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Carl Chiarella
Nadima El-Hassan

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EVALUATION OF DERIVATIVE SECURITY PRICES IN THE HEATH JARROW-MORTON FRAMEWORK AS PATH INTEGRALS USING FAST FOURIER TRANSFORM TECHNIQUES

by

Carl Chiarella & Nadima El-Hassan

School of Finance and Economics

University of Technology, Sydney

Sydney, Australia

Tel: + 61 2 9514 3631

Fax: + 61 2 9514 3636

E-mail: Carl.Chiarella@uts.edu.au

E-mail: nadima@ghostgum.itd.uts.edu.au

Abstract

This paper considers the evaluation of derivative security prices within the Heath-Jarrow-Morton framework of stochastic interest rates, such as bond options. Within this framework, the stochastic dynamics driving prices are in general non-Markovian. Hence, in principle the partial differential equations governing prices require an infinite dimensional state space. We discuss a class of forward rate volatility functions which allow the stochastic dynamics to be expressed in Markovian form and hence obtain a finite dimensional state space for the partial differential equations governing prices. By applying to the Markovian form, the transformation suggested by Eydeland (1994), the pricing problem can be set up as a path integral in function space. These integrals are evaluated using fast fourier transform techniques. We apply the technique to the pricing of American bond options and compare the computational time with other methods currently employed such as the method of lines and more traditional partial differential equation solution techniques.

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1 Introduction

There are currently two popular approaches to the modelling of the term structure of interest rates (and related contingent claims) based on arbitrage arguments. The first (and earliest developed) descends from Vasicek (1977) and the second from the work of Heath-Jarrow-Morton (1992). Both approaches impose on the economy under consideration the condition of no riskless arbitrage between bonds of differing maturities. The Vasicek approach results in a partial differential equation for the price of bonds and related contingent-claims. A widely perceived disadvantage of this approach is its dependence on investor preferences in the form of the market price of interest rate risk. The HJM (Heath-Jarrow-Morton) approach, on the other hand, expresses prices as expectation operators of payoffs calculated with respect to a risk neutral measure and hence it has the advantage that it is independent of investor preferences. The HJM approach has a further advantage over the Vasicek approach in that it matches the currently observed yield curve.

A computational disadvantage of the HJM model is that its most general form is non-Markovian which means that the dynamical state variables describing the evolution of prices are path dependent. So that in principal the partial differential equations governing prices require an infinite dimensional state space, in fact they will be integro-partial differential equations. Hence the computational task in calculating derivative security prices within the HJM framework is of an order of magnitude higher than that associated with the solution of Vasicek type partial differential equations.

It is shown by Bhar and Chiarella (1995) that by choosing the forward rate volatility function as a product of a certain class of time to maturity functions with a function of the instantaneous spot rate of interest, the state space for the representation of the price dynamics in the HJM model becomes finite dimensional. Similar results were also shown by Cheyette

(1992), Ritchken and Sankarasubramanian (1995) and Carverhill (1994). Chiarella and El-Hassan (1996) show how to use this finite-dimensional representation to obtain a preference-free partial differential equation for prices of contingent claims in the HJM framework and investigate the use of the method of lines (see Carr and Faguet (1994), Goldenberg and Schmidt (1994), Meyer and Van der Hoek (1994)) as a solution technique.

In this paper we continue our investigation of the evaluation of contingent claim prices in the above finite-dimensional state space representation of the HJM model. In particular we apply the path integral method proposed by Linetsky (1996) using repeated applications of the Chapman Kolmogorov equation and take advantage of a transformation suggested by Eydeland (1994) which allows the use of fast fourier transform techniques to evaluate the integrals concerned. We apply the techniques to the pricing of European and American bond options and compare computational efficiency with the method of lines.

The plan of the paper is as follows. Section 2 reviews the HJM approach. This section covers familiar ground however we feel it is necessary to provide the necessary framework in a consistent notation for the sake of clarity. Section 3 reviews the results that allow the HJM dynamics to be reduced to Markovian from and so obtain the preference-free partial differential operator for the pricing of interest rate contingent claims. Section 4 reviews the path integral method and its application to our representation of the HJM model. Section 5 describes the algorithm used and its application to our problem. The results obtained are presented in section 6 along wth some comparison of the compuational efficiency of the method with other techniques such as the method of lines. Section 7 draws some conclusions.

2 The HJM Model

In this discussion of HJM we assume only one noise process, and draw on the intuitive derivation of Bar and Chiarella (1995). For a proper technical discussion the reader should refer to HJM (1992).

The driving state variable of the HJM approach is $f(t, T)$, the forward rate at time t for instantaneous borrowing at time $T(\geq t)$, which is assumed to be driven by a stochastic integral equation of the form

$$f(t,T) = f(0,t) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma^f(s,T)dw(s), \qquad (1)$$

or, in differential form,

$$df(t,T) = \alpha(t,T)dt + \sigma^f(t,T)dw(t), \qquad (0 \leq t \leq T). \qquad (2)$$

Here $\alpha(t, T)$ and $\sigma^f(t, T)$ are the drift and volatility of the forward rate respectively. In general these could depend on $f(t, T)$ or on $r(t)$. However, here, we merely assume that they are dependent on time, maturity and possibly on r(t).

Since the instantaneous spot rate of interest $r(t) \equiv f(t, t)$ it is a simple matter to derive

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma^f(s,t)dw(s) \qquad (3)$$

or, in differential form

$$dr(t) = \left[f_2(0,t) + \alpha(t,t) + \int_0^t \alpha_2(v,t)dv + \int_0^t \sigma_2^f(v,t)dw(v) \right]dt + \sigma^{(f)}(t,t)dw(t), \qquad (4)$$

where the subscript i denotes the partial derivative with respect to the i[th] argument and $f(0, t)$ can be obtained from the currently observed yield curve. It is the second integral in the drift term of equation (4) which renders the HJM framework non-Markovian, as it depends on the history of the noise process up to time t, and hence in general is path dependent. However, as discussed below certain choices of $\sigma^f(t, T)$ allow us to express the stochastic dynamics of the HJM economy in Markovian form.

