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Abstract

The aim of this paper is to investigate the nexus between demand patterns and innovation as it stems from research efforts and the extent of specialization. In the proposed model an innovation race conducted by entrants investing in research and development against established incumbents raises productivity at the industry level and leads to a shift in the aggregate demand pattern and consequently to a redistribution of the profit fund among industries and a restructuring of the production process in each industry. The paper argues that the degree of development as reflected in a demand share distribution is characterized by a corresponding distribution of specialized sectors that becomes more even across industries as the development process proceeds and investigates the consequences in terms of economic growth.

Keywords: Innovation; development; structural change

JEL: O10; O30; O40

1 Introduction

The history of the last two centuries and a half of economic development cogently suggests that the increase in productivity due to technical advancement has played a major role in fostering a growth process that has then become largely self-sustaining. It is a well documented fact that the progress in technology and science applied to industry has brought about an extraordinary expansion of implements assisting labour in producing final commodities. This expansion has, of course, been made possible by the deepening of specialization, itself the outcome of ever-increasing division of labour. Yet, it is also equally well recorded that, concomitant to this trend, entire industrial sectors have been subject to sometimes radical restructuring: production processes have become leaner, the input structure simplified and real costs streamlined. The growth of the size of the market, in a Smithian sense, lies at the heart of this movement but evidence also suggests that there is a narrow relationship between increasing productivity and the pattern of final demand. This observation vouchsafes the view that economic development has progressed through stages hallmarked by patterns of final demand matching productivity as well as income per head levels. Amongst other possible criteria, a development stage taxonomy can, indeed, be fashioned by identifying typical consumption standards. Broadly speaking, low productivity and low income economies are by necessity constrained to afford only staple consumption goods required to support a basic livelihood: food, shelter and the bare means to entertain social intercourse. As productivity rises and income per head is augmented, the economy is reshaped to accommodate a demand pattern made up of goods of lesser priority, possibly allowing greater comfort and affluence. In this sense, a whole sequence of steps each characterizing a specific stage can be envisaged, from mere subsistence to mass consumption of durable goods (cars, home appliances, holidays) to luxury goods. The link between demand patterns and income levels has been stylized by the well-known Engel's curves showing that as income rises the weight of some commodities increases whilst that of others declines as demand is driven to saturation for the latter and as acceleration for the former occurs. A simple and stylized way to formalize this approach, however, is to assume a hierarchy of preferences that ranks goods from very high to very low priority ones, the share of the latter rising as productivity and income per head rise. These demand share shifts subject the economic system to structural changes that produce breaks in the pattern of long term growth and reshape the production structure.[1]

[1] These type of events have been studied in two strains of relevant literature; for the demand driven approach centered on hierarchical preferences see Matsuyama (2002), Foellmi and Zewilmueller (2006), (2008); for the one that considers different income elasticities see Kongsamut, Rebelo and Xie (2001) and Laitner (2000); for the supply driven approach

The aim of this paper is to investigate the nexus between demand patterns, hallmarks of development stages, and innovation as it stems from research efforts and the extent of specialization. Accordingly, the model we consider strives to put the latter at the centrestage of our analysis to determine the productivity growth allowing for income per head increases generating demand shifts and hence structural change. This is an implication-laden event that leads to both further specialization and to rationalization, the former where demand expands while the latter where it contracts. Thus, they are processes that are endogenously determined, see also Romer (1987) and Ciccone (2002). On the demand side, we consider the consumer's problem to be a quantity and a variety choice problem, and describe how changes in prices eventually lead to changes in aggregate demand pattern. On the production side, the economy is structured by vertically integrated industries the final sectors of which manufacture consumption goods whilst a number of specialized sectors provide the intermediate goods that are required by the former as inputs. Given this framework, the main source of growth is, of course, innovation the burden of which is mainly laid upon these manufacturers. Innovations, however, are idiosyncratic events that require investment in specific resources. For simplicity's sake but, we believe, without loss of generality, we restrict the resources to be employed for the purpose to highly specialized manpower. Successful innovation leads to the acquisition of exclusive rights to the production and sale of these goods to final users but once gained a monopoly position, innovators enjoy quasi rents and cease the research effort leaving this task to followers who, by taking up the challenge, attempt to oust them (see, for example, Aghion and Howitt, 1992). Successful efforts by innovators do not remain confined within the sector in which they occur: it has often and persuasively been observed that as a consequence of their innovation technological imbalances arise involving all the complementary inputs that make up the concerned technology. These imbalances, however, although involving only the industry to which the relevant sectors belong to, open the way to seek the solutions that are necessary to heal them by acting as focusing devises (Rosenberg, 1990; David, 1976; Mokyr, 1990) thus spreading the productivity increase to all sectors within the industry with which they are technologically bound. These considerations lead us to envisage productivity growth as an industry-wide phenomenon and not just as a sector-confined event. It is the promise of future productivity-linked profits that motivate the protagonists of technological advance but it must also be recognized that the size of profitability is set by the size of the market for each final good which, in turn, depends on the demand pattern prevailing in the economy in each point in time. The extent

see Ngai and Pissarides (2007) and Acemoglou and Guerrieri (2008). See Matsuyama (2008) and Buera and Kaboski (2009) for a survey.

of the profit fund, we hold, is relatively larger where demand and in equilibrium supply are also relatively larger defining an opportunity domain that can be exploited by contributing to the technical profile of each final good. This domain of opportunity acts by signalling new, would-be producers of intermediate goods that they can appropriate a share of the total profits made available by demand by introducing new specialized inputs that will then increase productivity throughout the industry. This has historically been another major source of innovations. New capital goods, in this paper new intermediates, are not however conjured up in a technological vacuum. On the contrary, they are the actual result of a process of learning and searching construing the awareness that it is technologically not only feasible but also profitable to further specialize a production process. The protagonists are those agents that at various levels are directly involved in the current manufacturing of already existing inputs: they are those who possess the knowledge, the know-how and the skills to realize how the production process can be further perfected and complemented. This is basically why new specialized inputs proceed from within the industry as spin-offs of already operating firms. They may be the very entrepreneurs but also workers, engineers and whoever is therein employed. If this leads to lengthening the chain, or string, of specialized inputs, it may also lead to its rationalization, meaning by this term the trimming and shortening of this chain, if the need and opportunity arise, by easing the industry cost structure. These further observations lead us to argue that as the relative size and the distribution of demand shares identifies a stage of development this is also identified by a corresponding extent of specialization in each industry. We deem this to be an important point since it allows to associate to the evolution of innovation-led productivity and income per head an evolution of demand shares and, therefore, of a pattern of specialization.

The economy we analyze is normalized by a constant total employment that accordingly defines the extent of the market. Furthermore, the time scale that the model applies is two-faceted. Demand shocks that engender structural change are deemed to be relatively rare events hence occurring on a slow time scale in contrast to innovations improving existing intermediate goods, theirs being accordingly a faster one. Thus, the economy is assumed to have the time required to settle onto stationary states therefore marking as many stages of development. It will be shown that the expected economy's growth rate in the stationary state, that is for any given size of total employment and demand share distribution, is eventually determined by the rate at which innovations arrive as a direct outcome of followers' investment in R&D employment. Yet, a crucial ingredient is the distribution of demand shares. It will be shown that as this distribution becomes less lopsided and more even, the expected growth rate decreases. This is due to the fact that the contribution to growth is higher where arrivals are higher as a consequence of each sector innovation effort. Economies in which there is a greater number of such sectors grow faster. Thus, the more intermediate-producing sectors concentrate on fewer industries the higher is the average growth rate and as they get more evenly distributed the lower it becomes. Specialization in our model is demand driven: as the growth process continues, consumers' demand is distributed on a wider range of goods stimulating an innovative process that is diluted on a larger number of industries thus reducing the growth rate. This carries an important implication: as an economy progresses towards higher stages of development keeping the same extent of the market and the same number of intermediate-producing sectors its growth rate slows down. By the same effect and conditions, a less developed country manages to have a higher growth rate than a developed one, exhibiting a process of convergence. The same effect applies if, thanks to international trade, it is able to concentrate its innovative effort on a fewer number of industries. An important question arises at this stage of our investigation since what remains to be seen is what happens when demand shares actually change. This event forebodes momentous adjustments as some industries are witness to demand increase whilst others are likely to be involved in the opposite event. Would-be investors attempting to innovate have to take due notice of these occurrence and modify their expectations, differently according to whether the change over is likely to involve them in an increasing demand environment or in a decreasing one. We formally analyze a simplified version of the model and then simulate a more complex version to show that results that have been obtained do hold more generally. By studying a traverse path over a fixed time period, we are able to show that the number of specialized sectors does increase where demand shares increase and decrease where the latter fall: thus, industries where demand expansion occurs are subject to deepening specialization while industries where it diminishes are subject to rationalization by lessening the number of intermediate sectors. An important side effect is that owing to uncertainty connected with the eventual disappearance, concomitantly coupled with the appearance, of some sectors generated by demand shifts, the level of innovation-oriented employment contracts during the traverse period to be restored when it is expected to be finally over.

The paper is structured as follows. Section 2 describes the demand side of the economy, Section 3 sets out the production structure while Section 4 describes the innovative process that takes place within the sphere of intermediate goods producers as well as specialization and rationalization that there occur. In Section 5 we calculate the stationary state expected growth rate when demand shares are fixed and then analyze the traverse path when demand shares change as a consequence of the innovation process. Section 6 draws the paper to a close. Proofs are placed in the appendix.

2 Consumption pattern

2.1 The individual demand function

Consider an economy with $1,...,j,...,J$ differentiated goods with prices at time t $p_{1,t},...,p_{j,t},...,p_{J,t}$ and populated by $\mathcal{L}$ consumers. We conceive the consumer's problem as a problem where, for given prices and income R_i, $i = 1, 2...\mathcal{L}$, an optimal decision is to be taken concerning how many goods, and how much of each, are to be consumed: a variety and quantity choice problem. Moreover, as prices change, a revision is to be made on whether to change the variety and quantity proportions of the consumption bundle. To solve this quantity and variety choice problem, it is stipulated that consuming $y_{i,1,t}$, $y_{i,2,t}$, ..., $y_{i,j,t}$ of j goods at time t yields to the individual i an instantaneous utility rendered by the following logarithmic function

$$u_i(j,t) = Log(C_{i,j}) + \alpha_j \sum_{h=1}^{j} Log(y_{i,h,t})$$

where $C_{i,j}$ is a weight that accounts for individual i's impatience to consume good j. Formally, for given prices $p_{1,t},...,p_{J,t}$, and a constant income R_i, and assuming that the individual's intertemporal discount factor coincides with the constant interest rate r, the individual i's problem can be written as a sequence of static problems

$$\begin{aligned} U_i(j,t) = \quad & \max_{y_{i,1,t}\cdots,y_{i,j,t}} u_i(j,t) \\ & \text{s.t. } \sum_{h=1}^{j} p_{h,t} y_{i,h,t} \leq R_i \end{aligned} \tag{1}$$

and

$$max_{j\in\{1,...,J\}} U_i(j,t) \tag{2}$$

In (1) individual i solves the quantity choice problem while in (2) the variety choice problem is solved. We postulate the following Assumption.

