms and a stronger version of independence of non-connected coalitions requiring that the solution does not depend on the worth of any non-connected coalition. Similarly as in the classical case, it turns out that each of the five axioms is logically independent of the remaining axioms and that an infinite universe of potential players is necessary. Moreover, a suitable definition and characterization of a prekernel for games with communication structures is presented.

Keywords: TU game · Solution concept · Communication and conference structure · Nucleolus

JEL Classification: C71

1 Introduction

In the classical theory of cooperative games one assumes that all players may cooperate, i.e., any coalition may form. However, a more general model for TU games is necessary in order to describe situations in which cooperation is restricted. This model requires to allow restricting the coalition function of a TU game to a set of *feasible* coalitions. E.g., Faigle (1989) has analyzed the cores of games with restricted cooperation in general. Moreover, in many situations there is a structural restriction on cooperation. E.g., the cooperation may be restricted by some social, economical, hierarchical, or some biological structure. In the present paper we adopt the model of Myerson (1977) who introduces TU games with communication structures. A *communication structure* on a finite set N, a graph with vertex set N, only allows two players to communicate if they are linked by an edge of the graph. Hence, it is assumed that only members of connected coalitions are able to sign binding agreements via a series of agreements of the linked players in the coalition. As the worth of a non-connected coalition may not be realized,

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Myerson replaced it by the sum of the worths of the connected components of this coalition (see (6.1) for the precise definition of the corresponding coalition function) in order to define the Shapley value of the arising so-called "Myerson restricted game" as a solution for the game with communication structure. Several other "value-related" solution concepts for this class of games have been introduced and analyzed (see, e.g., Herings, van der Laan, Talman, and Yang (2010)). In this paper we investigate two famous "core-related" solution concepts, namely the prenucleolus and the prekernel.

In order to generalize the aforementioned core-related solution concepts to a game with communication structure, it is not necessary to extend its coalition function to non-connected coalitions, e.g., by considering the Myerson restricted game. In fact, just the classical definition of the prenucleolus and the prekernel may be directly applied to the coalition function that is restricted to the connected coalitions.

It is shown that a suitable modification of Sobolev's (1975) famous classical characterization of the prenucleolus is still valid for games with communication structures. Only one additional axiom has to be added that we call *independence of non-connected coalitions* (INC). It requires not only that the solutions for two games with coinciding coalition structures and coinciding worths of connected coalitions coincide. In fact INC requires that the solution to a game with communication structure coincides with the solution to the game that allows internal unrestricted communication in each of the components provided that each of the former non-connected coalitions receives a sufficiently small worth.

Moreover, suitable versions of the determining axioms in Peleg's (1986) characterization allow to axiomatize the prekernel for games with coalition structures, even without INC.

Both solution concepts may easily be generalized to TU games with conference structures as introduced by Myerson (1980).

The paper is organized as follows. In Section 2 we recall the basic definitions of the general nucleolus, TU games with coalition and communication structures, and related concepts, and propose our definition of the prenucleolus of a TU game with communication structure that is entirely based on the possible cooperation inside the connected coalitions. Indeed, we consider the prenucleolus of a game restricted to the connected coalitions (cf. Katsev and Yanovskaya (2010) who investigated the prenucleolus for games restricted to systems of coalitions that contain the grand coalition). In Section 3 we show that the prenucleolus may be characterized by properties that are similar to those of Kohlberg (1971) for the (pre)nucleolus of classical games. Section 4 is devoted to the axiomatization of the prenucleolus for games with communication structures that is similar to Sobolev's (1975) axiomatization in the classical case. Only one additional new property, the aforementioned axiom INC, has to be employed in addition. Section 5 shows by means of examples that each of the axioms is logically independent of the remaining axioms. Peleg (1986) defines and axiomatizes the prekernel for games with coalition structures. In a completely analogous way we define and axiomatize the prekernel for games with communication structures in Section 6. Finally, in Section 7 it turns out that both concepts, the prenucleolus and the prekernel, for games with communication structures may easily be extended to games with conference structures.

2 Notation, Definitions, and Preliminaries

Let U, $|U| \geq 3$, be a set, the universe of players, containing, without loss of generality, $1, \ldots, k$ whenever $|U| \geq k$. A *coalition* is a finite nonempty subset of U. Let N be a coalition, $X \subseteq \mathbb{R}^N$, let D be a finite nonempty set, let $h : X \to \mathbb{R}^D$, and $d := |D|$. Define $\theta : X \to \mathbb{R}^d$ by

$$\theta_t(x) = \max_{T \subseteq D, |T|=t} \min_{i \in T} h_i(x) \text{ for all } x \in X \text{ and all } t = 1, \ldots, d,$$

that is, for any $x \in X$, $\theta(x)$ is the vector, whose components are the numbers $h_i(x)$, $i \in D$, arranged in non-increasing order. Let $\geq_{lex}$ denote the lexicographical order of $\mathbb{R}^d$. The *nucleolus* of h with respect to (w.r.t.) X, $\mathcal{N}(h, X)$, is defined by

$$\mathcal{N}(h, X) = \{x \in X \mid \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in X\}.$$

Remark 2.1 Justman (1977) proved the following statements.

(1) If X is nonempty and compact and if all $h_i, i \in D$, are continuous, then $\mathcal{N}(h, X) \neq \emptyset$.

(2) If X is convex and all $h_i, i \in D$, are convex, then $\mathcal{N}(h, X)$ is convex and $h_i(x) = h_i(y)$ for all $i \in D$ and all $x, y \in \mathcal{N}(h, X)$.

A (cooperative TU) *game* is a pair (N, v) such that N is a coalition and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Let N be a coalition. A *coalition structure* for N is a partition of N. A *game with coalition structure* (see Aumann and Drèze (1974)) is a triple $(N, v, \mathcal{R})$ such that (N, v) is a game and $\mathcal{R}$ is a coalition structure for N. We identify a game (N, v) with the game with coalition structure $(N, v, \{N\})$. The subgame on a coalition $\emptyset \neq S \subseteq N$ is denoted by (S, v). For any game with coalition structure $(N, v, \mathcal{R})$ let

$$\begin{aligned} X^*(N, v, \mathcal{R}) &= \{x \in \mathbb{R}^N \mid x(R) \leq v(R) \text{ for all } R \in \mathcal{R}\} \text{ and} \\ X(N, v, \mathcal{R}) &= \{x \in \mathbb{R}^N \mid x(R) = v(R) \text{ for all } R \in \mathcal{R}\} \end{aligned}$$

denote the set of *feasible* and *Pareto efficient* feasible payoffs (*preimputations*), respectively. We use $x(S) = \sum_{i \in S} x_i$ ($x(\emptyset) = 0$) for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$, and we write $x = (x_S, x_{N \setminus S})$. For $x \in \mathbb{R}^N$ and $S \subseteq N$ let $e(S, x, v) = v(S) - x(S)$ denote the *excess* of S at x w.r.t. (N, v). Let $\mathcal{S} \subseteq 2^N$ such that $\{i\} \in \mathcal{S}$ for all $i \in N$. The *nucleolus* of the game (N, v) w.r.t. $X \subseteq \mathbb{R}^N$ and $\mathcal{S}$, denoted by $\mathcal{N}(N, v, X, \mathcal{S})$, is the set $\mathcal{N}(h, X)$ where $h = (e(S, \cdot, v))_{S \in \mathcal{S}}$.

