Equilibrium Correlation of Asset Price and Return
*

Charles Ka Yui Leung[†]

November 12, 2005

*Acknowledgement: Converation with Min Hwang and Kelvin Wong motivates this research. The author is also grateful to the comments from Brent Ambrose, Ashok Bardhan, Yongheng Deng, John Quigley, Miki Seko and Chor-yiu Sin, participants of the Berkeley Finance Lunch, Japanese Association of Real Estate Financial Engineering Meeting, and especially Pierre Collin-Dufresne, an annoymous referee, James Kau, for many helpful discussion. The financial support from Chinese University of Hong Kong Direct Grant, RGC Earmark Grant, Fulbright Foundation are gratefully acknowleged. The usual disclaimer applies.

†Correspondence: Leung, Department of Economics, Chinese University of Hong Kong, Shatin, Hong Kong; (Phone) 852-2609-8194; (Fax) 852-2603-5805; (E-mail) charlesl@cuhk.edu.hk. (to be added)

Abstract

Two empirical questions concerning the equity and housing have been studied extensively: (1) Are the price and return serially correlated, and (2) What is the optimal weight of housing in the portfolio? The answer to the second question crucially depends on the cross-correlation of assets. This paper complements the literature by building a simple dynamic general equilibrium with fully rational agents, and obtain closed form solutions for the implied auto- and cross-correlations. The length of time horizon, as well as the persistence of economic shock matter. Implications and future research directions are then discussed.

Keywords: *rational expectation, price and return, serial and cross correlation, market efficiency, predictability*

JEL Classification Number: *E30, G10, R20*

1 Introduction and Motivation

This paper attempts to shed light on two empirical questions concerning the equity and housing by building a dynamic general equilibrium model: (1) Are the price and return of stock, as well as housing, serially correlated, or, at least partly *predictable*, (for instance, see Case and Shiller (1989, 1990))? and (2) What is the optimal weight of housing in the household portfolio? As observed by Hwang and Quigley (2003), the serial correlations of asset prices (the first question) are often quoted as evidence for market inefficiency, or even a basis to question the rationality of economic agents. Cochrane (2001, especially chapter 20) reviews the literature and concludes that "*returns are predictable*".[1]

The second question concerns the optimal portfolio allocation in the presence of housing. Behind this question is an attempt to justify why residential housing possess such a large share in a typical household portfolio.[2] A typical practice in the literature is to take the correlations of assets from the data, and then calculate the optimal weight of housing in the portfolio. Clearly, the answer crucially depends on the cross-correlation of assets. Quan and Titman (1999) find that the statistical significance of the relationship between the return of housing and equity depends on whether the data across countries are pooled together, and whether the measurement intervals are long enough.[3]

In response to the first question, this paper builds a tractable, unifying framework where agents have rational expectation and that the production of goods, and accumulation of physical and household capital are endogenized.[4] We find that even with *i.i.d.* shocks, the equilibrium correlations of asset prices and returns are in principle non-zero. With reasonable parameter values, our numerical work confirm this result. Thus, *serial correlations of asset prices and returns cannot be used as evidence against market efficiency.* The idea is similar to the partial equilibrium model of Wheaton (1999). Wheaton

[1] See also Gatzlaff and Tirtiroglu (1995) for a survey on the housing literature.

[2] Again, that literature is too large to be reviewed here. Among others, see Flavin and Yamashita (2002), Hwang and Quigley (2003).

[3] Recent works such as Englund, Hwang and Quigley (2002) on Swedish data, Iacoviello and Ortalo-Magne (2003) on London data suggest that there are large potential gains if institutions or housing derivatives are established to allow houseowners to hedge the risk.

[4] Throughout the paper, the terms "residential capital", "household capital", "housing" will be used interchangably.

(1999) shows that when the housing stock adjusts sluggishly, the housing price and return will be serially correlated. He however only considers the case of rental housing. In his model, the residents only have consumption motives, while the investors have only investment interests. This paper shared the intuition about sluggish adjustment of different capital stock, and extend the analysis to the case with forward-looking owner-occupiers-investors, and take into account the stochastic structure of the shocks.[5] The positive correlation of asset price and return partly reflects the correlation of output in adjacent periods, and partly reflect that the general equilibrium nature. When everyone tries to sell at a given price, no one can actaully sell anything. The asset price and return would adjust and by that time, some would maintain the portfolio unchanged.

In response to the second question, this paper demonstrates an alternative methodology: instead of taking the asset price or return correlation as given and compute the optimal portfolio, this paper *instead derives* the equilibrium asset prices and returns in a dynamic general equilibrium environment, in which the portfolio problem is solved by the representative agent in *each* period. In fact, since the focus of this paper is the dynamic behavior of prices and returns, it seems appropriate to model them as endogenous rather than exogenous variables: partly because it promotes the self-discipline in the asset-price modelling, and partly because it reflects the fact that the asset prices and the fundamental are correlated, as suggested by some recent empirical research (especially over a longer horizon).[6] For instance, Benzoni, Collin-Dufresne and Goldstein (2005) show that the correlation of labor income and asset return are increasing with the horizon, and suggest that stock returns and labor income are cointegrated. In this model, both asset return and labor income would endogenously arise from the individuals' maximization problem and market clearing condition, and would therefore be naturally correlated.[7]

[5] Notice that the homeownership in many countries are high. For instance, the homeownership rate in the USA is about two-third and in Hong Kong it is about a half. In some other countries, this number can be even higher. For instance, the homeownership rates in both Singapore and Taiwan are about 80%.

[6] For instance, see McQueen and Roley (1993), Wongbangpo and Sharma (2002), Parker and Julliard (2005).

[7] On the other hand, the life-cycle consideration emphasized by Benzoni, Collin-Dufresne and Goldstein (2005), the idiosyncratic housing return faced by a typical household emphasized by Englund, Hwang and Quigley (2002), are abstracted away here and can only be left to future research.

Clearly, this paper is relate to a recent literature on reconciling the aggregate economy on the one hand, and the stock market and/or the housing sector on the other hand, such as Jermann (1998), Krusell and Smith (1997), Ang, Piazzesi and Wei (2005), Diebold, Piazzesi and Rudebusch (2005), among others.[8] Piazzesi, Schneider and Tuzel (2003) introduces housing into an otherwise standard asset pricing framework and attempt to improve the model's explanatory power. While Piazzesi, Schneider and Tuzel (2004) is trying to match the data through a calibrated model, this paper attempts to produce some analytical results within a dynamic general equilibrium framework. Thus, some simplifying assumptions have to be made and some generality will be sacrificed. In return, this paper is able to provide some closed form solutions, and hence making the economic intuition more transparent. Moreover, since the model is not calibrated but rather analytically solved, it is exempted from the criticisms of Hansen and Heckman (1996).[9] Thus, the model itself may be of independent interest to researchers. In sum, the two papers should be viewed as complements rather than substitutes.

The organization of this paper is simple. The next section will present the basic model, followed by analytical and numerical results. The last section concludes.

2 A Simple Model

Our model is built on Greenwood and Hercowitz (1991), Hercowitz and Sampson (1991), Benassy (1995), Kan et. al. (2004), and hence the description will be brief.[10] Time is discrete in this model and the horizon is infinite. The economy is populated by a continuum of infinite-lived agents. The population is fixed over time. In each period t, $t = 1, 2, 3,...$, the representative agent derives utility $u(C_t, H_t + H_t^r, L_t)$ from non-durable consumption goods C_t , the stock of housing (or residential property) owned (rented) by the agent H_t (H_t^r), as well as the amount of leisure enjoyed $(1 - L_t)$, where L_t is the amount of time the agent supplied in the market, $0 \leq L_t \leq 1$. H_t is broadly defined to include the residential structure, as well as the associated

[8]See Leung (2004) for a review on the literature relating macroeconomics and housing.

[9]For more details, see the exchange between Hansen and Heckman (1996), Kydland and Prescott (1996).

[10]See also Kwong and Leung (2000), Leung (2001) for related studies.

amenities. Following Greenwood and Hercowitz (1991), it is assumed that

$$u(C_t, H_t + H_t^r, L_t) = \ln C_t + \omega_1 \ln (H_t + H_t^r) + \omega_2 \ln (1 - L_t), \qquad (1)$$

where $\omega_1,\ \omega_2 > 0$. The representative agent is assumed to maximize the expected value of the discounted sum of life time utility $E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, H_t + H_t^r, L_t)$, participate in the production of consumption goods C_t, and accumulate business capital stock K_t, and residential property H_t. β is the time discount factor, $0 < \beta < 1$.[11]

The representative agent is a price taker in all industries and he/she is subject to a series of constraints. (To ease the notations, time subscripts are suppressed unless there is a risk of confusion). First, the total value of non-durable consumption C, and investment in business capital, residential property, I_k, I_h respectively, the expenditure on the residential property purchased from the market, $P_h H^m$, the expenditure on renting residential property, $R_h H^r$, the expenditure on purchasing new equity, $P_s (S_{t+1} - S_t)$, cannot exceed the total value of rental income from capital, RK, labor income WL, and dividend $S\Pi^d$, where P_h is the relative price of housing (in terms of consumption goods), P_s is the relative price of equity, R is the factor return for capital, W is the real wage rate, Π^d is the total dividend.

Residential investment I_h includes not only new construction, but also maintenance, renovation, purchase of new furniture, appliance, etc.[12] Analogous interpretation applies to I_k as well. Whether it pays to invest or where to invest depends crucially on the payoff of the investment. Following Hercowitz and Sampson (1991) and Benassy (1995), we assume a specific form of law of motion for different types of capital, which will generate closed forms of the solution,[13]

$$K_{t+1} = (K_t)^{1-\delta_k} (I_{kt})^{\delta_k}, \qquad (2)$$

$$H_{t+1} = (H_t + H_t^m)^{1-\delta_h} (I_{ht})^{\delta_h}, \qquad (3)$$

[11] See Stokey, Lucas and Prescott (1989, esp. chapter 3) for more discussion on the role of the time discount factor.

[12] Downing and Wallace (2002a, b) show that the expenditures for home improvements and re-modelling total at least 2% of the GDP and is comparable to the expenditure on new housing construction. It also varies systematically over the business cycle.

[13] An alternative approach is to adopt a more general framework and then use loglinearization as in Campbell (1994). The reduced forms of the dynamics, however, would be similar. See Lau (2002).

where $0 < \delta_k,\ \delta_h < 1$. The dynamic programming problem of the representative agent is now:

$$V(K_t, H_t, S_t) = \max .u(C_t, H_t + H_t^r, L_t) \ + \beta E_t V(K_{t+1}, H_{t+1}, S_{t+1}) \quad (4)$$

$$\text{s.t. } R_t K_t + W_t L_t + S_t \Pi_t^d \geq I_{kt} + I_{ht} + C_t + P_{ht} H_t^m + R_{ht} H_t^r + P_{st} \left(S_{t+1} - S_t \right), \quad (5)$$

and (2), (3), where . It is implicitly assumed in (5) that the representative agent observes the current period productivity A_t first, and then decides how much raw materials are to be imported from the "rest of the world", given the amount of capital and property the representative agent owns.

The production side of the economy is simple. Output are produced by combining capital and labor through a concave function,

$$Y_t = A_t \left(K_t \right)^{\alpha_1} \left(L_t \right)^{\alpha_2} \quad (6)$$

where $A_t > 0$, $\forall t$, $0 < \alpha_1,\ \alpha_2$, and $\alpha_1 + \alpha_2 < 1$. The productivity is assumed to have finite mean and variance, $0 < E\left(A_t\right),\ Var(A_t) < \infty$. The factor markets are assumed to be competitive and the factor returns are equal to the marginal product,

$$R_t = \frac{\partial Y_t}{\partial K_t},\ W_t = \frac{\partial Y_t}{\partial L_t}. \quad (7)$$

It is further assumed that the dividend is equal to the profit $\Pi_t^d = \Pi_t$, which is the output net of factor payment,

$$\Pi_t = Y_t - R_t K_t - W_t L_t. \quad (8)$$

As it is standard in the growth model, the objective of the representative firm is to maximize the profit and hires capital and labor from the factor market accordingly.