Assumption 1 *1. The difference $\alpha_{j+1} - \alpha_j$ is positive and non-decreasing in j ;*

2. constants $C_{i,j}$ for $i = 1,...,\mathcal{L}$ and $j = 1,...,J$ satisfy the condition that $\chi_i(j) \equiv \frac{C_{i,j+1}}{C_{i,j}} \frac{\left(\frac{R_i}{j+1}\right)^{(j+1)\alpha_{j+1}}}{\left(\frac{R_i}{j}\right)^{j\alpha_j}}$

is non-increasing in j.

3. *prices are such that $p_{j+1,t} > p_{j,t} > 1$ for each $t \geq 0$.*

Condition 1. requires that the degree of concavity of the individual's utility function does not increase as variety j is increased. Condition 2. constrains the rate of change of the indirect utility obtainable by equally distributing income on the number of goods in a given basket, hence independently of prices, not to increase in j.[2] This property generates a hierarchy such that high priority goods weigh more in determining indirect utility than those of lesser priority. Condition 3. makes this hierarchy explicit by stating that prices of lower priority goods, the luxury ones, are higher than those that apply to higher priority goods, the basic ones. The solution to the consumer's problem is as follows. Let $j^*_{i,t}$ be the optimal variety at time t, that is $U_i\left(j^*_{i,t}, t\right) > U_i\left(j, t\right)$ for each j other than $j^*_{i,t}$, then

$$y_{i,j,t} = \begin{cases} \frac{1}{j^*_{i,t}} \frac{R_i}{p_{j,t}} & \text{for each } j \leq j^*_{i,t} \\ 0 & \text{for each } j > j^*_{i,t} \end{cases} \tag{3}$$

from which the indirect utility can be obtained. Assumption 1 states sufficient conditions for the existence of $j^*_{i,t}$. More particularly, it guarantees that the utility is monotonically increasing in j for the consumption bundles with variety lower than j^*_t and monotonically decreasing for baskets with variety larger than j^*_t, that is,

$$U_i\left(j, t\right) < U_i\left(j+1, t\right) \text{ for each } j = 1, ..., j^*_{i,t} - 1 \tag{4}$$

and

$$U_i\left(j+1, t\right) < U_i\left(j, t\right) \text{ for each } j = j^*_{i,t}, ..., J - 1 \tag{5}$$

To see this, taking into account (3), write $U\left(j, t\right) \lesseqgtr U_i\left(j+1, t\right)$, i.e. the inequality of indirect utilities, as $\Omega\left(j, t\right) \lesseqgtr \chi_i\left(j, t\right)$, where $\Omega\left(j, t\right) \equiv p^{\alpha_{j+1}}_{j+1,t}\left(\Pi^j_{h=1} p^{\alpha_{j+1}-\alpha_j}_{h,t}\right)$. Assumption 1 (i) and (iii) guarantee that $\Omega\left(j, t\right)$ is increasing in j. Conditions (4) and (5) can be written as follows

$$\Omega\left(j, t\right) < \chi_i\left(j, t\right) \text{ for each } j = 1, ..., j^*_{i,t} - 1$$

[2] To better grasp this point, consider that should a decrease of indirect utility occur by adding one more good in the equally distributed basket, a further increase of such goods would yield an even larger drop in indirect utility.

and

$$\Omega\left(j,t\right) > \chi_i\left(j,t\right) \text{ for each } j = j^*_{i,t}, ..., J-1$$

Since at each time period t, $\Omega\left(j,t\right)$ and $\chi_i\left(j,t\right)$ cross at most once, an unique $j^*_{i,t}$ exists.

Changes in prices or income lead to a revision of the individual's quantity choice and eventually also of the variety one. Suppose that at time t price $p_{j,t}$, $j \leqslant j^*_i + 1$, declines. This event decreases $\Omega\left(j^*_{i,t}, t\right)$ and may trigger an increase in the variety of the consumption bundle. In this case, a rebalancing of the individual's consumption pattern occurs, decreasing the consumption of some or all high priority goods to accommodate the consumption of an additional variety of lower priority. If the variety consumed remains unchanged, the individual's consumption of good j simply increases. Note that, as long as $p_{j+1,t} > p_{j,t} > 1$, a decrease in price $p_{j,t}$, for $j > j^*_{i,t} + 1$, does not affect consumption decisions since it does not affect $\Omega\left(j^*_{i,t}, t\right)$. In a similar fashion, a larger income, by increasing $\chi_i\left(j^*_{i,t}\right)$, leads individuals to revise their quantity choice and may eventually make lower priority goods affordable.

2.2 Aggregate demand function

Consumers are heterogeneous since they have a heterogenous income (R) and are heterogenous in their impatience to consume good j. Let Y_t be the economy's aggregate income, where $Y_t = \sum_{i=1}^{\mathcal{L}} R_i = \sum_{i=1}^{\mathcal{L}} \sum_{j=1}^{J} y_{i,j,t} p_{j,t}$, then we can write aggregate demand for good j as

$$y^d_{j,t} = \beta_{j,t} \frac{Y_t}{p_{j,t}} \tag{6}$$

with $\beta_{j,t} = \sum_{i=1}^{\mathcal{L}} I_{j \leq j^*_{i,t}} \frac{R_i}{j^*_{i,t} Y_t}$, where $I_{j \leq j^*_{i,t}}$ is an indicator function, indicating 1 if $j \leq j^*_{i,t}$. $\beta_{j,t}$ are expenditure shares that indicate the fraction of aggregate income consumers spend on a given good j at time t, comprising their income and impatience heterogeneity. Price changes may trigger changes in the aggregate consumption pattern. For instance, a fall in price may lead to increases in the number of goods j^*_i for some individuals, as discussed in the previous section, and thus to lower shares for some product and higher for others. As real income rises as a consequence of productivity gains, the share of the goods that are less essential increases whilst that of those that are more so, say staple foods, concomitantly decrease; hence, goods placed in the higher part of the product sequence ordering become progressively weightier. This process is meant to capture an essential feature of the development process: the record of demand share evolution indeed tells a cogent history about this process.

3 The production structure

We discuss an economy in which there are J industries, each producing a differentiated final good. They are composite entities of technologically vertically integrated production: each such industry manufactures a final good but it also encompasses all the sectors that provide it with the necessary inputs. Thus, in each industry j, $j = 1, 2....J$, k_j sectors engage in producing an industry specific intermediate good which embodies the latest technology and are the carriers of innovation. The critical approach that we take in this paper is to assume that these J industries are each composed by a final sector populated by a large number of firms. Thus, these firms operate in a freely competitive environment such that the prevailing prices drive their profits to zero. By contrast, the k_j intermediate sectors in each industry, being the exclusive owners of the extant technology, feature an incumbent monopolist earning a positive profit and a searching-to-innovate follower. The latter carry out a search and development effort that requires investment in specifically employed manpower. The economy's nominal wage rate is kept constant throughout.

3.1 The final sectors

The J industries final goods producing sectors avail themselves of a simple supply structure. The *ensemble* of intermediate inputs and the labour that is directly required make up an unique, up-to-date technique of production. The following is the linear production function featuring inputs that are all strictly complementary and that all firms in any industry j apply to produce their specific output:

$$y^s_{j,t} = \min\left\{a^1_{j,t}x^1_{j,t}, a^2_{j,t}x^2_{j,t}....a^{k_{j,t}}_{j,t}x^{k_{j,t}}_{j,t}, b_{j,t}l^y_{j,t}\right\} \quad ; \tag{7}$$

$a^k_{j,t}$ is the productivity of input k while $x^k_{j,t}$ is its input $k = 1, ..., k_{j,t}$. Furthermore, $b_{j,t}$ is the productivity of labour and $l^y_{j,t}$ its employment; all magnitudes being defined at time t. It is important to note that $k_{j,t}$ is an indicator that plays an important role in defining how production is structured. It sets, in fact, the number of specialized sectors that at time t are required to produce intermediate inputs. It signals, therefore, how far specialization has progressed in industry j as a result of restructuring processes that both specialize and rationalize. The extent of $k_{j,t}$ will endogenously be determined in a following section. Note, finally, that the use of this function avoids any increasing returns to scale effect.

Efficient utilization of all inputs implies that

$$y^s_{j,t} = a^1_{j,t} x^1_{j,t} = a^2_{j,t} x^2_{j,t} = ... = a^{k_{j,t}}_{j,t} x^{k_{j,t}}_{j,t} = b_{j,t} l^y_{j,t} \quad . \tag{8}$$

Calling $p_{j,t}$ the price of final good j, $u^k_{j,t}$ that of the intermediate input k entering final output j and w the manufacturing labour force wage rate, profits accruing to final good producing firms at time t are notionally defined as

$$\pi_{j,t} = p_{j,t} y^s_{j,t} - w l^y_{j,t} - \sum_{k=1}^{k_{j,t}} u^k_{j,t} x^k_{j,t} \tag{9}$$

and are assumed to be equal to zero on account of perfect competition: $\pi_{j,t} = 0$. The price of such goods is, therefore,

$$p_{j,t} = \frac{w}{b_{j,t}} \left(1 + \sum_{k=1}^{k_{j,t}} \frac{u^k_{j,t}}{w} \frac{b_{j,t}}{a^k_{j,t}} \right) \quad . \tag{10}$$

3.2 Intermediate goods producing sectors

The supply structure of these sectors is simpler. To eschew cumbersome formal complications, it is befitting to normalize the production process of intermediate goods in such a manner that they all require the same amount of labour for one unit to be produced: let it be designated by $\frac{1}{\eta}$. Thus, the following production function holds, $\forall k, j$:

$$x^k_{j,t} = \eta l^k_{j,t} \quad , \tag{11}$$

$l^k_{j,t}$ being the amount of manpower employed for the purpose of producing $x^k_{j,t}$ units of the intermediate good k in industry j at time t. In the simple portrait of the economy that we are sketching out, the process of producing intermediates is the backward lying one in the final output time profile. It is, nevertheless, the true protagonist of innovation, raising productivity in the final sector of any of the industries that compose the economy because of the systematic research and development effort that takes place therein. As mentioned above, the number, $k_{j,t}$, of sectors contributing the input requirement to each industry is the result of a specialization and rationalization process that, in time, has determined the length of its string. Monopolists, owners of up-to-date technologies, enjoy a positive profit that equals:

$$\pi^k_{j,t} = u^k_{j,t} x^k_{j,t} - w l^k_{j,t} \quad . \tag{12}$$

The price of an intermediate good k , $u^k_{j,t}$, is set according to a mark-up that we later determine.[3]

Assumption 2 *Intermediate good producer k of industry j sets the price of such goods according to a mark up $c^k_{j,t}$.*

$$u^k_{j,t} = (1 + c^k_{j,t}) \frac{1}{\eta} w \quad . \tag{13}$$

4 Innovations in the Intermediate Good Sectors

This simple production structure is stressed by the occurrence of innovations. The latter are introduced by old and new producers of intermediates and increase final sectors productivity. To simplify, it is assumed that when an innovation occurs, its effects on productivity spread equally on both intermediate and labour requirements. Innovations are idiosyncratic events. Indeed, they are the consequence of the researching process carried out by would-be innovators and crown their costly efforts to attempt entering a particular sector and industry. Whenever sector specific innovations are introduced they create a technological imbalance that concerns all complementary inputs that, at each point in time, are necessary to the final sector output. Neither innovations nor their effects are short-term occurrences; historians of technology and factual observation indicate that an innovative event that upsets a technological equilibrium becomes a *focusing device* that prompts adjustments wherever, along the complementarity string, frictions and mishaps happen in consequence. Therefore, an innovation causes a ripple of concomitant improvements and up-dating that spread the productivity increase. This remark justifies the assumption that whenever it takes place, a sector-specific innovation raises the productivity level of all inputs and becomes an industry-wide phenomenon[4]. It is important, at this stage, to distinguish between different kinds of innovative events.