By Remark 2.1, $\mathcal{N}(N, v, X, \mathcal{S})$ is a singleton whenever X is nonempty, compact, and convex. The *prenucleolus* of a game with coalition structure $(N, v, \mathcal{R})$ w.r.t. $\mathcal{S}$, denoted by $\mathcal{N}(N, v, \mathcal{R}, \mathcal{S})$, is the set $\mathcal{N}(N, v, X^*(N, v, \mathcal{R}), \mathcal{S})$. Now, let $z \in X^*(N, v, \mathcal{R}), \mu = \max_{S \in \mathcal{S}} e(S, z, v)$, and $X = \{x \in X^*(N, v, \mathcal{R}) \mid e(S, x, v) \leq \mu \text{ for all } S \in \mathcal{S}\}$. Then X is nonempty, compact, and convex so that $\mathcal{N}(N, v, X, \mathcal{S})$ is a singleton. Clearly, $\mathcal{N}(N, v, X, \mathcal{S}) = \mathcal{N}(N, v, X^*(N, v, \mathcal{R}), \mathcal{S})$ so that this set is a singleton whose unique element, denoted by $\nu(N, v, \mathcal{R}, \mathcal{S})$, is the *prenucleolus* (point) of $(N, v, \mathcal{R})$ w.r.t. $\mathcal{S}$.

A *graph* is a pair (N, g), where N is a coalition, called the set of *vertices*, and g is a set of 2-element subsets of N. An element of g is called *link*. A *game with communication structure* is a triple (N, v, g) such that (N, v) is a game and (N, g) is a graph. Let (N, v, g) be a game with communication structure, let $\emptyset \neq S \subseteq N$, and $i, j \in S$. The vertices i and j are *connected in* S *by* g if there exists a path in S that connects i and j, that is, if there exist $\ell \in \mathbb{N}$ and $k_1, \ldots, k_\ell \in S$ such that $i = k_1$, $j = k_\ell$, and $\{k_t, k_{t+1}\} \in g$ for all $t \in \mathbb{N}$ with $1 \leq t < \ell$. Let S/g denote the set of components of S w.r.t. g, that is,

$$S/g = \{\{i \in S \mid i \text{ and } j \text{ are connected in } S \text{ by } g\} \mid j \in S\}.$$

We say that S is *connected by* g if $|S/g| = 1$. Moreover, let $\mathcal{S}_{N,g}$ denote the set of all coalitions in N that are connected by g, that is,

$$\mathcal{S}_{N,g} = \{S \in 2^N \setminus \{\emptyset\} \mid S \text{ is connected by } g\}. \tag{2.1}$$

Games with communication structures and the related definitions of the foregoing paragraph are due to Myerson (1977). Now we are ready to define the prenucleolus of a game with communication structure.

Definition 2.2 *Let* (N, v, g) *be a game with communication structure. The* prenucleolus *of* (N, v, g)*, denoted by* $\mathcal{N}(N, v, g)$*, is the set* $\mathcal{N}(N, v, X^*(N, v, N/g), \mathcal{S}_{N,g})$*. The unique element of* $\mathcal{N}(N, v, g)$ *is denoted by* $\nu(N, v, g)$ *and is called* prenucleolus *(point) of* (N, v, g)*.*

Note that for classical games Schmeidler (1969) introduced the *nucleolus*, the individually rational modification of the prenucleolus.

Let (N, v, g) be a game with communication structure, $\emptyset \neq S \subseteq N$, and $x \in \mathbb{R}^N$. The *reduced graph* w.r.t S (see Albizuri and Zarzuelo (2009)) is the graph (S, g^S) defined by

$$g^S = \{\{i, j\} \subseteq S \mid i \neq j \text{ and } i \text{ and } j \text{ are connected in } \{i, j\} \cup N \setminus S \text{ by } g\}. \tag{2.2}$$

Hence, two players of S are linked in g^S if they are either linked already in g or if there is a path in g via players outside S connecting them.

Remark 2.3 Let (N, v, g) be a game with communication structure, $x = \nu(N, v, g)$, and $R \in N/g$. Then $\nu(R, v, g^R) = x_R$.

The *reduced game with communication structure* w.r.t. S and x is the game with communication structure $\left(S, v_g^{S,x}, g^S\right)$ whose coalition function, for $\emptyset \neq T \subseteq S$, is defined by

$$v_g^{S,x}(T) = \begin{cases} v(R) - x(R \setminus T) & \text{, if } T = S \cap R \text{ for some } R \in N/g, \\ \max_{Q \subseteq N \setminus S, T \cup Q \in \mathcal{S}_{N,g}} v(T \cup Q) - x(Q) & \text{, if } T \in \mathcal{S}_{S,g^S}, T \notin S/g^S, \\ \max_{Q \subseteq N \setminus S} v(T \cup Q) - x(Q) & \text{, otherwise.} \end{cases} \tag{2.3}$$

The definition of the reduced game with communication structure is similar to the definition of the Davis and Maschler (1965) reduced game with coalition structure as given by Peleg and Sudhölter (2007, Definition 3.8.8): If $\mathcal{R}$ is a coalition structure for N, then $\mathcal{R}^S = \{R \cap S \mid R \cap S \neq \emptyset, R \in \mathcal{R}\}$ and the coalition function of the reduced game $\left(N, v_{\mathcal{R}}^{S,x}, \mathcal{R}^S\right)$ is defined, for $\emptyset \neq T \subseteq N$, by

$$v_{\mathcal{R}}^{S,x}(T) = \begin{cases} v(R) - x(R \setminus T) & \text{, if } T = S \cap R \text{ for some } R \in \mathcal{R}, \\ \max_{Q \subseteq N \setminus S} v(T \cup Q) - x(Q) & \text{, otherwise.} \end{cases} \tag{2.4}$$

In order to interpret (2.3), we compare it with (2.4) for $\mathcal{R} = N/g$. In a game with coalition structure each of the coalitions in $\mathcal{R}$ may distribute its worth among its members. However, cooperation in any other coalition is still possible. Now, in the reduced game the players of S play their reduced game assuming that the players in $N \setminus S$ are ready to cooperate. In a game with communication structure only the members of a connected coalition may cooperate so that the worth of a non-connected coalitions may be regarded as *virtual* only. Hence, in the reduced game the worth of any disconnected coalition is still virtual (there is no need to change the definition), whereas the members of any coalition T that may be connected with the help of players in $Q \subseteq N \setminus S$ (i.e., $T \cup Q$ is connected by g) may in fact cooperate with the members of Q, thereby receiving $v(T \cup Q)$. This difference in the interpretations is reflected by the difference of the definitions of the reduced games.