To solve the model, it is necessary to impose market clearing conditions. Following Lucas (1978), the net trade of housing is assumed to be zero (both ownership market and rental market) and the total amount of equity is assumed to be unity in every period,

$$S_{t+1} = S_t = 1,\ H_t^m = H_t^r = 0. \quad (9)$$

In the appendix, we show that the problem can be simplified and prove the following proposition: (all proofs are contained in the appendix)

Proposition 1 *In this model economy, if*

$$\beta \alpha_1 \delta_k \left(1 - \beta \left(1 - \delta_k\right)\right)^{-1} < 1, \tag{10}$$

then the amount of working hours, the consumption and different kinds of investment shares of the output are constant,

$$L_t = L,\ C_t = \mathcal{S}^c Y_t,\ I_{j,t} = \mathcal{S}^j Y_t,\ j = k,\ h. \tag{11}$$

The asset prices depend on the output, investment and housing stock,

$$P_{ht} = \left(\frac{1 - \delta_h}{\delta_h}\right)\left(\frac{I_{ht}}{H_t}\right), \tag{12}$$

$$R_{ht} = \left(\frac{1 - \beta \left(1 - \delta_h\right)}{\beta \left(1 - \delta_h\right)}\right) P_{ht}, \tag{13}$$

$$P_{st} = \frac{\beta \left(1 - \alpha_1 - \alpha_2\right)}{1 - \beta} \cdot Y_t. \tag{14}$$

With reasonable market-clearing conditions imposed, this proposition enables to characterize the equilibrium quantities (and prices) *in each period* as functions of exogenous variables. In other words, we can trace the evolution of the whole system.

It is convenient to rewrite in log form, i.e., we write $c_t = \ln C_t$, $y_t = \ln Y_t$, $p_{ht} = \ln P_{ht}$, $p_{st} = \ln P_{st}$, $s^j = \ln \mathcal{S}^j$, $j = c, k, h$, etc. The economy is hence represented by the following *linear* equations:

$$y_t = \theta_y + \alpha_1 k_t + a_t, \tag{15}$$

$$k_{t+1} = \left(1 - \delta_k\right) k_t + \delta_k i_{kt}, \tag{16}$$

$$h_{t+1} = \left(1 - \delta_h\right) h_t + \delta_h i_{ht}, \tag{17}$$

$$c_t = \eta_c + y_t, \tag{18}$$

$$i_{kt} = \eta_k + y_t, \tag{19}$$

$$i_{ht} = \eta_h + y_t, \tag{20}$$

for some constants θ_y, and given the initial conditions a_0, k_0, h_0. We also have the following corollary:

Corollary 2 *In log form, the stock price is linearly correlated to the output,*

$$p_{st} = \theta_s + y_t. \tag{21}$$

In fact, we can also obtain an expression for the housing price in this model economy.

Lemma 3 *The (log) housing price can be written as a function of the current and previous period output, or productivity shocks,*

$$\begin{aligned} p_{ht} &= \theta_h + y_t + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i \, y_{t-1-i} \right), && (22) \\ &= \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i}, && (23) \end{aligned}$$

for some function of parameter, $\delta_p(i)$, $i = 0, 1, 2, ...$

Now, it is clear that the stochastic structure of the productivity shocks $\{a_t\}$ is crucial in determining all the key correlations. It would be no surprise if a serially correlated productivity shock leads to serially correlated prices. Therefore, for expositional purpose, we will first examine the case with serially *un*-correlated productivity shocks. Interestingly, even in that case, the asset prices will display serial correlation. In other words, there is a *internal propagation mechanism behind the asset prices.* Again, the appendix contains the proofs.

Proposition 4 *Even with serially uncorrelated productivity shocks, the serial correlation of asset prices will in general be non-zero. The formula for covariance and variance are given by the following expressions:*

$$\begin{aligned} & cov(p_{st}, p_{s,t+j}) \\ = \; & \sigma_a^2 \left(\alpha_1 \delta_k \left(1 - \delta_k + \alpha_1 \delta_k\right)^{j-1} + \frac{\left(\alpha_1 \delta_k\right)^2 \left(1 - \delta_k + \alpha_1 \delta_k\right)^j}{1 - \left(1 - \delta_k + \alpha_1 \delta_k\right)^2} \right) > 0, \quad (24) \end{aligned}$$

where $\sigma_a^2 \equiv var(a_t)$ and for $j = 1, 2, 3, ...$ And for housing prices,

$$cov(p_{ht}, p_{h,t+j}) = \sigma_a^2 \left(\sum_{i=0}^{\infty} \delta_p(i) \cdot \delta_p(i+j) \right), \tag{25}$$

for $j = 1, 2, 3, \ldots$. *Moreover, the two assets are correlated contemporarily ,*

$$cov(p_{st}, p_{ht}) = \sigma_a^2 \cdot \left[\sum_{i=0}^{\infty} \delta_p(i) \delta_p^s(i) \right], \tag{26}$$

where $\delta_p^s(i) > 0$, $\forall i$, *are functions of parameters. In general, the correlation between the current stock price and the subsequent period housing prices are non-zero. For* $j = 1, 2, 3, \ldots$.

$$cov(p_{st}, p_{h,t+j}) = \sigma_a^2 \left[\sum_{i=0}^{\infty} \delta_p(j+i) \delta_p^s(i) \right], \tag{27}$$

Conversely, the correlation between the current period stock price and previous period housing prices is

$$cov(p_{st}, p_{h,t-j}) = \sigma_a^2 \left[\sum_{i=0}^{\infty} \delta_p(i) \delta_p^s(j+i) \right], \tag{28}$$

where the variance of equity price and housing price are given by

$$var(p_{st}) = \sigma_a^2 \cdot \left(\sum_{i=0}^{\infty} \left(\delta_p^s(i) \right)^2 \right), \; var(p_{ht}) = \sigma_a^2 \cdot \left(\sum_{i=0}^{\infty} \left(\delta_p(i) \right)^2 \right).$$

For economic and financial research, it is natural to study not only the dynamic behavior of prices, but also of the rate of returns. We define the rate of return of stock and housing as follows.

$$\widetilde{R}_{s,t+1} = \frac{P_{s,t+1} + \Pi_{t+1}^d}{P_{st}}, \; \widetilde{R}_{h,t+1} = \frac{P_{h,t+1} + R_{h,t+1}}{P_{ht}}. \tag{29}$$

With these definitions, we can prove the following proposition.

Proposition 5 *In log form, the rate of return of stock can be written as a function of output,*

$$\widetilde{r}_{s,t+1} = -\ln \beta + y_{t+1} - y_t, \tag{30}$$

and that of the housing can be written as a function of housing price,

$$\widetilde{r}_{h,t+1} = -\ln\left(\beta\left(1 - \delta_h\right)\right) + p_{h,t+1} - p_{ht}. \tag{31}$$

In addition, we can show that, for $j = 1, 2, 3, \ldots$

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1},\widetilde{r}_{s,t+1+j}\right) \\ = \; & \sigma_a^2 \cdot \left\{ \delta_p^s(0)\, \delta_r^s(j) + \sum_{i=0}^{\infty} \delta_r^s(i+1) \cdot \delta_r^s(j+i+1) \right\}, \end{aligned} \quad (32)$$

$$\begin{aligned} & cov\left(\widetilde{r}_{h,t+1},\widetilde{r}_{h,t+1+j}\right) \\ = \; & \sigma_a^2 \cdot \left\{ \delta_p(0)\, \delta_r(j) + \sum_{i=0}^{\infty} \delta_r(i+1) \cdot \delta_r(j+i+1) \right\}, \end{aligned} \quad (33)$$

where $\delta_r^s(i) \equiv \left[\delta_p^s(i) - \delta_p^s(i-1)\right]$, $\delta_r(i) \equiv \left[\delta_p(i) - \delta_p(i-1)\right]$. *In addition, we can examine the cross-correlation of the two assets' return. In particular, the contemporaneous cross-correlation is given by the following formula,*

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1},\widetilde{r}_{h,t+1}\right) \\ = \; & \sigma_a^2 \cdot \left\{ \left(\delta_p^s(0) \cdot \delta_p(0)\right) + \sum_{i=0}^{\infty} \delta_r^s(i+1) \cdot \delta_r(i+1) \right\}. \end{aligned} \quad (34)$$

And for $j = 1, 2, 3, \ldots$ *the covariance between the current period stock return and the subsequent period housing return is*

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1},\widetilde{r}_{h,t+1+j}\right) \\ = \; & \sigma_a^2 \cdot \left\{ \left(\delta_p^s(0) \cdot \delta_r(j)\right) + \sum_{i=0}^{\infty} \delta_r^s(i+1) \cdot \delta_r(i+j+1) \right\}, \end{aligned} \quad (35)$$

and the covariance between the current period stock return and the previous period housing return is

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1},\widetilde{r}_{h,t+1-j}\right) \\ = \; & \sigma_a^2 \cdot \left\{ \left(\delta_r^s(j) \cdot \delta_p(0)\right) + \sum_{i=0}^{\infty} \delta_r^s(j+i+1) \cdot \delta_r(i+1) \right\}, \end{aligned} \quad (36)$$

where the variance of the stock and housing returns are respectively

$$\begin{aligned} var\left(\widetilde{r}_{s,t+j+1}\right) &= \sigma_a^2 \cdot \left\{ \left(\delta_p^s(0)\right)^2 + \sum_{i=0}^{\infty} \left(\delta_r^s(i+1)\right)^2 \right\}, \\ var\left(\widetilde{r}_{h,t+j+1}\right) &= \sigma_a^2 \cdot \left\{ \left(\delta_p(0)\right)^2 + \sum_{i=0}^{\infty} \left(\delta_r(i+1)\right)^2 \right\}. \end{aligned}$$

This section has provided the model, and the closed form solution for the difference covariance and variance terms of different asset prices and returns. However, as argued by Cooley (1995), an analytical model alone may not provide enough restrictions on the equilibrium quantity and should be complemented by numerical calculation or simulation, which is the focus of the next section.

3 Numerical Results

To gain a (quantitative) sense of the model, we substitute in parameter values that have been used in the literature and then compute the equilibrium correlations of the asset prices and returns. The following table provide a summary (on quarterly basis):

(Table 1 about here)

Notice that putting together α_1 and α_2 implies that "profit" is about 10% of the output, which is consistent with Gort, Greenwood and Rupert (1999), and a recent survey by *Economist* magazine. The other parameter values are pretty standard in the macroeconomics literature and we refer the readers to the original articles for more discussion. It should be noticed that this is a stationary model economy, and hence all prices are stationary.[14] In practice, however, asset prices are typically *non*-stationary in practice, even measured in real terms. Thus, the results here should be compared to the "*de-trended*" asset prices (in real terms) in reality rather than the raw (nominal) prices observed in data.[15] An analogous interpretation is applied to the asset returns. In addition, the correlation formula derived earlier presumed an infinite time series, while the correlation coefficients estimated in the literature are based on a finite sample. The potential small sample bias might be especially serious for housing data. As surveyed by Quan and Titman (1999), some empirical estimates are based on less than 20 years of data. On top of that, this paper intends to be a theoretical exploration, with some numerical examples for illustrations, rather than being a full-scale

[14] Even for models exhibit economic growth, the common practice for finding the solution is always to first convert the model economy to a stationary counterpart by dividing all variables by some appropriate "growth factor". See King and Rebelo (1999) for more details.

[15] In practice, "detrending" can be a very subtle issue. For instance, see Burnside (1998).

data-matching exercise. Thus, discerpencies between the empirical literature and the calculations here should be expected. Now we present the results for the case with *i.i.d.* shock. (The numbers in the table have been rounded up).

(Table 2a, b about here)

Several observations are in order. First, with *i.i.d.* shocks, both the auto-correlations and cross-correlations of the asset prices are non-zero but very small in magnitude. Second, there is a very strong "reverting" tendency of asset return. Notice that both equity and housing returns displays about -0.5 for one period of lag, and virtually zero correlation after. The intuition is clear. If the model economy is surprised by a positive shock this period, agents will increase their holdings in assets. As the short-run supply is always limited, the asset prices will immediately go up, which gives a surprisingly good return of asset holding this period. On the other hand, a positive shock also means that the investment in stock (capital as well as housing) increases, which tend to depress the future price. Thus, asset prices are expected to fall in later periods, and hence the future return and current return are negatively correlated. Third, the cross-correlation of asset prices are also weak, and the cross-period, cross-asset correlation are numerically significant only for one period, and then virtually zero afterwards. The similarity of pattern of auto-correlation and cross-correlation is due to the fact that this model is driven only by the technological shock.

3.1 Does time horizon matter?

Notice that the previous results are all in quarterly basis, as in the macro-economics literature. Recent studies such as Parker and Julliard (2005), Benzoni, Collin-Dufresne and Goldstein (2005), among others, seem to suggest that the time horizon matters. To examine whether it is the case in this framework, we adjust the depreciation rate, discount factor accordingly and present the result for the case when each period represents a year.[16]

(Table 3a, b about here)

[16] Notice that the utility function is maintained to be in log form, and thus the intertemporal elasticity of substitution is implictly fixed when we change the length of a period.