4.1 Vertical innovations

A first kind of innovations occurs as a consequence of efforts made by erstwhile monopolists who were ousted by current incumbents. Their activity aims at dislodging the latter by achieving improvements on existing intermediates and is driven by the profit outlook implied by the eventual gain of a monop-

[3] In Section , the mark-up is determined as a function of monopoly power.
[4] For simplicity's sake, we abstract from spill-overs between different industries.

olist's position. These innovations build upon the knowledge of outdated technologies that, although beaten by the ones achieved and owned by current monopolists in charge, are the basis for further improvements that can be had only through new searching and learning. Nevertheless, they can still pose a threat since they could be revived should the opportunity arise: more specifically, these outdated technologies could be combined with additions of new and specialized intermediates that reorganize the production process according to improved productivity standards. In order to avoid unnecessary algebraic complications, we shall hereafter introduce, without loss of generality, the following simplifications. As discussed in greater details in following paragraphs, productivity ensuing from innovations increases by raising both that of direct labour employed in final output and that of all intermediates used to produce it.

Assumption 3 *An innovation occurring at time t in sector k of any industry j raises the productivity of all inputs $b_{j,t}$ and $a^k_{j,t}$, $\forall k$ by a factor λ.*

$$\begin{aligned} a^k_{j,t} &= a^k_{j,t-}e^{\lambda} \quad, k = 1, 2, .., k_{j,t}; \\ b_{j,t} &= b_{j,t-}e^{\lambda} \quad . \end{aligned} \tag{14}$$

The current inputs $a^k_{j,t}$ and $b_{j,t}$ are recorded to be the result of past innovations at each time step increasing productivity by a constant λ. As a consequence of this assumption, the sector-specific intermediate good-labour ratio is defined as $\frac{b_{j,t}}{a^k_{j,t}} = \frac{b_{j,0}}{a^k_{j,0}}$ and is constant in time. We further define an index that plays an important role in the following analysis: it is the average ratio of indirect to direct labour requirement that is specific to each industry j

$$\delta_{j,t} = \frac{1}{k_{j,t}} \sum_{k=1}^{k_{j,t}} \frac{b_{j,0}}{\eta a^k_{j,0}} \quad . \tag{15}$$

4.2 Innovations by specialization and rationalization

The innovation process, however, is not confined to the enhancement of existing inputs productivity. The history of industrialization has shown that while production has evolved by deepening the utilization of means of production entering final goods, the process on which it has been grounded has been hallmarked by intertwined events of both specialization and restructuring. Although occurring in a variety of ways, in the simple model that we have construed, events of this kind can be accounted for, in the former case, by the lengthening in some industries of the $k_{j,t}$ strings of intermediate inputs and, in the latter one, by shortening in some others. They are effective acts of innovation that lead to an increase in productivity. It is interesting to note that both processes have been shown as taking place through innovations that stem from within the sectorial structure of any of the industries that make up the economy. When specialization occurs, stringent conditions necessarily apply as to the adoption of the new technique: they are investigated below.The following assumption will hold throughout the paper.

Assumption 4 *Let t denote the time of specialization or rationalization, then:*

(i) specialization:

a) at each step, the deepening of specialization brings about an overall productivity increase equal to λ.

$$
\begin{aligned}
k_{j,t} &= k_{j,t-} + 1 \\
a^{k}_{j,t} &= a_{j,t-} e^{\lambda} \ , \ k = 1, 2, .., k_{j,t} - 1 \\
b_{j,t} &= b_{j,t-} e^{\lambda} \ \ ;
\end{aligned}
\tag{16}
$$

b) the average ratio of indirect to direct labour requirement,(15), does not increase as a more specialized technology is introduced[5]

$$\delta_{j,t} \leq \delta_{j,t-} \quad ; \tag{18}$$

c) as soon as a specialization conducive spin-off appears, followers are able to adopt it and adapt their outdated technology thus keeping their gap from widening;

(ii) rationalization:

a) the process of rationalization is merely cost-cutting: when it occurs the number $k_{j,t}$ is reduced;

b) as soon as rationalization occurs, followers are able to adopt it and adapt their outdated technology thus keeping their gap from widening.

[5] This means that

$$\frac{1}{a^{k+1}_{j,0}} \leq \frac{1}{k} \sum_{k'=1}^{k} \frac{1}{a^{k'}_{j,0}} \tag{17}$$

for $k = 1, 2, ...$ and $j = 1, ..., J$. In particular (17) holds if the sequence $\left\{a^{k}_{j,0}\right\}_{k=1}^{\infty}$ is non decreasing in which case the new entrants' mark up, see (22) in a following page, is not higher than that of older producers.

When specialization is involved, because of the complex processes of learning-by-doing and by-using that constantly take place within the entire industry and more specifically in capital goods production, opportunities to further specialize are caught through spin-offs that lead to the setting up of new sectors, deepening the industry's capital structure. Thus, the extant length of the input string catches the extent to which the process of specialization has gone in a specific industry j. As later discussed, owing to the possibility of earning monopoly profits, there is an outstanding incentive to devise yet more specific implements in the industry which appears to insure larger profits. As shown by (18), the existence of a profit fund that can further accommodate more intermediate producers provides incentives for spin-offs. It is quite clearly the case that this opportunity is greater where total profits are higher. An event such as this is not without consequences. By the very fact that specialization is indeed furthered, tasks that were previously carried out by the joint contribution of existing inputs are now performed in a different and innovative way by new intermediates. The upshot is that all the technical norms that are incorporated in the already-in-use $k_{j,t-}$ intermediates are changed: it is an adjustment process that is driven by the focusing device brought about by the new element of production (Assumption 4 (ia)); for the sake of simplicity we are assuming that the productivity increase brought by specialization is the same as that brought by the innovation (14). Assumption 4 (ib) indicates that the productivity of the new input improves the average productivity in the relevant industry[6]. Until the appearance of a spin-off, researching followers remain the owners of an outdated but still effective technology, the productivity gap of which, relatively to monopolists in charge, is given by (14). Because of a specializing spin-off, this gap can potentially double (see (16)). Again for the sake of simplicity, we assume (see Assumption 4 (ic)) that followers can, by imitation and possibly reverse engineering, seize the opportunity provided by the new pervasive change to leap-frog on a technology with $k_{j,t}$ intermediates and bridge one of the two productivity gaps. As soon as a specialization conducive spin-off appears, followers are able to adopt it and adapt their outdated technology thus keeping their gap from widening (see Assumption 4 (ic)).

On the contrary, history has recorded rationalization processes through which the introduction of innovations has actually simplified and cut short the string of intermediate sectors. These events have generally been the consequence of adverse effective demand shifts that have jeopardized their profitability although creating an incentive to introduce cost-reducing innovations. The pruning of intermediate sectors has therefore had the effect of forcing those that have managed to remain to

[6] In the Appendix we show that by this assumption the mark-up does not increase after the occurrence of specialization.

restructure and become more productive. This restructuring process is the result of a profit decline that happens to hit some industries as a consequence of a relative demand shortfall that leads to a shrinking of the available profit fund. In this case, the length of the intermediate good string that characterizes the indirect cost structure becomes a burden that requires some leaning if profitability is to be restored. Yet, as in the case of specialization, this is a process that involves the whole industry leading to a change of the overall production technique. In both cases, the consequence is that an entirely new technique becomes available (Assumption 4 (iia)). Likewise, followers are able to replicate this process keeping the gap from widening (see Assumption 4 (iib)).

4.3 Profits and competitive threats

Given the assumptions mentioned above the profit to be earned in the k-th sector of the j-th industry is

$$\pi^k_{j,t} = \frac{1}{\eta} c^k_{j,t} x^k_{j,t} w = \frac{1}{\eta} c^k_{j,t} \frac{b_{j,0}}{a^k_{j,0}} l^y_{j,t} w \tag{19}$$

Assumption 5 *The absence of arbitrage opportunities among different sectors of the same industry insures that the employment of the same quantity of labour affords an equal profit flow*

$$\pi^k_{j,t} = \pi^{k'}_{j,t} \qquad k, k' = 1, ..., k_{j,t} \ , \ j = 1, .., J \quad .$$

Assumption 5 and (19) allow us to distinguish two different mark-up components: an industry wide term $c_{j,t}$ and a sector specific correcting factor $a^k_{j,0}$ which is linked to the k-good productivity. It is, indeed, $c_{j,t}$ that enables producers to earn monopoly profits. Yet, in order that this be actually the case it must be tuned as to forbid previous incumbents to remain in the industry as competitors. Thus, the size of $c_{j,t}$ is to be such that the productivity increase be entirely appropriated by the entrant and the old incumbent's profits be driven to zero to oust him or her out of the market. The critical size of $c_{j,t}$ can be derived by considering that the price of the final good is the same no matter who produces the intermediate good to be employed. Taking Assumptions 3 and 5 into account, we obtain the following.

Lemma 1 *The mark-up of intermediate good producer k of industry j is*

$$c^k_{j,t} = c_{j,t} a^k_{j,0} \quad , \tag{20}$$

while the industry wide mark-up is

$$c_{j,t} = \frac{\eta}{b_{j,0}k_{j,t}} \left(1 + k_{j,t}\delta_{j,t}\right) \left(e^{\lambda} - 1\right) \quad . \tag{21}$$

Proof. In the Appendix. ■

By (10) and (21), the price of a final good j becomes simply

$$p_{j,t} = \frac{w}{b_{j,t}} \left(1 + \delta_{j,t}k_{j,t}\right) e^{\lambda} \tag{22}$$

Remark 1 *On account of the non-arbitrage Assumption 5, the profit of an intermediate good producer in any industry j depends only on the specific industry, but not on the specific intermediate good k*

$$\pi_{j,t} = \pi^{k}_{j,t} = \frac{1}{k_{j,t}} \left(1 + \delta_{j,t}k_{j,t}\right) \left(e^{\lambda} - 1\right) wl^{y}_{j,t} \quad . \tag{23}$$

The appearance of new sectors carries with it the burden of a new production process, no matter how simple, encumbering the economy with more employment, a new technique and yet another monopolist enjoying exclusive ownership rights upon it. It follows that some conditions must be satisfied for the lengthening of the process to be feasible and further specialization take place. As mentioned, because of specialization the whole process becomes more productive. On account of (16), a productivity increase, say at time t, must translate into lower prices; it follows that furthering specialization is feasible if and only if

$$p_{j,t} \leq p_{j,t-} \tag{24}$$

The price $p_{j,t}$ that rules when such an event occurs is determined by a new mark-up that successful incumbents charge on their production cost and which depends on the competitive threats on their monopoly position. The adoption of a new specialized technology in which a new incumbent joins the existing ones lengthening the input string to $k_{j,t}$ intermediate goods must reckon with a twofold competitive threat. On the one hand, erstwhile monopolists and now followers, although availing themselves of an outdated technology, are in a position, see Assumption 4 (ic), to incorporate the new specializing input and narrow the gap separating them from the more productive one. On the other hand, the $k_{j,t-} = k_{j,t} - 1$ incumbents in charge [7] are likely to resist the adoption of the new technology

[7] Since specialization lowers the mark-up (see Assumption 4 (ib)), previous incumbents will not freely adopt the new technology.

implied by specialization that would, in fact, decrease their share of the industry total profits. The first threat may be averted by limiting[8] the mark-up to $c_{j,t}$ since the gap between the two technologies is still e^{λ}. As to the second one, the new technology with $k_{j,t}$ intermediate inputs will be adopted only if this new mark-up $c_{j,t}$ is larger or equal to the bottom-line one, denote it by $c'_{j,t}$, that the previous $k_{j,t}-1$ incumbents can charge without allowing their followers to enter the market (again, see Assumption 4 (ic)).