3 Kohlberg's Characterization

This section is devoted to present a suitable modification of Kohlberg's (1971) characterization of the prenucleolus of games with coalition structures by balanced collections of coalitions.

First, we recall his characterization: Let $(N, v, \mathcal{R})$ be a game with coalition structure. Denote by $\nu(N, v, \mathcal{R})$ the prenucleolus of this game, i.e., the unique element of $\mathcal{N}(N, v, X^*(N, v, \mathcal{R}), 2^N)$. For every $x \in \mathbb{R}^N$ and any $\alpha \in \mathbb{R}$ denote

$$\mathcal{D}(\alpha, x, v) = \{S \in 2^N \mid e(S, x, v) \geq \alpha\}.$$

Moreover, $\mathcal{B} \subseteq 2^N$ is called *balanced* (over N) if there are $\delta^S > 0, S \in \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \delta^S \chi^S = \chi^N$, where for any $T \subseteq N$, χ^T is the indicator vector of T, i.e., $\chi^T \in \mathbb{R}^N$ is defined by $\chi_i^T = 1$ if $i \in T$ and $\chi_i^T = 0$ if $i \in N \setminus T$.

Let $x \in X(N, v, \mathcal{R})$. Then the following statements are equivalent:

$$x = \nu(N, v, \mathcal{R}). \tag{3.1}$$

$$\alpha \in \mathbb{R}, y \in \mathbb{R}^N, y(R) = 0 \forall R \in \mathcal{R}, y(S) \geq 0 \forall S \in \mathcal{D}(\alpha, x, v) \Rightarrow y(S) = 0 \forall S \in \mathcal{D}(\alpha, x, v). \tag{3.2}$$

$$\alpha \in \mathbb{R} \Rightarrow \mathcal{D}(\alpha, x, v) \cup \mathcal{R} \text{ is balanced.} \tag{3.3}$$

This characterization is an immediate generalization of Kohlberg's characterization of the nucleolus. For an explicit proof see Peleg and Sudhölter (2007, Theorems 5.2.6 and 6.4.1).

Now, the foregoing characterization of the prenucleolus of a game with coalition structure is modified. Let (N, v, g) be a game with communication structure. For every $x \in \mathbb{R}^N$ and any $\alpha \in \mathbb{R}$ denote

$$\mathcal{D}(\alpha, x, v, g) = \{S \in \mathcal{S}_{N,g} \mid e(S, x, v) \geq \alpha\}.$$

Proposition 3.1 *Let (N,v,g) be a game with communication structure and $x \in X(N,v,N/g)$. Then the following statements are equivalent:*

$$x = \nu(N,v,g). \tag{3.4}$$

$$\alpha \in \mathbb{R}, y \in \mathbb{R}^N, y(R) = 0 \forall R \in N/g, y(S) \geq 0 \forall S \in \mathcal{D}(\alpha,x,v,g) \Rightarrow y(S) = 0 \forall S \in \mathcal{D}(\alpha,x,v,g). \tag{3.5}$$

$$\alpha \in \mathbb{R} \Rightarrow \mathcal{D}(\alpha,x,v,g) \cup N/g \textit{ is balanced.} \tag{3.6}$$

The proof of Proposition 3.1 is similar to the proof of the equivalence of the statements (3.1) - (3.3) and, hence, it is omitted.

In view of Remark 2.3, (3.5) and (3.6) may be reformulated as follows. For any $R \in N/g$,

$$\alpha \in \mathbb{R}, y \in \mathbb{R}^R, y(R) = 0, y(S) \geq 0 \forall S \in \mathcal{D}(\alpha,x_R,v,g^R) \Rightarrow y(S) = 0 \forall S \in \mathcal{D}(\alpha,x_R,v,g^R). \tag{3.7}$$

$$\alpha \in \mathbb{R} \Rightarrow \mathcal{D}(\alpha,x_R,v,g^R) \cup \{R\} \text{ is balanced over } R. \tag{3.8}$$

Here the function x in $\mathcal{D}(\alpha,x_R,v,g^R)$ is the coalition function of the subgame (R,v) of (N,v).

Let Δ_U^{cmm} and Δ_U^{clt} be the sets of games with communication structures and of games with coalition structures, respectively, and let N be a coalition. To any coalition structure $\mathcal{R}$ for N we may assign its associated graph $g = g(\mathcal{R})$ defined by $g = \{\{i,j\} \mid i,j \in R, i \neq j, \text{ for some } R \in \mathcal{R}\}$. Thus, to any game with coalition structure, $(N,v,\mathcal{R})$, we may assign its associated game with communication structure, $(N,v,g(\mathcal{R}))$. We conclude that

$$\Delta_U^{\text{clt}} \hookrightarrow \Delta_U^{\text{cmm}} \text{ defined by } (N,v,\mathcal{R}) \mapsto (N,v,g(\mathcal{R}))$$

is an embedding and we write "$\Delta_U^{\text{clt}} \subseteq \Delta_U^{\text{cmm}}$".

The following example shows that there is $(N,v,\mathcal{R}) \in \Delta_U^{\text{clt}}$ such that $\nu(N,v,\mathcal{R})$ does not coincide with $\nu(N,v,g(\mathcal{R}))$.

Example 3.2 Let $N = \{1,2,3\}, \mathcal{R} = \{\{1,2\},\{3\}\}$, and (N,v) be defined by $v(\{1,3\}) = 2$ and $v(S) = 0$ for all $S \in 2^N \setminus \{\{1,3\}\}$. By applying (3.1) - (3.3) and (3.4) - (3.6), respectively, it is straightforward to verify that

$$\nu(N,v,g(\mathcal{R})) = (0,0,0) \neq (1,-1,0) = \nu(N,v,\mathcal{R}).$$

Remark 3.3 Nevertheless, there is the following relation between the prenucleoli of games with communication structures and games with coalition structures. Let $(N,v,\mathcal{R}) \in \Delta_U^{\text{clt}}$. Define (N,w) by

$$w(S) = \sum_{R \in \mathcal{R}} v(S \cap R) \text{ for all } S \subseteq N. \tag{3.9}$$

Then

$$\nu(N,w,\mathcal{R}) = \nu(N,v,g(\mathcal{R})). \tag{3.10}$$

In order to show (3.10), we apply the well-known *reduced game property* for the prenucleolus of games with coalition structures:

$$(N,v,\mathcal{R}) \in \Delta_U^{\text{clt}}, x = \nu(N,v,\mathcal{R}), \emptyset \neq S \subseteq N \Rightarrow x_S = \nu(S, v_{\mathcal{R}}^{S,x}, \mathcal{R}^S). \tag{3.11}$$