From the instantaneous forward rate we can express the price at t of the pure discount bond maturing at T as

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right). \tag{5}$$

By use of Fubini's theorem and Ito's lemma, HJM show that the bond price must satisfy the stochastic differential equation

$$dP(t,T) = [r(t) + b(t,T)]P(t,T)dt + a(t,T)P(t,T)dw(t), \tag{6}$$

where

$$a(t,T) \equiv -\int_t^T \sigma^f(t,v)dv, \tag{7}$$

and

$$b(t,T) = -\int_t^T \alpha(t,v)dv + \frac{1}{2}a(t,T)^2. \tag{8}$$

As in the Vasicek approach, portfolios of bonds of differing maturities can be set up with bond dynamics now driven by equation (6). The condition that there be no riskless arbitrage opportunities between bonds of differing maturities here reduces to

$$b(t,T) + \phi(t)a(t,T) = 0, \tag{9}$$

where $\phi(t)$ is the market price of interest rate risk. This latter equation can be manipulated to yield

$$\alpha(t,T) = -\sigma^f(t,T)\left[\phi(t) - \int_t^T \sigma^f(t,v)dv\right]. \tag{10}$$

Equation (10) essentially states that in an arbitrage free economy the drift of the forward rate is determined by the forward rate volatility and the market price of interest rate risk.

Using equation (10), the stochastic integral equation for $r(t)$ becomes

$$r(t) = f(0,t) + \int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)dydu + \int_0^t \sigma^f(u,t)d\tilde{w}(u), \tag{11}$$

or, in differential form

$$dr(t) = \left[f_2(0,t) + \frac{\partial}{\partial t}\int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)\ dy\ du - \int_0^t \sigma_2^f(u,t)\ d\tilde{w}(u)\right]dt$$
$$+\ \sigma^f(t,t)\ d\tilde{w}(t) \tag{12}$$

whilst the stochastic differential equation for P(t,T) is

$$dP(t,T) = r(t)P(t,T)dt + \left[-\int_t^T \sigma^f(t,u)du\right]P(t,T)d\tilde{w}(t), \tag{13}$$

where the new Wiener process $\tilde{w}(t)$ is defined by

$$\tilde{w}(t) = w(t) - \int_0^t \phi(s)ds. \tag{14}$$

HJM further show that the conditions of Girsanov's theorem are satisfied under fairly unrestrictive assumptions on the $\sigma^f(t,T)$. This in turn implies that the probability measures under w and $\tilde{w}$ are equivalent. Furthermore under the equivalent measure $\tilde{w}(t)$ is a standard Wiener process.

From equations (11) and (13) it is then a fairly simple matter to show that the relative bond price, $Z(t,T) = P(t,T)\exp\left\{-\int_t^T r(y)dy\right\}$, satisfies the stochastic differential equation

$$dZ(t,T) = \left\{-\int_t^T \sigma^f(t,u)du\right\}Z(t,T)d\tilde{w}(t). \tag{15}$$

Clearly Z(t, T) is a martingale, hence the bond price may be expressed as

$$P(t,T) = \tilde{E}_t\left[\exp\left\{-\int_t^T r(y)dy\right\}\right], \tag{16}$$

where $\tilde{E}_t$ denotes the mathematical expectation calculated with respect to information at time t under the probability measure induced by $\tilde{w}$ (t).

The principal difficulty in calculating derivative security prices under the HJM approach is that it is not a simple matter to re-express $P(t, T)$ in (16) as the solution of a Vasicek type partial differential equation. Although, as we have stated in the introduction, it is in principle possible to obtain an integro-partial differential equation for $P(t, T)$ from (16). We show in the next section that under appropriate assumptions on the forward rate volatility $\sigma^f(t, T)$ it is possible to express $P(t, T)$ as the solution of a Vasicek type partial differential equation.

3. A Preference Free Partial Differential Equation

This section summarises the discussion of Chiarella and El-Hassan (1996). As we have stated in the previous sections, the system dynamics of the HJM framework are in general non-Markovian and the equation expressing prices will in general be some kind of integro-partial differential equation. However, Bhar and Chiarella (1995) show that if one assumes that the forward rate volatility function has the general form

$$\sigma^f(t,T) = p_n(T-t)e^{-\lambda(T-t)}G(r(t)), \tag{17}$$

where $p_n(u)$ is the polynomial

$$p_n(u) = a_0 + a_1u + \ldots + a_nu^n, \tag{18}$$

and G is some reasonably well behaved function, then the system dynamics may be expressed in Markovian form. The cost of this reduction is the expansion of the state space by the introduction of some supplementary state variables which summarise various statistical properties of the path history. Similar results are also reported by Cheyette (1992), Carverhill (1994) and Ritchken and Sankarasubramanian (1995).

3.1 Time Dependent Forward Rate Volatility

Here we restrict our attention to almost the simplest possible version of equation (17), namely

$$\sigma^f(t,T) = a_0 e^{-\lambda(T-t)}, \tag{19}$$

where a_0 and λ are constants.

