VARIANCE OPTIMAL CAP PRICING MODELS

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Abstract

We propose new closed-form pricing formulas for interest rate options which guarantee perfect compatibility with volatility smiles. These cap pricing formulas are computed under variance optimal measures in the framework of the market model or the Gaussian model and achieve an exact calibration of observed market prices. They are presented in a general setting allowing to study model and numéraire choice effects on the computed prices. Numéraire dependence is particularly emphasized. A numerical example and an empirical application on market data are given to illustrate the practical use of the calibration procedure.

keywords : discount bond option, cap pricing formula, volatility smile, variance optimal measure, implied pricing model.

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Introduction

For accounting and regulatory purposes, assessing the value of a portfolio of interest rate derivative assets is one of the tasks given to financial institutions. Such financial assets are routinely used to tailor the interest rate risk exposure induced by existing deposits and loans, and are now part of standard financial products such as capped mortgage loans. Financial institutions such as investment banks must valuate them in accordance with observed prices of liquid financial products, following the "mark-to-market" principle. For banks managing large portfolios, only a few option prices are observed and unobserved prices must thus be infered in order to get the current market value of the book. Indeed the relevant information is difficult and costly to obtain and only a small part of portfolios corresponds to assets whose prices are easily available. The unobserved option prices are in general rebuilt through the use of pricing models.

A first typical aspect of interest rate pricing models is their multiplicity. Furthermore none of those can really be considered as a benchmark (unlike the BLACK- SCHOLES (1973) paradigm for stocks). This calls for unifying approaches. The framework of HEATH, JARROW and MORTON (hereafter HJM) (1992) with its applications in the Gaussian case (see e.g. EL KAROUI, MYNENI and VISWANATHAN (1992), EL KAROUI and LACOSTE (1993), JAMSHIDIAN (1993)), in the log-normal case (see e.g. MILTERSEN, SANDMANN and SONDERMAN (1994, 1997), JAMSHIDIAN (1997), BRACE, GATAREK and MUSIELA (1997)), and in the square root case (RITCHKEN and SANKARASUBRAMANIAN (1995)) provides an illustration. Another example is given by models with Markov state variables (LANGETIEG (1980), COX, INGERSOLL and ROSS (1985), DUFFIE and KAN (1996)).

A second characteristic of interest rate modelling is the importance of the change of numéraire technique (EL KAROUI and ROCHET (1989), GEMAN, EL KAROUI and ROCHET (1995), JAMSHIDIAN (1993, 1997)). Forward measures, Libor and dual swap measures prove to be a key tool for computing pricing formulas for interest rate options such as caps and swaptions.

There is also a mounting concern about the calibration of pricing models to observed prices : perfect calibration to the initial yield curve (the HJM framework), calibration of the term structure of implied volatilities (e.g. HULL and WHITE (1990), BRACE and MUSIELA (1994)).

More recently some authors have raised the problem of calibrating models to volatility smiles. In the case of options on stocks or on exchange rates (MADAN and MILNE (1994), RUBINSTEIN (1994), BUCHEN and KELLY (1996), JACKWERTH and RUBINSTEIN (1996), MAGNIEN, PRIGENT and TRANNOY (1996)) one can extract risk neutral densities from a finite set of observed option prices. The aim of this paper is to provide a presentation of this approach for interest rate options taking into account the special features of interest rate modelling (multiplicity of pricing models and numéraires). Our approach allows to recover in a simple way caplet pricing formulas given by well known models (the market model and the Gaussian model) and also to derive new explicit formulas consistent with observed volatility smiles. Thus we will be able to forecast the unobserved price of a cap contract from a given sequence of observed cap prices while achieving a perfect calibration of these prices (mark-to-market principle). As already mentioned, this cross price forecast technique is particularly useful for the valuation of large portfolios of interest rate derivative products.

The paper is organized in two main parts.

The first part goes from Sections 1 to 3. It provides the theoretical background regarding numéraires, price operators, state price measures and caplet pricing formulas. This framework is static and the state space can either be discrete or continuous. These sections are intended both to provide a review and pave the way towards the more applied sections by providing suitable mathematical tools and notations. A close look at the spaces of risk neutral measures is given in order to ease the choice of a specific one which is done in the second part.

In Section 1, we recall market practice regarding Libor rates and present payoffs and numéraires that are commonly used in interest rate modelling such as FRAs, caps, ...

In Section 2, we briefly review the Harrison and Kreps (1979) setting (price operators, state price measures) and adapt this framework to the presence of different numéraires used in interest rate modelling. We review the state price measure invariance property of Geman, El Karoui and Rochet (1995) which allows to link the forward measure to the Libor measure. We also detail the spaces of risk neutral measures and provide nonemptiness conditions.

Section 3 is dedicated to the presentation of general caplet pricing formulas and their relations with risk neutral measures. We restate and generalize previous results obtained by Merton (1973), Breeden and Litzenberger (1978), and El Karoui and Rochet (1989). We are then able to propose simple formulas for options on discount bonds and caps in a static framework without relying on stochastic calculus. Several examples illustrate this integrated framework which facilitates the implementation and shifts between models from a computerized point of view (see Brace (1996)).

In the second part, the more practically oriented Sections 4 to 7 deal with the implementation of caplet pricing models consistent with observed caplet prices based on an a priori model.

In Section 4 we describe the implied model approach with an L^2 distance. We focus on the variance optimal signed measures and on the variance optimal probability measures of Schweizer (1992), Delbaen and Schachermayer (1996), Gouriéroux, Laurent and Pham (1998) and we study the numéraire dependence effect. We link the induced price with the approximation price, i.e. the price of the best mean-variance hedging portfolio.

Section 5 is the main contribution of the paper. We consider closed-form pricing formulas for caps and options on discount bonds that are consistent with observed volatility smiles. These pricing formulas are simple and easy to compute.

Section 6 studies bounds on the prices provided by the theoretical pricing formulas thanks to the concept of super-replication used by El Karoui and Quenez (1995). In the presence of traded caps, the bounds for the unobserved cap prices are narrowed.

Eventually, Section 7 contains a numerical example and an empirical application based on real market data (DEM caplet prices). This section aims to show that the calibration procedure is straightforward to implement and can be used in real time applications.

Concluding remarks are given in Section 8.

1 The market

We begin with the description of what forms the basement of our analysis. For concreteness, we focus on Libor rates (i.e. money market rates quoted among banks in London) and later consider asset payoffs that depend on these Libor rates. FRAs, Interest rate swaps and caps on Libor rates are the main products on the OTC interest rate derivatives market. A typical plain vanilla cap on three month Libor will pay every three months the positive part of the difference between the current three month Libor and a predetermined rate called the exercise (or strike) rate until the expiration of the cap. Typical cap maturities may vary from six months to ten years or more. The reference Libor is often a three month Libor, although other references are not scarce. This means that caps are long term options based on short term money market rates. The Libor may correspond to USD rates but other active markets exist in other currencies. Each individual payment of a cap is a separate financial contract, namely a caplet (see e.g. Brace, Gatarek and Musiela (1997), Jamshidian (1997), Brace (1996), Musiela and Rutkowski (1997) for illustrations). Thus a cap is a collection of caplets. If one is able to price a caplet (the building block of a cap) the valuation of the cap is straightforward. For the sake of simplifying notations, we will further consider the valuation of caplets whose valuation is similar to the valuation of discount bond options (Briys, Crouhy and Schoebel (1991)).

Closely related products (and competitors to caplets) are options on Eurodollar type contracts. These products are traded on exchanges and based on futures contracts written on Libor rates (Abken, Madan and Ramamurtie (1996)). Caplets also have to be distinguished from options on continuously compounded discount bond yields since the caplet payoffs are based on money market conventions (LONGSTAFF (1990), LEBLANC and SCAILLET (1998)).

1.1 Libor rates and forward prices

We adopt the usual international conventions in terms of interest rate fixing and delivery dates for interbank deposits, three-month Libor swaps, and three-month Libor caps and swaptions. Typically, this time period has three dates, the first date τ_0 corresponding to the fixing of the Libor which prevails between the dates τ_1 and τ_2 ($\tau_0 < \tau_1 < \tau_2$). The date τ_2 i.e. the payment date of the Libor. In London, date τ_1 is two trading days after date τ_0. Time spaces between these dates may vary because of the presence of nontrading days and generally differ when one shifts to another underlying interest rate reference for the contracts.

Along the paper, we assume that the discount bonds which mature at dates τ_2 and τ_1 are traded assets and that their prices are observed at date 0 and date τ_0. Libor rates and forward prices are defined in Table 1 where $B(\tau_0, \tau_2)$ is the price at date τ_0 of the discount bond delivering one money unit, say a franc or a Euro, at date τ_2, $\delta = J(\tau_1, \tau_2)/360$ and $J(\tau_1, \tau_2)$ are the number of years and the number of days between τ_1 and τ_2, respectively.