Indeed, let $R \in \mathcal{R}$ and $x = \nu(N, w, \mathcal{R})$. By (3.11), $x_S = \nu(R, w^{R,x}_{\mathcal{R}})$. Now, $w^{R,x}_{\mathcal{R}}(R) = v(R)$ and, for any $\emptyset \neq S \subsetneqq R$, $w^{R,x}_{\mathcal{R}}(S) = v(S) + c$, where $c = \max_{Q \subseteq N \setminus R} w(Q) - x(Q)$. Hence, for any $\alpha \in \mathbb{R}$, $\mathcal{D}(\alpha + c, x_R, w^{R,x}_{\mathcal{R}}) \cup \{\emptyset, R\} = \mathcal{D}(\alpha, x_R, v) \cup \{\emptyset, R\}$ so that the proof is finished by (3.3) and Remark 2.3.

4 Axiomatization of the Prenucleolus for Games with Communication Structures

Let $\Delta \subseteq \Delta^{\text{cmm}}_U$. A *solution* on Δ is a function σ which associates with any $(N, v, g) \in \Delta$ a subset $\sigma(N, v, g)$ of $X^*(N, v, N/g)$. A solution σ on Δ satisfies:

(1) *Efficiency* (EFF) if

$$(N, v, g) \in \Delta, x \in \sigma(N, v, g) \Rightarrow x(R) = v(R)\ \forall R \in N/g.$$

(2) *Covariance under strategic equivalence* (COV) if

$$(N, v, g), (N, w, g) \in \Delta, \alpha > 0, \beta \in \mathbb{R}^N, w = \alpha v + \beta \Rightarrow \sigma(N, w, g) = \alpha\sigma(N, v, g) + \beta.$$

(3) *Anonymity* (AN) if

$$(N, v, g) \in \Delta, \pi : N \to U \text{ is injective }, (\pi(N), \pi v, \pi(g)) \in \Delta \Rightarrow \sigma(\pi(N), \pi v, \pi g) = \pi(\sigma(N, v, g)),$$

where $(\pi v)(\pi(S)) = v(S)\ \forall S \subseteq N, \pi g = \{\{\pi(i), \pi(j)\} \mid \{i, j\} \in g\}$, and $\pi(x) = y \in \mathbb{R}^{\pi(N)}$ is defined by $y_{\pi(i)} = x_i \forall x \in \mathbb{R}^N, \forall i \in N$.

(4) *Singlevaluedness* (SIVA) if $|\sigma(N, v, g)| = 1\ \forall (N, v, g) \in \Delta$.

(5) The *reduced game property* (RGP^{cmm}) if

$$(N, v, g) \in \Delta, x \in \sigma(N, v, g), \emptyset \neq S \subseteq N \Rightarrow (S, v^{S,x}_g, g^S) \in \Delta, x_S \in \sigma(S, v^{S,x}_g, g^S).$$

We recall that a solution σ on a set $\Delta \subseteq \Delta^{\text{clt}}_U$ satisfies the *reduced game property in the sense of Davis and Maschler* (RGP^{clt}) if it satisfies the property that differs from RGP^{cmm} only inasmuch as g is replaced by $\mathcal{R}$ in the displayed part of (5) wherever it occurs (cf. (3.11)).

Remark 4.1 The prenucleolus on Δ^{clt}_U is the unique solution that satisfies COV, AN, SIVA, and RGP^{clt}, provided that $|U| = \infty$. Sobolev (1975) proved this famous result for the restricted set of games "without coalition structures", that is, for $\Gamma_U = \{(N, v, \mathcal{R}) \in \Delta^{\text{clt}}_U \mid \mathcal{R} = \{N\}\}$, but his proof may easily be extended to Δ^{clt}_U (see Peleg and Sudhölter (2007)).

Lemma 4.2 *The prenucleolus on* Δ^{cmm}_U *satisfies* COV, AN, SIVA, *and* RGP^{cmm}.

Proof: COV, AN, and SIVA are immediate. In order to show $\mathrm{RGP}^{\mathrm{cmm}}$, let $(N,v,g) \in \Delta_U^{\mathrm{cmm}}$, $\emptyset \neq S \subseteq N$, $x = \nu(N,v,g)$, $w = v_g^{S,x}$, $R' \in S/g^S$, $\alpha \in \mathbb{R}$, and $y_{R'} \in \mathbb{R}^{R'}$ such that $y(R') = 0$ and $y(T) \geq 0$ for all $T \in \mathcal{D}(\alpha, x_{R'}, w, g^{R'})$. Let $R \in N/g$ such that $R' = R \cap S$. Then

$$\{T \cap R' \mid T \in \mathcal{D}(\alpha, x_R, v, g^R), \emptyset \neq T \cap R \neq R'\} = \mathcal{D}(\alpha, x_{R'}, w, g^{R'}) \setminus \{\emptyset, R'\}. \tag{4.1}$$

Let $y_R = (y_{R'}, 0_{R \setminus R'})$. Then $y_R \in \mathbb{R}^R, y(R) = 0$, and, by (4.1), $y(T) \geq 0$ for all $T \in \mathcal{D}(\alpha, x_R, v, g^R)$. By (3.5) of Proposition 3.1, $y(T) = 0$ for all $T \in \mathcal{D}(\alpha, x_R, v, g^R)$. Therefore $y(T) = 0$ for all $T \in \mathcal{D}(\alpha, x_{R'}, w, g^{R'})$ and Proposition 3.1 completes the proof. **q.e.d.**

Lemma 4.3 *If σ is a solution on Δ_U^{cmm} that satisfies* COV, SIVA, *and* $\mathrm{RGP}^{\mathrm{cmm}}$*, then σ satisfies* EFF.

The proof of Lemma 4.3 is similar as in the "classical" case (where Δ_U^{cmm} is replaced by Δ_U^{clt} and $\mathrm{RGP}^{\mathrm{cmm}}$ is replaced by $\mathrm{RGP}^{\mathrm{clt}}$, see, e.g., Peleg and Sudhölter (2007, Lemma 6.2.11)) and, hence, it is skipped.