Since this functional form has the property

$$\sigma_2^f(t,T) = -\lambda\sigma^f(t,T),$$

we see from equation (12) that the non-Markovian term may be expressed as

$$\int_0^t \sigma_2^f(u,t)d\tilde{w}(u) = -\lambda \int_0^t \sigma^f(u,t)d\tilde{w}(u).$$

However, from equation (11) we have

$$\int_0^t \sigma^f(u,t)d\tilde{w}(u) = r(t) - f(0,t) - \int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)dydu.$$

Thus, it is obvious that the non-Markovian term may be expressed as

$$\int_0^t \sigma_2^f(u,t)d\tilde{w}(u) = \lambda\left[f(0,t) + \int_0^t \sigma^f(u,t)\int_u^t \sigma^f(u,y)dydu - r(t)\right] \tag{20}$$

Finally, substituting equation (20) into equation (12), we see that the stochastic differential equation for *r(t)* may thus be written as

$$dr(t) = [D(t) - \lambda\ r]dt + \sigma d\tilde{w}(t), \tag{21}$$

where

$$D(t) = f_2(0,t) + \lambda\ f(0,t) + \int_0^t \sigma(v,t)^2 dv. \tag{22}$$

The stochastic differential equation (21) may be regarded as a preference free version of the one employed by Vasicek. The function *D(t)* is the (time dependent) long run mean which is determined by the current yield curve and the parameters σ and λ of the forward rate volatility function. This is in contrast to the original Vasicek formulation where D(t) is exogenously specified.

The expectation operator $\tilde{E}_t$ in equation (16) is induced by the stochastic differential equation (14) which under the forward rate volatility function in equation (19) becomes the stochastic differential equation (21). This stochastic differential equation has associated with it the Kolmogorov backward equation (24) for the transition probability density of the distribution of r_T conditional on r(t) = r, denoted by $\pi(r_T, T|r, t)$ (i.e. the probability of observing r_T at time T conditional on r at time $t<T$),

$$\frac{1}{2}\sigma^2\frac{\partial^2\pi}{\partial r^2}+[D(t)-\lambda r]\frac{\partial\pi}{\partial r}+\frac{\partial\pi}{\partial t}=0, \tag{24}$$

with $t_0 \le t \le T$.

The initial time t_0 could be the point in time that we are seeking to value the bond which in turn could be the maturity date of an option on a bond maturing at time T.

Introducing the elliptic partial differential operator K, given by

$$K=\frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2}+[D(t)-\lambda r]\frac{\partial}{\partial r} \tag{25}$$

we can write equation (24) as

$$K\pi+\frac{\partial\pi}{\partial t}=0, \qquad t_0 \le t \le T, \tag{26}$$

which must be solved subject to the initial condition

$$\pi(r_T, T|r, t)=\delta(r_T-r),$$

where δ is the Dirac delta function.

Given the transition probability density function π, we can in principle calculate the expectation in equation (16). By use of the Feynman-Kac formula we find that the expectation of the functional in equation (16) satisfies the partial differential equation

$$KP + \frac{\partial P}{\partial t} - rP = 0, \qquad t_0 \le t \le T, \tag{27}$$

subject to the initial condition $P(T,T) = 1$. This partial differential equation is of the same form as the Vasicek (1977) one, from which it may be obtained by replacing Vasicek's constant mean with the yield curve and volatility dependent D(t) and by setting Vasicek's market price of interest rate risk to zero. It is in fact the preference free HJM partial differential equation for the assumed forward rate volatility function.

3.2 More General Volatility Functions

In this subsection we allow the forward rate volatility to also be a function of the instantaneous spot rate of interest in the form

$$\sigma_f(t,T) = a_0 e^{-\lambda(T-t)} G(r(t)), \tag{28}$$

where G is a suitably well behaved function. Typically we might take

$$G(r) = r^\gamma, \qquad (0 \le \gamma). \tag{29}$$

The motivation for including a dependency on $r(t)$ in equation (28) is to capture the effect on volatility of a general movement in the level of market rates. Ideally we would also like to allow a dependency on $f(t, T)$ but it then becomes impossible to reduce the driving stochastic dynamics to Markovian form.

Bhar and Chiarella (1995) show that for the forward rate volatility (28) the stochastic dynamics for $r(t)$ may be expressed in the Markovian form,

$$dr = \left[f_2(0,t) + \lambda f(0,t) + a_0^2 \phi - \lambda r \right] dt + \sigma G(r) d\tilde{w}, \tag{30}$$

$$d\phi = \left[G(r)^2 - 2\lambda\,\phi\right]dt. \tag{31}$$

We note in passing that the quantity $\phi(t)$ is given by

$$\phi(t) = \int_0^t G(r(s))^2 e^{-2\lambda(t-s)} ds, \tag{32}$$

which can be calculated form the history of the r process up to any point in time t.

The Kolmogorov operator for stochastic differential equation system (32) may formally be written

$$K = \frac{1}{2}\sigma^2 G(r(t))^2 \frac{\partial^2}{\partial r^2} + \left[f_2(0,t) + \lambda f(0,t) + a_0^2\phi(t) - \lambda r\right]\frac{\partial}{\partial r} + \left[G(r)^2 - 2\lambda\phi\right]\frac{\partial}{\partial\phi} \tag{33}$$

The partial differential equation for the bond price may thus again be written

$$KP + \frac{\partial P}{\partial t} - rP = 0, \qquad t_0 \le t \le T. \tag{34}$$

The partial differential operator K now involves the two state variables r and ϕ. The early exercise boundary in the American bond pricing problem becomes an early exercise surface. This partial differential equation may be viewed as the continuous time equivalent of the lattice model of Li, Ritchken and Sankarasubramanian (1995).