Table 1 : Libor rates and forward prices

Libor at date τ_0	$x = x(\tau_0)$	$\delta^{-1}\dfrac{B(\tau_0,\tau_1) - B(\tau_0,\tau_2)}{B(\tau_0,\tau_2)}$
Forward Libor (date 0)	$x(0)$	$\delta^{-1}\dfrac{B(0,\tau_1) - B(0,\tau_2)}{B(0,\tau_2)}$
Forward price at date τ_0	$z^{-1} = B(\tau_0,\tau_1,\tau_2)$	$\dfrac{B(\tau_0,\tau_2)}{B(\tau_0,\tau_1)}$
Forward price at date 0	$B(0,\tau_1,\tau_2)$	$\dfrac{B(0,\tau_2)}{B(0,\tau_1)}$

The forward Libor $x(0)$ is the Libor which one is able to lock at date 0. The forward Libor $x(0)$ is thus equivalent to the fixed rate of an FRA (forward rate agreement) on the Libor with maturity date τ_0. The forward price at date τ_0 of a discount bond with maturity $\tau_2 - \tau_1$ at date τ_1 is given by the ratio : $B(\tau_0, \tau_2)/B(\tau_0, \tau_1)$, and we deduce the relation between the inverse of the forward price z and Libor rate x :

$$z = 1 + \delta x. \tag{1.1}$$

In some cases writing numéraires, asset payoffs or risk neutral measures in terms of the inverse of the forward price will lead to simpler computations.

1.2 Numéraires and asset payoffs

An asset payoff is characterized by an amount paid in a given numéraire at a given date. For obvious practical purposes, there is often a delay between the time when the asset payoff is determined (the so-called fixing date) and its payment date. This can be viewed (through a standard discounting argument) as receiving at the fixing date a given amount of discount bonds maturing at the payment date. This means that the numéraire can precisely be this discount bond. In our context the fixing date is τ_0, and we consider amounts known at that time which depend on the Libor rate x. These amounts can be expressed in one one of the three following numéraires useful in interest rate modelling :

- the discount bond maturing at τ_1 [the first numéraire U_1].
- the discount bond maturing at τ_2 [the second numéraire U_2].
- the "exchange principal asset" : one discount bond maturing at τ_1 in long (buy) position and one discount bond maturing at τ_2 in short (sell) position [the third numéraire U_3].

We denote by $U_i(\tau)$, the price at time τ, $\tau = 0, \tau_0$, of numéraire U_i :

$$\begin{cases} U_1(\tau) &= B(\tau, \tau_1), \\ U_2(\tau) &= B(\tau, \tau_2), \\ U_3(\tau) &= B(\tau, \tau_1) - B(\tau, \tau_2). \end{cases}$$

We will denote by $U_{i/j}(\tau)$ the exchange rate between numéraires U_i and U_j (the relative price of U_i w.r.t. U_j) at date τ :

$$U_{i/j}(\tau) = \frac{U_i(\tau)}{U_j(\tau)}, \; \tau = 0, \tau_0, \; i, j = 1, 2, 3. \tag{1.2}$$

From Table 1 we can directly deduce the exchange rates between the numéraires at date τ_0 (see Table 2) as functions of x, the Libor rate at date τ_0.

Table 2 : Exchange rates between numéraires (Libor)

	numéraire 1	numéraire 2	numéraire 3
numéraire 1	$U_{1/1}(\tau_0) = 1$	$U_{1/2}(\tau_0) = 1 + \delta x$	$U_{1/3}(\tau_0) = \dfrac{1 + \delta x}{\delta x}$
numéraire 2	$U_{2/1}(\tau_0) = \dfrac{1}{1 + \delta x}$	$U_{2/2}(\tau_0) = 1$	$U_{2/3}(\tau_0) = \dfrac{1}{\delta x}$
numéraire 3	$U_{3/1}(\tau_0) = \dfrac{\delta x}{1 + \delta x}$	$U_{3/2}(\tau_0) = \delta x$	$U_{3/3}(\tau_0) = 1$

We are thus able to express the payoffs of standard financial contracts, such as caplets and digital caplets, in units of these three numéraires. Note that digital caplets are also traded on interest rate markets though less frequently than caplets. They are often used when customizing financial asset payoffs involving an interest rate guarantee. The payoff expressions are gathered in Table 3 where $(x-c)^+ = \max(0, x-c)$, $\mathbb{1}_{x(\tau_0)\geq c} = 1$ if $x(\tau_0) \geq c$ and 0 otherwise, g_{U_i} $(i = 1, 2, 3)$ is a real function, and c is the exercise rate.

Table 3 : Numéraires and asset payoffs (Libor)

	numéraire 1	numéraire 2	numéraire 3
Discount bond τ_1	1	$1+\delta x$	$\frac{1+\delta x}{\delta x}$
Discount bond τ_2	$\frac{1}{1+\delta x}$	1	$\frac{1}{\delta x}$
FRA	$\frac{(x-c)\delta}{1+\delta x}$	$(x-c)\delta$	$\frac{(x-c)}{x}$
Caplet	$\frac{(x-c)^+\delta}{1+\delta x}$	$(x-c)^+\delta$	$\frac{(x-c)^+}{x}$
Digital Caplet	$\frac{\mathbb{1}_{x\geq c}}{1+\delta x}$	$\mathbb{1}_{x\geq c}$	$\frac{\mathbb{1}_{x\geq c}}{\delta x}$
General asset	$g_{U_1}(x)$	$g_{U_2}(x) = (1+\delta x) g_{U_1}(x)$	$g_{U_3}(x) = \frac{1+\delta x}{\delta x} g_{U_1}(x)$

These payoffs share the remarkable property that they depend only on the Libor rate x. It is clear that whatever the numéraire, the same amount of cash will be received at date τ_2. Let us also remark that for any given asset, the following numéraire invariance property holds :

$$g_{U_i} = U_{j/i}(\tau_0) g_{U_j}, \ \forall i, j = 1, 2, 3. \tag{1.3}$$

From equation (1.1), we can express the exchange rates between the three numéraires at date τ_0 as functions of the inverse of the forward price z (see Table 4).

Table 4 : Exchange rates between numéraires (inverse of forward price)

	numéraire 1	numéraire 2	numéraire 3
numéraire 1	1	z	$\frac{z}{z-1}$
numéraire 2	$\frac{1}{z}$	1	$\frac{1}{z-1}$
numéraire 3	$\frac{z-1}{z}$	$z-1$	1

Similarly payoffs may be rewritten using the inverse of the forward discount bond price instead of the Libor. They are shown in Table 5 where :

$$\bar{g}_{U_i}(z) = g_{U_i}((z-1)/\delta), \quad i = 1, 2, 3. \tag{1.4}$$

Table 5 : Numéraires and asset payoffs (inverse of forward price)

	numéraire 1	numéraire 2	numéraire 3
Discount bond τ_1	1	z	$\frac{z}{z-1}$
Discount bond τ_2	$\frac{1}{z}$	1	$\frac{1}{z-1}$
FRA	$\frac{z-(1+\delta c)}{z}$	$z-(1+c\delta)$	$\frac{z-(1+c\delta)}{z-1}$
Caplet	$\frac{(z-(1+\delta c))^+}{z}$	$(z-(1+c\delta))^+$	$\frac{(z-(1+c\delta))^+}{z-1}$
Digital Caplet	$\frac{\mathbb{I}_{z\geq 1+\delta c}}{z}$	$\mathbb{I}_{z\geq 1+\delta c}$	$\frac{\mathbb{I}_{z\geq 1+\delta c}}{z-1}$
General asset	$\bar{g}_{U_1}(z)$	$\bar{g}_{U_2}(z)=z\bar{g}_{U_1}(z)$	$\bar{g}_{U_3}(z)=\frac{z}{z-1}\bar{g}_{U_1}(z)$

Let us remark that for any given asset, the following numéraire invariance property holds :

$$\bar{g}_{U_i} = U_{j/i}(\tau_0)\bar{g}_{U_j},\ \forall i,j=1,2,3. \tag{1.5}$$

2 Viable price operators and numéraires

Here we briefly recall results of HARRISSON and KREPS (1979) but taking into account explicitly the presence of different numéraires. Our framework is static, i.e. does not allow for dynamic trading. The state space may either be discrete or continuous. In most applications Libor rates are continuously distributed. Thus we do not emphasize the discrete case based on discrete time pricing models (see e.g. HO and LEE (1986), TURNBULL and MILNE (1991), GOURIÉROUX and SCAILLET (1997)). The space of attainable claims is made of linear combinations of traded interest rate derivatives and is thus of finite dimension. The market is here highly incomplete unlike the standard case where the prices are uniquely determined and equal to the prices of dynamic self-financing replicating portfolios.

There are both technical, economic and practical advantages to this static approach. Firstly our approach only relies on measure theory and does not require the sophisticated apparatus of semimartingale theory. From an economic point of view, there are some advantages to allow for uncertainty in derivative prices. Indeed, in complete markets, the deterministic relationships between derivative and underlying asset prices are rejected by the data leading to the immediate conclusion that pricing models are misspecified. Besides dynamic portfolios are more sensitive to transaction costs than static ones. Lastly in practice our approach is very easy to implement and only requires to solve linear equations.