In order to characterize the prenucleolus, one further property of a solution is needed. For $(N,v,g) \in \Delta_U^{\mathrm{cmm}}$ and $\beta \in \mathbb{R}$ let $\Delta_{N,v,g}^{\beta}$ denote the set of all $(N, w, g(N/g)) \in \Delta_U^{\mathrm{cmm}}$ that satisfy, for all $\emptyset \neq S \subseteq N$,

$$w(S) = v(S) \text{ for all } S \in \mathcal{S}_{N,g} \text{ and } w(S) \leq \beta, \text{ otherwise.}$$

Definition 4.4 *A solution σ on Δ_U^{cmm} satisfies* independence of non-connected coalitions *(*INC*) if for any $(N,v,g) \in \Delta_U^{\mathrm{cmm}}$ there exists $\beta \in \mathbb{R}$ such that $\sigma(N,v,g) = \sigma(N,w,g(N/g))$ for all $(N,w,g(N/g)) \in \Delta_{N,v,g}^{\beta}$.*

INC requires that x is a member of the solution to a game with communication structure if and only if x is a member of the solution to the game that allows unrestricted cooperation within all components but sufficiently punishes those coalitions that have not been connected before. In particular, if two games with communication structures (N,v,g) and (N,v',g) coincide on all connected coalitions, then $\sigma(N,v,g) = \sigma(N,v',g)$ for any solution σ that satisfies INC. This fact may explain the name of the property.

Lemma 4.5 *The prenucleolus on Δ_U^{cmm} satisfies* INC.

Proof: Let $(N,v,g) \in \Delta_U^{\mathrm{cmm}}$, $x = \nu(N,v,g)$,

$$\beta < \min_{S \in \mathcal{S}_{N,g}} e(S,x,v) + \min_{S \subseteq N} x(S), \tag{4.2}$$

and (N,w) satisfy $w(S) = v(S)$ for all coalitions that are connected by g and $w(S) \leq \beta$ for all non-connected coalitions. Let

$$\gamma = \min_{S \in \mathcal{S}_{N,g}} e(S,x,v). \tag{4.3}$$

Then $e(S,x,w) \geq \gamma$ for all $S \in \mathcal{S}_{N,g} \cup \{\emptyset\}$ and $e(T,x,w) < \gamma$ for all other $T \in 2^N$. Hence, for any $\alpha \geq \gamma$, $\mathcal{D}(\alpha,x,w) = \mathcal{D}(\alpha,x,v,g)$ so that, by (3.6), $\mathcal{D}(\alpha,x,w) \cup N/g$ is balanced. Moreover, all singletons are connected by definition so that, for any $T \subseteq N$, χ_T is in the linear span of $\{\chi_S \mid S \in \mathcal{D}(\gamma, x, w)\}$. It is straightforward to show (see, e.g., Peleg and Sudhölter (2007, Lemma 6.1.2)) that $\{T\} \cup \mathcal{D}(\gamma, x, w)$ is balanced. We conclude that $\mathcal{D}(\alpha, x, w) \cup N/g$ is balanced for any $\alpha \in \mathbb{R}$ so that the proof is finished by (3.3). **q.e.d.**

Now we are able to prove the main result of this section.

Theorem 4.6 *Let $|U|$ be infinite. Then there is a unique solution σ on Δ_U^{cmm} that satisfies* COV, AN, SIVA, RGP$^{\text{cmm}}$*, and* INC*, and it is the prenucleolus.*

Proof: By Lemma 4.2 and Lemma 4.5 the prenucleolus satisfies the desired properties. Thus, it remains to prove the uniqueness part. Let σ be a solution on Δ_U^{cmm} that satisfies COV, AN, SIVA, RGP$^{\text{cmm}}$, and INC, let $(N, v, g) \in \Delta_U^{\text{cmm}}$ and $x = \nu(N, v, g)$. It remains to prove that $\sigma(N, v, g) = \{x\}$. By COV, we may assume that $x = 0 \in \mathbb{R}^N$. By INC, we may assume that g is component-complete, i.e., $g = g(N/g)$. By RGP$^{\text{cmm}}$ we may assume that $|N/g| = 1$. Let $\mathcal{R} = N/g$ and (N, w) be defined by (3.9). As $N/g = \{N\}$, $w = v$. By Remark 2.3, $\nu(N, v, g) = \nu(N, v, \mathcal{R}) = \nu(N, w)$. Now, according to Sobolev (1975) there is a game (M, u) with the following properties (see Peleg and Sudhölter (2007, Sections 6.3) for a detailed proof in English):

(1) (M, u) is transitive, i.e., the symmetry group of (M, u) is transitive.

(2) $N \subseteq M, u(M) = 0$.

(3) With $y = 0 \in \mathbb{R}^M$, $u^{N,y} = u^{N,y}_{\{M\}} = v$.

By SIVA, $\sigma(M, u, g(\{M\})) = z$ for some $z \in \mathbb{R}^M$. By AN and the transitivity of the symmetry group of $(M, u, g(\{M\}))$, $z_i = z_j$ for all $i, j \in M$. By Lemma 4.3, $z(M) = u(M) = 0$, hence $z = y$. As $g(\{M\})^N = g(\{N\})$ is complete, RGP$^{\text{cmm}}$ and SIVA yield $\{y_N\} = \sigma(N, v, g)$. **q.e.d.**

5 Logical Independence of the Axioms

This section serves to show that each of the axioms in Theorem 4.6 is logically independent of the remaining axioms and that this theorem is no longer valid if the infinity assumption on the cardinality of U is deleted. For each $k = 1, \ldots, 5$, we construct a solution σ^k that exclusively violates the k-th axiom.

The "equal split solution" σ^1 that assigns to each $(N, v, g) \in \Delta_U^{\text{cmm}}$ the unique element $x \in \mathbb{R}^N$ defined by $x_i = \frac{v(R)}{|R|}$ for all $i \in R \in N/g$ satisfies AN, SIVA, RGP$^{\text{cmm}}$, and INC, but does not coincide with the prenucleolus, hence violates COV.

We now construct an example of a solution that satisfies SIVA, COV, RGP$^{\text{cmm}}$, and INC, but violates AN. For this purpose we use the notation $t_+ = \max\{0, t\}$ for any $t \in \mathbb{R}$ and define, for any $(N, v, g) \in \Delta_U^{\text{cmm}}$ (see Section 2 for the definition of the general nucleolus),

$$\mathcal{C}_+(N, v, g) = \mathcal{N}(((e(S, \cdot, v)_+)_{S \in \mathcal{S}_{N,g}}, X^*(N, v, N/g)). \tag{5.1}$$

That is, with $x = \nu(N,v,g)$ we have $y \in \mathcal{C}_+(N,v,g)$ iff $y \in \mathbb{R}^N$ and $e(S,y,v)_+ = e(S,x,v)_+$ for all coalitions S that are connected by g. Applied to a game (N,v), this solution, i.e., $\mathcal{C}_+(N,v) = \mathcal{C}_+(N,v,g(\{N\}))$, is called the *positive core* (see Orshan and Sudhölter (2010)) of (N,v). As singletons are connected, $\mathcal{C}_+(N,v,g)$ is a nonempty compact convex polyhedral set. Select any total order $\succeq$ of U and define

$$\sigma^2(N,v,g) = \{x \in \mathcal{C}_+(N,v,g) \mid x \succeq_{lex} y \text{ for all } y \in \mathcal{C}_+(N,v,g)\},$$

where $\succeq_{lex}$ is the lexicographic order on $\mathbb{R}^N$ induced by $\succeq$, i.e., if $x,y \in \mathbb{R}^N$, then $x \succeq_{lex} y$ is defined by

$$i \in N, y_i > x_i \Rightarrow \text{ There exists } j \in N \text{ with } x_j > y_i \text{ and } j \succeq i.$$

Clearly σ^2 satisfies COV, and SIVA, and it is straightforward to verify INC. The proof of RGP$^{\text{cmm}}$ is a straightforward generalization of the proof in the classical case (see, e.g., Peleg and Sudhölter (2007, Lemma 6.3.15)), and hence, it is skipped.