Several observations are in order. First, *in percentage terms, the auto-correlations of asset prices dramatically increase, though the level of those numbers are still small.* Notice that we obtain closed form solutions for all the covariance and variance terms and hence the difference of results in quarterly and annual basis are not due to approximation errors. Rather, it demonstrates that in longer horizon, the asset prices are more correlated, and are consistent with some recent research that asset pricing models perform much better in longer horizon (for instance, see Parker and Julliard (2005), Benzoni, Collin-Dufresne and Goldstein (2005)). Second, *the pattern of auto- as well as cross-correlations of asset return are dramatically similar.* It seems that as arbitrage is at work, the length of the time period does not matter. Readers should bear these observations in mind, as we are going to compare with the case with persistent shocks.

3.2 Does the persistence of shocks matter?

Thus far the results presented are restricted to the case with i.i.d. shocks. In reality, however, one can argue that shocks tend to be persistent. For instance, a major change in government policy is often followed by more minor amendments. A major technological breakthrough often takes time to diffuse across sectors and firms, and often followed by a wave of further improvement. Therefore, it is important to consider the case with persistent shocks. To facilitate the comparison with the macroeconomics literature, we follow Cooley (1995) by assuming that the shock takes a simple AR(1) form:

$$a_t = \rho a_{t-1} + u_t, \tag{37}$$

where u_t is i.i.d., with $E(u_t) = 0$, $var(u_t) = \sigma_u^2 < \infty$, $\forall t$ and $cov(u_t, u_s) = 0$, $\forall s \neq t$. According to Cooley, setting the value of ρ as 0.95 is found common and indeed useful in the macroeconomic literature. This simple structure tends out to be very tractable. In the appendix, we show how the previous formula can be modified to adopt the change in assumption. It suffices to say that by appropriately re-define certain parameters, the closed form solutions for all the variance and covariance terms maintain. And we tabulate all the results in table 4.

(Table 4a, b about here)

Several observations are in order. First, *equity price becomes very persistent.* Housing price is more persistent than before as well. In order words, the persistence of shock matters. Second, in terms of auto-correlation of asset returns, *only the one-period lag equity return for one period is affected. The effect quickly dies out.* For housing, the auto-correlations of returns are almost identical. Thus, the persistence of shock would have different impact on different assets in the short run, and almost no impact in the long run. Third, the *cross-correlations of asset prices, whether it is the equity price leading the housing price, or vice versa, have dramatically increased.* It may not be surprising as the auto-correlations of asset prices have increased significantly. What may be interesting, however, is that with any given period of time lag, the cross-correlation between equity and housing is noticeably larger than the auto-correlation of different periods of housing prices. Notice that this phenomenon is consistent with competitive markets and rational agents, and should not used as evidence against the efficiency of the market. Fourth, *the cross-correlations of returns are affected only in the short-run.* It is similar to the case of the auto-correlations of asset returns.

We have also adjusted different parameter values for the case of annual frequency under persistent shocks, including the persistence parameter. The results are reported in table 4c and 4d.

(Table 4c, d about here)

Clearly, there are many qualitative difference with the quarterly case. What should be noticed, however, is that while the auto-correlation of equity is only slightly affected by the change of frequency, the auto-correlation of housing price is significantly changed. It means that we should be cautious as we compare results using different time frequency, perhaps especially for housing price. In addition, while the auto-correlation with one period lag of equity are similar for quarterly and annual data, the numerical values for later periods auto-correlation are different. In particular, while auto-correlation with two or three periods of lag may be detected as statistically *in*-significant in quarterly data with 1% significant level, it will be detected as significant with annual data. Again, it is another demonstration that the frequency of data matters.

4 Concluding Remarks

The efficiency of asset markets, or markets in general, have long been questioned. For instance, Lamont and Thaler (2003) review some recent literature on the violation of the "Law of One Price" in asset markets. Even for the internet, where the information cost should be minimal, Baye, Morgan and Scholten (2004) do not find empirical support for the "Law of One Price". The price dispersion phenomenon identified by Leung, Leong and Wong (2005) also suggest that the housing market may be inefficient. On the other hand, the research agenda of establishing the *in*-efficiency of the market by studying the asset price or asset return correlation may need to be refined. As shown in Wheaton (1999), sluggish adjustment in (rental) housing stock itself is enough to generate housing price auto-correlation. This paper generalizes this insight in several dimensions. Analytically, we find that even when agents have rational expectation, the equilibrium prices and return of assets (housing and equity) will be correlated. Furthermore, the cross-correlation of the prices and return of the two assets will be non-zero. The numerical exercises confirm that the equilibrium correlations of asset prices (both equity and housing) are very sensitive to the length of period and the persistence of shock. When the shocks are persistent, both the auto- and cross-correlation of asset prices are can be very significant. The equilibrium predictions of asset return correlations, except for the first period, seem to be relatively invariant to the persistence of shock, as well as to the length of a period.

Clearly, there are much rooms for improvement for this research. For one thing, the model is very stylized. There is only one shock in this model. An ongoing project extends this model to include monetary and other shocks in the model and examine how the asset price/return correlations change as more "market frictions" are added into the model. Nevertheless, the results reported in this research seem to encourage further effort to understand how asset prices and returns would be correlated in a general equilibrium setting.

The model developed in this paper may also be of independent interest. In fact, our model is so tractable that in the case of *i.i.d.* and AR(1) shocks, closed form solutions for asset price and return correlations are delivered. It can be extended and modified in different directions for the investigation of other issues in financial economics. For instance, this paper can be extended to include risk-free bond, following the formulation of Lucas (1978). A more interesting research direction would be to include different sources of uncertainty in the model and see how that impact the asset prices and returns at the equilibrium. And to make the portfolio choice more interesting, this paper should extend to the case with heterogeneous agents. Research along these lines may deepen our understanding of the asset markets.

Table 1: parameter values

Parameter	Values	Source
ω_1	1	GH
ω_2	1	GH
β	0.99	Cooley (1995)
α_1	0.3	Cooley (1995), GGR
α_2	0.6	Cooley (1995), GGR
δ_k	2%	GH, KKL
δ_h	2%	GH, KKL

where GH denotes Greenwood and Hercowitz (1991), GGR denotes Gort, Greenwood and Rupert (1999), KKL denotes Kan, Kwong and Leung (2004).

Table 2a: Auto-Correlation of Asset Prices and Returns
(Quarterly, *i.i.d.* shock)

Lag	Equity Price	Equity Return	Housing Price	Housing Return
1	0.0073	−0.5000	−0.0069	−0.5000
2	0.0072	0.0000	−0.0070	−0.0001
3	0.0071	0.0000	−0.0069	0.0000
4	0.0070	0.0000	−0.0068	0.0000
5	0.0069	0.0000	−0.0067	0.0000

Table 2b: Cross-Correlation of Asset Prices and Returns
(Quarterly, *i.i.d.* shock)

Lag	EP leads HP	EP lags HP	ER leads HR	ER lags HR
1	−0.0169	0.0029	−0.5049	−0.5000
2	−0.0169	0.0029	−0.0001	0.0000
3	−0.0167	0.0029	0.0000	0.0000
4	−0.0164	0.0028	0.0000	0.0000
5	−0.0161	0.0028	0.0000	0.0000

where Lag denote the number of lags, EP denotes Equity Price, HP denotes Housing Price, ER denotes Equity Return, HR denotes Housing Return.

Table 3a: Auto-Correlation of Asset Prices and Returns

(Annual, *i.i.d.* shock)

Lag	Equity Price	Equity Return	Housing Price	Housing Return
1	0.0289	−0.4992	−0.0269	−0.4992
2	0.0272	0.0000	−0.0285	−0.0016
3	0.0257	0.0000	−0.0267	0.0000
4	0.0243	0.0000	−0.0251	0.0000
5	0.0229	0.0000	−0.0235	0.0000

Table 3b: Cross-Correlation of Asset Prices and Returns

(Annual, *i.i.d.* shock)

Lag	EP leads HP	EP lags HP	ER leads HR	ER lags HR
1	−0.0169	0.0029	−0.5049	−0.5000
2	−0.0169	0.0029	−0.0001	0.0000
3	−0.0167	0.0029	0.0000	0.0000
4	−0.0164	0.0028	0.0000	0.0000
5	−0.0161	0.0028	0.0000	0.0000

where Lag denote the number of lags, EP denotes Equity Price, HP denotes Housing Price, ER denotes Equity Return, HR denotes Housing Return.

Table 4a: Auto-Correlation of Asset Prices and Returns
(Quarterly, *AR(1)* shock)

Lag	Equity Price	Equity Return	Housing Price	Housing Return
1	0.9995	−0.9881	0.5879	−0.4927
2	0.9982	0.0053	0.5819	−0.0001
3	0.9961	0.0049	0.5760	0.0000
4	0.9933	0.0045	0.5701	0.0000
5	0.9898	0.0041	0.5641	0.0000

Table 4b: Cross-Correlation of Asset Prices and Returns
(Quarterly, *AR(1)* shock)

Lag	EP leads HP	EP lags HP	ER leads HR	ER lags HR
1	−0.7736	−0.7595	−0.7166	−0.6932
2	−0.7725	−0.7522	−0.0092	0.0000
3	−0.7708	−0.7448	−0.0081	0.0000
4	−0.7689	−0.7373	−0.0075	0.0000
5	−0.7657	−0.7291	−0.0069	0.0000

where Lag denote the number of lags, EP denotes Equity Price, HP denotes Housing Price, ER denotes Equity Return, HR denotes Housing Return.

Table 4c: Auto-Correlation of Asset Prices and Returns
(Annual, *AR(1)* shock)

Lag	Equity Price	Equity Return	Housing Price	Housing Return
1	0.9944	−0.9561	0.3782	−0.4744
2	0.9900	0.0143	0.3463	−0.0053
3	0.9591	0.0105	0.3210	−0.0019
4	0.9333	0.0075	0.2980	−0.0017
5	0.9040	0.0050	0.2772	−0.0015

Table 4d: Cross-Correlation of Asset Prices and Returns
(Annual, *AR(1)* shock)

Lag	EP leads HP	EP lags HP	ER leads HR	ER lags HR
1	−0.6603	−0.5686	−0.7416	−0.6538
2	−0.6494	−0.5292	−0.0292	0.0011
3	−0.6337	−0.4933	−0.0178	0.0010
4	−0.6147	−0.4605	−0.0124	0.0009
5	−0.5935	−0.4305	−0.0080	0.0008

where Lag denote the number of lags, EP denotes Equity Price, HP denotes Housing Price, ER denotes Equity Return, HR denotes Housing Return.

References

[1] Ang, A.; M. Piazzesi and M. Wei (2005), "What does the yield curve tell us about GDP growth?" *Journal of Econometrics*, forthcoming.

[2] Baye, M.; J. Morgan and P. Scholten (2004), "Price Dispersion in the Small and in the Large: Evidence from an Internet Price Comparison Site," *Journal of Industrial Economics*, 52(4): 463-96.

[3] Benassy, Jean-Pascal (1995), "Money and wage contracts in an optimizing model of the business cycle," *Journal of Monetary Economics*, 35, 303-315.

[4] Benzoni, L.; P. Collin-Dufresne and R. Goldstein (2005), "Portfolio choice over the life-cycle in the presence of 'trickle down' labor income," *NBER Working Paper 11247*.

[5] Burnside, C. (1998), "Detrending and Business Cycle Facts: A Comment," *Journal of Monetary Economics*, 41(3), 513-32.

[6] Campbell, J. (1994), "Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model," *Journal of Monetary Economics*, 33, 463-506.

[7] Case, K. and R. Shiller (1989), "The efficiency of he market for single-family homes," *American Economic Review*, 79(1), 125-137.

[8] Case, K. and R. Shiller (1990), "Forecasting prices and excess returns in the housing market," *AREUEA Journal*, 18(3), 253-273.

[9] Cochrane, J. (2001), **Asset Pricing**, Princeton: Princeton University Press.

[10] Cochrane, J. (2005), "Financial Markets and the real economy," *NBER Working Paper 11193*.

[11] Collin-Dufresne, P. (2005), Lecture notes on dynamic asset pricing, University of California, Berkeley, mimeo.

[12] Cooley, T. ed. (1995), **Frontiers of Business Cycle Research**, Princeton: Princeton University Press.

[13] Diebold, F.; M. Piazzesi and G. Rudebusch (2005), "Modeling bond yields in finance and macroeconomics," *American Economic Review Papers and Proceedings*, forthcoming.

[14] Downing, C. and N. Wallace (2002a), "A real options approach to housing investment," University of California, Berkeley, mimeo.