Of course, the major drawback of such a static approach is the narrowness of the space of attainable claims due to the absence of dynamic hedging. For approaches aiming to introduce such dynamic aspects we refer to AVELLANEDA et al. (1997) in a diffusion setting and LEISEN and LAURENT (1998) in a Markov chain setting.

2.1 Price operators and probability measures

The aim of this section is to provide a precise definition of pricing models (through the use of price operators) within an interest rate setting and to show the close links between pricing models and probability measures. Moreover we recall conditions for the set of risk neutral measures to be non empty so that the search of specific ones is meaningful.

2.1.1 Libor rate

Let us introduce a probability measure μ on $\mathcal{B}_{[a,b]}$, the borel algebra for the interval $[a, b] \subset \mathbb{R}$. μ represents an a priori probability measure on the Libor rate x. This probability measure can be discrete or continuous without loss of generality. The bounds a and b may be seen as lower and upper bounds for Libor rates and b may be infinite. The measure μ may be viewed as a technical reference measure as in MADAN and MILNE (1994) and ELLIOTT and MADAN (1998). It may be taken equal to the Lebesgue measure or a Gaussian measure in order to ease computations. It may also be chosen in order to reflect any expectations based on historical observations, and will then be related to the so-called historical measure.

We have defined some exchange rates between the numéraires : $1 + x\delta$, $x\delta$, $1/(1 + x\delta)$, ... For these exchange rates to be well behaved we assume that they are strictly positive, μ a.s.. Hence we will not necessarilly consider all numéraires when building pricing models and we will restrict ourselves in practical applications to a subset of numéraires whose exchange rates are strictly positive. Furthermore we assume that the numéraires have finite moments of order p ($p \geq 1$) under μ. This guarantees that all caplet and digital caplet payoffs are in $L^p(\mu)$. If μ is the historical measure on Libor rates, the tail index tells us whether Libor rates exhibit too heavy tails (Levy distributions) to ensure that caplet payoffs are in $L^p(\mu)$. If μ is another reference measure (such as the Gaussian one), it usually involves finite moments of any order. Besides it is often convenient to restrict oneself to $L^2(\mu)$ in order to exploit its special properties (projection theorem) due to its Hilbertian nature.

Remember that a price operator is a strictly positive linear functional associating a price to a given payoff expressed in some account unit (say U_1), and belonging to some vector space (say $L^p(\mu)$, $p \geq 1$) (see HARRISON and KREPS (1979), HARRISON and PLISKA (1981)). We require that a price operator is compatible with the traded prices of numéraires. The current date is by convention set to zero. If we denote by π_{U_1} such a price operator, the previous restriction means (see LONGSTAFF (1995)) :

$$\pi_{U_1}[U_{i/1}(\tau_0)] = U_i(0),\ i = 1, 2, 3. \tag{2.1}$$

Let us remark that $\pi_{U_1}[1]$ has the interpretation of the price of the riskless asset corresponding to numéraire U_1 since we get for sure one unit of U_1 at date τ_2. As an application, we can state a simple expression for the price of the floating leg of a FRA contract, i.e. the asset delivering δx at date τ_2. The associated cash-flow is equal to δx units of numéraire U_2, or equivalently to $\delta x/(1+\delta x)$ units of numéraire U_1, thus :

$$\pi_{U_1}\left[\frac{\delta x}{1+\delta x}\right] = \pi_{U_1}\left[U_{3/1}(\tau_0)\right] = U_3(0).$$

It means that the price of the floating leg of a FRA contract is equal to the price of the "exchange principal asset" of the practioners' terminology.

The continuity of a positive operator on $L^p(\mu)$ (see DUFFIE (1988 p. 63)) and Riesz Representation Theorem in $L^p(\mu)$, $p \in [1, \infty[$, (see RUDIN (1974), Theorem 6.16) lead to the following statement which closely relates price operators and probability measures.

Proposition 1 (price operators and probability measures)
Let us take the account unit U_1 and the price operator corresponding to this numéraire π_{U_1}. There exists a unique (μ a.s.) strictly positive function f_{U_1} of $L^q(\mu)$ ($\frac{1}{p}+\frac{1}{q}=1$) with $\int f_{U_1} d\mu = 1$ such that :

$$\pi_{U_1}[g_{U_1}] = U_1(0)\int g_{U_1} f_{U_1} d\mu, \quad \forall g_{U_1} \in L^p(\mu). \tag{2.2}$$

The equality $\int f_{U_1} d\mu = 1$ is due to the compatibility of the price operator with the numéraire price ($\pi_{U_1}[1] = U_1(0)$). Moreover $f_{U_1} > 0$ μ a.s. since $\pi_{U_1} > 0$, i.e. $\pi_{U_1}[g_{U_1}] > 0$ for $g_{U_1} > 0$ ($g_{U_1} \geq 0$ and $\mu(g_{U_1} > 0) > 0$). Hence $f_{U_1} d\mu$ defines a new *equivalent probability measure*. f_{U_1} is the density of this probability measure w.r.t. μ and has the interpretation of a *risk premium*. $U_1(0) f_{U_1} d\mu$ will be further called a *state price measure*. Note that the Riesz Representation Theorem states an isometry between the set of price operators and the set of probability measures equivalent to μ, and thus a simple way to elaborate a price operator is to pick up a suitable density f_{U_1}.

For a given payoff g_{U_1} corresponding to numéraire U_1 and belonging to $L^p(\mu)$, the associated payoffs g_{U_2}, g_{U_3} corresponding to numéraires U_2 and U_3 belong to $L^p\left(\frac{d\mu(x)}{(1+x\delta)^p}\right)$ and $L^p\left(\left(\frac{x}{1+x\delta}\right)^p d\mu(x)\right)$. These two spaces contain $L^p(\mu)$ under the standing assumption that $x > 0$, μ a.s. (this assumption implies that $U_{2/1}(\tau_0)$ and $U_{3/1}(\tau_0)$ are in $L^\infty(\mu)$). We can then define on (at least) $L^p(\mu)$ the two strictly positive linear operators :

$$\pi_{U_i}[g_{U_i}] = \pi_{U_1}[U_{i/1}(\tau_0) g_{U_i}],\ i = 2, 3. \tag{2.3}$$

π_{U_i}, $i = 1, 2, 3$ may be viewed as different representations (associated to different account units) of the same pricing model. We can see that π_{U_2} and π_{U_3} admit the following representation as a direct consequence of their definition :

$$\pi_{U_i}[g_{U_i}] = U_i(0) \int g_{U_i} f_{U_i} d\mu, \qquad i = 2, 3,$$

with :

$$U_i(0) f_{U_i} = U_{i/1}(\tau_0) U_1(0) f_{U_1},\ \mu \text{ a.s., } \forall i = 1, 2, 3. \tag{2.4}$$

Moreover, we can notice that for $i = 2, 3$: $f_{U_i} > 0$ μ a.s., $\int f_{U_i} d\mu = 1$ (due to the compatibility with numéraire prices), and if $f_{U_1} \in L^q(\mu)$, $f_{U_i} \in L^q(\mu)$ (under the assumption of a positive Libor rate, $U_{2/1}(\tau_0)$ and $U_{3/1}(\tau_0)$ are in $L^\infty(\mu)$).

$$U_i(0) f_{U_i} = U_{i/j}(\tau_0) U_j(0) f_{U_j},\ \mu \text{ a.s., } \forall i, j = 1, 2, 3, \tag{2.5}$$

is a *state price measure invariance property* w.r.t. the choice of numéraire. Such an invariance property is also well known in the dynamic case (e.g. DUFFIE (1992), BAJEUX and PORTAIT (1994), GEMAN, EL KAROUI and ROCHET (1995), GOURIÉROUX, LAURENT and PHAM (1998)). State price measures associated with numéraires U_2 and U_1 are known as forward measures (GEMAN, EL KAROUI and ROCHET (1995)) and state price measure associated to numéraire U_3 is known as the Libor measure (BRACE (1996), JAMSHIDIAN (1997)).

Finally we denote by Q_{U_i} the probability measure on Libor rate whose density w.r.t. μ is equal to f_{U_i} :

$$Q_{U_i}(E) = \int_E f_{U_i} d\mu,\ i = 1, 2, 3, \tag{2.6}$$

where E is some μ-mesurable set, and by F_{U_i} the cumulative density function on the Libor rate, defined by :

$$F_{U_i}(c) = Q_{U_i}([a, c[) = 1 - \int_c^b f_{U_i} d\mu,\ i = 1, 2, 3,\ c \in [a, b]. \tag{2.7}$$

2.1.2 Inverse of the forward price

Previous quantities may be expressed as functions of the inverse of the forward discount bond price. Since $z = 1 + \delta x$, we can readily derive a probability measure $\bar{\mu}$ on $\mathcal{B}_{[1+\delta a, 1+\delta b]}$ related to the inverse of the forward discount bond price z.