The positive core $\sigma^3 = \mathcal{C}_+$ satisfies all axioms except SIVA.

In order to give an example of a solution σ^4 that satisfies COV, AN, SIVA, and INC, but violates RGP$^{\text{cmm}}$, we generalize a well-known solution concept for TU games (Driessen and Funaki (1991) called it the *center of the imputation set*) to TU games with communication structures. For $(N,v,g) \in \Delta_U^{\text{cmm}}$ let $\sigma^4(N,v,g) = \{x\}$ be defined by the requirement that $x \in \mathbb{R}^N$ is given by

$$x_i = v(\{i\}) + \frac{v(R) - \sum_{j \in R} v(\{j\})}{|R|} \text{ for all } i \in R \text{ and all } R \in N/g.$$

Clearly, σ^4 satisfies COV, AN and SIVA. As $w(\{i\}) = v(\{i\})$ and $w(R) = v(R)$ for all $i \in R \in N/g$ for all $(N,w,N,g(N/g)) \in \Delta^{\beta}_{N,v,g}$, it also satisfies and INC. Three-person examples show that $\sigma^4(N,v,g)$ may not coincide with $\nu(N,v,g)$ so that σ^4 violates RGP$^{\text{cmm}}$.

We now construct an example of a solution σ^5 that satisfies SIVA, COV, AN, and RGP$^{\text{cmm}}$, but violates INC. For this purpose we define, for any $(N,v,g) \in \Delta_U^{\text{cmm}}$

$$\sigma^0(N,v,g) = \mathcal{N}((e(S,\cdot,v))_{S \in 2^N \setminus \mathcal{S}_{N,g}}, \mathcal{C}_+(N,v,g)). \tag{5.2}$$

Note that $\sigma^0(N,v,g)$ is a nonempty compact convex polyhedral set. Now, we are ready to define our solution by

$$\sigma^5(N,v,g) = \mathcal{N}((e(S,\cdot,v)_{S \in 2^N}, \sigma^0(N,v,g)). \tag{5.3}$$

By Remark 2.1, σ^5 satisfies SIVA and it is straightforward to show that σ^5 satisfies AN and COV. The following lemma is useful to show RGP$^{\text{cmm}}$.

Lemma 5.1 *Let $(N,v,g) \in \Delta_U^{\text{cmm}}$ and $x \in X(N,v,N/g)$. Then $\{x\} = \sigma^5(N,v,g)$ if and only if the following conditions are satisfied:*

(1) If $\alpha > 0$ and $y \in \mathbb{R}^N$ satisfies $y(R) = 0$ for all $R \in N/g$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha,x,v,g)$, then $y(S) = 0$ for all $S \in \mathcal{D}(\alpha,x,v,g)$.

(2) If $\alpha \in \mathbb{R}$ and $y \in \mathbb{R}^N$ satisfies $y(R) = 0$ for all $R \in N/g$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(0, x, v, g) \cup (\mathcal{D}(\alpha, x, v) \setminus \mathcal{S}_{N,g})$, then $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v) \setminus \mathcal{S}_{N,g}$.

(3) If $\alpha < 0$ and $y \in \mathbb{R}^N$ satisfies $y(R) = 0$ for all $R \in N/g$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, then $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$.

Proof: Necessity. Assume that $\{x\} = \sigma^5(N, v, g)$. In order to show any of the conditions (1), (2), or (3), let α and y have the requested properties, define $z^\varepsilon = x + \varepsilon y$, observe that

$$z^\varepsilon \in X(N, v, N/g) \text{ and } e(S, z^\varepsilon, v) = e(S, x, v) - \varepsilon y(S) \text{ for all } S \subseteq N, \tag{5.4}$$

and choose $\varepsilon > 0$ small enough so that $e(S, z^\varepsilon, v) > e(T, z^\varepsilon, v)$

(1) for all $S \in \mathcal{D}(\alpha, x, v, g)$ and $T \in \mathcal{S}_{N,g} \setminus \mathcal{D}(\alpha, x, v, g)$. If $y(S) > 0$ for some $S \in \mathcal{D}(\alpha, x, v, g)$ is assumed, then $x \notin \mathcal{C}_+(N, v, g)$ by (5.4).

(2) for all $S \in \mathcal{D}(0, x, v, g) \cup (\mathcal{D}(\alpha, x, v) \setminus \mathcal{S}_{N,g})$ and $T \in 2^N \setminus \mathcal{D}(0, x, v, g) \cup (\mathcal{D}(\alpha, x, v) \setminus \mathcal{S}_{N,g})$. If $y(S) > 0$ for some $S \in \mathcal{D}(0, x, v, g) \cup (\mathcal{D}(\alpha, x, v) \setminus \mathcal{S}_{N,g})$ is assumed, then $x \notin \sigma^0(N, v, g)$ by (5.4).

(3) for all $S \in \mathcal{D}(\alpha, x, v)$ and $T \in 2^N \setminus \mathcal{D}(\alpha, x, v)$. If $y(S) > 0$ for some $S \in \mathcal{D}(\alpha, x, v)$ is assumed, then $x \notin \sigma^5(N, v, g)$ by (5.4).

Sufficiency: Let $x \in X(N, v, N/g)$ satisfy (1), (2), and (3), let $\{z\} = \sigma^5(N, v, g)$, and define $y = z - x$. As $z \in \mathcal{C}_+(N, v, g)$, it may be easily deduced recursively on the possible excesses, starting at connected coalitions of maximal excess, that y satisfies the assumptions of (1). Hence, $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v, g)$ and all $\alpha > 0$. Similarly, as $z \in \sigma^0(N, v, g)$, y satisfies the assumptions of (2), $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v) \setminus \mathcal{S}_{N,g}$ and all $\alpha \in \mathbb{R}$. Finally, as $z \in \sigma^5(N, v, g)$, it satisfies the assumptions of (3), i.e., $y(S) = 0$ for all $S \in 2^N$ so that $z = x$. **q.e.d.**

Corollary 5.2 *The solution σ^5 satisfies* RGP^{cmm}.

With the help of Lemma 5.1 the proof of Corollary 5.2 is straightforward and skipped.