4. Path/Functional Integrals

As a prelude to the solution algorithm used in this article, this section provides a brief heuristic overview to the technique of path (or functional) integrals widely used in statistical physics as expounded by Feynman (1972), Kac (1980), Simon (1979) and Muldowney (1987). The subject has wide applications in financial modelling and pricing of path-dependent financial contracts. However, as stated earlier the use of path integrals in financial applications has been limited to a few articles[1], starting with Dash (1986, 1988, 1989) where many financial applications are described in the path integral formalism. These include option pricing under various classical models such as Black-Scholes, Merton, Courtadon, and Brennan-Schwartz. Also treated are options under stochastic volatility assumptions and bond options consistent with various term structure models. Linetsky (1996) provides a detailed exposition of path integral applications in finance with particular emphasis on the formulation of exotic options in this framework. The aforementioned research focused on examples with explicit solutions, hence discussion of numerical techniques for the evaluation of path integrals has been limited to Eydeland (1994) who provides an algorithm for evaluating functional integrals arising in financial applications. Eydeland's algorithm will be applied to the evaluation of American bond options in the HJM framework as outlined in section 3 of this paper. Here we review the subject of path integrals in brief from a heuristic point of view.

Wiener employed functional[2] integration for solving partial differential equations of stochastic nature in the early 1920s, obtaining the fundamental solution (propagator) of the diffusion equation. Feynman's (1943) treatment of path integrals in his thesis on the space-time approach to non-relativistic quantum mechanics resulted in a new formulation of quantum mechanics and in much of the theory of path integrals in its current form.

A path (or functional) integral can be defined as a limit of a sequence of finite dimensional integrals. Suppose that the points $y, x_1, \ldots, x_{N-1}, x$ are connected by lines, such that a broken line or piecewise *path* connects y to x. *The integrals over the quantities* $x_1, \ldots, x_{N-1}$ *can be interpreted as summing over all possible broken line paths connecting y and x.* Assuming that any continuous path can be approximated by a broken line path, in the limit $N \to \infty$, the integral can be interpreted as a sum over all paths or as a *path integral.*

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[1]To the best of our knowledge

[2]A function of infinitely many variables labelled by a continuous index.

Generally speaking, integration of a functional over all its variables, requires the introduction of a weighting factor[3] per variable of integration, or a joint weighting function. The weighting function represents a measure of integration, usually in the form of a probability density function. The fundamental solution of certain partial differential equations in mathematical physics[4] can be obtained with the appropriate choice of the measure of integration of special functionals. When the measure of integration is a probability density then the functional integral represents the mean (expected) value of the functional, a concept central to the solution algorithm used below. Thus, expectation values (averages) of various quantities dependent upon paths are given by path integrals over all the possible paths from an initial state to the final state of a system (sum over histories). This concept has many applications in stochastic financial models where the value of a path-dependent contract is given as an expectation over an appropriate probability measure. This expectation can be expressed as a path integral over the set of all paths of the underlying stochastic variable and defined as a limit of a sequence of multiple integrals. A slightly more formal explanation of the functional integral is presented in Appendix I.

It is also possible to formulate the expectation of a functional as a path integral by repeated use of the Chapman-Kolmogorov equation for transition probabilities (Risken, 1989). The Chapman-Kolmogorov equation in one-dimension is given by

$$\pi(x_n,t_n|x_1,t_1) = \int \pi(x_n,t_n|x_2,t_2)\pi(x_2,t_2|x_1,t_1)dx_2 \tag{35}$$

where $t_n \geq t_2 \geq t_1$ and $\pi(x_j,t_j|x_i,t_i)$ denotes the transition probability (i.e. the probability of observing x_j at time t_j conditional x_i at time t_i).

For small time intervals of length Δt, the transition probability density is given by

[3]To avoid divergence of the solution.
[4]Such as Schrodinger equation and Fokker-Plank equation.

$$\pi(x,t+\Delta t|x',t)=\frac{1}{\sqrt{2\pi\sigma^2(x',t)\Delta t}}\exp\left(\frac{-[(x-x')-m(x',t)\Delta t]^2}{2\sigma^2(x',t)\Delta t}\right) \tag{36}$$

where $m(x', t)$ and $\sigma^2(x', t)$ are the drift and diffusion coefficients respectively of the underlying stochastic differential equation.

For example, consider the path integral representation of

$$E[h(x(t))]=\int_{-\infty}^{\infty}h(x)\pi(x,t|x_0,t_0)dx \tag{37}$$

where $\pi(x, t| x_0, t_0)$ is the probability distribution function and h is a function of the state variable at time t. The transition probabilities required for the path integral formulation are derived by repeated use of the Chapman-Kolmogorov equation. Dividing the time difference $(t - t_0)$ into N small intervals of length $\Delta t = (t - t_0)/N$, and let $t_n = t_0 + n\Delta t$, then

$$\pi(x,t|x_0,t_0)=\int dx_{N-1}\cdots\int dx_1\pi(x,t|x_{N-1},t_{N-1})$$
$$\times\pi(x_{N-1},t_{N-1}|x_{N-2},t_{N-2})\ldots\pi(x_1,t_1|x_0,t_0) \tag{38}$$

Substituting expression (36) for the transition probability $\pi(x_{i-1},t_{i+1}|x_i,t_i)$ into the above equation yields

$$\pi(x,t|x_0,t_0)=\frac{1}{\sqrt{[2\pi\sigma^2(x_0,t_0)\Delta t]}}\int_{-\infty}^{\infty}\underset{N-1}{\cdots}\int_{-\infty}^{\infty}\prod_{i=1}^{N-1}\left\{\frac{dx_i}{\sqrt{[2\pi\sigma^2(x_i,t_i)\Delta t]}}\right\}$$
$$\times\exp\left(-\sum_{i=0}^{N-1}\frac{[(x_{i+1}-x_i)-m(x_i,t_i)\Delta t]^2}{2\sigma^2(x_i,t_i)\Delta t}\right). \tag{39}$$