The payoffs $\bar{g}_{U_1}$ are then in $L^p(\bar{\mu})$. To any probability density f_{U_1} (w.r.t μ) on the Libor rate, we can derive $\bar{f}_{U_1}$ the associated probability density on the inverse of the forward discount bond price z by :

$$\bar{f}_{U_1}(z) = f_{U_1}((z-1)\delta^{-1}), \tag{2.8}$$

and therefore the corresponding state price measure invariance property is :

$$U_i(0)\bar{f}_{U_i} = U_{i/1}(\tau_0)U_1(0)\bar{f}_{U_1},\ \bar{\mu}\text{ a.s.},\ i = 2,3, \tag{2.9}$$

where $U_{i/1}(\tau_0)$ is now viewed as a function of z.

The probability measure $\bar{f}_{U_i}d\bar{\mu}$ defined on $\mathcal{B}_{[1+\delta a,1+\delta b]}$ will be denoted by $\bar{Q}_{U_i}$. We also denote by $\bar{F}_{U_i}$ the cumulative density function on the inverse of the forward price. It is such that :

$$\bar{F}_{U_i}(1+\delta c) = \bar{Q}_{U_i}([1+\delta a, 1+\delta c[) = 1 - \int_{1+\delta c}^{1+\delta b} \bar{f}_{U_i}d\bar{\mu} = F_{U_i}(c),\ i = 1,2,3,\ c \in [a,b]. \tag{2.10}$$

2.2 Viable price operators

In the framework of Harrison and Kreps (1979) a *viable price operator* is compatible with the prices of attainable claims. This is of course a highly desirable feature for a price operator. A subset of attainable claims is usually made of "traded assets". In the first option pricing models, the traded assets considered were the "underlying asset" and the "riskless asset". These pricing models, like the Black-Scholes model are indeed compatible with the prices of these assets (for an empirical investigation see Longstaff (1995)). Let us notice that in our setting, we have considered pricing models compatible with the traded prices of numéraires. Currently, a lot of options are liquid traded assets and pricing operators should be compatible with these option prices. Here, we detail the constraints induced by the observed prices and we recall that the absence of arbitrage opportunities guarantees the existence of a viable price operator.

This leads us to introduce a sequence of *observed prices*, i.e. a finite sequence of pairs : $(g_{U_i,j}, P_j), g_{U_i,j} \in L^p(\mu), P_j \in \mathbb{R},\ j \in J,\ i = 1,2,3,$ where P_j stands for the observed price of the asset delivering $g_{U_i,j}$ units of U_i at date τ_2, and where the payoffs $g_{U_i,j}$ are related by the numéraire invariance property (1.3). The sequence of observed prices may in particular include observed caplet prices with different exercise rates. We will further assume that the sequence of observed prices includes two out of the three numéraires (the third one beeing deduced by linear combination). We denote by G_{U_i}, the subspace of $L^p(\mu)$ spanned by $(g_{U_i,j}),\ j \in J$; it is the (static) *investment opportunity set*. We further assume (without loss of generality) that traded assets are not redundant, i.e.

$$\sum_{j\in J} \lambda_j g_{U_i,j} = 0,\ \mu\text{ a.s.} \Longrightarrow \lambda_j = 0,\ \forall j.$$

Due to the assumed positivity of exchange rates, this assumption holds simultaneously for the three numéraires. Under this assumption, $(g_{U_i,j}),\ j \in J$ is a basis of G_{U_i}. We are now able to state :

Definition 1 *A pricing measure $f_{U_i}d\mu$, $f_{U_i} \in L^q(\mu)$, is said to be risk-neutral if it is compatible with the sequence of observed prices, i.e. :*

$$U_i(0) \int g_{U_i,j} f_{U_i} d\mu = P_j, \ \forall j \in J. \tag{2.11}$$

We insist on the fact that according to Harrison and Kreps (1979) a probability measure is said to be risk-neutral if it is compatible with all observed prices. The only compatibility with traded numéraires (the martingale restriction on the underlying assets) is not sufficient to get a risk-neutral pricing measure.

We will denote by $\mathcal{F}_{U_i}^{q,e}$ the set of equivalent risk-neutral probability densities associated to numéraire U_i, i.e.

$$\mathcal{F}_{U_i}^{q,e} = \left\{ f_{U_i} \in L^q(\mu),\, f_{U_i} > 0,\, U_i(0) \int g_{U_i,j} f_{U_i} d\mu = P_j, \ \forall j \in J \right\}.$$

Since the numéraires are assumed to be observed, $(g_{U_i,j} = 1, U_i(0))$ belongs to the sequence of observed prices which implies : $\int f_{U_i} d\mu = 1$.

We will further need to choose among risk neutral measures. Therefore we have to carefully describe all involved spaces, and introduce $\mathcal{F}_{U_i}^{q}$ the set of risk-neutral probability densities associated to numéraire U_i absolutely continuous w.r.t. μ :

$$\mathcal{F}_{U_i}^{q} = \left\{ f_{U_i} \in L^q(\mu),\, f_{U_i} \geq 0,\, U_i(0) \int g_{U_i,j} f_{U_i} d\mu = P_j, \ \forall j \in J \right\},$$

and $\mathcal{F}_{U_i}^{q,s}$ the set of risk-neutral signed-measure densities associated to numéraire U_i :

$$\mathcal{F}_{U_i}^{q,s} = \left\{ f_{U_i} \in L^q(\mu),\, U_i(0) \int g_{U_i,j} f_{U_i} d\mu = P_j, \ \forall j \in J \right\}.$$

A price operator π_{U_i} is said to be viable if it is associated to an equivalent risk-neutral probability measure. A celebrated result in finance states that the absence of arbitrage opportunities and the existence of an equivalent risk-neutral probability measure (or of a viable price operator) are equivalent. In our framework, the assumption of the absence of (static) arbitrage opportunities translates as :

$$\sum_{j \in J} \lambda_j g_{U_i,j} > 0, \ \mu \text{ a.s.} \Longrightarrow \sum_{j \in J} \lambda_j P_j > 0. \tag{2.12}$$

We can notice that this assumption is numéraire invariant since exchange rates are strictly positive μ a.s.. We are then able to give the useful property :

Proposition 2 (risk neutral density existence)
Under the assumption of no arbitrage opportunities, there exists a triplet $(f_{U_1}, f_{U_2}, f_{U_3}) \in \mathcal{F}_{U_1}^{q,e} \times \mathcal{F}_{U_2}^{q,e} \times \mathcal{F}_{U_3}^{q,e}$ that satisfies the state price invariance property, i.e. $U_i(0) f_{U_i} = U_{i/j}(\tau_0) U_j(0) f_{U_j}$, μ a.s., $\forall i, j = 1, 2, 3$.

Proof : see Appendix.

Under the assumption of no arbitrage opportunities, $\mathcal{F}^{q,e}_{U_i}$ are non empty convex sets of $L^q(\mu)$. We can remark that $\mathcal{F}^{q,e}_{U_i} \subset \mathcal{F}^{q}_{U_i} \subset \mathcal{F}^{q,s}_{U_i}$ and that the later sets are thus non-empty closed convex sets of $L^q(\mu)$.

3 Caplet pricing formulas

It is a well known feature that option pricing formulas and risk-neutral probability measures are closely related (see Breeden and Litzenberger (1978), Dupire (1992)). We review and extend these relations by taking into account for the multiplicity of numéraires. Since various caplet pricing formulas are used in practice, we want to disclose hereafter an integrated framework. A benchmark pricing model does not really exist but a series of interest rate models live side by side. Very little has in fact to be known about the interest rate dynamics to derive pricing formulas (see examples below). One only needs the (marginal) probabilities of the exercise region and a repeated use of the change of numéraire technique.

We first present fairly general properties of caplet pricing formulas. Then under weak distributional assumptions we provide more specific expressions which include as special cases the widely used market and Gaussian model formulas.

Let us assume that we are given a price system, i.e. risk neutral measures. This allows to write caplet prices under more familiar forms, i.e. either as an interest rate option pricing formula (Brace, Gatarek and Musiela (1997), Brace (1996), Jamshidian (1997)) or as a discount bond option pricing formula (El Karoui and Rochet (1989), Briys, Crouhy and Schoebel (1991)).

We call a mapping : $c \in [a,b] \subset \mathbb{R} \longrightarrow Cap(c) = U_2(0)\int_a^b (x-c)^+ \delta f_{U_2}(x)d\mu(x)$, $f_{U_2} \in \mathcal{F}^{q,e}_{U_2}$, a *caplet pricing formula* or a caplet model. The choice of f_{U_2} in $\mathcal{F}^{q,e}_{U_2}$ guarantees the consistency with observed prices. By analogy we denote by $Dig(c)$ the digital caplet pricing formula. As it will be seen, the latter is closely related to the state price measures.

We begin by considering the caplet pricing formula.

Property 1 (interest rate option pricing formula)
The caplet pricing formula $Cap(c)$ may be written under the three following forms :

$$(a) \qquad (B(0,\tau_1) - B(0,\tau_2))(1 - F_{U_3}(c)) - c\delta B(0,\tau_2)(1 - F_{U_2}(c)),$$

$$(b) \qquad U_3(0)Q_{U_3}(E) - c\delta U_2(0)Q_{U_2}(E),$$

$$(c) \qquad \delta B(0,\tau_2)\left(x(0)Q_{U_3}(E) - cQ_{U_2}(E)\right),$$

where $E = [c,b]$ is the caplet exercise region and $c \in [a,b]$.