Example 5.3 Let $N = \{1, 2, 3\}$, (N, v) be defined by $v(\{1, 3\}) = 10, v(\{1, 2\}) = v(\{2, 3\}) = 2, v(N) = 6$, and $v(S) = 0$ for all other $S \in 2^N$. Moreover, let $g = \{\{1, 2\}, \{2, 3\}\}$ so that $\{1, 3\}$ is the unique non-connected coalition. It is easily verified that

$$\nu(N, v) = (4, -2, 4), \nu(N, v, g) = (2, 2, 2), \text{ and } \sigma^5(N, v, g) = \{(3, 0, 3)\}.$$

By Example 5.3, σ^5 does not satisfy INC.

Finally, for completeness reasons, it should be remarked that Theorem 4.6 is no longer valid if the infinity assumption of U is deleted. Indeed, if $4 \leq |U| < \infty$, then the example σ defined by Peleg and Sudhölter (2007, Remark 6.3.3) may easily be extended to as solution on Δ_U^{cmm} that satisfies all five axioms but does not coincide with the prenucleolus on games with communication structures.

6 The Prekernel

The prekernel introduced by Maschler, Peleg, and Shapley (1972) (see also Davis and Maschler (1965)) may also be generalized to games with communication structures. For $(N, v, g) \in \Delta_U^{\text{cmm}}$, $k, \ell \in N, k \neq \ell$, and $x \in \mathbb{R}^N$ let $s_{k\ell}(x, v, g)$ denote the *maximum surplus* of k over ℓ at x, i.e.,

$$s_{k\ell}(x, v, g) = \max\{e(S, x, v) \mid S \in \mathcal{S}_{N,g}, k \in S \not\ni \ell\}.$$

Then the *prekernel* of (N, v, g) is the set

$$\mathcal{PK}(N, v, g) = \{x \in X(N, v, N/g) \mid s_{k\ell}(x, v, g) = s_{\ell k}(N, v, g) \text{ for all } k, \ell \in R \in N/g, k \neq \ell\}.$$

Remark 6.1 By literally copying the proof for games with coalition structures (see Peleg and Sudhölter (2007, Theorem 5.1.17)) it may be shown that $\nu(N, v, g) \in \mathcal{PK}(N, v, g)$ for any game with communication structure (N, v, g). Moreover, similarly to the classical context it may be shown that $\mathcal{PK}$ satisfies COV, AN, and RGP$^{\text{cmm}}$.

Proposition 6.2 *The prekernel on the set of games with communication structures satisfies* INC.

Proof: Let $(N, v, g) \in \Delta_U^{\text{cmm}}$, and $x \in \mathcal{PK}(N, v, g)$. Let β satisfy (4.2), let γ be defined by (4.3), and let $(N, w, g(N/g)) \in \Delta_{N,v,g}^{\beta}$ so that $e(S, x, w) \geq \gamma$ for all coalitions S that are connected by g and $e(T, x, w) < \gamma$ for all other coalitions. Note that $N/g = N/g(N/g)$. Hence, for any $k, \ell \in R \in N/g$ with $k \neq \ell$, $s_{k\ell}(x, w, g(N/g))$ can only be attained by a connected coalition so that $s_{k\ell}(x, w, g(N/g)) = s_{k\ell}(x, v, g)$. Thus, $x \in \mathcal{PK}(N, w, g(N/g))$.

Conversely, let $x \in \mathbb{R}^N$ be such that there exists $\widehat{\beta} \in \mathbb{R}$ such that $x \in \mathcal{PK}(N, w, g(N/g))$ for all $(N, w, g(N/g)) \in \Delta_{N,v,g}^{\widehat{\beta}}$. Again, let β and γ satisfy (4.2) and (4.3), respectively. Then, for $k, \ell \in R \in N/g$ with $k \neq \ell$, $s_{k\ell}(x, w, g(N/g))$ is only attained by some coalitions that are connected by g for all $(N, w, g(N/g)) \in \Delta_{N,v,g}^{\beta}$. As $\Delta_{N,v,g}^{\widehat{\beta}} \cap \Delta_{N,v,g}^{\beta} = \Delta_{N,v,g}^{\min\{\widehat{\beta},\beta\}}$, we may conclude that $x \in \mathcal{PK}(N, v, g)$. **q.e.d.**

It is possible to adjust Peleg's (1986) axiomatization of the prekernel on games with coalition structures to games with communications structures. To this end let σ be a solution on Δ_U^{cmm}. Then σ satisfies

(1) *non-emptiness* (NE) if $\sigma(N, v, g) \neq \emptyset$ for all $(N, v, g) \in \Delta_U^{\text{cmm}}$;

(2) the *restricted equal treatment property* (RETP) if the following property holds for all $(N, v, g) \in \Delta_U^{\text{cmm}}$, for all $k, \ell \in N$: If, for all $S \subseteq N \setminus \{k, \ell\}$, $v(S \cup \{k\}) = v(S \cup \{\ell\})$ and $S \cup \{k\} \in \mathcal{S}_{N,g}$ if and only if $S \cup \{\ell\} \in \mathcal{S}_{N,g}$, then $x_k = x_\ell$ for all $x \in \sigma(N, v, g)$;

(3) the *converse reduced game property* (CRGP$^{\text{cmm}}$) if the following condition holds for any $(N, v, g) \in \Delta_U^{\text{cmm}}$ with $|N| \geq 2$: If $x \in X(N, v, N/g)$ and $x_S \in \sigma(S, v_g^{S,x}, g^S)$ for all $S \subseteq R$ with $|S| = 2$ for all $R \in N/g$, then $x \in \sigma(N, v, g)$.

The proof of the following theorem is similar to Peleg's proof so that it is skipped.

Theorem 6.3 *There is a unique solution on* Δ_U^{cmm} *that satisfies* NE, EFF, COV, RETP, $\mathrm{RGP}^{\mathrm{cmm}}$, *and* $\mathrm{CRGP}^{\mathrm{cmm}}$*, and it is the prekernel.*

As in the classical case, each of the employed axioms in Theorem 6.3 is logically independent of the remaining axioms, provided $|U| \geq 4$. Indeed, the empty solution exclusively violates NE, the "non-efficient prekernel" (defined by

$$(N,v,g) \mapsto \{x \in X^*(N,v,N/g) \mid s_{k\ell}(x,v,g) = s_{\ell k}(N,v,g) \text{ for all } k,\ell \in R \in N/g, k \neq \ell\})$$

exclusively violates EFF, the equal split solution σ^1 exclusively violates COV, and the "preimputation solution" (defined by $(N,v,g) \mapsto X(N,v,N/g))$ exclusively violates RETP.