[15] Downing, C. and N. Wallace (2002b), "Housing investment dynamics and the estimation of hedonic price indexes," University of California, Berkeley, mimeo.

[16] Englund, P.; M. Hwang and J. Quigley (2002), "Hedging housing risk," *Journal of Real Estate Finance and Economics*, 24, 167-200.

[17] Flavin, M. and T. Yamashita (2002), "Owner-occupied housing and the composition of the household portfolio," *American Economic Review*, 345-362.

[18] Gatzlaff, D. and D. Tirtiroglu (1995), "Real Estate Market Efficiency: Issues and Evidence," *Journal of Real Estate Literature*, 3 (2), 157-189.

[19] Gort, M., J. Greenwood and P. Rupert. (1999), "Measuring the rate of technological change in structures," *Review of Economic Dynamics*, 2, 207-30.

[20] Greenwood, J. and Z. Hercowitz (1991) "The allocation of capital and time over the business cycle," *Journal of Political Economy*, 99, 1188-1214.

[21] Hansen, L. P. and J. Heckman (1996), "The Empirical Foundations of Calibration," *Journal of Economic Perspectives,* 10(1), 87-104.

[22] Hercowitz, Z. and M. Sampson (1991), "Output, growth, the real wage, and employment fluctuations," *American Economic Review*, 81, 1215-37.

[23] Hwang, M. and J. M. Quigley (2003), "Price discovery in time and space: the course of condominium prices in Singapore," UC Berkeley, mimeo.

[24] Hwang, M., J. M. Quigley and J. Y. Son (2005), "The dividend pricing model: new evidence from the Korean housing market," forthcoming in *Journal of Real Estate Finance and Economics.*

[25] Iacoviello, M. and F. Ortalo-Magne (2003), "Hedging housing risk in London," *Journal of Real Estate Finance and Economics*, 27, 191-209.

[26] Jermann, U. (1998), "Asset Pricing in Production Economies," *Journal of Monetary Economics*, 41(2): 257-75.

[27] Kan, K., S. K. S. Kwong and C. K. Y. Leung (2004) "The dynamics and volatility of commercial and residential property prices: theory and evidence," *Journal of Regional Science,* 44(1), 95-123.

[28] King, R. and S. Rebelo (1999), "Resuscitating Real Business Cycles," inTaylor, John and Woodford, Michael, eds. **Handbook of Macroeconomics**. Volume 1B. New York: North-Holland, 927-1007.

[29] Krusell, P. and A. Smith (1997), "Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns," *Macroeconomic Dynamics*, 1(2), 387-422.

[30] Kydland, F. and E. Prescott (1996), "The Computational Experiment: An Econometric Tool," *Journal of Economic Perspectives*, 10(1), 69-85.

[31] Lamont, O. and R. Thaler (2003), "Anomalies: The Law of One Price in Financial Markets," *Journal of Economic Perspectives*, 17(4), 191-202.

[32] Lau, S.-H. P. (1997), "Using stochastic growth models to understand unit roots and breaking trends," *Journal of Economic Dynamics and Control*, 21, 1645-1667.

[33] Lau, S.-H. P. (2002). "Further inspection of the stochastic growth model by an analytical approach," *Macroeconomic Dynamics*, 6, 748-757.

[34] Leung, C. K. Y. (2004) "Macroeconomics and Housing: a review of the literature," *Journal of Housing Economics*, 13, 249-267.

[35] Leung, C. K. Y.; Y. Leong and K. Wong (2005), "Housing price dispersion: an empirical investigation," forthcoming in *Journal of Real Estate Finance and Economics.*

[36] Ljungqvist, Lars and Thomas Sargent (2000), **Recursive Macroeconomic Theory**, Cambridge: MIT Press.

[37] Lucas, R. (1978), "Asset Prices in an exchange economy," *Econometrica*, 46, 1426-45.

[38] Lütkepohl, H. (1993), **Introduction to Multiple Time Series Analysis**, 2nd edition. New York: Springer-Verlag.

[39] McQueen, G. and V. Roley (1993), "Stock prices, news and business conditions," *Review of Financial Studies*, 6(3), 683-707.

[40] Parker, J. and C. Julliard (2005), "Consumption risk and the cross section of expected returns," *Journal of Political Economy*, 113 (1), 185-222.

[41] Piazzesi, M.; M. Schneider and S. Tuzel (2004), "Housing, consumption and asset pricing," University of Chicago, mimeo.

[42] Quan, D. and S. Titman (1999), "Do real estate prices and stock prices move together? an international analysis," *Real Estate Economics*, 27, 183-207.

[43] Rebelo, S. (1991), "Long-Run policy analysis and long- run growth," *Journal of Political Economy*, 99, 500-21.

[44] Sargent, T. (1987), **Macroeconomic Theory**, 2nd ed. New York: Academic Press.

[45] Stokey, N.; R. Lucas and E. Prescott (1989), **Recursive Methods in Economic Dynamics**, Cambridge: Harvard University Press.

[46] Wheaton, W. (1999), "Real estate "cycles": some fundamentals," *Real Estate Economics*, 27(2), 209-230.

[47] Wongbangpo, P. and S. Sharma (2002), "Stock market and macroeconomic fundamental dynamic interactions: ASEAN-5 countries," *Journal of Asian Economics*, 13(1), 27-51.

A Proofs

A.1 Proof of (11), (12), (14)

To start, we first provide the first order conditions of the maximization for household, i.e. maximizing (4), subject to (5), and (2), (3).

Now we need to derive the first order conditions of the representative agent. Let λ_{1t}, λ_{2t}, λ_{3t} be the Lagrangian multipliers of the constraints (5), (3) and (2) respectively. The first order conditions are standard:

$$\begin{aligned}
\lambda_{1t} &= (C_t)^{-1}, && (38)\\
\omega_2 (1 - L_t)^{-1} &= \lambda_{1t} W_t, && (39)\\
\lambda_{3t} &= \beta E_t \left[\lambda_{1,t+1} R_{t+1} + (1 - \delta_k) \left(\frac{\lambda_{3,t+1} K_{t+2}}{K_{t+1}} \right) \right], && (40)\\
\lambda_{1t} &= \lambda_{3,t} (\delta_k) (K_{t+1}/I_{kt}), && (41)\\
P_{ht} \lambda_{1t} &= \lambda_{2t} (1 - \delta_h) (H_{t+1}/(H_t + H_t^m)), && (42)\\
R_{ht} \lambda_{1t} &= \omega_1 / (H_t + H_t^m) && (43)\\
P_{st} \lambda_{1t} &= \beta E_t \left[\lambda_{1,t+1} \left(\Pi_{t+1}^d + P_{s,t+1} \right) \right], && (44)\\
\lambda_{2t} &= \beta E_t \left[\left(\frac{\omega_1}{H_{t+1}} \right) + (1 - \delta_h) \left(\frac{\lambda_{2,t+1} H_{t+2}}{\left(H_{t+1} + H_{t+1}^m \right)} \right) \right], && (45)\\
\lambda_{1t} &= \lambda_{3t} (\delta_h) (H_{t+1}/I_{ht}), && (46)
\end{aligned}$$

and the condition that profit is equal to dividend $\Pi_t^d = \Pi_t$.

To solve this system of equations, the aggregate production function, (6), market clearing conditions, (9), and factor market conditions, (7) and (8), need to be imposed. Thus, (5) is simplified as

$$I_{kt} + I_{ht} + C_t \leq Y_t. \tag{47}$$

In addition, we will need to make conjecture about the policy functions of the agent, as in Ljungqvist and Sargent (2000). Our conjecture is simply that the consumption and investment shares are constant over time,

$$C_t = \mathcal{S}^c Y_t,\ I_{j,t} = \mathcal{S}^j Y_t,\ j = k,\ h,$$

where $\mathcal{S}$ are all constants, i.e. (11). Combining that with (47), we have

$$\mathcal{S}^c + \mathcal{S}^h + \mathcal{S}^k = 1. \tag{48}$$

Equipped with these, we are ready to prove our results. For instance, (39) becomes

$$\begin{aligned} \frac{\omega_2}{1 - L_t} &= \frac{\alpha_2}{L_t} \cdot \frac{1}{\mathcal{S}^c} \\ \text{or } L_t &= \frac{\alpha_2}{\alpha_2 + \mathcal{S}^c \omega_2} \equiv L, \end{aligned} \tag{49}$$

which is constant over time, as long as $\mathcal{S}^c$ is indeed a constant. Clearly, $0 < L < 1$.

Now, we want to get an expression for $\mathcal{S}^k$. Combining the conjecture, (11), (40), (7), we get

$$\lambda_{3t} K_{t+1} = \frac{\alpha_1 \beta}{\mathcal{S}^c} + \beta \left(1 - \delta_k\right) E_t \left[\lambda_{3,t+1} K_{t+2}\right]. \tag{50}$$

Notice that K_{t+1} is pre-determined at time t and hence can be extracted from the expectation operator. Notice also that (50) is of the form of a forward-looking, stochastic difference equation,

$$x_t = a_0 + a_1 E_t \left(x_{t+1}\right),$$

where $x_t = \lambda_{3t} K_{t+1}$, $a_1 = \beta \left(1 - \delta_k\right)$, and $0 < a_1 < 1$. Following Ljungqvist and Sargent (2000), we impose the no-bubble condition (which is also one of the transversality conditions in this model),

$$\lim_{s \to \infty} \left[\beta \left(1 - \delta_k\right)\right]^s E_t \left[\lambda_{3,t+s} K_{t+s+1}\right] = 0,$$

and solving iteratively, it is easy to see that (50) implies

$$\lambda_{3t} K_{t+1} = \frac{\alpha_1 \beta}{\mathcal{S}^c} \cdot \frac{1}{1 - \beta \left(1 - \delta_k\right)}.$$

From (41), with the conjecture, we have

$$\lambda_{3t} K_{t+1} = \left(\delta_k\right)^{-1} \left(\mathcal{S}^k / \mathcal{S}^c\right).$$

Equating these two expressions, we have

$$\mathcal{S}^k = \frac{\alpha_1 \beta \delta_k}{1 - \beta \left(1 - \delta_k\right)}, \tag{51}$$

which is indeed a constant. Clearly, $\mathcal{S}^k > 0$. The only additional assumption we need is that $\mathcal{S}^k < 1$, which is indeed (10).

Similarly, with (45), (9), (7), we can show that

$$\lambda_{2t} H_{t+1} = \omega_1 \beta + \beta (1 - \delta_h) E_t [\lambda_{2,t+1} H_{t+2}], \tag{52}$$

which is analogous to (50). Thus, we impose an analogous no bubble condition,

$$\lim_{s\to\infty} [\beta (1 - \delta_h)]^s E_t [\lambda_{2,t+s} H_{t+s+1}] = 0,$$

and solving iteratively, it is easy to see that (52) implies

$$\lambda_{2t} H_{t+1} = \frac{\omega_1 \beta}{1 - \beta (1 - \delta_h)}. \tag{53}$$

From (46), with the conjecture, we have

$$\lambda_{2t} H_{t+1} = (\delta_h)^{-1} \left(\mathcal{S}^h / \mathcal{S}^c\right).$$

Equating these two expressions, we have

$$\mathcal{S}^h = \frac{\omega_1 \beta \delta_h}{1 - \beta (1 - \delta_h)} \mathcal{S}^c. \tag{54}$$

Now, combine this with (48) and (51), we have

$$\begin{aligned} \mathcal{S}^c &= \left(1 - \frac{\alpha_1 \beta \delta_k}{1 - \beta (1 - \delta_k)}\right) \left(\frac{1 - \beta (1 - \delta_h)}{1 - \beta (1 - \delta_h) + \omega_1 \beta \delta_h}\right), \\ \mathcal{S}^h &= \left(1 - \frac{\alpha_1 \beta \delta_k}{1 - \beta (1 - \delta_k)}\right) \left(\frac{\omega_1 \beta \delta_h}{1 - \beta (1 - \delta_h) + \omega_1 \beta \delta_h}\right), \end{aligned} \tag{55}$$

which are clearly in between 0 and 1, and it completes the verification of (11). (Recall that the constancy of L, i.e. (49), actually depends on the constancy of $\mathcal{S}^c$.)

Now, combining (42) and (46) will deliver (12).