Proof : see Appendix.

The last form (c) tells us that the caplet pricing formula may be written as δ times the discounted value of the difference between the forward Libor and the exercise rate, multiplied by the exercise probabilities corresponding to the exchange principal asset (U_3) and the discount bond maturing at the end of the option (U_2), respectively. This form has thus the interpretation of the pricing formula of a call option on an interest rate as in the market model for caps (see below). Forms (a) and (b) may be interpreted as the pricing formula of an exchange option between the asset U_3 and the asset $c\delta U_2$ (this last asset is often called the "coupon asset" by practitioners).

Property 2 (discount bond option pricing formula)
The caplet pricing formula $Cap(c)$ may be written under the three following forms :

$$\begin{array}{ll} (a) & B(0,\tau_1)(1-\bar{F}_{U_1}(1+c\delta)) - B(0,\tau_2)(1+c\delta)(1-\bar{F}_{U_2}(1+c\delta)), \\ (b) & U_1(0)Q_{U_1}(E) - U_2(0)(1+c\delta)Q_{U_2}(E), \\ (c) & (1+c\delta)B(0,\tau_1)\left(\frac{1}{1+c\delta}Q_{U_1}(E) - B(0,\tau_1,\tau_2)Q_{U_2}(E)\right), \end{array}$$

where $E=[c,b]$ is the caplet exercise region and $c \in [a,b]$.

Proof : see Appendix.

The last form (c) is equal to $(1+c\delta)$ times the discounted value of the difference between the strike price $(1+c\delta)^{-1}$ and the forward price $B(0,\tau_1,\tau_2)$ multiplied respectively by the exercise probabilities $Q_{U_1}(E)$ and $Q_{U_2}(E)$ associated with the two discount bond numéraires. The form (a) (or (b)) is the pricing formula of an exchange option between discount bonds. The expression in (b) is similar to El Karoui and Rochet (1989) general formula for a put option on a discount bond.

The interesting point from a forecast point of view is to draw a link between the pricing formula for caplets, i.e. the function $Cap(c)$ and the functions F_{U_i}, $i=1,2,3$ obtained from the price operator π_{U_i}. It passes through the digital caplet pricing formula.

Property 3 (digital caplet pricing formula)
The digital caplet pricing formula $Dig(c)$ is equivalently given by the four following forms :

$$\begin{array}{ll} (a) & \delta U_2(0)Q_{U_2}(E), \\ (b) & -(Cap)'_g(c), \\ (c) & \delta U_2(0)(1-F_{U_2}(c)), \\ (d) & \delta U_2(0)\int_c^b f_{U_2}d\mu, \end{array}$$

where $E = [c, b]$ is the caplet exercise region and $c \in]a, b]$ and $(Cap)'_g$ stands for the left-hand derivative.

Proof : see Appendix.

As Q_{U_2} completely defines the price operator π_{U_2} (cf. Proposition 1), the equality between (a) and (b) implies that π_{U_2} is embodied in $Cap(c)$. Property 3 thus states that it is equivalent to be given the caplet pricing formula, i.e. a function of the exercise rate, or the price operator. In particular, if μ admits a density w.r.t Lebesgue measure, we get : $\delta U_2(0) f_{U_2}(c) \frac{d\mu}{dx}(c) = Cap''(c)$. This result is an adaptation of the result of BANZ and MILLER (1978) and BREEDEN and LITZENBERGER (1978) to interest rate models. From Properties 1 and 3, we deduce that a caplet pricing formula associated with a price system is convex, decreasing and fulfills the boundary properties :

$$\begin{aligned} Cap(b) &= 0, \\ Cap(a) &= \delta U_2(0)\,(x(0) - c)\,, \\ -(Cap)'_g(b) &\geq 0, \\ \lim_{c \longrightarrow a} -(Cap)'_g(c) &\leq \delta U_2(0). \end{aligned}$$

Let us note that the probability of the exercice region $Q_{U_2}(E) = Q_{U_2}(x \geq c)$ is conditioned by information available at time 0. We now restrict ourselves to models where this relevant information may be summarized by $(U_2(0), U_3(0))$, i.e. we can write $Q_{U_2}(E) = Q_{U_2}(x \geq c \mid U_2(0), U_3(0))$. This assumption will allow us to derive an homogeneity property for the probabilities Q_{U_i}, $i = 1, 2, 3$, in the discount bond prices. First, let us suppose that we are paid in dollars instead of francs and that the exchange rate USD/FRF is equal to e. If we want to receive francs, discount bond and option prices have to be multiplied by e from which we deduce the homogeneity of degree one of the option pricing formulas. This simple homogeneity property will be used in the proof of the next one.

Property 4 (money neutrality)
The probabilities Q_{U_i}, $i = 1, 2, 3$, are homogeneous of degree zero in $(U_2(0),\ U_3(0))$ and in $(U_2(0),\ U_1(0))$, and the caplet pricing formulas may be equivalently written :

$$U_2(0)\left[U_{3/2}(0) Q_{U_3}\left(x \geq c \mid U_{3/2}(0)\right) - c\delta Q_{U_2}\left(x \geq c \mid U_{3/2}(0)\right)\right],$$

as an interest rate option pricing formula or :

$$(1 + c\delta) U_1(0)\left[\frac{1}{1 + c\delta} Q_{U_1}\left(x \geq c \mid U_{1/2}(0)\right) - U_{2/1}(0) Q_{U_2}\left(x \geq c \mid U_{1/2}(0)\right)\right],$$

as a discount bond option pricing formula, with $U_{i/i'}(0) = U_i(0)/U_{i'}(0)$, $i, i' = 1, 2, 3$.

Proof : see Appendix.

Let us remark that $U_{3/2}(0) = x(0)$ and $U_{1/2}(0) = B^{-1}(0,\tau_1,\tau_2)$. The caplet pricing formulas may further be simplified if we assume that the Libor rate x (respectively the inverse of the forward discount bond price z) can be written as $x = U_{3/2}(0)Z$ (resp. $z = U_{1/2}(0)Z$) where Z is some random variable not depending on the conditioning variable $U_{3/2}(0)$ (the so-called proportionality assumption). A caplet pricing formula satisfying this condition is homogeneous of degree one in $(c, U_{3/2}(0))$ (resp. in $(1 + c\delta, U_{1/2}(0))$) (see also MERTON (1973), GARCIA and RENAULT (1998)).

Property 5 (proportional models)
Under the proportionality assumption, the caplet pricing formula may be written :

$$U_2(0)\left[U_{3/2}(0)Q_{U_3}\left(Z \geq \frac{c}{U_{3/2}(0)}\right) - c\delta Q_{U_2}\left(Z \geq \frac{c}{U_{3/2}(0)}\right)\right], \tag{3.1}$$

or :

$$(1+c\delta)U_1(0)\left[\frac{1}{1+c\delta}\bar{Q}_{U_1}\left(Z \geq \frac{1+c\delta}{U_{1/2}(0)}\right) - U_{2/1}(0)\bar{Q}_{U_2}\left(Z \geq \frac{1+c\delta}{U_{1/2}(0)}\right)\right], \tag{3.2}$$

as an interest rate or discount bond option pricing formula respectively.

Let us remark that the previous pricing formulas rely on the no arbitrage assumption and on state price invariance property but not on the assumption of completeness. In usual interest rate modelling, one starts with a specific dynamics of the spot rate (or on forward rates or on bond prices), then deduces the risk-neutral distributions of Libor rates and discount bond prices for this dynamics and at last computes caplet prices. Here, we directly start with an arbitrary risk-neutral distribution of Libor rates and compute caplet prices, thus avoiding the use of stochastic calculus and some computational burden. We can also notice that in order to get our pricing formulas, we have relied both on Libor risk-neutral and forward probability measures.

As an example, we rederive in our framework two famous pricing formulas. The first one is known as the pricing formula computed under the market model (see e.g. MILTERSEN, SANDMANN and SONDERMAN (1994, 1997), BRACE (1996), JAMSHIDIAN (1997)). It is a BLACK-SCHOLES formula on interest rate. The second one is the pricing formula obtained in the Gaussian model (for a description of this model see e.g. EL KAROUI and ROCHET (1989), EL KAROUI, MYNENI and VISWANATHAN (1992), EL KAROUI and LACOSTE (1993), JAMSHIDIAN (1993), BRACE and MUSIELA (1994)).