In the limit as $N \to \infty$ equation (39) would become the formal path integral expression for $\pi(x,t|x_0,t_0)$. Hence, using equation (39), it is possible to represent (37) as a formal path integral (in the limit $N\to\infty$) or for practical implementation as a sequence of finite dimensional integrals (i.e. $N < \infty$)

5. Algorithm[5] For Evaluating Path (or Functional) Integrals

The value of an American bond option in the stochastic interest rate environment of section 3.1 is given by the expected value of the product of two stochastic quantities, the discount factor and the option payoff, both evaluated at the optimal stopping time. Let V(r, t, T, T_B) denote the price at time t of an American put option on a discount bond with exercise price X and expiry[6] T and when the instantaneous spot rate of interest is r. The maturity of the underlying discount bond is T_B where $T \leq T_B$ and its current price is given by P(r, t, T_B)[7]. Hence, the price of the American put option is given by[8]

$$V(r,t,T,T_B) = \max_{t \leq \tau \leq T} \widetilde{E}_t \left[\exp\left\{ -\int_t^{\tau} r(y)dy \right\} (X - P(r,\tau,T_B))^+ \right], \qquad (40)$$

where $\widetilde{E}$ is the expectation over the probability distribution described by the diffusion process in (21). The expectation in (40) can be evaluated using either probabilistic methods or numerical techniques for solving the partial differential equation (27) subject to the appropriate boundary conditions for the American option. The probabilistic methods include the simulation of the driving stochastic diffusion process (21) or the numerical integration of the expectation (40).

The solution algorithm used in this paper to evaluate American options on zero-coupon bonds in the HJM framework of section 3 is based on the technique described by Eydeland [1994]. The algorithm provides an efficient technique for evaluating path integrals in function spaces, which as discussed above, can be expressed as sequences of finite dimensional integrals. The efficiency of the algorithm is the result of the particular problem formulation and the use of the Fast Fourier Transform technique. The application of the algorithm depends on the ability

[5]See Eydeland [1994] for a more detailed exposition of the algorithm

[6]Option expiry date

[7]Which satisfies the partial differential equation (27).

[8]This is the solution to the partial differential equation (24) subject to the boundary condition $[X - P(r, t, T_B)]^+$ and is facilitated by the use of the Feynman-Kac formula.

to find a change of variable which will result in a stochastic differential equation with a constant diffusion coefficient. The main features of the algorithm are summarised below, closely following the original article by Eydeland [1994].

Within the framework of the time dependent volatility function of section 3.1, the price of a zero coupon bond can be expressed as the mean value along all paths of the state variable, the spot rate of interest r(t), driven by the stochastic differential equation (21), i.e.

$$P(r,t,T) = \widetilde{E}\left(e^{-\int_t^T r(s)ds} \right) \tag{41}$$

From the previous discussion on path integrals, the expectation can be determined as an integral in the function space. The evaluation of the functional integral requires the provision of a weighting factor, or a probability measure on the space of paths of the short term interest rate.

The form of the stochastic differential equation (21) driving the short term interest rate is of the general form given in Eydeland (1994), namely

$$dr = \alpha(r,t)dt + \beta(r,t)d\widetilde{W}, \qquad r(0) = r_0,$$

where

$$\alpha(r,t) = D(t) - \lambda r = f_2(0,t) + \lambda f(0,t) + \frac{\sigma^2}{2\lambda}[1 - e^{-2\lambda t}] - \lambda r,$$

and

$$\beta(r, t) = \sigma,$$

are the drift and volatility coefficients respectively.

In summary the algorithm used can be described as follows[9]:

Step1:

"Normalisation" of the stochastic differential equation to avoid negative rates in the numerical procedure. This involves a redefinition of the volatility coefficient in the neighbourhood of zero, according to

$$dr = \alpha(r,t)dt + \beta_{\varepsilon}(r,t)dW, \qquad r(0) = r^{0} \tag{43}$$

where

$$\beta_{\varepsilon}(r,t) = \frac{\beta(\varepsilon,t)}{\varepsilon} r, \qquad \text{if} \qquad 0 < r \le \varepsilon, \tag{44}$$

$$= \beta(r,t), \qquad \text{if} \qquad r > \varepsilon \tag{45}$$

Step 2:

Using an appropriate change of variable, transform the stochastic process in (42) so that the volatility coefficient is constant under the new process. Given that the state variable, in this case, the short term interest rate, is driven by a stochastic differential equation of the form in (42), define a change of variable of the form $(r,t) \leftrightarrow (\xi,t)$:

$$\xi(r,t) = \int_{r_0(t)}^{r} \frac{1}{\beta_{\varepsilon}(r,t)} dr \qquad r > 0,\ t > 0 \tag{46}$$

where $r_0(t)$ is chosen to correspond to the origin for ξ. Note that the function $\xi(r, t)$ is defined on the space on which r can take values.

Assume that $\xi(r, t)$ is monotonic and twice continuously differentiable in r. Using Ito's lemma, we express the stochastic differential equation in terms of the new function $\xi(r, t)$, defining the distribution of the process $\xi(r(t), t)$, as

$$d\xi(r,t) = \left(\frac{\alpha(r,t)}{\beta_{\varepsilon}(r,t)} - \frac{1}{2}\beta'_{\varepsilon}(r,t) \right) dt + dW(t), \tag{47}$$

[9] For a more detailed exposition, see Eydeland [1994].

where $\beta_\xi'(r,t) = \dfrac{\partial \beta_\xi(r,t)}{\partial r}$

That is, the change of variable in (46) is applied to stochastic process describing the dynamics of the instantaneous spot rate given by (21) and reduces to

$$d\xi(r,t) = M(r,t)dt + d\tilde{w}(t), \qquad (47\text{ a})$$

where

$$M(r,t) = \frac{[D(t) - \lambda r]}{\sigma},$$

and

$$D(t) = f_2(0,t) + \lambda f(0,t) + \frac{a_0^2}{2\lambda}[1 - e^{-2\lambda t}],$$

to normalise the spot rate.