Lemma 2 *(24) holds if and only if*

$$e^{\lambda}-1 \geq \frac{\frac{1}{a_{j,0}^{k_{j,t}}}}{\frac{\eta}{b_{j,0}}+\sum_{k'=1}^{k_{j,t-}}\frac{1}{a_{j,0}^{k'}}} \tag{25}$$

Moreover, if (25) holds, then

$$c'_{j,t} \leq c_{j,t} \tag{26}$$

Proof. In the Appendix. ■

This lemma simply states that the increase in productivity λ must be sufficiently high as to more than offset the increase in real direct and indirect labour costs implied by the lengthening of the specialization string. If this is the case, then the productivity of the new technology is sufficiently high to force the old incumbents to fine tune their own technology, allow a specializing spin-off to introduce a new intermediate input and share[9] their monopoly position with this new producer.

Remark 2 *We explicitly remark that the right hand side of inequality (25) is a decreasing function of $k_{j,t}$. Thus, if $k_{j,t}$ is above a certain threshold, then (25) holds and this is all the more true at every successive specialization event. If this is not the case, then specialization may not be viable.*

5 Innovation, growth and structural change

In this section we specify the innovation processes and determine the economy's stationary state as well as its expected growth rate. Furthermore, we characterize a traverse process as demand undergoes a structural shift. Three different innovation events are actually dealt with, two of them being concomitant. The first is a vertical innovation, a firm-specific occurrence, that results from

[8] See Lemma 1 and (21).

[9] We stress again that previous incumbents will not freely adopt the new technology (see (48) in the proof of Lemma 2 in the appendix).

followers' researching efforts and that comes to pass with a Poisson arrival rate. Since this rate is increasing with the number of employees (h) hired to carry out this process, it is expedient to normalize it as h such that the probability of an innovation event in a period dt of time is hdt. These efforts in the same time period imply a cost that for simplicity's sake is rendered by $C(h) = \frac{a}{2}h^2 + \frac{F}{2}$.

The second and third are a specialization spin-off and a rationalization process occurring in consequence of a demand shift that is assumed to happen according to an arrival rate μ_d: the former where the demand share becomes larger, the latter where it contracts. As discussed in the Section 2, demand shifts depend on relative price changes. Reference is made to a consumption pattern which arranges goods according to their priority in terms of the quality of life that they can afford and thus in accordance to income effective purchasing power. The price decline is made possible by technology-driven productivity increases being the result of firms' innovative efforts. In this context, it is expedient to assume that the Poisson arrival rate μ_d depends on the research efforts of firms engaged in high priority good production [10].

The value of an innovation to those who attempt it depends on the profit flow and, crucially, on the occurrence of these events according to whether they happen to be in expanding or contracting industries. To an innovator in an expanding industry, the value of an innovation depends on the flow of profits and on the likelihood of being ousted by the next generation of vertical innovators who will be employing $h_{f,t}$ workers for this purpose defining an arrival rate of equal magnitude. To one in an industry that stands a chance of contracting, the value, besides on the profit flow, depends on the likewise probability of being ousted and on the probability of being involved in a rationalization process that makes his sector redundant. Let $V_{j,t}$ designate the value of an innovation in the j-th industry. If equilibrium prevails, innovators in industry j will reckon that their expected flow of profits based on the likely value of their innovation is, given a discount rate r,

$$rV_{j,t} = \pi_{j,t}w - h_{f,t}V_{j,t} - \mu_d V_{j,t}\frac{\max\{\Delta k_{j,t}, 0\}}{k_{j,t}} \tag{27}$$

where $\Delta k_{j,t} = k_{j,t} - k'_{j,t}$ is the difference between the number $k_{j,t}$ of intermediate producers in industry j at time t and their expected number $k'_{j,t}$ after the demand shock. Thus, $\frac{\max\{\Delta k_{j,t},0\}}{k_{j,t}}$ indicates the probability of an intermediate producer in that industry of being caught in the rationalization process, given the probability of demand shock $\mu_d dt$: this probability is positive if the demand shock is negative and zero otherwise. Solving (27):

[10] A more precise formulation of this assumption will be given below (see Assumption 7).

$$V_{j,t} = \frac{\pi_{j,t}}{r + h_{f,t} + \mu_d \frac{\max\{\Delta k_{j,t}, 0\}}{k_{j,t}}} \tag{28}$$

Maximization of the net expected value of a vertical innovation for a follower in the intermediate sector of industry j reads[11]

$$W_{j,t} = \max_h \left[hV_{j,t} - \frac{a}{2}h^2 - \frac{F}{2} \right] \text{ for } j = 1..., J \ ; \tag{29}$$

here we assume that firms do not internalize the effect of their own research efforts on the arrival rate μ_d.

The J first order conditions resulting from the maximization problems (29) yield the followers' innovative efforts $h_{j,t}$ as functions of the number of monopolists in each industry and of the expected efforts of future and present competitors. Since we are assuming that there is no entry barrier, new spin-offs occur as long as the net expected value of a vertical innovation remains positive; this means that, after a demand shock, the number of monopolists $k_{j,t}$ in industry j will increase [decrease] if the share $\beta_{j,t}$ increases [decreases], as long as $W_{j,t} > 0$. Since in equilibrium expected values equal current ones, the free entry condition determines the size of $k_{j,t}$, for each j and t.

In the following subsections we are going to characterize the stationary state levels of both employment and output as well as stationary state growth rates. In subsection 5.1 the structure of demand shares is assumed as given and coinciding with a level of per capita income denoting a corresponding development level. In subsection 5.2 shares evolve due to price changes and the stationary states thereof implied are considered. Furthermore, a traverse from a stationary state to another is investigated as a consequence of a demand share shift. For the following we introduce the simplifying assumption that $r = 0$ and consider $k_{j,t}$ to be real valued.

5.1 Stationary state

We first characterize the static product and labor market equilibrium equilibrium. Given aggregate demand for good j at time t (6), equilibrium requires that at any point of time t, real demand and supply match:

$$y^s_{j,t} = y^d_{j,t} = \beta_{j,t} \frac{Y_t}{p_{j,t}} \quad . \tag{30}$$

[11] We recall that, thanks to Assumption 5 of no arbitrage, the expected value depends only on the industry j and not on sector k. Therefore all followers in the same industry j will choose the same optimal effort $h_{j,t}$.

From this the supply of intermediate good k and the employment in the final sector of industry j follow at once:

$$x^k_{j,t} = \beta_{j,t}\frac{1}{a^k_{j,t}}\frac{Y_t}{p_{j,t}} \ , \tag{31}$$

$$l^y_{j,t} = \beta_{j,t}\frac{1}{b_{j,t}}\frac{Y_t}{p_{j,t}} \ . \tag{32}$$

The k-th sector realized profits (23) in this industry are, in consequence, a mere proportion of aggregate output:

$$\pi_{j,t} = \left(1 - e^{-\lambda}\right)\frac{1}{k_{j,t}}\beta_{j,t}Y_t \ , \tag{33}$$

showing that the sector-wise flow of profits, given the productivity rate of increase and the demand share, depends only on the size of aggregate output and the extent of specialization. As mentioned above, the total industry profit fund $\Pi_{j,t} = \pi_{j,t}k_{j,t} = \left(1 - e^{-\lambda}\right)\beta_{j,t}Y_t$ is a function of share $\beta_{j,t}$ and aggregate output; a fact that indicates that deepening specialization implies dividing up in smaller slices the same volume of profits, for a constant productivity rate of increase. As discussed below in greater detail, $\Pi_{j,t}$ has the further implication that, ceteris paribus, profits are higher where the demand share is higher. As it is to be expected, incentives clearly lie where demand for final goods is relatively higher and it pays to further specialize in that process of final production the demand for which is proportionately higher than the relatively longer string length.

We assume that the available labour force does not change over time.

Assumption 6 *The size of overall employment $\mathcal{L}$ is kept constant*

$$\mathcal{L} = L^y_t + L^x_t + H_t \ .$$

Here $L^y_t = \sum_{j=1}^{J} l^y_{j,t}$ is the total final-good employment while $L^x_t = \sum_{j=1}^{J}\sum_{k=1}^{k_{j,t}} l^k_{j,t}$ is the total employment of intermediate goods producing sectors across all industries, finally $H_t = \sum_{j=1}^{J} h^k_{j,t}$ is the employment of manpower that followers use to conjure up the next round of innovations where $h^k_{j,t}$ measures the innovative effort of firm k in industry j at time t.

Lemma 3 *Profits in each intermediate sector of any industry j are proportional to the final goods* *industries' total employment:*

$$\pi_{j,t} = \frac{\frac{\beta_{j,t}}{k_{j,t}}}{\sum\limits_{j'=1}^{J} \frac{\beta_{j',t}}{1+\delta_{j',t}k_{j',t}}} \left(e^{\lambda}-1\right) w_t L_t^y \tag{34}$$

Proof. In the appendix. ■

We next define the stationary state.

Definition 1 *A stationary state is the state in which demand shares do not and are not expected to change over time.*

The following proposition characterizes the economy's stationary state.

Proposition 1 *In the stationary state the number of intermediate goods sectors in industry j amounts to:*

$$k_j = \beta_j \left(1 - e^{-\lambda}\right) \frac{Y}{F} \quad , \tag{35}$$

from which the overall total number of sectors:

$$\bar{k} = \sum_{j=1}^{J} k_j = \left(1 - e^{-\lambda}\right) \frac{Y}{F} \tag{36}$$

and aggregate output is

$$Y = \frac{\mathcal{L}}{\frac{1}{we^{\lambda}} + \sqrt{\frac{1}{Fa}\left(1 - e^{-\lambda}\right)}} \quad . \tag{37}$$

Proof. In the Appendix. ■

Note that Y remains constant over time and that it is an increasing function of $\mathcal{L}$. Furthermore $k_j = \beta_j f(\mathcal{L})$ while $\bar{k} = f(\mathcal{L})$ remains constant over time, $f(\mathcal{L})$ increasing in $\mathcal{L}$ and decreasing in F.

Remark 3 *Inserting (33) into (37) we find a simple expression for the aggregate nominal profit $\Pi_t = \sum_{j=1}^{J} \Pi_{j,t}$*

$$\Pi_t = \frac{\mathcal{L}}{\frac{1}{w(e^{\lambda}-1)} + \sqrt{\frac{1}{Fa}}}$$

Π_t is also constant in time.

It is interesting to note that, in the stationary state, while the extent of specialization in each sector depends on the extant demand share and hence on the current stage of development, aggregate profits depend only on the extent of the market as measured by total employment. Having characterized stationary state levels, the ensuing subsection will address the issue of the economy's long-run growth rate highlighting some of its properties concerning the role of specialization and the size of the market.