Example 1 : Market model

The Libor rate is assumed here to be log-normal both under μ and under $Q_{U_2} = f_{U_2} d\mu$ the pricing measure associated with numéraire U_2. This leads to :

$$x = \frac{U_3(0)}{U_2(0)\delta} \exp\left(\sigma\sqrt{\tau_0}\,\varepsilon - \frac{\sigma^2\tau_0}{2}\right), \tag{3.3}$$

where ε is a standard Gaussian variable with respect to Q_{U_2}. We can notice that the homogeneity condition is satisfied and Property 5 can be applied. The computation of $Q_{U_2}(x \geq c)$ is straightforward. For the computation of $Q_{U_3}(x \geq c)$, we can notice that : $Q_{U_3}(x \geq c) = \delta U_{2/3}(0) E^{Q_{U_2}}[x\mathbf{I}_{x \geq c}]$. The last expectation can be calculated using Cameron Martin formula (the discrete time version of Girsanov Theorem see Karatzas and Shreve (1991) p. 190). This leads to the following pricing formula :

$$\left\{\begin{array}{lll} Cap(c) & = & U_3(0)\phi(d_1) - c\delta U_2(0)\phi(d_2), \\ \text{with :} & & \\ & d_1 = & \frac{1}{\sigma\sqrt{\tau_0}} \log\left(\frac{U_3(0)}{U_2(0)c\delta}\right) + \frac{\sigma}{2}\sqrt{\tau_0}, \\ \text{and :} & & \\ & d_2 = & d_1 - \sigma\sqrt{\tau_0}. \end{array}\right. \tag{3.4}$$

The formula (3.4) is parametrized through the volatility σ of the forward Libor.

Example 2 : Gaussian model

In that case, we assume that the inverse of the forward price is log-normal under $\bar{Q}_{U_2}$ and under $\bar{\mu}$:

$$z = \frac{U_1(0)}{U_2(0)} \exp\left(\bar{\sigma}\sqrt{\tau_0}\,\varepsilon - \frac{\bar{\sigma}^2\tau_0}{2}\right),$$

where ε is a standard Gaussian variable under $\bar{Q}_{U_2}$. $\bar{Q}_{U_2}(E)$ with $E = \{z : z > c\}$ is written readily and $\bar{Q}_{U_1}(E)$ is computed using again Cameron-Martin formula.

$$\left\{\begin{array}{lll} Cap(c) & = & U_1(0)\phi\left(\bar{d}_1\right) - (1 + c\delta)U_2(0)\phi\left(\bar{d}_2\right), \\ \text{with :} & & \\ & \bar{d}_1 = & \frac{1}{\bar{\sigma}\sqrt{\tau_0}} \log\left(\frac{U_1(0)}{U_2(0)(1 + c\delta)}\right) + \frac{\bar{\sigma}}{2}\sqrt{\tau_0}, \\ \text{and :} & & \\ & \bar{d}_2 = & \bar{d}_1 - \bar{\sigma}\sqrt{\tau_0}. \end{array}\right. \tag{3.5}$$

In this model, the parameter in the caplet pricing formula (3.5) is $\bar{\sigma}$. In the Vasicek model (Vasicek (1977)), $\bar{\sigma}^2\tau_0$ specializes to :

$$\bar{\sigma}^2\tau_0 = \frac{\sigma^2}{2\lambda^3}\left[\left(e^{-\lambda(\tau_2-\tau_0)} - e^{-\lambda(\tau_1-\tau_0)}\right)^2 - \left(e^{-\lambda\tau_2} - e^{-\lambda\tau_1}\right)^2\right], \tag{3.6}$$

which involves two parameters σ and λ, the volatility and the mean reversion coefficient of the instantaneous interest rate, respectively.

4 Implied pricing models with an L^2-distance

4.1 Implied approaches

It is a well known feature that the standard BLACK-SCHOLES model may not be consistent with observed option prices of different exercise prices and lead to the presence of volatility smiles (see e.g. BATES (1996), BAKSHI, CAO and CHEN (1997)). Similar departures occur between standard caplet pricing models and quoted market prices (see data below). We thus briefly review the procedure for building viable pricing models from a set of observed prices and an a priori pricing model (see e.g. DUPIRE (1992), SHIMKO (1993), DERMAN and KANI (1994), RUBINSTEIN (1994), DUMAS, FLEMING and WHALEY (1995), BUCHEN and KELLY (1996), JACKWERTH and RUBINSTEIN (1996)).

The basic idea in the implied approach is to find a probability measure as close as possible to an a priori probability measure among all risk neutral probability measures. The a priori measure usually comes from a caplet pricing model derived according to some theoretical considerations (either in discrete or continuous time, from equilibrium or arbitrage arguments). This model can also be a model adopted by market practitioners. In other words the a priori model is a kind of benchmark or structural model such as the widely used market and Gaussian models. The implied approach relies on modifying the a priori pricing model in order to achieve compatibility with the set of observed prices. The difference between the a priori and a posteriori models is similar to an error term due to omission of various effects and variables in the structural initial model.

Various criteria have been proposed in order to measure proximities between probability measures such as the quadratic, cross-entropy (Kullback-Leibler) or goodness-of-fit criteria. On economic grounds, we show that the quadratic criterion is related to the standard mean-variance hedging problem. Indeed the derived option price appears to be equal to the solution of a static mean-variance portfolio choice. This price is called the approximation price and is the price of the hedge portfolio which minimizes the residual risk. This economic interpretation prompts to use a quadratic criterion. Since SOLNIK (1974) it is common knowledge that the solutions of mean-variance problems are numéraire dependent. Therefore we study hereafter in detail these effects thanks to the state price invariance property.

On practical grounds, since tractability also matters, the quadratic criterion reveals to be very attractive. It is by far the most simple and leads to explicit expressions for the a posteriori pricing measure and for caplet price forecasts thanks to the use of convenient numéraires. Furthermore the robustness of the predicted option prices w.r.t. to choices concerning numéraires, proximity criteria, or a priori models is a very convincing argument in favor of our implied approach. This appears in several empirical papers (JACKWERTH and

RUBINSTEIN (1996), JONDEAU and ROCKINGER (1997), FRACHOT, LAURENT and PICHOT (1998)) and is confirmed here in the empirical section.

For these different reasons we prefer to rely on an $L^2(\mu)$-approach (the observed payoffs $g_{U_i,j}$, $j \in J, i = 1,2,3$ are square integrable : $p = 2$) in the remaining of the paper.

4.2 Definition and existence of variance optimal measure densities

Let us choose a numéraire (say U_1) and consider the set $\mathcal{F}_{U_1}^{2,s}$ of (square integrable) risk-neutral signed measure densities, the set $\mathcal{F}_{U_1}^{2}$ of (square integrable) risk-neutral densities absolutely continuous w.r.t to μ, and the set $\mathcal{F}_{U_1}^{2,e}$ of (square integrable) risk-neutral densities equivalent to μ. These different sets have been precisely defined in the first part of the paper. By taking any element of such sets, we can build a pricing model consistent with observed prices.

Let us consider an a priori pricing model (say the market model). We denote by $f_{U_1}^{0\sigma}$, the density of this priori model w.r.t μ, associated to numéraire U_1. If the a priori model is not consistent with observed option prices, $f_{U_1}^{0\sigma} \notin \mathcal{F}_{U_1}^{2,e}$. We examine the two minimisation problems which differ in their optimisation sets :

$$\min_{f \in \mathcal{F}_{U_1}^{2,s}} \int \left(f - f_{U_1}^{0\sigma}\right)^2 d\mu,$$

$$\min_{f \in \mathcal{F}_{U_1}^{2}} \int \left(f - f_{U_1}^{0\sigma}\right)^2 d\mu.$$

Since the two minimisation sets are non-empty, closed and convex subsets of $L^2(\mu)$, there exist a unique minimisation element (by projection theorem) for each minimisation problem.

The solutions are called the *variance optimal signed measure density associated to numéraire* U_1 *and a priori model* $f_{U_1}^{0\sigma}$ (and denoted by $\tilde{f}_{U_1}^{1\sigma}$) and *variance optimal probability measure density associated to numéraire* U_1 *and a priori model* $f_{U_1}^{0\sigma}$ (and denoted by $f_{U_1}^{1\sigma}$). $\tilde{f}_{U_1}^{1\sigma}$ and $f_{U_1}^{1\sigma}$ mirror some risk premium updated thanks to information provided by observed prices.

Of course, if the a priori pricing model is already consistent with observed prices (no volatility smile), then $\tilde{f}_{U_1}^{1\sigma} = f_{U_1}^{0\sigma}$.

Let us notice that there does not always exist a minimal distance element between $f_{U_1}^{0\sigma}$ and $\mathcal{F}_{U_1}^{2,e}$, since the later set is not closed. However if $\tilde{f}_{U_1}^{1\sigma}$ happens to be in $\mathcal{F}_{U_1}^{2,e}$, it is clearly a minimal distance element between $f_{U_1}^{0\sigma}$ and $\mathcal{F}_{U_1}^{2,e}$. In a continuous-time framework and when asset prices are continuous semimartingales, it has been proved by DELBAEN and SCHACHERMAYER (1996) that the variance optimal signed measure density is always strictly positive and thus equivalent to μ (see also GOURIÉROUX, LAURENT and PHAM (1998) for a discussion, and LAURENT and PHAM (1998) for applications).