To prove (14), we start with (44). With (11), it can be re-written as

$$P_{st} \lambda_{1t} = \frac{(1 - \alpha_1 - \alpha_2) \beta}{\mathcal{S}^c} + \beta E_t [P_{s,t+1} \lambda_{1,t+1}]. \tag{56}$$

Once again, we impose the no-bubble condition,

$$\lim_{j\to\infty} (\beta)^j E_t [\lambda_{1,t+j} P_{s,t+j+1}] = 0,$$

and solving iteratively, it is easy to see that (56) implies

$$\lambda_{1t} P_{st} = \frac{(1-\alpha_1-\alpha_2)\,\beta}{\mathcal{S}^c} \cdot \frac{1}{1-\beta}.$$

However, by (11) and (38),

$$\lambda_{1t} = \frac{1}{\mathcal{S}^c Y_t}.$$

Combining the two expressions deliver (14).

Now, we will prove (13). Combining (42), (43) and (9), we get

$$\frac{P_{ht}}{R_{ht}} = \frac{(1-\delta_h)}{\omega_1} \lambda_{2t} H_{t+1}.$$

And by (53), this expression can be re-written as

$$\frac{P_{ht}}{R_{ht}} = \frac{\beta\,(1-\delta_h)}{1-\beta\,(1-\delta_h)},$$

which can in turn be written as (13).

A.2 Proof of equations, from (15) to (20)

To prove (15), we start with (6). Since

$$\begin{aligned} Y_t &= A_t\,(K_t)^{\alpha_1}\,(L_t)^{\alpha_2} \\ \Rightarrow\ y_t &= a_t + \alpha_1 k_t + \alpha_2 l_t. \end{aligned}$$

However, by (49), $l_t = \ln L_t = \ln L \equiv l$, which is a constant. Thus, we define

$$\theta_y \equiv \alpha_2 l_t = \alpha_2 \ln L,$$

where the formula of L is given by (49).

(16) and (17) can be obtained by taking natural log from (2) and (3) respectively.

(18) to (20) can indeed be obtained by taking natural log from (11), with

$$\eta_j \equiv \ln \mathcal{S}^j,\ j = c, k, h,$$

where the formulae for $\mathcal{S}^j$, $j = c, k, h$ are given by (51), (55).

A.3 Proof of (21)

From (12), (14), we have the following corollary:

$$p_{st} = \theta_s + y_t, \ \forall t.$$

Also, from the above expression, we can have

$$cov(p_{st}, p_{s,t+j}) = cov(y_{st}, y_{s,t+j}), \ j = 1, 2, 3, ... \tag{57}$$

We assume that the economic environment is stationary, in the sense that

$$var(p_{st}) = var(p_{s,t+j}), \ j = 1, 2, 3, ...$$

By definition,

$$cor(p_{st}, p_{s,t+j}) = \frac{cov(p_{st}, p_{s,t+j})}{\sqrt{var(p_{st})}\sqrt{var(p_{s,t+j})}},$$

and analogous expression applies to $cor(y_{st}, y_{s,t+j})$, $j = 1, 2, 3, ...$

It is clear that the dynamical system is block-recursive and once we can dictate the joint dynamics of y_t and k_t, we will also be able to pin down the dynamics of all other variables. Since this (intermediate) result is important in deriving the other results, we first present a formal statement of the result and then put the proof as a subsection.

Lemma 6 *If* $0 < E\left(a_t\right), Var\left(a_t\right) < \infty$*, the joint dynamics of* $\overrightarrow{y_{t+1}}' = (y_t, k_t)$ *can be described by the following vector equation,*

$$\overrightarrow{y_{t+1}} = \mathcal{M}_0 + \mathcal{M}_1 \overrightarrow{y_t} + \mathcal{M}_2 \overrightarrow{a_{t+1}}, \tag{58}$$

where $\overrightarrow{a_{t+1}}' = \left(a_{t+1}, \ 0\right)$*, and for some constant matrices* $\mathcal{M}_0, \ \mathcal{M}_1, \ \mathcal{M}_2$*.*

With (58), we will be able to dictate the dynamics of the (log) housing price. Therefore, we would first prove (58).

A.4 Proof of (58) and closed form solution for $(y_t, \ k_t, \ h_t)$

It is clear that the dynamical system from (15) to (20) is block-recursive and once we can dictate the joint dynamics of y_t and k_t, we will also be able to pin down the dynamics of all other variables. To start with, we observe that (15), (16) and (19) can be reduced as

$$M_1\overrightarrow{y_{t+1}} = N_1 + M_2\overrightarrow{y_t} + \overrightarrow{a_{t+1}}, \tag{59}$$

where

$$M_1 = \begin{pmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{pmatrix}, \; N_1 = \begin{pmatrix} \theta_y \\ \delta_k\eta_k \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 0 \\ \delta_k & (1-\delta_k) \end{pmatrix},$$

$$\overrightarrow{y_{t+1}} = \begin{pmatrix} y_{t+1} \\ k_{t+1} \end{pmatrix}, \; \overrightarrow{a_{t+1}} = \begin{pmatrix} a_{t+1} \\ 0 \end{pmatrix},$$

with the additional assumption that

$$0 < E(a_t), Var(a_t) < \infty.$$

It is easy to see that the solution of (59) can be rewritten as

$$\overrightarrow{y_{t+1}} = (M_1)^{-1}N_1 + (M_1)^{-1}M_2\overrightarrow{y_t} + (M_1)^{-1}\overrightarrow{a_{t+1}}, \tag{60}$$

$$\begin{aligned} \text{where } (M_1)^{-1} &= \begin{pmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} = \mathcal{M}_2, \end{aligned}$$

$$\begin{aligned} (M_1)^{-1}N_1 &= \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \theta_y \\ \delta_k\eta_k \end{pmatrix} = \begin{pmatrix} \theta_y + \alpha_1\delta_k\eta_k \\ \delta_k\eta_k \end{pmatrix} = \mathcal{M}_0, \\ (M_1)^{-1}M_2 &= \begin{pmatrix} \alpha_1\delta_k & \alpha_1(1-\delta_k) \\ \delta_k & 1-\delta_k \end{pmatrix} = \mathcal{M}_1, \end{aligned}$$

which completes the proof of (58). With (58), or more explicitly (60), it is now possible to simulate (*numerically*) the evolution of output and business capital $(y_t, k_t)'$.

Alternatively, we can also obtain a "closed form solution" for this simple two-variable system. We can re-write (60) as

$$\overrightarrow{y_{t+1}} = \mathcal{M}_0 + \mathcal{M}_1(B)\overrightarrow{y_{t+1}} + \mathcal{M}_2\overrightarrow{a_{t+1}} \tag{61}$$

$$\text{where } \mathcal{M}_1(B) = \begin{pmatrix} \alpha_1\delta_k B & \alpha_1(1-\delta_k)B \\ \delta_k B & (1-\delta_k)B \end{pmatrix}$$

where B is the backshift operator (or Lag operator), $B^n x_t = x_{t-n}$, for all variable x, $n = 1, 2, 3, ...$, and $Bc = c$ for all constant c (see Sargent (1987), Lütkepohl (1993), Lau (1997) for more details). In addition, Sargent (1987), Lütkepohl (1993) show that

$$\begin{aligned}(1 - cB)^{-1} x_t &= \left(1 + cB + (cB)^2 + (cB)^3 + ...\right) x_t \\ &= x_t + cx_{t-1} + c^2 x_{t-2} + ... \\ &= \sum_{i=0}^{\infty} (c)^i x_{t-i}. \end{aligned} \tag{62}$$

Thus, (61) can be re-arranged as

$$\begin{aligned}[I - \mathcal{M}_1(B)] \overrightarrow{y_{t+1}} &= \mathcal{M}_0 + \mathcal{M}_2 \overrightarrow{a_{t+1}}, \\ \text{or } \overrightarrow{y_{t+1}} &= [I - \mathcal{M}_1(B)]^{-1} [\mathcal{M}_0 + \mathcal{M}_2 \overrightarrow{a_{t+1}}], \\ \text{or } \overrightarrow{y_t} &= [I - \mathcal{M}_1(B)]^{-1} [\mathcal{M}_0 + \mathcal{M}_2 \overrightarrow{a_t}], \end{aligned} \tag{63}$$

where I is the identity matrix. Note,

$$\text{where } I - \mathcal{M}_1(B) = \begin{pmatrix} 1 - \alpha_1 \delta_k B & -\alpha_1 (1 - \delta_k) B \\ -\delta_k B & 1 - (1 - \delta_k) B \end{pmatrix}.$$

We let

$$[I - \mathcal{M}_1(B)]^{-1} = \begin{pmatrix} a_{11}(B) & a_{12}(B) \\ a_{21}(B) & a_{22}(B) \end{pmatrix},$$

and by definition, $[I - \mathcal{M}_1(B)] [I - \mathcal{M}_1(B)]^{-1} = I$, where I is the identity matrix. In other words, we have

$$\begin{aligned}\begin{pmatrix} 1 - \alpha_1 \delta_k B & -\alpha_1 (1 - \delta_k) B \\ -\delta_k B & 1 - (1 - \delta_k) B \end{pmatrix} \begin{pmatrix} a_{11}(B) \\ a_{21}(B) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \text{and } \begin{pmatrix} 1 - \alpha_1 \delta_k B & -\alpha_1 (1 - \delta_k) B \\ -\delta_k B & 1 - (1 - \delta_k) B \end{pmatrix} \begin{pmatrix} a_{21}(B) \\ a_{22}(B) \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

By Cramer′s Rule, it is easy to show that

$$\begin{aligned} & [I - \mathcal{M}_1(B)]^{-1} \\ = \; & (1 - (1 - \delta_k + \alpha_1 \delta_k) B)^{-1} \begin{pmatrix} 1 - (1 - \delta_k) B & \alpha_1 (1 - \delta_k) B \\ \delta_k B & 1 - \alpha_1 \delta_k B \end{pmatrix}. \end{aligned}$$

In fact, we can obtain "closed form solutions" for the physical capital stock, aggregate output and the housing stock. We first state the results as a lemma.

Lemma 7 *In log form, the physical capital stock, output and housing stock can be written as a summation of previous period productivity shocks,*

$$
\begin{aligned}
k_t &= \frac{\theta_y + \delta_k \eta_k}{(1-\alpha_1)} + \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i \, a_{t-1-i}, && (64) \\
y_t &= \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha_1)} + a_t + \alpha_1 \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i \, a_{t-1-i}, && (65) \\
h_t &= \eta_h' + \delta_h a_{t-1} + \delta_h \sum_{i=0}^{\infty} \delta_h(i) a_{t-2-i}, && (66)
\end{aligned}
$$

for some constant η_h', and some positive function of parameter, $\delta_h(i) > 0$, $\forall i$.

Now, we will prove this lemma. Recall from (63) that

$$
\begin{aligned}
\overrightarrow{y_t} &= [I - \mathcal{M}_1(B)]^{-1} [\mathcal{M}_0 + \mathcal{M}_2 \overrightarrow{a_t}] \\
&= \mathcal{M}_0' + \mathcal{M}_2'(B) \overrightarrow{a_t} && (67)
\end{aligned}
$$

where

$$
\mathcal{M}_0' = \frac{1}{(1-\alpha)} \begin{pmatrix} \theta_y + \alpha_1 \eta_k \\ \theta_y + \delta_k \eta_k \end{pmatrix},
$$

$$
\mathcal{M}_2'(B) = (1 - (1 - \delta_k + \alpha_1 \delta_k) B)^{-1} \begin{pmatrix} 1 - (1 - \delta_k) B & \alpha_1 \\ \delta_k B & 1 \end{pmatrix}.
$$

We can elaborate the equation (67), we have

$$
\begin{aligned}
y_t &= \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha)} + a_t + \alpha_1 \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i B^{i+1} a_t \\
&= \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha)} + a_t + \alpha_1 \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i a_{t-1-i},
\end{aligned}
$$

which is (65), and

$$
\begin{aligned}
k_t &= \frac{\theta_y + \delta_k \eta_k}{(1-\alpha)} + \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i B^{i+1} a_t \\
&= \frac{\theta_y + \delta_k \eta_k}{(1-\alpha)} + \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i a_{t-1-i},
\end{aligned}
$$

which is (64).