Step 3:

Like lattice models, the algorithm[10] is applied in a grid framework of discrete time and space variables. Hence, this step involves the definition of an equidistant rectangular mesh of the space and time variables, ξ and T respectively, as[11] ($t^k = k\Delta t$)

$$\{\xi_j^k\}, \qquad j = 1,\ldots,m, \qquad k = 1,\ldots,n,$$

The expectation (41) is an integral in the space of functions, ξ(r, t), and can be represented by a finite dimensional integral by repeated use of the Chapman-Kolmogorov equation. The integral in the exponent of (41) is approximated using the trapezoidal rule. Hence,

---

[10]For numerical integration

[11]j is the index of the space variable, ξ, and k is the index of the time variable, t. So ξ_j^k denotes the point (jΔξ, kΔt) on the space -time grid. Furthermore, ξ^k denotes all values of ξ at time kΔt.

$$P(r,t,T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{-\frac{1}{2}r(\xi^0,t^0)\Delta t - r(\xi^1,t^1)\Delta t - \ldots - r(\xi^{n-1},t^{n-1})\Delta t - \frac{1}{2}r(\xi^n,t^n)\Delta t}$$

$$\times\; p^1(\xi^1|\xi^0)p^2(\xi^2|\xi^1)p^3(\xi^3|\xi^2)\ldots p^n(\xi^n|\xi^{n-1}).d\xi^1 d\xi^2 d\xi^3 \ldots d\xi^{n-1} \qquad (48)$$

where $p^k(\xi^k|\xi^{k-1})$ denotes the transition probability density at $t = t^k$ for the process described by Geometric Brownian motion. $p^k(\xi^k|\xi^{k-1})$ is the probability that $\xi(t^k) = \xi^k$ given that $\xi(t^{k-1}) = \xi^{k-1}$ and can be regarded as a weighting factor, or probability measure on the space of paths of the short term interest rate. An approximation to the Gaussian transition probability function in this case is obtained from the stochastic differential equation (47a) and is of the form[12]

$$p^k(\xi^k|\xi^{k-1}) = \frac{1}{\sqrt{2\pi\Delta t}}\exp\left(\frac{-[(\xi^k - \xi^{k-1}) - M(\xi^{k-1},t^{k-1})\Delta t]^2}{2\Delta t}\right). \qquad (49)$$

Step 4:

Employ an efficient quadrature procedure to evaluate the integral (48).

First consider the case of a constant drift coefficient: $M\{\xi_j^k\} \equiv M$. Time stationary and equidistant spacing of the integration mesh result in a Toeplitz structure for the discrete matrix of transition probabilities Π^k. That is, the entries of Π^k along each diagonal parallel to the main diagonal are identical. A *symmetric* Toeplitz matrix is uniquely characterised by its first column $\left[\pi_0^k\ \pi_1^k\ \pi_2^k\ \ldots\pi_{m-2}^k\ \pi_{m-1}^k\right]^T$.

Thus for a given k,

$$\prod{}^k = \begin{bmatrix} \pi_0^k & \pi_1^k & . & . & \pi_{m-1}^k \\ \pi_1^k & \pi_0^k & \pi_1^k & . & \pi_{m-2}^k \\ \pi_2^k & \pi_1^k & \pi_0^k & . & \pi_{m-3}^k \\ . & . & \ddots & \ddots & . \\ \pi_{m-1}^k & . & . & \pi_1^k & \pi_0^k \end{bmatrix}, \qquad (50)$$

[12] Let $\Delta t = t^k - t^{k-1}$

and the integral (45) can be represented in vector form as follows:

$$P(r,t,T) = \delta_0^T . D^0 \pi^1 D^1 . D^1 \pi^2 D^2 \ldots D^{n-1} \pi^n D^n . \Sigma^n \tag{51}$$

where:

- δ_0 is the vector of initial probability distribution for ξ^0 and usually has the form
- $\delta_0^T = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$ with the 1 corresponding to the initial rate, or ξ^0_{j0} .
- D is a diagonal matrix of discount factors corresponding to the grid ξ_j^k ;
- Σ^n is the vector of ones required for summing: $\Sigma^T = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \end{bmatrix}$

All lattice models perform some type of expectation calculation on the discrete space of paths. In this technique, the result of the product,

$$\delta_0^T . D^0 \pi^1 D^1 . D^1 \pi^2 D^2 \ldots D^{n-1} \pi^n D^n , \tag{52}$$

is a vector of probability weighted discount factors along all piecewise linear paths between the initial point ξ^0 and the grid of nodes at the final time T. Determining the expectation (41) involves summing the probability weighted quantities and is achieved by multiplying (52) by Σ^n.

An advantage of this technique over other numerical techniques such as some lattice models[15] and Monte Carlo simulation, is that all broken line paths which connect the initial node to all final time nodes, in the space of paths, are represented in determining the expectation (41).

Step 5:

The computational speed of the algorithm depends on the ability to efficiently perform the matrix multiplication in (51). This is achieved by exploiting the Toeplitz structure of the transition probability matrix, Π, in determining the product of the matrix with a vector. The basic idea is to embed the Toeplitz matrix, Π, in a larger matrix with symmetric circulant

[15] Except bushy tree models

structure of dimensions 2m x 2m. In a circulant matrix, the successive columns are obtained by successive cyclic permutations of the first column (Barnett, 1990). This circulant structure of the matrix is the key to the speed and efficiency of the procedure as it enables the application of the Fast Fourier Transform (Papoulis, 1977, NAG 1990).