4.3 Characterization of variance optimal measure densities

The solution of these standard convex optimization problems can be written through the first order conditions as :

$$\begin{array}{rcl} \tilde{f}^{1\sigma}_{U_1} & = & f^{0\sigma}_{U_1} + \sum_{j \in J} \tilde{\lambda}^{\sigma}_{U_1,j} g_{U_1,j}, \\ f^{1\sigma}_{U_1} & = & \left(f^{0\sigma}_{U_1} + \sum_{j \in J} \lambda^{\sigma}_{U_1,j} g_{U_1,j} \right)^{+}, \ \mu \text{ a.s..} \end{array} \tag{4.1}$$

where $\tilde{\lambda}^{\sigma}_{U_1,j}, \lambda^{\sigma}_{U_1,j}$ are real numbers uniquely (because of non redundancy) determined by the price constraints. These forms are obtained by applying the Lagrange Multiplier Theorem (see LUTTMER (1996), Proposition 2 and HANSEN and JAGANNATHAN (1997), Proposition A.2).

The variance optimal signed measure of SCHWEIZER (1992) corresponds to the case where $f^{0\sigma}_{U_1} = 1$. This variance optimal signed measure also appears in a portfolio context in HANSEN and RICHARD (1987), HANSEN and JAGANNATHAN (1991) and BANSAL, HSIEH and VISWANATHAN (1993). The variance optimal measures depend on the a priori density $f^{0\sigma}_{U_1}$. If the a priori pricing model corresponds to the market model or to the Gaussian model, the variance optimal measures depend on the volatility parameter σ.

The variance optimal measures also *depend on the choice of μ but only through the Lagrange multipliers* $\tilde{\lambda}^{\sigma}_{U_1,j}, \lambda^{\sigma}_{U_1,j}$ as can be seen in equation (4.1). MAGNIEN, PRIGENT and TRANNOY (1996) use the Lebesgue measure on a finite length interval $[a, b]$. In section 5, we will use log-normal measures (i.e. x (resp. z) will be log-normal under μ (resp. $\bar{\mu}$)).

When μ is the Lebesgue measure on some finite interval $[a, b]$, MICHELLI, SMITH, SWETITS and WARD (1985), IRVINE, MARIN and SMITH (1986)[1], MAGNIEN, PRIGENT and TRANNOY (1996) characterize and compute the variance optimal probability measure when observed prices are call option prices. In that framework, the variance optimal probability measure is related to B-splines (a common interpolation technique).

We have already noticed that the variance-optimal measures may not be associated to an equivalent (to μ) risk-neutral measure. Therefore, the associated price operator is not always (strictly) positive. We can however state the following result :

Property 6 (variance optimal measure positivity)
If the variance optimal signed measure and the variance optimal probability measure differ, then the variance optimal probability measure is not equivalent to μ and does not lead to a (strictly) positive price operator.

Proof : see Appendix.

[1]We are grateful to F. Magnien for providing the two last references.

4.4 Dependence on the choice of numéraire

In the previous section, we took as a benchmark the numéraire U_1. We now show that the variance optimal measure densities give rise to different pricing models when we change our reference numéraire from U_1 to U_i. Here the density $\tilde{f}^j_{U_i}$ (resp. $f^j_{U_i}$) will be the variance optimal signed (resp. probability) measure densities associated with numéraire U_i when the numéraire U_j is used as reference (for notational simplification, we drop here the dependence on σ).

In order to make a comparison, let us consider some other numéraire, U_i, $i \neq 1$ while keeping U_1 as reference numéraire for the moment. By the state price invariance property, the density associated to numéraire U_i can be written as :

$$\tilde{f}^1_{U_i} = \frac{U_{i/1}}{U_{i/1}(0)} \tilde{f}^1_{U_1}, \tag{4.2}$$

where $U_{i/1}$ is the exchange rate between numéraires U_i and U_1 at time τ_0 (and is a μ-measurable function). Let us remark that $\tilde{f}^1_{U_i}$ is in $L^2(\mu)$ under the standing assumption that the Libor rate is strictly positive (which in turn implies that $U_{i/1} \in L^\infty(\mu)$). The density f_{U_i} of the a priori pricing model, associated to numéraire U_i can also be obtained by state price invariance property :

$$f^0_{U_i} = \frac{U_{i/1}}{U_{i/1}(0)} f^0_{U_1}. \tag{4.3}$$

From the characterisation of $\tilde{f}^1_{U_1}$ in equation (4.1) and from the relations between payoffs under different numéraires, $g_{U_1,j} = U_{i/1} g_{U_i,j}$, we obtain by (4.2) and (4.3) :

$$\tilde{f}^1_{U_i} = f^0_{U_i} + \frac{U_{i/1}}{U_{i/1}(0)} \sum_{j \in J} \tilde{\lambda}_{U_1,j} U_{i/1} g_{U_i,j}. \tag{4.4}$$

On the other hand, we can directly compute the variance optimal signed measure density associated to numéraire U_i while choosing as reference numéraire U_i. By adapting the characterization result (4.1) to numéraire U_i instead of U_1 there exist some real numbers $\tilde{\lambda}_{U_i,j}$, $j \in J$ such that :

$$\tilde{f}^i_{U_i} = f^0_{U_i} + \sum_{j \in J} \tilde{\lambda}_{U_i,j} g_{U_i,j}. \tag{4.5}$$

Now, since in the usual cases $U_{i/1}$ is a non constant random variable and the a posteriori model differs from the a priori model (some of the Lagrange multipliers are different from zero), we clearly see from (4.4) and (4.5) that :

$$\tilde{f}^1_{U_i} \neq \tilde{f}^i_{U_i},$$

which states that the variance optimal measures will differ if we start with numéraire U_i ($\tilde{f}^i_{U_i}$) instead of U_1 ($\tilde{f}^1_{U_i}$). The reason for this dependence of the a posteriori pricing model on the chosen numéraire will be a consequence of the property given in the next subsection.

4.5 Variance optimal signed measures and approximation prices

In this section we show that option prices computed under the variance optimal signed measure correspond to the "approximation price" of the option introduced by SCHWEIZER (1992).

We first consider the mean-variance hedging problem :

$$\min_{\lambda_{U_i}} \int (\sum_{j\in J} \lambda_{U_i,j} g_{U_i,j} - g_{U_i})^2 d\mu.$$

where g_{U_i} is a square integrable payoff (for $i = 1, 2, 3$). This problem consists in finding a (static) portfolio $\sum_{j\in J} \lambda_{U_i,j} g_{U_i,j}$ which minimizes the square of the hedging residual : $\sum_{j\in J} \lambda_{U_i,j} g_{U_i,j} - g_{U_i}$, or equivalently the $L^2(\mu)$ distance to payoff g_{U_i}. This problem has been introduced in a dynamic framework by DUFFIE and RICHARDSON (1991) and further studied among others by SCHWEIZER (1992) and GOURIÉROUX, LAURENT and PHAM (1998).

A direct application of the projection theorem guarantees that under non redundancy and no arbitrage, there exists a unique $\lambda^*_{U_i}$ to the previous minimization problem.

The mapping : $g_{U_i} \longrightarrow P^*_{U_i}[g_{U_i}] = \sum_{j\in J} \lambda^*_{U_i,j} P_j$ is a continuous linear functional on $L^2(\mu)$ consistent with observed prices and $\sum_{j\in J} \lambda^*_{U_i,j} P_j$ is called the *approximation price* of g_{U_i}. The price of the approximating portfolio $\sum_{j\in J} \lambda^*_{U_i,j} g_{U_i,j}$ is thus equal to a linear combination of the asset prices. As it is well known in international portfolio management (SOLNIK (1974)), the approximating portfolio and thus its price are numéraire dependent (since $P^*_{U_i}[g_{U_i}]$ is obtained by taking as reference numéraire U_i). We are now able to state the following :

Property 7 (approximation price)
The approximation price $P^*_{U_i}[g_{U_i}]$ of some payoff g_{U_i} is equal to the price of this payoff under the variance optimal signed measure associated to numéraire U_i, $U_i(0) \int \tilde{f}^i_{U_i} g_{U_i} d\mu$, where $\tilde{f}^i_{U_i}$ is the variance optimal signed measure density associated to numéraire U_i and to $f^{0\sigma}_{U_i} = 1$ (i.e. no risk premia).

Proof : see Appendix.

Now, the dependence of the variance optimal signed measure on the choice of numéraire is a consequence of the dependence of the approximation price on the choice of numéraire.

5 Explicit cap pricing formulas under the variance optimal measure

In order to shed light on the model building procedure, some of the previous points are now illustrated in interest rate modelling. It is then easy to derive closed-form caplet pricing formula consistent with observed prices in the variance optimal measure setting.