5.2 The stationary state growth rate

Although the view we hold of this economy is one in which a development process continuously reshapes the pattern of final demand, we nevertheless assess these long-run properties in the stationary state, namely when demand shares remain constant and no drive to both specialization and rationalization occurs. Thus, the focus will lie on a comparison of growth rates for different but constant demand shares. Because of the assumptions that have been made (see Assumption 3 and 6 and (48)), the economy growth is essentially due to productivity growth which is, in turn, explained by the innovations that followers conjure up in their strive to oust reigning monopolists, a feat that is achieved thanks to investment in research and development.

Proposition 2 *The stationary state growth rate is*

$$g_{Y_R} = \sqrt{\frac{F}{a}} \sum_{j=1}^{J} \beta_j \left(e^{\lambda \beta_j f(\mathcal{L})} - 1 \right) \quad . \tag{38}$$

Proof. In the Appendix. ■

This result lends itself to some interesting interpretations. The first observation is that g_{Y_R} depends on total employment $\mathcal{L}$, effectively a proxy of the extent of the market. In an economy in which it is held constant, the sum of the intermediate sectors, $\overline{k}$, also remains constant, specialization having gone further where demand shares had increased and restructuring eased sectorial costs where they had declined. Thus, since the growth rate in each point of a stationary state sequence depends on a specific distribution of demand shares, a stage of development as denoted by the corresponding demand shares is accordingly identified. Furthermore, it follows that a larger $\mathcal{L}$ implies a greater number of intermediate sectors hence a higher aggregate growth rate: a larger extent of the market in a Smithian sense as measured by $\mathcal{L}$ deepens overall specialization and enhances the economy's long-run growth rate.

The second observation is, as noted above and given $\mathcal{L}$, that the growth rate depends on the distribution of the intermediate sectors amongst the various industries. It is indeed straightforward to see that the aggregate growth factor is an average of the various industries' own factor weighted by aggregate demand shares, the latter having been shaped by the very development process. As this process unfolds assigning greater weights to goods that are less essential with real income growth, the impact of innovations owing to monopolists' followers' research and development efforts is spread more evenly over a larger number of industries. The implication is that the overall growth factor is lessened on account of a more balanced distribution of $\overline{k}$ over the entire number J of industries. Notice that the comparatively lowest growth factor occurs when $\beta_j = \beta = \frac{1}{J}$. This point has an interesting implication. If the development process gets under way, overall specialization being furthered and the size of the market increased, the growth rate of the less developed economies rises whilst that of the more developed ones slows down generating a process of convergence.

A third observation follows immediately from the previous two. An economy that manages to concentrate its aggregate output on fewer industries, other things being equal, achieves higher aggregate growth for the simple reason that specialization is also more concentrated: the same $\bar{k}$ distributed on fewer $j's$. This configuration may, for instance, occur in economies that, in spite of possessing a high real income per capita, through foreign trade have specialized in the production of a relatively small number of goods that it exports while importing many more allowing comparative advantage and higher growth. It is, furthermore, a well established result of the relevant theory that international trade by increasing the demand for the goods subject to relative specialization enlarges the size of the market, $\mathcal{L}$, leading to an yet higher long-run growth rate. It must, however, be stressed that graduating from a stage of development to the next depends crucially on the research and development process that finally yields innovations, productivity growth and ultimately the increase of income per head that reshapes the distribution of demand shares. It is on this logical sequence of events that the development process hinges upon, the sooner the innovation-led virtuous circle of productivity growth is ignited, the faster will growth be and the more effective the catching-up path.

What remains to be seen are the implications of passing from a demand pattern to another as a consequence of the above stated process. This is the topic of the following subsection.

5.3 Traverse dynamics

We analyze the traverse dynamics between two different demand patterns in two ways. Firstly, by introducing some simplifying assumptions we describe the main properties in an analytical way. Secondly, we confirm the main result through numerical simulations.

The level of generality that has so far been found expedient to illustrate the model does not easily lend itself to a detailed, general solution. We wish, nevertheless, to establish the pattern of specialization and rationalization as demand shares are subject to shifts on account of productivity and real income increases. This task basically involves determining the evolution of the $k_{j,t}$ strings as they lengthen in the former case and shorten in the latter. In order to illustrate the likely behavior of these processes, we proceed according to the following plan. We propose, first, a simplified version of the model featuring only two industries with homogeneous sectors. Simulations are then run to show that the analytical solutions that have been obtained are borne out by numerical results.

5.3.1 A simplified case

We begin by assuming the existence of only two industries, industry 2 initially producing luxury goods and industry 1 producing basic ones, and by normalizing the initial magnitudes of their sector productivities as to be equal at $t = 0$. Hence

$$a_{j,0}^{k} = a_0 \text{ for } k = 1, 2, \ldots \text{ and } j = 1, 2 \tag{39}$$

and

$$b_{j,0} = b_0 \text{ for } j = 1, 2 \tag{40}$$

While being initially the same at the conventional date, $t = 0$, when the innovation process begins, they will then follow their technological trajectory as and when their industry is involved in innovation events. Given this assumption, δ_j no longer depends on j [12] (see (15)) and $\delta = \frac{1}{\eta}\frac{b_0}{a_0}$. It is, therefore, the reach of specialization that crucially establishes how much of the profit fund each monopolist gets. Since demand shocks are more frequent the more rapid the decline of the basic good price $p_{1,t}$ is and eventually the more frequent vertical innovations are, a suitable formalization of this causal chain can be rendered as follows:

Assumption 7 *The arrival rate of demand shocks μ_d directly depends on the (expected) average research efforts $\overline{h}_1$ of firms producing the high priority good $\mu_d(h) = \overline{\mu}_d \overline{h}_1$.*

[12] Simple expressions for the mark-up, prices and profits, output and employment can be obtained inserting (40), (39) and $\delta = \frac{1}{\eta}\frac{b_0}{a_0}$ into (21), (22), (34), (51), (52) and (54).

The first order condition for problem (29) yields $h^I_{j,t} = \frac{V_{j,t}}{a}$ and given the free entry condition $W_{j,t} = 0$ in industry j, (28) becomes simply

$$\frac{\pi_{j,t}}{1+\overline{\mu}_d \frac{\max\{\Delta k_{j,t},0\}}{k_{j,t}}} = F \ , \ j = 1,2 \tag{41}$$

The optimal research effort $h^I_{j,t}$ is constant in time and throughout industry (see the third of (53)).

In the limit for $\delta \to 0$, (41) for $j = 2$ becomes

$$\frac{\beta_{2,t}}{k_{2,t}} \left(e^{\lambda} - 1\right) wL_t = F \quad , \tag{42}$$

while (41) for $j = 1$ is

$$\frac{\beta_{1,t}}{k_{1,t}} \left(e^{\lambda} - 1\right) wL_t = \left(1 + \overline{\mu}_d \frac{\Delta k_{1,t}}{k_{1,t}}\right) F \quad , \tag{43}$$

where $\Delta k_{1,t} = k_{1,t} - k'_{1,t}$.

(42) and (43) yield the following values for $k_{1,t}$ and $k_{2,t}$

$$k_{2,t} = \beta_{2,t} \frac{\left(e^{\lambda} - 1\right) wL_t}{F} \quad , \tag{44}$$

$$k_{1,t} = \beta_{1,t} \frac{\left(e^{\lambda} - 1\right) wL_t}{\left(1 + \overline{\mu}_d\right) F} + \frac{\overline{\mu}_d}{1 + \overline{\mu}_d} k'_{1,t} \quad . \tag{45}$$

We now proceed to study a system in which several demand shocks, say T, sequentially occur. In consequence the economy evolves as a sequence $\tau = 0, 1, ..., T$ of successive meta-stationary state equilibria , each beginning and ending with a demand shock.

For the sake of tractability, we assume that all workers employed in the economy earn the same wage rate w and that $C_{i,j}$ is distributed across individuals $i \in \mathcal{L}$ such that at time 0 some already consume both goods, i.e. $j^*_{i,0} = 2$, while others consume only good one , ie. $j^*_{i,0} = 1$. Thus, changes in consumption patterns and demand shares are due to changes in relative prices and not to changes in income distribution. Accordingly, as prices p_1 and p_2 vary over time, more and more workers start consuming good 2 following the dynamics outlined in Section 2, with a corresponding increase in $\beta_{2,t}$ and a decrease in $\beta_{1,t}$.

Definition 2 *A meta-stationary state equilibrium is a state in which the demand shares does not change but in which it is expected to change over time.*

In this particular setting, which implicitly defines a stage of development, it is possible to calculate explicitly all the relevant variables of the model. Since the value of an innovation (43) depends on the number $\Delta k_{1,t}$ of intermediate sectors, which are expected to be severed to restructure industry 2, it is necessary to specify the assumptions on expectation formation. We assume that at each stage agents are able to exactly forecast the number of intermediate sectors in the subsequent stage. Under these assumptions we can prove the following results.

Proposition 3 *(a) The sequence* $\{k_{1,\tau}\}_{\tau=1}^{T}$ *is strictly decreasing. (b) The sequence* $\{k_{2,\tau}\}_{\tau=1}^{T}$ *is strictly increasing. (c) For every* $\tau = 1, ..., T-1$, $L_\tau > L_T$.

Proof. In the Appendix.

It is important to note that this proposition implies, see Assumption 5, that the level of manpower employed in supporting the innovation process declines during the traverse. This effect is due to the uncertainty that is generated by demand shifts and thus by the likelihood that rationalization may wipe out some intermediate producing sectors. ■

5.3.2 Simulation results

In this section we report a simulation experiment that bears out the above obtained algebraic results. The exercise is conducted by choosing an arrival rate of innovations accomplished by monopolists' followers which is ten times higher than that of demand shocks; thus, the former is set to be $\mu_V = 10$ while $\mu_d = 1$. The basic idea is that the structural change that owes to demand shifts prompted by income per head increases is an event that occurs less frequently than innovations to be ascribed to the researching process, the true engine of growth. The productivity growth that is assumed to take place when the latter come to pass is set to be equal to four per cent, $\lambda = 0.04$, while the number of demand shifts totals 20. As far as the narrative of this economy goes, what is being portrayed is a long process that allows for a substantial productivity growth and a significant structural change. There are two industries; hence the economy is simple: the initial share of the low priority good is 10% but it is allowed to rise to 50% as the process unfolds. The population size is neither huge nor exceedingly small, one million people are assumed to be employed and distributed between actual manufacturing, both final goods and intermediate ones, and highly skilled and qualified research and development activities. The experiment highlights the following topical results.

The impact of demand shifts on the number of specialized sectors producing intermediate goods: Figure 1 and 2 are an illustration. It can clearly be seen that as the share of demand (β_2) of the second industry rises the number of intermediate sectors that therein become active increases enhancing its specialization profile whilst (continuous line in Figure 1), symmetrically, the number of sectors in the industry where the share contracts ($\beta_1 = 1 - \beta_2$) fall in consequence of a restructuring process that cuts them short (dashed line in Figure 1).

(FIGURE 1 HERE)

The impact of demand shifts on employment. Fig 2. As it can be seen, manufacturing employment, both downstream in final goods sectors and upstream in intermediate goods ones, rises during the change-over process to fall back to its stationary level as the latter peters out. The opposite, of course, occurs for employment devoted to the research activity. This graph underscores an important occurrence. Demand shifts generate a restructuring process with winners and losers. In the industry where demand falls, would be innovators must take into account the probability of being the victims of this process; namely, they stand the chance of being evicted from the market. This is a likely event that lowers the value of an innovation and leads to underinvestment in innovation-dedicated employment: thus, its magnitude immediately falls and remains at a lower level until a stationary state is restored.