Finally, we want to prove (66). By combining (17) and (20), we have

$$\begin{aligned} h_{t+1} &= \delta_h \eta_h + (1-\delta_h) h_t + \delta_h y_t \\ \text{or } \left(1-(1-\delta_h) B\right) h_{t+1} &= \delta_h \eta_h + \delta_h y_t \\ \text{or } h_{t+1} &= \left(1-(1-\delta_h) B\right)^{-1} (\delta_h \eta_h + \delta_h y_t) \\ h_{t+1} &= \eta_h + \left(1-(1-\delta_h) B\right)^{-1} (\delta_h y_t), \\ \text{or } h_t &= \eta_h + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i B^{i+1} y_t \right), \\ \text{or } h_t &= \eta_h + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i y_{t-1-i} \right), \qquad (68) \end{aligned}$$

and by (65), it can be re-written as

$$h_t = \eta_h + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i \left(\theta'_y + a_{t-1-i} + \alpha_1 \delta_k \sum_{j=0}^{\infty} (1-\delta_k+\alpha_1\delta_k)^j a_{t-2-i-j} \right) \right)$$

where $\theta'_y = \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha)}$. Clearly, it is equal to

$$\begin{aligned} h_t &= \left(\eta_h + \theta'_y\right) + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i a_{t-1-i} \right) \\ &\quad + \alpha_1 \delta_h \delta_k \sum_{i=0}^{\infty} (1-\delta_h)^i \sum_{{}^i j=0}^{\infty} (1-\delta_k+\alpha_1\delta_k)^j a_{t-2-i-j}, \\ \text{or } h_t &= \left(\eta_h + \theta'_y\right) + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i a_{t-1-i} \right) \\ &\quad + \alpha_1 \delta_h \delta_k \sum_{i=0}^{\infty} \sum_{{}^i j=0}^{i} (1-\delta_h)^{i-j} (1-\delta_k+\alpha_1\delta_k)^j a_{t-2-i-j}, \\ \text{or } h_t &= \left(\eta_h + \theta'_y\right) + \delta_h a_{t-1} \\ &\quad + \delta_h \sum_{i=0}^{\infty} \left[(1-\delta_h)^{i+1} + \alpha_1 \delta_k \sum_{{}^i j=0}^{i} (1-\delta_h)^{i-j} (1-\delta_k+\alpha_1\delta_k)^j \right] a_{t-2-i}, \end{aligned}$$

which is (66), with $\eta_h' = \left(\eta_h + \theta_y'\right)$, and

$$\delta_h(i) \equiv \left[(1-\delta_h)^{i+1} + \alpha_1 \delta_k \sum_{i\, j=0}^{i} (1-\delta_h)^{i-j} (1-\delta_k + \alpha_1 \delta_k)^j \right].$$

Clearly, $\delta_h(i) > 0$, $\forall i$.

A.5 Proof of (22), (23)

By taking log of (12), we also have the following equation:

$$p_{ht} = \theta_h + i_{ht} - h_t, \tag{69}$$

where $\theta_s = \ln(1-\alpha_1-\alpha_2)$, $\theta_h = \ln\left(\frac{1-\delta_h}{\delta_h}\right)$. From (13), we also get

$$r_{ht} = \theta_{rh} + p_{ht}. \tag{70}$$

Thus, to understand the dynamics of rent, it suffices to characterize the dynamics of housing price. Now, by (69), we have

$$\begin{aligned} p_{ht} &= \theta_h + i_{ht} - h_t \\ &= (\theta_h + \eta_h) + y_t - h_t. \end{aligned} \tag{71}$$

and by (68), we have

$$h_t = \eta_h + \delta_h \left(\sum_{i=0}^{\infty} (1-\delta_h)^i y_{t-1-i} \right).$$

Substituting the last expression in (71) gives (22).

Alternatively, we can also substitute (65) and (66) in (71), which will give

$$
\begin{aligned}
p_{ht} &= (\theta_h + \eta_h) + y_t - h_t \\
&= (\theta_h + \eta_h) + \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha)} + a_t \\
&\quad + \alpha_1 \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i a_{t-1-i} \\
&\quad - \left(\eta_h + \theta_y'\right) - \delta_h a_{t-1} \\
&\quad - \delta_h \sum_{i=0}^{\infty} \left[(1-\delta_h)^{i+1} + \alpha_1 \delta_k \sum_{i j=0}^{i} (1-\delta_h)^{i-j} (1 - \delta_k + \alpha_1 \delta_k)^j \right] a_{t-2-i} \\
&= \theta_h + a_t + (\alpha_1 \delta_k - \delta_h) a_{t-1} \\
&\quad + \sum_{i=2}^{\infty} \Big\{ \alpha_1 \delta_k (1 - \delta_k + \alpha_1 \delta_k)^{i-1} \\
&\quad - \delta_h \left[(1-\delta_h)^{i-1} + \alpha_1 \delta_k \sum_{i j=0}^{i-2} (1-\delta_h)^{i-j} (1 - \delta_k + \alpha_1 \delta_k)^j \right] \Big\} a_{t-i},
\end{aligned}
$$

where $\sum_{i=0}^{0} x(i) = 1$ for any $\{x(i)\}_i$. Thus, $p_{ht} = \theta_h + a_t + (\alpha_1 \delta_k - \delta_h) a_{t-1} + \sum_{i=2}^{\infty} \delta_p(i) a_{t-i}$, $\forall t$, where

$$
\begin{aligned}
\delta_p(i) &= \Big\{ \alpha_1 \delta_k (1 - \delta_k + \alpha_1 \delta_k)^{i-1} \\
&\quad - \delta_h \left[(1-\delta_h)^{i-1} + \alpha_1 \delta_k \sum_{j=0}^{i} (1-\delta_h)^{i-j} (1 - \delta_k + \alpha_1 \delta_k)^j \right] \Big\}, \\
\text{for } i &= 2, 3, 4, ...
\end{aligned}
$$

And, for the sake of convenience, let us define

$$
\begin{aligned}
\delta_p(0) &= 1, \\
\delta_p(1) &= (\alpha_1 \delta_k - \delta_h).
\end{aligned}
$$

Thus,

$$
p_{ht} = \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i}. \tag{72}
$$

It completes the proof of (23).

Notice that the sign of the term $\left\{\alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^{i-1}\right.$

$\left.-\delta_h\left[(1-\delta_h)^{i-1}+\alpha_1\delta_k\sum_{j=0}^{i-2}(1-\delta_h)^{i-j}(1-\delta_k+\alpha_1\delta_k)^j\right]\right\}$ is uncertain in general.

A.6 Proof of (24) to (28)

We will first prove the serial correlation of the stock price to be positive. Recall from (57) that $cov(p_{st}, p_{s,t+j}) = cov(y_t, y_{t+j})$, $j = 1, 2, 3, ...$ Thus, it suffices to investigate the covariance of output in different periods, $cov(y_t, y_{t+j})$. Now, recall from (65), we have $y_{t+j} = \frac{\theta_y+\alpha_1\eta_k}{(1-\alpha)} + a_{t+j}$ $+\alpha_1\delta_k\sum_{i=0}^{\infty}(1-\delta_k+\alpha_1\delta_k)^i a_{t+j-1-i}$. By definition, $cov(a_t, a_{t+j}) = 0$, $\forall j \neq 0$. Therefore,

$$
\begin{aligned}
& cov(y_t, y_{t+j}) \\
= \;& cov\left(\frac{\theta_y+\alpha_1\eta_k}{(1-\alpha)} + a_t + \alpha_1\delta_k\sum_{i=0}^{\infty}(1-\delta_k+\alpha_1\delta_k)^i a_{t-1-i}\right., \\
& \left.\frac{\theta_y+\alpha_1\eta_k}{(1-\alpha)} + a_{t+j} + \alpha_1\delta_k\sum_{i=0}^{\infty}(1-\delta_k+\alpha_1\delta_k)^i a_{t+j-1-i}\right) \\
= \;& cov\left(a_t, \alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^{j-1}a_t\right) \\
& +cov\left(\alpha_1\delta_k a_{t-1}, \alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^j a_{t-1}\right) \\
& +cov\left(\alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)a_{t-2}, \alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^{j+1}a_{t-2}\right) \\
& +... \\
= \;& \alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^{j-1}var(a_t) \\
& +(\alpha_1\delta_k)^2(1-\delta_k+\alpha_1\delta_k)^j var(a_{t-1}) \\
& +(\alpha_1\delta_k)^2(1-\delta_k+\alpha_1\delta_k)^{j+2} var(a_{t-2}) + ... \\
= \;& \sigma_a^2\left(\alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^{j-1}\right. \\
& \left.+(\alpha_1\delta_k)^2(1-\delta_k+\alpha_1\delta_k)^j\left[1+(1-\delta_k+\alpha_1\delta_k)^2+...\right]\right) \\
= \;& \sigma_a^2\left(\alpha_1\delta_k(1-\delta_k+\alpha_1\delta_k)^{j-1} + \frac{(\alpha_1\delta_k)^2(1-\delta_k+\alpha_1\delta_k)^j}{1-(1-\delta_k+\alpha_1\delta_k)^2}\right) > 0
\end{aligned}
$$

as $(1-\delta_k+\alpha_1\delta_k)$ and other terms are positive, and by definition, $var(a_t) = var(a_{t-j}) \equiv \sigma_a^2$, $j = 1, 2, 3, ...$, which completes the proof of (24).

Now, we want to prove that the serial correlation of the housing price is also non-zero in general. Recall from (23) that $p_{ht} = \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i}$, and for $j = 1, 2, 3,...,$

$$\begin{aligned}
& cov(p_{ht}, p_{h,t+j}) \\
= \; & cov\left(\theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i} \; , \; \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t+j-i}\right) \\
= \; & cov\left(\delta_p(0) a_t, \delta_p(j) a_t\right) \\
& + cov\left(\delta_p(1) a_{t-1}, \delta_p(j+1) a_{t-1}\right) + ... \\
= \; & \sigma_a^2 \left(\delta_p(0)\delta_p(j) + \delta_p(1) \cdot \delta_p(j+1) + \sum_{i=2}^{\infty} \delta_p(i) \cdot \delta_p(i+j)\right) \\
= \; & \sigma_a^2 \left(\sum_{i=0}^{\infty} \delta_p(i) \cdot \delta_p(i+j)\right),
\end{aligned}$$

which completes the proof of (25).

Notice further that

$$var(p_{st}) = var(y_t).$$

Since $y_t = \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha)} + a_t + \alpha_1 \delta_k \sum_{i=0}^{\infty} \left(1 - \delta_k + \alpha_1 \delta_k\right)^i a_{t-1-i}$, and $\{a_t\}$ is serially uncorrelated,

$$\begin{aligned}
var(p_{st}) &= var(y_t) \\
&= var\left(a_t + \alpha_1 \delta_k \sum_{i=0}^{\infty} \left(1 - \delta_k + \alpha_1 \delta_k\right)^i a_{t-1-i}\right) \\
&= \sigma_a^2 \cdot \left[1 + (\alpha_1 \delta_k)^2 \left(\sum_{i=0}^{\infty} \left(1 - \delta_k + \alpha_1 \delta_k\right)^{2i}\right)\right] \\
\text{or} \quad &= \sigma_a^2 \cdot \left[\sum_{i=0}^{\infty} \left(\delta_p^s(i)\right)^2\right].
\end{aligned}$$

Similarly, as $p_{ht} = \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i}$, $\forall t$,

$$var(p_{ht}) = \sigma_a^2 \cdot \left(\sum_{i=0}^{\infty} \left(\delta_p(i)\right)^2\right).$$

Now, we can also prove (26). Recall that $p_{st} = (cons \tan t) + a_t$

$+\alpha_1\delta_k \sum_{i=0}^{\infty} (1-\delta_k+\alpha_1\delta_k)^i a_{t-1-i}$, and $p_{ht} = \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i}$. And for convenience, we define

$$\begin{aligned} \delta_p^s(0) &= 1 \\ \delta_p^s(i) &= \alpha_1\delta_k (1-\delta_k+\alpha_1\delta_k)^i, \ i = 1, 2, 3, ... \end{aligned}$$

Note $\delta_p^s(i) > 0$, $\forall i$, and as $i \to \infty$, $\delta_p^s(i) \to 0$. Then

$$p_{st} = (cons\tan t) + \sum_{i=0}^{\infty} \delta_p^s(i)\, a_{t-i}, \tag{73}$$

and hence

$$\begin{aligned} & cov(p_{st}, p_{ht}) \\ = \;& cov\left(\sum_{i=0}^{\infty} \delta_p^s(i)\, a_{t-i},\ \sum_{i=0}^{\infty} \delta_p(i) a_{t-i}\right) \\ = \;& \sigma_a^2 \cdot \left[\sum_{i=0}^{\infty} \delta_p(i)\delta_p^s(i)\right]. \end{aligned}$$

Note that since $0 < \alpha_1, \delta_k, \delta_h < 1$, $|\alpha_1\delta_k - \delta_h| < 1$. Thus, $1 + (\alpha_1\delta_k)(\alpha_1\delta_k - \delta_h) > 0$. However, we do not know the magnitude of the expression $\delta_p(i)\delta_p^s(i)$, as $\delta_p(i)$ can be negative for some i.