Hence, the embedding of the transition probability matrix, results in a 2m x 2m circulant matrix[14] C characterised by its *first column* c as follows:

$$c = \left[\pi_0, \pi_1, \pi_2, \ldots, \pi_{m-1}, 0, \pi_{m-1}, \pi_{m-2}, \ldots, \pi_m\right]^T$$

The first m elements of c are exactly the first column of Π and Π is exactly embedded in C.

The product of the Toeplitz matrix Π and a vector[15] v is determined as the first m co-ordinates of the vector $C\upsilon'$, $\upsilon' \in R^{2m}$, $\upsilon' = \begin{bmatrix} \upsilon \\ 0 \end{bmatrix}$. The vector $C\upsilon'$ is calculated using the Fast Fourier Transform which exploits the structure of C.

This algorithm can be extended to handle a general drift function $M(\xi^k, t^k)$ in which case the structure of transition probability matrix is not Toeplitz. However,

$$\pi_{ij}^k = p^k(\xi_j^k | \xi_i^{k-1} - M(\xi_j^k, t_i^{k-1})\Delta t)$$

results in a Toeplitz matrix.

Under the constant drift case, the expectation (39) is determined by repeatedly integrating the function $d(\xi^k)$ multiplied by a transition probability density function $p^k(\xi^k | \xi^{k-1})$, where

$$d(\xi^{k-1}) = \int_{\xi^k} e^{-\frac{1}{2}r(\xi^k, t^{k})\Delta t} p^k(\xi^k | \xi^{k-1}) d(\xi^k) d\xi^k \qquad k = n-1, n-2, \ldots, 1$$

In the case of general drift function $M(\xi^k, t^k)$, it becomes necessary to interpolate from points on the grid $\{\xi_j^{k-1} + M(\xi_j^{k-1}, t^{k-1})\Delta t\}$ to points on the grid $\{\xi_j^{k-1}\}$. This operation can be included in the numerical quadrature formula via an interpolation operator denoted by V^{k-1}.

[14]This will also have a symmetric Toeplitz structure

[15]The product of $D^n \Sigma^n$ is a vector.

The algorithm as described above will evaluate the price of a zero-coupon bond given by the expectation (41). The technique has many interesting applications, as discussed in Eydeland [1994], in the area of derivative security pricing. Of particular interest here is the application of the technique to pricing American bond options, in which case the algorithm is applied in a similar way to other lattice models. At expiry date, the option value at each of the points on the grid can be determined as a function of state and time. The option value is determined by backward recursion through time on the state space grid. At each point on the grid, the early exercise condition must be checked,

$$Max\left[Payoff(t),\ \frac{E[V(r(t+\Delta t),t+\Delta t,,T,T_B)]}{Exp[\int_t^{t+\Delta t} r(y)dy]}\right],$$

where *payoff(t)* is given by

$$\max\ [(X - P(r(t),t,T_B),\ 0].$$

This essentially involves checking at each node to see whether early exercise is preferable to holding the option for a further period, Δt. $Exp[\int_t^{t+\Delta t} r(y)dy]$ is the discount factor used to determine the present value of the option price over the period [t, t + Δt].

6. Results

This section summarises the results obtained using the path integral solution algorithm described in the previous section. The algorithm was used to evaluate American bond options in the HJM framework using the results of section 3 and the expectations in (40) and (41). In particular, Table 1 and Table 2 summarise the results obtained from the path integral technique applied to the stochastic differential equation (21) with the time dependent forward rate volatility given in (19) for different discretisations of the space variable $\xi(r, t)$. As mentioned in section 5, the value of the American put option on a zero-coupon bond can be determined using the probabilistic methods on the expectation given in (40) or by numerically solving the partial differential equation subject to appropriate boundary conditions[16] for the American put option. Chiarella and El-Hassan (1996) value the American bond option in the HJM framework as the solution of the free boundary problem using the method of lines with invariant imbedding (see Chiarella and El-Hassan (1996) for more details). Finite differences were also used to compare accuracy and efficiency of the method of lines algorithm. The results and computation time of the path integral technique are compared with the results of Chiarella and El-Hassan (1996) and summarised in Table 3 below.

The values of a 1-year American put option on a 3-year discount bond with face value 100 determined for various exercise prices are shown in Tables 1 and 2.

Table 1: *1 - Year American Put Option on a 3-Year Bond*[17] *(m = 800)*

Step Size (Δt)	*EXERCISE PRICES* 95	100	105
1/40	0.6571	1.2764	1.7496
1/60	0.6590	1.2782	1.7515
1/100	0.6607	1.2799	1.7537

[16] See Chiarella and El-Hassan (1996)

[17] See comment under Table 2 for details of input data.

Table 2. *1 - Year American Put Option on a 3-Year Bond (m = 2000)*

Step Size (Δt)	*EXERCISE PRICES* 95	100	105
1/40	0.6616	1.2783	1.7603
1/60	0.6619	1.2827	1.7617
1/100	0.6624	1.2865	1.7633
1/400	0.6626	1.2868	1.7636

The input data for these examples is as follows:
Bond face value = 100, σ = 0.08334 and λ = 0.16034. The initial forward rate was determined using a polynomial expansion as suggested in Bhar and Hunt (1993). The values of the β coefficients are $\beta_0 = 0.08485$, $\beta_1 = -0.03178$, $\beta_2 = -0.02327$, $\beta_3 = 0.00312$.