(FIGURE 2 HERE)

Figure 3 illustrates the trend of the real growth rate as the economy undergoes adjustments caused by demand share shifts. As it is to be expected, the growth rate slows down. This is due to the evening out of demand share contributions. Since specialized sectors become less concentrated, the growth contribution they supply is better distributed but has less impact on each industry. The simulation results that we obtain bear out the conclusion that as goods of lesser priority, initially with a small weight in total demand, gain relevance in the consumption basket while those of greater priority lose it the growth rate slows down other things remaining equal.

(FIGURE 3 HERE)

6 Conclusions

The inception of this paper is that technical progress is a major engine of growth and furthermore that it is endogenous to economic activity owing to the search-for-innovation efforts made by firms in quest of profits to be had only by ousting incumbent ones. This is the basic proposition that we exploit to construe a model that features vertically integrated industries in each of which competitive, profitless firms avail themselves of several technical progress embodying intermediate goods produced by specialized monopolist firms. It is the latter that have been the achievers of innovations that had served the purpose of ousting previous incumbents from the market; it is indeed the unfolding incumbent-entrant strife ensuing from the search for monopolists' *quasi* rents that drives the system to innovate and consequently to grow. This process does not come scot-free. Whilst incumbents confine themselves on manufacturing, keeping owners of outdated intermediate goods technologies at bay through a mark-up on costs that bars them from earning positive profits, entrants threaten them by attempting to innovate by hiring professional manpower. The outcome of innovations is a productivity increase which, in turn, lowers final goods prices. It is argued, however, that innovations that augment the performance of existing intermediate goods are not the sole source of productivity growth. Another source is identified in the concomitant processes of specialization and rationalization driven by the domain of profit opportunities pried open by the structure of demand for final goods. The profit fund arising from the relative demand in each industry, given the market forces at play, motivates a search to introduce new specialized intermediate goods that, likewise the vertical innovation case, enhance the whole industry's productivity. This drive towards specialization taking the shape of an addition to the string of inputs requires a deep and effective knowledge of the process of production as it is currently being carried out. This fact warrants the assumption that the protagonists of this innovative feat are those agents already engaged in intermediate good manufacturing and that this type of innovations be introduced as spin-offs of existing firms. The first implication is that the extent of specialization in each industry depends on the size of its profits. The second is that the occurrence of a relative demand shift fosters both the furthering of specialization in the expanding industries and the rationalization by reduction of the number of intermediate sectors in contracting ones. More specifically, since vertical innovators invest in manpower, we solve for the size of this professional employment by maximizing the expected value of an innovation depending on the probability of being themselves evicted by the next generation of followers as well as on the likelihood that a demand shift chasten them through rationalization. Once expected profits are determined given who probably contracts and who instead expands, the number of sectors in the various industries consequently follows. It is then shown that the distribution of sectors among industries depends on that of demand shares but that their total number depends solely on the market size that is here proxied by total employment. The solutions that are so derived allow to establish the long term growth rate as a function of these distributions and of this total. These results allow us to argue that the degree of development of an economy as reflected in a given distribution of demand shares is also characterized by a corresponding distribution of specialized sectors. Thus, given the market size and the total number of these sectors, as demand is reshaped by increasing productivity and incomes per head, they become more evenly distributed across industries. Since the growth rate hinges on the innovation contribution of each sector, the more evenly the distribution the slower the growth rate on account of a composition effect. A spate of interesting implications are then contrived. First, given an equal extent of the market and if an innovation process is indeed set off, growth rates tend to converge since less developed economies grow faster than more developed ones. Second, if economies entertain an international trade that effectively promotes division of labour among them, each may concentrate its specialized sectors on fewer industries thus achieving a higher growth rate. Third, the same occurs if international trade is allowed to expand since it enlarges the aggregate size of the market.

The final analysis of a traverse path from a demand pattern to another draws the paper to a close. By investigating what happens to a sector that expands and what to one that contracts, it is seen that it is indeed the case that the former increases the extent of its specialization through a higher number of intermediate sectors while the latter rationalizes it by diminishing them. Furthermore, owing to increased uncertainty over the possibility of being let out of the market, innovative investment in professional manpower falls while that of manufacturing rises until equilibrium is reestablished as soon as the stationary state is restored.

References

[1] Acemoglu, D. and V. Guerrieri, 2008. Capital Deepening and Nonbalanced Economic Growth. Journal of Political Economy, 116, 467 - 498.

[2] Aghion, P., Howitt, P., 1992. A model of growth through creative destruction. Econometrica, 60, 323–351.

[3] Buera, F. and J.P. Kaboski, 2009. Can Traditional Theories of Structural Change Fit the Data? Journal of the European Economic Association, 7, 469 - 477.

[4] Ciccone, A., 2002. Input Chains and Industrialization. Review of Economic Studies, 69, 565-587.

[5] David, P., 1976: Technical Choice, Innovation and Economic Growth. London, Cambridge University Press.

[6] Foellmi, R. and J. Zweimüller, 2006. Income Distribution and Demand-Induced Innovations. Review of Economic Studies, 73, 941-960.

[7] Foellmi, R. and J. Zweimüller, 2008. Structural change, Engel's consumption cycles and Kaldor's facts of economic growth. Journal of Monetary Economics, 55, 1317 - 1328.

[8] Kongsamut, P., S. Rebelo and D. Xie, 2001. Beyond Balanced Growth. Review of Economics Studies, 68, 869 - 882.

[9] Laitner, J., 2000. Structural Change and Economic Growth. Review of Economic Studies, 67, 545 - 561.

[10] Matsuyama, K., 2002. The Rise of Mass Consumption Societies. Journal of Political Economy, 110, 1035 - 1070.

[11] Matsuyama, K., 2008. Structural Change. In: Durlauf, S.N. and L.E. Blume (Eds.), The New Palgrave Dictionary of Eco- nomics, 2nd edition. Palgrave McMillan.

[12] Mokyr J.,1990: The Lever of Riches. New York, Oxford University Press.

[13] Ngai, L.R. and C.A. Pissarides, 2007. Structural Change in a Multisector Model of Growth. American Economic Review, 97, 429 - 443.

[14] Romer, P., 1987. Growth Based on Increasing Returns Due to Specialization. American Economic Review, 77, 56 - 63.

[15] Rosenberg, N.,1990. Inside the Black Box. Cambridge, Cambridge University Press.

Appendix

Proof of Lemma 1. Since the new technique is more productive than the preceding one, the innovator can push the price of the final good down to a level $p'_{j,t}$, driving the mark-up $c'_{j,t}$ of the previous incumbents to zero, thus ousting them from the market. By (10) and (13) with $c'_{j,t} = 0$ we obtain

$$
\begin{aligned}
p'_{j,t} &= \frac{w}{b_{j,t}e^{-\lambda}}\left(1+\sum_{k=1}^{k_{j,t}}\frac{b_{j,0}}{\eta a_{j,0}^{k}}\right)\\
p_{j,t} &= \frac{w}{b_{j,t}}\left(1+\sum_{k=1}^{k_{j,t}}\left(1+c_{j,t}a_{j,t}^{k}\right)\frac{b_{j,0}}{\eta a_{j,0}^{k}}\right)
\end{aligned}
$$

The equation $p'_{j,t}=p_{j,t}$ together with (15) gives (21). ■

Proof of Lemma 2. First we show that necessary and sufficient condition for (26) is

$$
e^{\lambda}-1\geq\frac{\frac{\omega}{\eta}\frac{b_{j,0}}{a_{j,0}^{k_{j,t}}}}{1+\omega\delta_{j,t}k_{j,t}}\quad . \tag{46}
$$

To see this, note that the maximum mark-up, which prevents followers to enter the market with an outdated but specialized technology, can easily be calculated by equating the latter unitary cost of production to the cost of production plus the mark-up that incumbents can charge by still using the shorter, pre-specialization one [13]

$$
\left(\frac{1}{b_{j,t-}}+\omega\sum_{k=1}^{k_{j,t}}\frac{1}{\eta a_{j,t-}^{k}}\right)w_t=\left(\frac{1}{b_{j,t-}}+\omega\sum_{k=1}^{k_{j,t}-1}\frac{1}{\eta a_{j,t-}^{k}}\left(1+c'_{j,t}a_{j,0}^{k}\right)\right)w_t\quad , \tag{47}
$$

whence $c'_{j,t}=\frac{1}{a_{j,0}^{k_{j,t}}k_{j,t}}$. Substituting this (21) and (15) in (26), we obtain (46). It is clear that if (25) holds, then also (46) does.

We next show that under Assumption 2, condition

$$
c_{j,t-}\geq c_{j,t} \tag{48}
$$

holds. Note that (48) is equivalent to $\frac{1}{k_{j,t-}}\geq\frac{\omega}{\eta}\left(\frac{b_{j,0}}{a_{j,0}^{k_{j,t}}}-\frac{1}{k}\sum_{k'=1}^{k_{j,t-}}\frac{b_{j,0}}{a_{j,0}^{k'}}\right)$; this may be written as $\frac{1}{a_{j,0}^{k_{j,t-}+1}}\leq\frac{1}{k_{j,t-}}\left(\frac{\eta}{\omega b_{j,0}}+\sum_{k'=1}^{k_{j,t-}}\frac{1}{a_{j,0}^{k'}}\right)$, which in turn holds iff $\frac{b_{j,0}}{\eta a_{j,0}^{k}}\leq\delta_{j,t-}$ for $k=1,2,...$ and $j=1,...,J$ (see 17), whence (18). ■

Proof of Lemma 3. Employment in both final and intermediate sectors can be characterized in

[13] We stress explicitly that in (47) the productivity $b_{j,t-}$ is the same for the incumbents' old technology and for that available to followers. This holds since we have assumed that specialization may also be exploited by followers and the productivity gain is the same in the case of pure innovations and of specialization Assumption (4(ic)), (14) and (16)).

terms of aggregate output. From (22), (31) , (32) workers engaged in producing a final good j number:

$$l^y_{j,t} = \beta_{j,t}\frac{Y_t}{we^{\lambda}\left(1+\delta_{j,t}k_{j,t}\right)} \tag{49}$$

Bearing in mind (49), L^y_t is seen to be a function, other things being equal, of the extent of specialization:

$$L^y_t = \frac{Y_t}{we^{\lambda_v}}\sum_{j=1}^{J}\left(\frac{\beta_{j,t}}{1+k_{j,t}\delta_{j,t}}\right) \tag{50}$$

From (50)

$$Y_t = \frac{e^{\lambda}}{\sum\limits_{j=1}^{J}\frac{\beta_{j,t}}{1+\delta_{j,t}k_{j,t}}}wL^y_t \tag{51}$$

The expression within the sum in (50) is the value of effective demand reaching industry j in terms of its real labour cost; as such it is a measure of demand in real terms. It follows that its sum is an index of real aggregate demand that divided by the productivity augmented wage rate yields the volume of employment it can afford.