Similarly, we can show (27), for $j = 1, 2, 3, 4, ...$

$$\begin{aligned} & cov(p_{st}, p_{h,t+j}) \\ = \;& cov\left(\sum_{i=0}^{\infty} \delta_p^s(i)\, a_{t-i},\ \sum_{i=0}^{\infty} \delta_p(i) a_{t+j-i}\right) \\ = \;& [\delta_p^s(0)\, \delta_p(j) var(a_t) \quad + \delta_p^s(1)\, \delta_p(j+1) var(a_{t-1}) \\ & + \delta_p^s(2)\, \delta_p(j+2) var(a_{t-2}) + ...] \\ = \;& \sigma_a^2 \left[\sum_{i=0}^{\infty} \delta_p(j+i)\delta_p^s(i)\right]. \end{aligned}$$

Conversely, the current period stock price and previous period housing prices have non-zero serial correlation as well, $j = 1, 2, 3, ...$, as (28) asserts,

$$\begin{aligned}
& cov(p_{s,t}, p_{h,t-j}) \\
= \; & cov\left(\sum_{i=0}^{\infty} \delta_p^s(i)\, a_{t-i},\ \sum_{i=0}^{\infty} \delta_p(i) a_{t-j-i}\right) \\
= \; & \delta_p^s(j)\, \delta_p(0) var\left(a_{t-j}\right) + \delta_p^s(j+1)\, \delta_p(1) var\left(a_{t-j-1}\right) \\
& + \delta_p^s(j+2)\, \delta_p(2) var\left(a_{t-j-2}\right) + ... \\
= \; & \sigma_a^2 \left[\sum_{i=0}^{\infty} \delta_p(i) \delta_p^s(j+i)\right].
\end{aligned}$$

A.7 Proof of (30) to (36)

We need to first simplify the expressions for the rate of returns. We first start with the rate of return for stock. By (7), (8), we have

$$\Pi_t^d = (1 - \alpha_1 - \alpha_2)\, Y_t.$$

And combining this with (14), we have

$$P_{s,t+1} + \Pi_{t+1}^d = \frac{(1 - \alpha_1 - \alpha_2)}{1 - \beta} \cdot Y_{t+1}.$$

Thus, by (29),

$$\begin{aligned}
\widetilde{R}_{s,t+1} &= \frac{P_{s,t+1} + \Pi_{t+1}^d}{P_{st}} \\
&= \frac{1}{\beta} \cdot \frac{Y_{t+1}}{Y_t},
\end{aligned}$$

and in log form,

$$\widetilde{r}_{s,t+1} = -\ln \beta + y_{t+1} - y_t,$$

which is (30).

Now, we can modify the expression for the rate of return for housing. Combining (12) with (13), we have

$$P_{h,t+1} + R_{h,t+1} = \frac{1}{\beta\,(1 - \delta_h)} \cdot P_{h,t+1},$$

and by (29),

$$\begin{aligned}\widetilde{R}_{h,t+1} &= \frac{P_{h,t+1}+R_{h,t+1}}{P_{ht}} \\ &= \frac{1}{\beta\left(1-\delta_h\right)}\cdot\frac{P_{h,t+1}}{P_{h,t}},\end{aligned}$$

which is, in log form,

$$\widetilde{r}_{h,t+1} = -\ln\left(\beta\left(1-\delta_h\right)\right)+p_{h,t+1}-p_{ht},$$

which is (31).

Note that by (73), (30) can be written as

$$\widetilde{r}_{s,t+1} = -\ln\beta+\delta_p^s\left(0\right)a_{t+1}+\sum_{i=0}^{\infty}\left[\delta_p^s\left(i+1\right)-\delta_p^s\left(i\right)\right]a_{t-i}. \tag{74}$$

Thus, the rate of return of stock in period $t+1+j$, $j=1$, 2, 3,... (in log form) is

$$\widetilde{r}_{s,t+j+1} = -\ln\beta+\delta_p^s\left(0\right)a_{t+j+1}+\sum_{i=0}^{\infty}\left[\delta_p^s\left(i+1\right)-\delta_p^s\left(i\right)\right]a_{t+j-i},$$

and the variance of the return of stock is

$$var\left(\widetilde{r}_{s,t+j+1}\right) = \sigma_a^2\cdot\left\{\left(\delta_p^s\left(0\right)\right)^2+\sum_{i=0}^{\infty}\left[\delta_p^s\left(i+1\right)-\delta_p^s\left(i\right)\right]^2\right\}.$$

Hence, the covariance of the rate of return of stock in period $t+1$ and $t+j+1$, $j=1$, 2, 3,... is

$$\begin{aligned}& cov\left(\widetilde{r}_{s,t+1},\widetilde{r}_{s,t+j+1}\right) \\ =\; & \delta_p^s\left(0\right)\left[\delta_p^s\left(j\right)-\delta_p^s\left(j-1\right)\right]var(a_{t+1}) \\ & +\left[\delta_p^s\left(1\right)-\delta_p^s\left(0\right)\right]\left[\delta_p^s\left(j+1\right)-\delta_p^s\left(j\right)\right]var(a_t) \\ & +\left[\delta_p^s\left(2\right)-\delta_p^s\left(1\right)\right]\left[\delta_p^s\left(j+2\right)-\delta_p^s\left(j+1\right)\right]var(a_{t-1}) \\ & +... \\ =\; & \sigma_a^2\cdot\left\{\delta_p^s\left(0\right)\left[\delta_p^s\left(j\right)-\delta_p^s\left(j-1\right)\right]\right. \\ & \left.+\sum_{i=0}^{\infty}\left[\delta_p^s\left(i+1\right)-\delta_p^s\left(i\right)\right]\left[\delta_p^s\left(j+1+i\right)-\delta_p^s\left(j+i\right)\right]\right\},\end{aligned}$$

and applies the definition $\delta_r^s(i) \equiv \left[\delta_p^s(i) - \delta_p^s(i-1)\right]$ will complete the proof of (32).

Now, we want to compute the covariance of different period housing return. First, we need to obtain an expression of the housing price in terms of the productivity shocks. By (72), (31), we have

$$\begin{aligned}
\widetilde{r}_{h,t+1} &= -\ln(\beta(1-\delta_h)) + p_{h,t+1} - p_{ht} \\
&= -\ln(\beta(1-\delta_h)) + \delta_p(0)\,a_{t+1} + \sum_{i=0}^{\infty}\left[\delta_p(i+1) - \delta_p(i)\right]a_{t-i} \\
&= -\ln(\beta(1-\delta_h)) + \delta_p(0)\,a_{t+1} + \sum_{i=0}^{\infty}\delta_r(i+1)\cdot a_{t-i}, \qquad (75)
\end{aligned}$$

where

$$\delta_r(i) \equiv \left[\delta_p(i) - \delta_p(i-1)\right]$$

Thus, the rate of return of stock in period $t+1+j$, $j = 1, 2, 3, \ldots$ (in log form) is

$$\widetilde{r}_{h,t+1+j} = -\ln(\beta(1-\delta_h)) + \delta_p(0)\,a_{t+1+j} + \sum_{i=0}^{\infty}\delta_r(i+1)\cdot a_{t-i+j}.$$

Hence, the covariance of the rate of return of stock in period $t+1$ and $t+1+j$, $j = 1, 2, 3, \ldots$ is

$$\begin{aligned}
& cov\left(\widetilde{r}_{h,t+1}, \widetilde{r}_{h,t+1+j}\right) \\
=\ & -(1-\alpha_1\delta_k+\delta_h)\,var(a_{t+1}) \\
& -(1-\alpha_1\delta_k+\delta_h)\left[\delta_p(0)-(\alpha_1\delta_k-\delta_h)\right]var(a_t) \\
& +\left[\delta_p(0)-(\alpha_1\delta_k-\delta_h)\right]\left[\delta_p(1)-\delta_p(0)\right]var(a_{t-1}) \\
& +\left[\delta_p(1)-\delta_p(0)\right]\left[\delta_p(2)-\delta_p(1)\right]var(a_{t-2}) \\
& +\ldots \\
=\ & \sigma_a^2\cdot\left\{-(1-\alpha_1\delta_k+\delta_h)\left[1+\delta_p(0)-(\alpha_1\delta_k-\delta_h)\right]\right. \\
& +\left[\delta_p(0)-(\alpha_1\delta_k-\delta_h)\right]\left[\delta_p(1)-\delta_p(0)\right] \\
& \left.+\sum_{i=0}^{\infty}\left[\delta_p(i+1)-\delta_p(i)\right]\left[\delta_p(i+2)-\delta_p(i+1)\right]\right\}.
\end{aligned}$$

$$
\begin{aligned}
& cov\left(\widetilde{r}_{h,t+1},\widetilde{r}_{h,t+j+1}\right) \\
= \; & \delta_p\left(0\right)\left[\delta_r\left(j\right)\right] var(a_{t+1}) \\
& +\left[\delta_r\left(1\right)\right]\left[\delta_r\left(j+1\right)\right] var(a_t) \\
& +\left[\delta_r\left(2\right)\right]\left[\delta_r\left(j+2\right)\right] var(a_{t-1}) \\
& +... \\
= \; & \sigma_a^2 \cdot \left\{\delta_p\left(0\right)\left[\delta_r\left(j\right)\right]\right. \\
& \left. + \sum_{i=0}^{\infty} \delta_r\left(i+1\right)\cdot\delta_r\left(j+1+i\right)\right\},
\end{aligned}
$$

It completes the proof of (33).

Notice also that $\widetilde{r}_{h,t+1} = -\ln\left(\beta\left(1-\delta_h\right)\right) + a_{t+1} - \left(1-\alpha_1\delta_k+\delta_h\right)a_t + \left[\delta_p\left(0\right)-\left(\alpha_1\delta_k-\delta_h\right)\right]a_{t-1} + \sum_{i=0}^{\infty}\left[\delta_p\left(i+1\right)-\delta_p\left(i\right)\right]a_{t-2-i}$, which means that

$$
\begin{aligned}
& var\left(\widetilde{r}_{h,t+1}\right) \\
= \; & \sigma_a^2 \cdot \left\{1+\left(1-\alpha_1\delta_k+\delta_h\right)^2 + \left[\delta_p\left(0\right)-\left(\alpha_1\delta_k-\delta_h\right)\right]^2\right. \\
& \left. + \sum_{i=0}^{\infty}\left[\delta_p\left(i+1\right)-\delta_p\left(i\right)\right]^2\right\}.
\end{aligned}
$$

Now, we are ready to compute the cross-correlations of the two assets. We will start with the contemporaneous correlation. Recall that $\widetilde{r}_{s,t+1} = -\ln\beta + \delta_p^s\left(0\right)a_{t+1} + \sum_{i=0}^{\infty}\delta_r^s\left(i+1\right)\cdot a_{t-i}$, where $\delta_r^s\left(i\right)\equiv\left[\delta_p^s\left(i\right)-\delta_p^s\left(i-1\right)\right]$, and $\widetilde{r}_{h,t+1} = -\ln\left(\beta\left(1-\delta_h\right)\right) + \delta_p\left(0\right)a_{t+1} + \sum_{i=0}^{\infty}\delta_r\left(i+1\right)\cdot a_{t-i}$, where $\delta_r\left(i\right)\equiv\left[\delta_p\left(i\right)-\delta_p\left(i-1\right)\right]$. Then,

$$
\begin{aligned}
& cov\left(\widetilde{r}_{s,t+1},\widetilde{r}_{h,t+1}\right) \\
= \; & \left(\delta_p^s\left(0\right)\cdot\delta_p\left(0\right)\right) var\left(a_{t+1}\right) \\
& + \sum_{i=0}^{\infty}\delta_r^s\left(i+1\right)\cdot\delta_r\left(i+1\right)\cdot var\left(a_{t-i}\right) \\
= \; & \sigma_a^2\cdot\left\{\left(\delta_p^s\left(0\right)\cdot\delta_p\left(0\right)\right) + \sum_{i=0}^{\infty}\delta_r^s\left(i+1\right)\cdot\delta_r\left(i+1\right)\right\},
\end{aligned}
$$

which is (34). Similarly, we can relate the covariance of the current period stock price to the subsequent period (or previous period) housing return.