The comparisons below use the results of Chiarella and El-Hassan (1996) from the finite difference technique with $\Delta t = \frac{1}{400}$ as an "accurate" value of the option[18]. American bond option values given in Table 1 and Table 2 show that option values converge to the "accurate" value as the discretisation of the space grid is made finer with a large value for m. The finer the space-time grid, the larger the sample paths accounted for in the evaluation of the expectation (40). However, even with a relatively crude discretisation (Table 1) of the space variable, the resulting option values are acceptable given the speed of the algorithm as reported in Table 3.

Chiarella and El-Hassan (1996) value the American put option on a discount bond in the HJM framework , assuming a forward rate volatility function as given in (19), as the solution of the free boundary problem. The solution technique used is the method of lines with invariant imbedding (see Chiarella and El-Hassan,1996 for more details). Finite differences

[18]The corresponding results using the finite difference method with $\Delta t = 1/400$ for the given input data and exercise value are 0.6626, 1.2869, 1.7636.

were also used to compare accuracy and efficiency of the algorithm. Table 3 summarises the computational time for the option value using these techniques.

Table 3. *Computational Times for the Different Solution Techniques***

Step Size (Δt)	*MOLII*	*SOLUTION TECHNIQUE Run Time (Secs)* FD	*PI1*	*PI2*
1/40	4.651	5.104	3.44	5.92
1/60	5.155	6.715	4.83	7.75
1/100	7.031	8.002	6.34	8.94
1/400	- -	19.632	-	22.04

** MOLII = Method of Lines with Invariant Imbedding
FD = Finite Differences
PI1 = Path Integrals Technique with m = 700 (Table 1)
PI2 = Path Integrals Technique with m = 2000 (Table 2)

Comparison of the computational times reported in Table 3 for the different solution techniques indicate that the efficiency of the path integral techniques depends on the discretisation of the space time grid. However, for the option values reported in Table 1, the computational times are much faster than both the finite differences and the method of line techniques. To improve the accuracy of the results to the values shown in Table 2, a considerably larger value of m was chosen, increasing the computational effort required considerably.

On comparing the method of lines with the path integral technique, some limitations of both methods should be mentioned. The implementation of the method of lines as a numerical solution to the partial differential equation requires considerably more implementation effort then the path integral algorithm described in section 5. However, the strengths of the method of lines include the determination of the time dependent critical asset price, the ability to handle general and complicated free boundary conditions and extendibility to multidimensional cases. However, a problem associated with numerically solving the partial differential equation approach is that it is not always possible to determine the relevant differential equation. This requires the application of Ito's lemma to the function describing the relation between the option price and the underlying state variable. However, this has proved difficult for interest rate options in the HJM framework using very general volatility functions. On the other hand, the path integral method is simpler to implement but depends on the ability to find a change of variable transforming the driving stochastic differential equation into one with a constant volatility coefficient. Also the backbone of the computational efficiency of the algorithm is due to the Toeplitz form of the transition probability matrix and the applicability of fast fourier transform methods.

7. Conclusion

Using a particular class of forward rate volatility functions, we have shown how the stochastic dynamics of the state variable, the short term interest rate, can be expressed in Markovian form and obtained a finite dimensional state space for the partial differential equation governing the security prices. The security prices are expressed as expectations of discounted payoffs under the probability distribution described by the Markovian form of the stochastic dynamics of the state variable. In particular, we consider the price of the American bond put option with forward rate volatility of the form described in section 3.1 We treated these expectations as path integrals which can be approximated by a sequence of finite dimensional integrals. The path integral is evaluated using the algorithm suggested by Eydeland [1994] and described in section 5 of this paper.

As an immediate extension to the results reported here, progress is underway on research into other techniques for evaluating derivative security pricing in a computationally efficient manner. We are particularly focusing on using techniques involving Fourier-Hermite series expansions and identifying recurrence relations associated with these expansions.

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Appendix I

Suppose that

$$\phi_{P_N} = \phi_{P_N}(f_0, f_1, \ldots, f_{N-1}), \tag{A1.1}$$

a function of N variables, is an approximation of the functional φ[f] associated with the partition P_N of the interval [t', t].

The multiple integral representation of a function of N variables over a certain region $\Re$ is:

$$J_{P_N} = \int_{\Re}\cdots\int \phi_{P_N}(f_0, f_1, \ldots, f_{N-1})df_0 df_1 \ldots df_{N-1}. \tag{A1.2}$$

As the partitions of the interval [t', t] get smaller, the number of variables in (A1.2) increase and the limit for the sequence J_{P_N} should tend to the value of the functional integral φ[f].

Choosing the measure of integration as

$$W_{P_N}\prod_{j=0}^{N-1} df_j = W_{P_N}(f_0, f_1, \ldots, f_{N-1})\prod_{j=0}^{N-1} df_j \quad , \tag{A1.3}$$

it is possible to form a new sequence of multiple integrals:

$$I_{P_N} = \int_{-\infty}^{\infty}\cdots\int \phi_{P_N}(f_0, f_1, \ldots, f_{N-1})W_{P_N}(f_0, f_1, \ldots, f_{N-1})\prod_{j=0}^{N-1} df_j \ .$$

If there exists

$$\lim_{N\to\infty} I_{P_N} = I[f] \ ,$$

then I[*f*] is the functional integral representation of the functional φ[*f*] given through the sequence (A1.1) with respect to the measure of integration supplied by the weighting sequence (99). *When the measure of integration* W_{P_N} *is a probability density then the functional integral I[f], is the mean (expected) value of the functional φ[f].*

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For copies of the above papers write to:

School Secretary,
U.T.S.,
School of Finance and Economics,
Kuring-gai Campus,
P.O. Box 222,
LINDFIELD 2070 N.S.W.
Fax: International + (612) 9514 5515
STD - (02) 9514 5515
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