Given (49) and (51) the industry j labour force can simply be viewed as a proportion of total employment where the proportional factor is the ratio of an index of real effective demand of industry j to the index of real aggregate demand:

$$l^y_{j,t} = \frac{\frac{\beta_{j,t}}{1+\delta_{j,t}k_{j,t}}}{\sum\limits_{j'=1}^{J}\frac{\beta_{j',t}}{1+\delta_{j',t}k_{j',t}}}L^y_t \tag{52}$$

Notice, finally, that (34) follows from (23), (52) and (11), (9). ■

Proof of Proposition 1. Given the stationary state definition, $k_{j,t}$ remains invariant, hence (28) becomes $V_{j,t} = \frac{\pi_{j,t}}{h}$; consequently the first order conditions[14] for problem (29) yield

$$\begin{aligned} h^k_{j,t} &= h_{j,t} = \frac{V_{j,t}}{a} \quad , \\ \pi_{j,t} &= F \quad , \\ \text{and } h_{j,t} &= \sqrt{\frac{F}{a}} \quad . \end{aligned} \tag{53}$$

While from (11):

[14]Since $\pi_{j,t}$ is decreasing in $k_{j,t}$ and increasing in $k_{i,t}$ for $i \neq j$, see (34), the solution is unique.

$$L_t^x = \frac{Y_t}{we^{\lambda_v}} \sum_{j=1}^{J} \left(\frac{k_{j,t}\delta_{j,t}}{1+k_{j,t}\delta_{j,t}} \beta_{j,t} \right) \tag{54}$$

Given Assumption 6 , (49) and (54) it follows that sectorial employment in each industry is

$$l_{j,t}^x = \sum_{k=1}^{k_{j,t}} l_{j,t}^k = \beta_{j,t} \sum_{k=1}^{k_{j,t}} \frac{b_{j,t}}{\eta a_{j,t}^k} \frac{Y_t}{we^{\lambda}\left(1+\delta_{j,t}k_{j,t}\right)} = \beta_{j,t} Y_t \frac{\delta_{j,t}k_{j,t}}{we^{\lambda}\left(1+\delta_{j,t}k_{j,t}\right)}$$

and total manufacturing employment :

$$L_t = \sum_{j=1}^{J} \left(l_{j,t}^y + l_{j,t}^x \right) = \frac{Y_t}{we^{\lambda}} \quad . \tag{55}$$

From (33) and the free entry condition in the stationary state (35) and (36) are obtained. The third of (53) and (36) allow us to calculate the employment H_t that followers hire to conjure up the next round of innovations.

Considering (50) and (54) , H_t is, then, equal to

$$\mathcal{L} - \frac{Y_t}{we^{\lambda_v}} = H_t \tag{56}$$

Finally from (56), (36) and the third of (53),

$$\mathcal{L} = Y \left\{ \frac{1}{we^{\lambda}} + \sqrt{\frac{1}{Fa}} \left(1-e^{-\lambda}\right) \right\} \quad . \tag{57}$$

aggregate final good output (37) follows. ■

Proof of Proposition 2. The accounting definition of final goods aggregate growth, given (33), demand and supply being in equilibrium[15], is:

$$\frac{\dot{Y}_t}{Y_t} = \sum_{j=1}^{J} \beta_{j,t} \left(\frac{\dot{p}_{j,t}}{p_{j,t}} + \frac{\dot{y}_{j,t}}{y_{j,t}} \right) \quad .$$

In this model, however, the nominal wage rate is kept constant and productivity gains translate into proportionally lower prices such that the real wage rate and likewise real monopolists' profits grow in step with productivity. Because of the stationary state assumption in which the economy is taken to be in a notional state in which neither demand shares nor specialization strings vary, we can account

[15] By (30) this means that $\frac{p_{j,t}y_{j,t}}{Y_t} = \beta_{j,t}$.

for real growth by assuming them to be β_j and k_j respectively, both remaining invariant at a base point in time. In this case, long-run real growth $Y_{R,t}$ turns out to be:

$$\frac{\dot{Y}_{R,t}}{Y_{R,t}} = \sum_{j=1}^{J} \beta_j \frac{\dot{y}_{j,t}}{y_{j,t}} \quad ,$$

from which we can characterize the expected long-run growth rate, recalling that the Poisson arrival rate is $h_{j,t} = \sqrt{\frac{F}{a}}$, as:

$$g_{Y_R} = E\left(\frac{\dot{Y}_R}{Y_R}\right) = \sqrt{\frac{F}{a}} \sum_{j=1}^{J} \beta_j \left(e^{k_j \lambda} - 1\right) \quad ,$$

since $k_j = \beta_j f(\mathcal{L})$, the expression in (38) follows. ■

Proof of Proposition 3. Thanks to the role of expectations in our model, the proofs are obtained beginning from the final stage $\tau = T$ and then proving backward recursion formulae for $k_{j,\tau}$ and L_τ.

In the final state $\Delta k_{1,\tau} = k_{1,\tau} - k'_{1,\tau} = 0$ because no further demand shock is expected. From (53) (15) and (28), (34)

$$h^I_{j,\tau} = \frac{V_{j,\tau}}{a} = \frac{\pi_{j,\tau}}{ah^I_{j,\tau}} = \frac{\frac{\beta_{j,\tau}}{k_{j,\tau}}\left(e^{\lambda} - 1\right) wL_\tau}{ah^I_{j,\tau}} \quad . \tag{58}$$

Substituting from (58) and (53) the total employment $\mathcal{L}$ becomes a function of L_τ

$$\mathcal{L} = L_\tau + k_1 h^I_{1,\tau} + k_1 h^I_{1,\tau} = \left(1 + \frac{\left(e^{\lambda} - 1\right) w}{\sqrt{aF}}\right) L_\tau \quad . \tag{59}$$

Thus the labour force employed in manufacturing diminishes when the productivity gain λ of an innovation increases or the wage rate w increases, whereas it diminishes when either the fixed cost of research F or the variable one a, or both, increase

$$L_\tau = \frac{\mathcal{L}}{1 + \frac{(e^{\lambda}-1)w}{\sqrt{aF}}} \quad . \tag{60}$$

The reverse is true for employment hired to search for innovations

$$H^I_\tau = k_1 h^I_{1,\tau} + k_2 h^I_{2,\tau} = \frac{\frac{(e^{\lambda}-1)w}{\sqrt{aF}}\mathcal{L}}{1 + \frac{(e^{\lambda}-1)w}{\sqrt{aF}}} \quad . \tag{61}$$

Consider the next to the last state, where $\Delta k_{1,\tau} = k_{1,\tau} - k'_{1,\tau} > 0$. From the first of (53), (28),

(15) and (34) with $r=0$ and $j=2$

$$h^I_{2,\tau} = \frac{V_{2,\tau}}{a} = \frac{\pi^k_{2,\tau}}{ah^I_{2,\tau}} = \frac{\frac{\beta_{2,\tau}}{k_{2,\tau}}\left(e^\lambda - 1\right) wL_\tau}{ah^I_{2,\tau}} \quad ; \tag{62}$$

again from (53), (28), (15) and (34) with $r=0$ and $j=1$

$$h^I_{1,\tau} = \frac{V_{1,\tau}}{a} = \frac{\pi^k_{1,\tau}}{ah^I_{1,\tau}} = \frac{\frac{\beta_{1,\tau}}{k_{1,\tau}}\left(e^\lambda - 1\right) wL_\tau}{a\left(1+\overline{\mu}_d\frac{\Delta k_{1,\tau}}{k_{1,\tau}}\right)h^I_{1,\tau}} \quad . \tag{63}$$

We conclude with total employment; from the last of (53)gain w

$$\mathcal{L} = L_\tau + k_1 h^I_{1,\tau} + k_2 h^I_{2,\tau} = L_\tau + (k_{1,\tau} + k_{2,\tau})\sqrt{\frac{F}{a}} \quad ; \tag{64}$$

substituting from the last of (53), (44), (45), rearranging terms and solving for L_τ

$$L_\tau = \frac{\mathcal{L} - \sqrt{\frac{F}{a}}\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k'_{1,\tau}}{1 + \frac{(e^\lambda-1)w}{\sqrt{Fa}}\left(1 - \frac{\overline{\mu}_d}{(1+\overline{\mu}_d)}\beta_{1,\tau}\right)} \quad . \tag{65}$$

Consider next the necessary and sufficient condition for the monotonicity of the sequence $\{k_{1,\tau}\}_{t=1}^T$. Since we are assuming that, at every period τ, agents correctly forecast the future value of k_1, putting $k'_{1,\tau} = k_{1,\tau+1}$, (44), (45), (65) become

$$k_{2,\tau} = \beta_{2,t}\sqrt{\frac{a}{F}}\Phi L_\tau \tag{66}$$

$$k_{1,\tau} = \beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi\frac{1}{1+\overline{\mu}_d}L_\tau + \frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1} \tag{67}$$

$$L_\tau = \frac{\mathcal{L} - \sqrt{\frac{F}{a}}\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1}}{1 + \Phi\left(1 - \frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)} \tag{68}$$

where $\Phi = \frac{e^\lambda - 1}{\sqrt{aF}}w$.

Now, $k_{1,\tau} > k_{1,\tau+1}$ if

$$k_{1,\tau+1} < \beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi\frac{1}{1+\overline{\mu}_d}L_\tau + \frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1} \quad , \tag{69}$$

substituting (68) into (69) we obtain

$$k_{1,\tau+1} < \beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi\frac{1}{1+\overline{\mu}_d}\frac{\mathcal{L}-\sqrt{\frac{F}{a}}\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)}+\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1} \quad . \tag{70}$$

Elementary simplifications give the following result. The necessary and sufficient condition for the sequence $\{k_{1,\tau}\}_{t=1}^{T}$ to be decreasing is

$$k_{1,\tau+1} < \mathcal{L}\frac{\Phi}{(1+\Phi)}\beta_{1,\tau}\sqrt{\frac{a}{F}} \quad . \tag{71}$$

Consider the recurrence formula for the sequence $\{k_{1,\tau}\}_{\tau=1}^{T}$. Substituting (68) into (67) we obtain

$$k_{1,\tau} = \beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi\frac{1}{1+\overline{\mu}_d}\frac{\mathcal{L}-\sqrt{\frac{F}{a}}\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)}+\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1} \quad ,$$

rearranging terms

$$k_{1,\tau} = \frac{\mathcal{L}\beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi\frac{1}{1+\overline{\mu}_d}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)}+\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1}\frac{1+\Phi-\Phi\beta_{1,\tau}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)} \quad . \tag{72}$$

Using the recurrence formula (72), it is possible to prove that, if (71) holds, then it holds also for $k_{1,\tau}$ and $\beta_{1,\tau-1}$;

$$\begin{aligned} k_{1,\tau} \quad < \quad & \frac{\mathcal{L}\beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi\frac{1}{1+\overline{\mu}_d}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)}+ \\ & \frac{\overline{\mu}_d}{1+\overline{\mu}_d}\mathcal{L}\frac{\Phi}{(1+\Phi)}\beta_{1,\tau}\sqrt{\frac{a}{F}}\frac{1+\Phi-\Phi\beta_{1,\tau}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,\tau}\right)} \ ; \end{aligned} \tag{73}$$

eliminating the common factor $\mathcal{L}\frac{\Phi}{1+\Phi}\beta_{1,\tau}\sqrt{\frac{a}{F}}$, rearranging terms and remembering that $\left\{\beta_{1,\tau}\right\}$ is a decreasing function, we obtain

$$k_{1,\tau} < \mathcal{L}\frac{\Phi}{1+\Phi}\beta_{1,\tau-1}\sqrt{\frac{a}{F}} \quad . \tag{74}$$

We are ready to prove part (a) of the Proposition. Since (71) implies (74) for every τ, it is sufficient to prove that (71) holds for $\tau = T-1$. Since $k_{1,T} = k_{1,T+1}$ (67) yields

$$k_{1,T} = \beta_{1,\tau}\sqrt{\frac{a}{F}}\Phi L_T \tag{75}$$

which, together with (68), gives

$$k_{1,T} = \mathcal{L}\frac{\Phi}{1+\Phi}\beta_{1,T}\sqrt{\frac{a}{F}} \quad ; \tag{76}$$

finally, since $\beta_{1,T} > \beta_{1,T-1}$

$$k_{1,T} < \mathcal{L}\frac{\Phi}{1+\Phi}\beta_{1,T-1}\sqrt{\frac{a}{F}} \quad .$$

Remark 4 *From (75) and (76) we immediately get*

$$L_T = \mathcal{L}\frac{1}{1+\Phi} \tag{77}$$

and

$$k_{1,T} = \mathcal{L}\frac{\Phi}{1+\Phi}\beta_{2,T}\sqrt{\frac{a}{F}} \quad . \tag{78}$$

Substituting (76) into (68) it is easy to calculate L_{T-1}

$$L_{T-1} = \mathcal{L}\frac{1}{1+\Phi}\frac{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,T}\right)}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\beta_{1,T-1}\right)} \quad ; \tag{79}$$

clearly $L_{T-1} > L_T$.