Note that, for any $(N,v,g) \in \Delta_U^{\mathrm{cmm}}$ with $|R| \leq 2$ for all $R \in N/g$, $\mathcal{PK}(N,v,g) = \{\nu(N,v,g)\}$. We say that a solution σ is a *standard solution* $\sigma(N,v,g) = \{\nu(N,v,g)\}$ for any game with communication structure that satisfies $|R| \leq 2$ for all $R \in N/g$. Note that the *Myerson value*, defined by $\mathcal{M}(N,v,g) = \{\phi(N,v/g)\}$ where ϕ denotes the Shapley value in the sense of Shapley (1953) and

$$(v/g)(S) = \sum_{T \in S/g} v(T) \tag{6.1}$$

is a standard solution as well as the solution defined by $(N,v,g) \mapsto \{\nu(N,v/g,N/g)\}$. Already for games with $|R| = 3$ for some connected component, $\nu(N,v,g)$ may differ from $\nu(N,v/g,N/g)$ as the following example shows. Let $N = \{1,2,3\}$, (N,v) be defined by $v(S) = 6$ for all $\emptyset \neq S \subseteq N$, and let $g = \{\{1,2\},\{2,3\}\}$. Then $N/g = \{N\}$ and $S = \{1,3\}$ is the unique non-connected coalition so that $(v/g)(S) = 12$ and $(v/g)(T) = v(T)$ for all $T \in 3^N \setminus \{S\}$. It is straightforward to verify that

$$\nu(N,v,g) = (2,2,2) \neq (3,0,3) = \nu(N,v/g) = \nu(N,v/g,N/g).$$

Clearly, both aforementioned standard solutions (i.e., $\mathcal{M}$ and $(N,v,g) \mapsto \{\nu(N,v/g,N/g)\}$) satisfy EFF, RETP, and COV. Hence, the union of the prekernel with any other standard solution that satisfies EFF, RETP, and COV, exclusively violates $\mathrm{RGP}^{\mathrm{cmm}}$, e.g., the solution defined by $(N,g,v) \mapsto \mathcal{PK}(N,v,g) \cup \{\nu(N,v/g,N/g)\}$.

Finally, the prenucleolus exclusively violates $\mathrm{CRGP}^{\mathrm{cmm}}$.

7 Extension to Games with Conference Structures

According to Myerson (1980) a *conference structure* is a pair (N,Q) where N is a coalition and $Q \subseteq 2^N$ satisfies $|R| \geq 2$ for all $R \in Q$. A *game with conference structure* is a triple (N,v,Q) such that (N,v) is a game and (N,Q) is a conference structure. Let Δ_U^{cnf} denote the set of games with conference structures. According to Myerson (1980, p. 178) "complete cooperation within the coalition S" is reflected by $\{\{i,j\} \mid i,j \in S, i \neq j\} = g$ for a graphical communication structure and by $\{T \subseteq S \mid |T| \geq 2\} = Q$ for a conference structure. So we identify a graph g with the conference structure of all coalitions that contain at least two elements and are connected by g. This identification is in contrast to that of Albizuri and Zarzuelo (2009) who identify any graph (N, g) with the conference structure (N, Q) defined by $Q = g$.

Thus, if (N, g) is a communication structure, then the conference structure *corresponding* to (N, g), $(N, Q(g))$, is defined by

$$Q(g) = \{S \subseteq N \mid |S| \geq 2 \text{ and } S \text{ is connected by } g\}. \tag{7.1}$$

Hence, "$\Delta_U^{\text{cmm}} \subseteq \Delta_U^{\text{cnf}}$" throughout means the embedding

$$\Delta_U^{\text{cmm}} \hookrightarrow \Delta_U^{\text{cnf}} \text{ defined by } (N, v, g) \mapsto (N, v, Q(g)).$$

Let (N, v, Q) be a game with conference structure. We adopt the notion of connectedness from the aforementioned articles: Let $S \subseteq N$. The elements $i, j \in S$ are *connected* in S by Q if there exits $S_1, \ldots, S_\ell \in Q$ such that $i \in S_1, j \in S_\ell, S_j \subseteq S$ for all $j = 1, \ldots, \ell$, and $S_t \cap S_{t+1} \neq \emptyset$ for all $t = 1, \ldots, \ell - 1$. Moreover, S/Q denotes the components of S w.r.t. Q, and a coalition S that has only one component is called *connected*. Furthermore, $\mathcal{S}_{N,Q}$ denotes the set of all coalitions in N that are connected by Q.

A *solution* on $\Delta \subseteq \Delta_U^{\text{cnf}}$ is a function σ which associates with any $(N, v, Q) \in \Delta$ a subset $\sigma(N, v, Q)$ of $X^*(N, v, N/Q)$. EFF, COV, AN, SIVA are defined similarly as in Section 4; just "graph (N, g)" has to be replaced by "conference structure (N, Q)".

In order to define RGP$^{\text{cnf}}$, we adopt the definition of the reduced conference structure of Albizuri and Zarzuelo (2009). Let (N, Q) be a conference structure and $\emptyset \neq S \subseteq N$. Then the *reduced conference structure* (S, Q^S) is defined by

$$Q^S = \left\{ S \cap \bigcup_{t=1}^{\ell} S_t \;\middle|\; \ell \in \mathbb{N}, S_t, S_\ell \in Q, S_t \cap S_{t+1} \neq \emptyset \forall t = 1, \ldots, \ell - 1, \left| S \cap \bigcup_{t=1}^{\ell} S_t \right| \geq 2 \right\}. \tag{7.2}$$

Note that (N, Q^N) may not coincide with (N, Q). Indeed, Q^N contains Q and all coalitions in N that are connected by Q and contain at least two elements. However, for any graph (N, g), $Q(g) = Q(g)^N$.

The coalition function of the *reduced game with conference structure* w.r.t. S and $x \in \mathbb{R}^N$, $(S, v_Q^{S,x}, Q^S)$, may now be defined analogously to (2.3) by replacing g with Q whenever it occurs. Now, again by replacing g with Q whenever this is needed, RGP$^{\text{cnf}}$ is defined analogously to RGP$^{\text{cmm}}$ and INC is generalized in a similar way. The prenucleolus of (N, v, Q), $\nu(N, v, Q)$, is defined by literally copying Definition 2.2 with the exception that "g" has to be replaced by "Q" wherever it occurs.

Now it is straightforward to generalize Theorem 4.6 to games with conference structures.

We should like to remark that there are examples of games with conference structures, (N, v, Q), such that $Q^N \neq Q$, whereas and $v_Q^{N,x} = v$, $v_Q^{S,x} = v_{Q^N}^{S,x}$, and $g^N = g$ for any $x \in \mathbb{R}^N$, $(N, v, Q) \in \Delta_U^{\text{cnf}}$, $\emptyset \neq S \subseteq N$, and any graph (N, g).

It should finally be remarked that the prekernel for games with communication structures together with Proposition 6.2 and Theorem 6.3 may be generalized to games with conference structures.

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