Formally, for $j = 1, 2, 3, ...$

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1}, \widetilde{r}_{h,t+1+j}\right) \\ = & \left(\delta_p^s(0) \cdot \delta_r(j)\right) var\left(a_{t+1}\right) \\ & + \sum_{i=0}^{\infty} \delta_r^s(i+1) \cdot \delta_r(i+j+1) \cdot var\left(a_{t-i}\right) \\ = & \sigma_a^2 \cdot \left\{ \left(\delta_p^s(0) \cdot \delta_r(j)\right) + \sum_{i=0}^{\infty} \delta_r^s(i+1) \cdot \delta_r(i+j+1) \right\}, \end{aligned}$$

which is (35). And

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1}, \widetilde{r}_{h,t+1-j}\right) \\ = & \left(\delta_r^s(j) \cdot \delta_p(0)\right) var\left(a_{t+1}\right) \\ & + \sum_{i=0}^{\infty} \delta_r^s(j+i+1) \cdot \delta_r(i+1) \cdot var\left(a_{t-i}\right) \\ = & \sigma_a^2 \cdot \left\{ \left(\delta_r^s(j) \cdot \delta_p(0)\right) + \sum_{i=0}^{\infty} \delta_r^s(j+i+1) \cdot \delta_r(i+1) \right\}, \end{aligned}$$

which is (36).

B The case of AR(1) productivity shock

In this section, we consider the case where $\{a_t\}$ is an AR(1), and examine how the results are affected. Formally,

$$a_t = \rho a_{t-1} + u_t, \tag{76}$$

where u_t is i.i.d., with $E(u_t) = 0$, $var(u_t) = \sigma_u^2 < \infty$, $\forall t$ and $cov(u_t, u_s) = 0$, $\forall s \neq t$. By (76), we have

$$a_t = \sum_{j=0}^{\infty} (\rho)^j u_{t-j}.$$

Thus, by above,

$$
\begin{aligned}
p_{st} &= \theta_s + \frac{\theta_y + \alpha_1 \eta_k}{(1-\alpha)} \\
&\quad + a_t + \alpha_1 \delta_k \sum_{i=0}^{\infty} (1 - \delta_k + \alpha_1 \delta_k)^i \, a_{t-1-i} \\
&= (const) + \sum_{i=0}^{\infty} \delta_p^s (i) \, a_{t-i} \\
&= (const) + \sum_{i=0}^{\infty} \delta_p^s (i) \sum_{j=0}^{\infty} (\rho)^j u_{t-i-j} \\
&= (const) + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \delta_p^s (i) \, (\rho)^{i-j} \right) u_{t-i} \\
&= (const) + \sum_{i=0}^{\infty} \delta_p^{sr} (i) \, u_{t-i}
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_p^s(0) &= 1 \\
\delta_p^s (i) &= \alpha_1 \delta_k \, (1 - \delta_k + \alpha_1 \delta_k)^i \,, \ i = 1, 2, 3, \ldots
\end{aligned}
$$

and

$$
\delta_p^{sr} (i) \equiv \left(\sum_{j=0}^{i} (\rho)^{i-j} \delta_p^s (i) \right).
$$

Similarly,

$$
\begin{aligned}
p_{ht} &= \theta_h + \sum_{i=0}^{\infty} \delta_p(i) a_{t-i} \\
&= \theta_h + \sum_{i=0}^{\infty} \delta_p(i) \sum_{j=0}^{\infty} (\rho)^j u_{t-i-j} \\
&= \theta_h + \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \delta_p (i) \, (\rho)^{i-j} \right) u_{t-i} \\
&= \theta_h + \sum_{i=0}^{\infty} \delta_p^r (i) \, u_{t-i}
\end{aligned}
$$

where

$$\begin{aligned}\delta_p(0) &= 1, \\ \delta_p(1) &= (\alpha_1\delta_k - \delta_h),\end{aligned}$$

and for $i = 2, 3, 4, ...$

$$\begin{aligned}\delta_p(i) &= \Big\{\alpha_1\delta_k\,(1-\delta_k+\alpha_1\delta_k)^{i-1} \\ &\quad -\delta_h\left[(1-\delta_h)^{i-1}+\alpha_1\delta_k\sum_{j=0}^{i}(1-\delta_h)^{i-j}(1-\delta_k+\alpha_1\delta_k)^{j}\right]\Big\},\end{aligned}$$

and

$$\delta_p^r(i) \equiv \left(\sum_{j=0}^{i}(\rho)^{i-j}\delta_p(i)\right).$$

Notice that after some algebraic manipulations, the case with AR(1) shocks is very similar to the case with iid shock, except that a_t is now replaced by u_t, $\delta_p^s(i)$ by $\delta_p^{sr}(i)$ and $\delta_p(i)$ by $\delta_p^r(i)$. Thus the formula are analogous and skipped due to the space limitation.

C Computational Appendix

In this section, we will describe how the numerical computations are conducted. In the text, there are total four cases: iid-quarterly, iid-annual, AR(1)-quarterly, AR(1)-annual. Since they are similar, we will only describe the case of AR(1)-quarterly and the other cases will be available upon request. (In fact, we can always see iid as a special case of AR(1), by setting the persistence coefficient ρ to be zero.

First, we set the parameter values:

$\alpha_1 = 0.3$ (capital share in production)

$\delta_k = 0.02$ (capital depreciation rate)

$\delta_h = 0.02$ (housing depreciation rate)

$\sigma_u^2 = 1$ (normalization)

Then, we write down the formula that would be used for housing-related variance and co-variance terms:

$$\begin{aligned}\delta_p(0) &= 1, \\ \delta_p(1) &= (\alpha_1\delta_k - \delta_h).\end{aligned}$$

$$\begin{aligned}\delta_p(i) &= \Big\{\alpha_1\delta_k\left(1-\delta_k+\alpha_1\delta_k\right)^{i-1} \\ &\quad -\delta_h\left[\left(1-\delta_h\right)^{i-1}+\alpha_1\delta_k\sum_{j=0}^{i}\left(1-\delta_h\right)^{i-j}\left(1-\delta_k+\alpha_1\delta_k\right)^{j}\right]\Big\}, \\ \text{for } i &= 2,3,4,....\end{aligned}$$

and define a new variable

$$\delta_p^r\left(i\right)\equiv\left(\sum_{j=0}^{i}(\rho)^{i-j}\delta_p\left(i\right)\right).$$

Simiarly, we write down the formula that would be used for equity-related variance and co-variance terms:

$$\begin{aligned}\delta_p^s(0) &= 1 \\ \delta_p^s\left(i\right) &= \alpha_1\delta_k\left(1-\delta_k+\alpha_1\delta_k\right)^{i},\ i=1,\ 2,\ 3,...\end{aligned}$$

and define a new variable

$$\delta_p^{rs}\left(i\right)\equiv\left(\sum_{j=0}^{i}(\rho)^{i-j}\delta_p^s\left(i\right)\right).$$

Now, we need to handle infinite sum. We use truncation and then check whether the results are sensitive to the particular truncations used. For intance, define $\boldsymbol{S}=\sum_{i=0}^{\infty}\delta_p(i)\cdot\delta_p(j+i)$, then $\boldsymbol{S(T)}\equiv\sum_{i=0}^{T}\delta_p(i)\cdot\delta_p(j+i)$, and then we try $T=500$, 1000, 5000, etc... and see if there is any noticeable difference as we change $\boldsymbol{T}$, (for instance, 0.01% difference).

Now, from the previous section, we know that for stock, $j=1,2,3,...$

$$\begin{aligned}&cov(p_{st},p_{s,t+j}) \\ &= \sigma_u^2\left(\sum_{i=0}^{\infty}\delta_p^{rs}(i)\cdot\delta_p^{rs}(i+j)\right)>0,\end{aligned}\tag{77}$$

and for housing prices,

$$\begin{aligned}&cov(p_{ht},p_{h,t+j}) \\ &= \sigma_u^2\left(\sum_{i=0}^{\infty}\delta_p^{r}(i)\cdot\delta_p^{r}(i+j)\right).\end{aligned}\tag{78}$$

Clearly, we can use the computer to directly compute these infinite sum by using the truncation method discussed earlier. Moreover,

$$\begin{aligned} & cov(p_{st}, p_{ht}) \\ = \; & \sigma_u^2 \cdot \left[\sum_{i=0}^{\infty} \delta_p^r(i) \delta_p^{rs}(i) \right], \end{aligned} \tag{79}$$

Now, we can also get the co-variance between the current stock price and the subsequent period housing prices. For $j = 1, 2, 3, ...$

$$\begin{aligned} & cov(p_{st}, p_{h,t+j}) \\ = \; & \sigma_u^2 \left[\sum_{i=0}^{\infty} \delta_p^r(j+i) \delta_p^{rs}(i) \right], \end{aligned} \tag{80}$$

Conversely, the co-variance betwen the current period stock price and previous period housing prices is

$$\begin{aligned} & cov(p_{s,t}, p_{h,t-j}) \\ = \; & \sigma_u^2 \left[\sum_{i=0}^{\infty} \delta_p^r(i) \delta_p^{rs}(j+i) \right]. \end{aligned} \tag{81}$$

To compute correlation, we also need variance terms,

$$var\,(p_{st}) = \sigma_u^2 \cdot \left(\sum_{i=0}^{\infty} \left(\delta_p^{rs}(i) \right)^2 \right), \tag{82}$$

$$var\,(p_{ht}) = \sigma_u^2 \cdot \left(\sum_{i=0}^{\infty} \left(\delta_p^{r}(i) \right)^2 \right). \tag{83}$$

And with the formulae for variance and covariance, it is trivial to compute the correlation coefficients.

Now we also want to examine the dynamic behavior of the rate of returns. We use the definition stated in the main text or earlier section of the appendix, we can calculate the co-variance terms of the rates of return. For $j = 1, 2, 3, ...$

$$\begin{aligned} & cov\left(\widetilde{r}_{s,t+1}, \widetilde{r}_{s,t+1+j}\right) \\ = \; & \sigma_u^2 \cdot \left\{ \delta_p^{rs}(0)\, \delta_r^{rs}(j) + \sum_{i=0}^{\infty} \delta_r^{rs}(i+1) \cdot \delta_r^{rs}(j+i+1) \right\}, \end{aligned} \tag{84}$$

$$
\begin{aligned}
& cov\left(\widetilde{r}_{h,t+1}, \widetilde{r}_{h,t+1+j}\right) \\
= \; & \sigma_u^2 \cdot \left\{ \delta_p^r(0)\, \delta_r^r(j) + \sum_{i=0}^{\infty} \delta_r^r(i+1) \cdot \delta_r^r(j+i+1) \right\},
\end{aligned} \tag{85}
$$

where for $i = 0,\ 1,\ 2,...$

$$
\begin{aligned}
\delta_r^{rs}(i) &\equiv \left[\delta_p^{rs}(i) - \delta_p^{rs}(i-1)\right], \\
\delta_r^{r}(i) &\equiv \left[\delta_p^{r}(i) - \delta_p^{r}(i-1)\right],
\end{aligned}
$$

In addition, we can examine the cross-co-variance of the two assets' return.

$$
\begin{aligned}
& cov\left(\widetilde{r}_{s,t+1}, \widetilde{r}_{h,t+1}\right) \\
= \; & \sigma_u^2 \cdot \left\{ \left(\delta_p^{rs}(0) \cdot \delta_p^r(0)\right) + \sum_{i=0}^{\infty} \delta_r^{rs}(i+1) \cdot \delta_r^r(i+1) \right\}.
\end{aligned} \tag{86}
$$

And for $j = 1,\ 2,\ 3,...$

$$
\begin{aligned}
& cov\left(\widetilde{r}_{s,t+1}, \widetilde{r}_{h,t+1+j}\right) \\
= \; & \sigma_u^2 \cdot \left\{ \left(\delta_p^{rs}(0) \cdot \delta_r^r(j)\right) + \sum_{i=0}^{\infty} \delta_r^{rs}(i+1) \cdot \delta_r^r(i+j+1) \right\},
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
& cov\left(\widetilde{r}_{s,t+1}, \widetilde{r}_{h,t+1-j}\right) \\
= \; & \sigma_u^2 \cdot \left\{ \left(\delta_r^{rs}(j) \cdot \delta_p(0)\right) + \sum_{i=0}^{\infty} \delta_r^{rs}(j+i+1) \cdot \delta_r^r(i+1) \right\},
\end{aligned} \tag{88}
$$

and

$$
\begin{aligned}
var(\widetilde{r}_{s,t+1}) &= \sigma_u^2 \cdot \left\{ \left(\delta_p^{rs}(0)\right)^2 + \sum_{i=0}^{\infty} \left[\delta_r^{rs}(i+1)\right]^2 \right\}, \\
var(\widetilde{r}_{h,t+1}) &= \sigma_u^2 \cdot \left\{ \left(\delta_p^{r}(0)\right)^2 + \sum_{i=0}^{\infty} \left[\delta_r^{r}(i+1)\right]^2 \right\}.
\end{aligned}
$$

And with the formulae for variance and covariance, it is trivial to compute the correlation coefficients.