Consider next part (b) of the Proposition. From (66) and (67) we get

$$k_{2,\tau} = \frac{\beta_{2,\tau}}{\beta_{1,\tau}}\left(1+\overline{\mu}_d\right)k_{1,\tau} + \frac{\beta_{2,\tau}}{\beta_{1,\tau}}\overline{\mu}_d k_{1,\tau+1} \quad . \tag{80}$$

Inserting the recurrence formula (72) into (80) and simplifying we obtain

$$k_{2,\tau} = \frac{\beta_{2,\tau}\Phi}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\left(1-\beta_{2,\tau}\right)\right)}\left(\mathcal{L}\sqrt{\frac{a}{F}}-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1}\right) \quad . \tag{81}$$

Since $\tau \to \left(\mathcal{L}\sqrt{\frac{a}{F}}-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}k_{1,\tau+1}\right)$ is positive and increasing by Proposition 3 and $\tau \to \frac{\beta_{2,\tau}}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\left(1-\beta_{2,\tau}\right)\right)}$ is positive and increasing as well, the same holds for $\tau \to k_{2,\tau}$.

Part (c) of the Proposition. First we show that the following recurrence formula holds for L_τ

$$L_\tau = \frac{\left(1+\beta_{2,\tau+1}\Phi\right) L_{\tau+1} + \mathcal{L}}{\frac{1+\overline{\mu}_d}{\overline{\mu}_d}\left(1+\Phi\right) - \beta_{1,\tau}\Phi} \quad . \tag{82}$$

From (64)

$$k_{2,\tau} + k_{1,\tau} = \sqrt{\frac{a}{F}}\left(\mathcal{L} - L_\tau\right) \quad . \tag{83}$$

From (81) we get

$$\frac{\overline{\mu}_d}{1+\overline{\mu}_d} k_{1,\tau+1} = \mathcal{L}\sqrt{\frac{a}{F}} - \frac{k_{2,\tau}}{\frac{\beta_{2,\tau}\Phi}{1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\left(1-\beta_{2,\tau}\right)\right)}} \quad , \tag{84}$$

substituting $k_{2,\tau}$ from (66) in (84) and adding to both sides the term $\frac{\overline{\mu}_d}{1+\overline{\mu}_d} k_{2,\tau+1}$ we obtain

$$\begin{aligned} \frac{\overline{\mu}_d}{1+\overline{\mu}_d} k_{2,\tau+1} + \frac{\overline{\mu}_d}{1+\overline{\mu}_d} k_{1,\tau+1} \quad = \quad & \mathcal{L}\sqrt{\frac{a}{F}} - \\ & \left(1+\Phi\left(1-\frac{\overline{\mu}_d}{1+\overline{\mu}_d}\left(1-\beta_{2,\tau}\right)\right)\right)\sqrt{\frac{a}{F}} L_\tau + \frac{\overline{\mu}_d}{1+\overline{\mu}_d} k_{2,\tau+1} \end{aligned} \tag{85}$$

substituting $k_{2,\tau+1}$ again from (66) and $k_{1,\tau+1} + k_{2,\tau+1}$ from (83) and arranging terms we obtain the recurrence equation

$$L_{\tau+1} = \frac{\left(\frac{1+\overline{\mu}_d}{\overline{\mu}_d}\left(1+\Phi\right) - \beta_{1,\tau}\Phi\right) L_\tau - \frac{1}{\overline{\mu}_d}\mathcal{L}}{1+\beta_{2,\tau+1}\Phi} \quad . \tag{86}$$

Inverting (86) we obtain the backward recurrence equation (82).

Next we show that the necessary and sufficient condition for $L_{\tau+1} < L_\tau$ is

$$L_{\tau+1} < \frac{\mathcal{L}}{1+\Phi-\overline{\mu}_d\left(\beta_{2,\tau+1}-\beta_{2,\tau}\right)} \quad . \tag{87}$$

From (82) $L_{\tau+1} < L_\tau$ means

$$L_{\tau+1} < \frac{\left(1+\beta_{2,\tau+1}\Phi\right) L_{\tau+1} + \mathcal{L}}{\frac{1+\overline{\mu}_d}{\overline{\mu}_d}\left(1+\Phi\right) - \beta_{1,\tau}\Phi} \quad . \tag{88}$$

Simplifying and rearranging terms we obtain (87).

By (82) and (77) $L_\tau > L_T$ becomes

$$\frac{\left(1+\beta_{2,\tau+1}\Phi\right) L_{\tau+1} + \mathcal{L}}{\frac{1+\overline{\mu}_d}{\overline{\mu}_d}\left(1+\Phi\right) - \beta_{1,\tau}\Phi} > \frac{\mathcal{L}}{1+\Phi} \quad . \tag{89}$$

Solving for $L_{\tau+1}$ (89) gives

$$L_{\tau+1} > \frac{\mathcal{L}}{1+\Phi}\frac{1+\beta_{2,\tau}\Phi}{1+\beta_{2,\tau+1}\Phi} \tag{90}$$

which implies $L_\tau > L_T$ since the sequence $\{\beta_{2,\tau}\}$ is increasing and $L_{T-1} > L_T$ by (79). ■

Figure 1

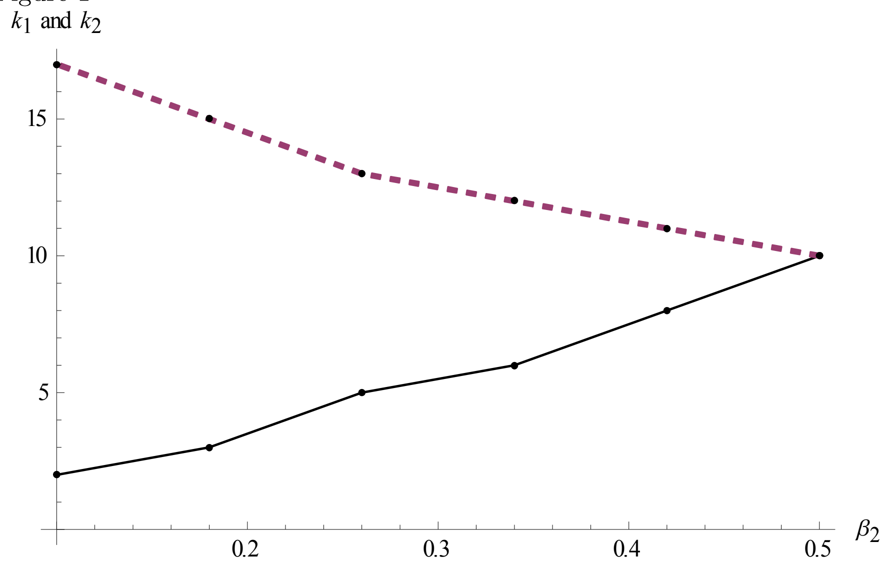


Figure 2

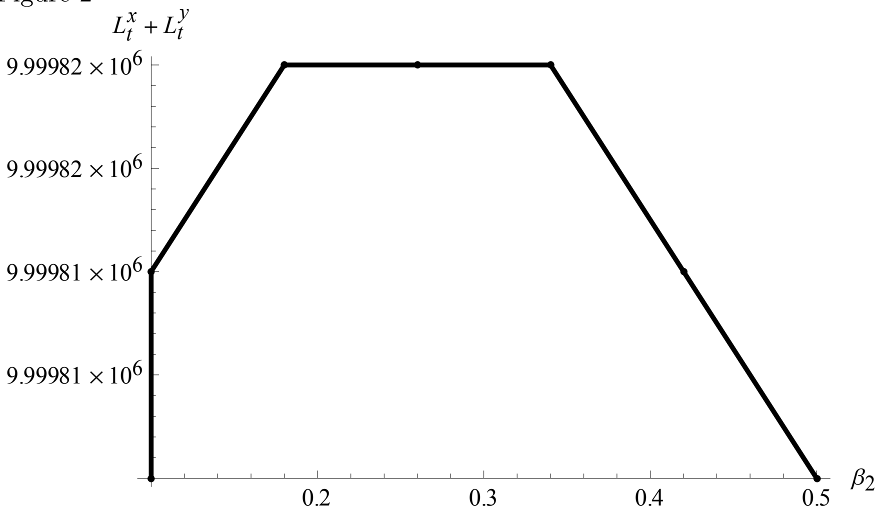
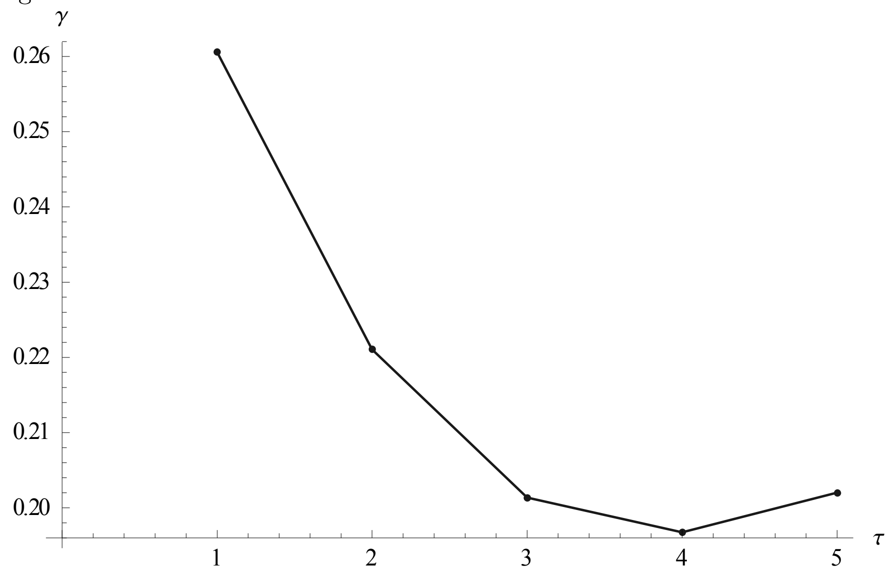


Figure 3



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