Let us consider a sequence of observed prices corresponding to numéraires and caplets, namely $(g_{U_2,0}(x) = 1, U_2(0))$, $(g_{U_2,j}(x) = (x-c_j)^+\delta, P_j)$, $c_1 = 0$, $j \geq 1, j \in J$. From equation (4.1), the variance optimal signed measure density associated to the a priori model $f^{0\sigma}_{U_2} = 1$ (i.e. the no risk premium case) and to numéraire U_2 can be written as :

$$\tilde{f}^2_{U_2}(x) = 1 + \lambda_0 + \sum_{j\geq 1, j\in J} \lambda_j (x - c_j)^+\delta, \quad \mu \text{ a.s.}, \tag{5.1}$$

which leads to :

$$Cap(c) = U_2(0) \int (x-c)^+\delta \left(1 + \lambda_0 + \sum_{j\geq 1, j\in J} \lambda_j (x-c_j)^+\delta\right) d\mu(x). \tag{5.2}$$

Let us introduce the density function $f^0_{U_4}$ defined by :

$$U_2(0) = \frac{U_4(0) f^0_{U_4}(x)}{x^2\delta^2}, \quad \mu \text{ a.s.},$$

where $U_4(0) = U_2(0) \int x^2\delta^2 d\mu(x)$ and $Q^0_{U_4}$ the probability associated to measure $f^0_{U_4} d\mu$. Since x is in $L^2(\mu)$ this density is well defined. Straightforward computations give the following property.

Property 8 (caplet price forecast)
A caplet pricing formula consistent with observed numéraires and caplet prices in the variance optimal signed measure is given by :

$$Cap(c) = (1+\lambda_0) Cap^0(c) + \sum_{j\geq 1, j\in J} \lambda_j Cap^j(c), \tag{5.3}$$

where $Cap^0(c)$ *is the a priori pricing formula (i.e.* $U_2(0) E^{Q^0_{U_2}}[(x-c)^+\delta]$*) and* $Cap^j(c)$ *is equal to :*

$$Cap^j(c) = U_4(0) Q^0_{U_4}(E_j) - U_3(0)\, \delta\, (c + c_j) Q^0_{U_3}(E_j) - U_2(0)\, c\, c_j\, \delta^2 Q^0_{U_2}(E_j), \tag{5.4}$$

E_j *the exercise region being equal to* $\{x \geq c \vee c_j\}$ *and* $c \vee c_j = \sup(c, c_j)$*. The Lagrange multipliers* λ_j *are determined by the linear equations :*

$$\begin{aligned} P_j &= (1+\lambda_0) Cap^0(c_j) + \sum_{i\geq 1, i\in J} \lambda_i Cap^i(c_j), \quad j \geq 1, j \in J, \\ U_2(0) &= U_2(0)(1+\lambda_0) + \sum_{i\geq 1, i\in J} \lambda_i Cap^0(c_i). \end{aligned}$$

Hence we get an explicit formula which only requires solving linear equations for its practical inplementation. This explicit formula is particularized in the next two examples.

Example 1 : Modified market model

When x is log-normal under $Q^0_{U_2}$ (see equation (3.3)), the caplet formula provided through equation (5.4) specializes to :

$$Cap^j(c) = U_2(0)\left[x^2(0)\delta^2 \exp(\sigma^2\tau_0)\phi(d_0^j) - x(0)\delta^2(c+c_j)\phi(d_1^j) + c\, c_j\, \delta^2\phi(d_2^j)\right], \tag{5.5}$$

with :

$$\begin{cases} d_1^j &= \frac{1}{\sigma\sqrt{\tau_0}}\log\frac{x(0)}{c\vee c_j} + \frac{\sigma}{2}\sqrt{\tau_0}, \\ d_2^j &= d_1^j - \sigma\sqrt{\tau_0}, \\ d_0^j &= d_1^j + \sigma\sqrt{\tau_0}. \end{cases}$$

This simple expression is due to the lognormality of the Libor rate under $Q^0_{U_2}$, $Q^0_{U_3}$ and $Q^0_{U_4}$. Let us remark that $Cap(c)$ is homogeneous of degree one in $(U_2(0), U_3(0))$ but is not homogeneous of degree one in $(c, x(0))$ as in the proportional models.

Example 2 : Modified Gaussian model

For the Gaussian model (z is log-normal under $\bar{Q}^0_{U_2}$), the same kind of explicit formulas can be derived under the measure that minimizes the $L^2(\mu)$-distance between $\bar{f}^{0\bar{\sigma}}_{U_2}$ and the set of densities for z compatible with observed prices. This measure takes the form :

$$\tilde{\bar{f}}^2_{U_2}(z) = 1 + \bar{\lambda}_0 + \sum_{j\geq 1, j\in J} \bar{\lambda}_j (z - (1 + c_j\delta))^+, \quad \bar{\mu} \text{ a.s.},$$

which gives :

$$\bar{C}ap(c) = (1+\bar{\lambda}_0)\bar{C}ap^0(c) + \sum_{j\geq 1, j\in J} \bar{\lambda}_j \bar{C}ap^j(c),$$

with :

$$\bar{C}ap^0(c) = U_1(0)\phi(\bar{d}_1) - \bar{c}U_2(0)\phi(\bar{d}_2), \tag{5.6}$$

$$\bar{C}ap^j(c) = U_2(0)\left[z^2(0)\exp(\bar{\sigma}^2\tau_0)\phi(\bar{d}_0^j) - z(0)(\bar{c}+\bar{c}_j)\phi(\bar{d}_1^j) + \bar{c}\,\bar{c}_j\,\phi(\bar{d}_2^j)\right], \tag{5.7}$$

with :

$$\begin{cases} \bar{c} &= 1 + c\delta, \\ \bar{d}_1 &= \frac{1}{\bar{\sigma}\sqrt{\tau_0}}\log\frac{z(0)}{\bar{c}} + \frac{\bar{\sigma}}{2}\sqrt{\tau_0} \\ \bar{d}_2 &= \bar{d}_1 - \bar{\sigma}\sqrt{\tau_0}, \\ \bar{c}_j &= 1 + c_j\delta, \\ \bar{d}_1^j &= \frac{1}{\bar{\sigma}\sqrt{\tau_0}}\log\frac{z(0)}{\bar{c}\vee\bar{c}_j} + \frac{\bar{\sigma}}{2}\sqrt{\tau_0}, \\ \bar{d}_2^j &= \bar{d}_1^j - \bar{\sigma}\sqrt{\tau_0}, \\ \bar{d}_0^j &= \bar{d}_1^j + \bar{\sigma}\sqrt{\tau_0}. \end{cases}$$

Let us remark that a convex combination of the two previous caplet formulas also lead to a viable price operator (due to the convexity of the set of viable price operator) and thus to another option pricing model.

Finally similar explicit caplet pricing formulas are also available under the variance optimal probability measure. This is due to the fact that the positive part of a piecewise linear function is still piecewise linear. Thus the densities w.r.t. μ (resp. $\bar{\mu}$) are piecewise linear functions of x (resp. z) and the computations go along the same lines as in the signed case.

6 Pricing bounds

Let us remark that one may still question our approach and the dependence of the rebuilt caplet prices on the retained risk-neutral measure. Indeed, if one only relies on arbitrage arguments in our highly incomplete market (in which lots of risk neutral pricing measures exist), it is true that any risk-neutral pricing models may be taken. In that case the set of admissible prices may be quite large as shown for example by MERTON (1973).

One may try to narrow these bounds either by restricting the set of admissible risk-neutral measures choosing for instance measures close to the log-normal, or by equilibrium arguments. Our previous approach which aims to choose a special pricing measure relies on a mean-variance argument. JOUINI (1997), PHAM and TOUZI (1996) among others show that the set of admissible prices cannot be narrowed if general preferences are considered. Assuming decreasing marginal utility w.r.t. the underlying price in a discrete framework, PERRAKIS and RYAN (1984) RITCHKEN (1985) exhibit bounds significantly more stringent that the MERTON's bounds. Now regarding pricing bounds unlike the aforementioned references, we have to take into account the existence of traded options and not only traded numéraires. This induces extra restrictions on pricing measures thus narrowing the pricing bounds. The bound we propose appears closely related to a dual portfolio choice problem introduced in continuous time by EL KAROUI and QUENEZ (1995), known as the super-replication problem.

The highest price at which the portfolio of assets with payoff g_{U_i} can be valuated is provided by :

$$\sup_{f_{U_i} \in \mathcal{F}_{U_i}^{q,e}} U_i(0) \int g_{U_i} f_{U_i} d\mu, \tag{6.8}$$

where $\mathcal{F}_{U_i}^{q,e}$ is the set of equivalent risk-neutral probability density measures. This set is non empty under the no arbitrage opportunity assumption. A duality approach allows to characterize this highest price. The previous problem reveals to be the dual problem of the so-called primal super-replication problem :

$$\inf_{\lambda} \sum_{j \in J} \lambda_j P_j \quad \text{s.t.} \quad \sum_{j \in J} \lambda_j g_{U_i,j} \geq g_{U_i}, \ \mu \text{ a.s..}$$

The solution of this problem is the minimum price of a portfolio based only on observed assets that dominates the portfolio payoff g_{U_i} and is called the super-replication price of the portfolio. This super-replication price is an obvious upper bound of the accounted value of the portfolio. We assume hereafter that this problem is consistent i.e. there exists at least one feasible solution (the set Λ of $\lambda = (\lambda_j)$ satisfying the constraints is non empty). This problem belongs to a particular class of linear programming problems. It is called the class of continuous semi-infinite linear programs because the set of constraints is uncountable (see Anderson and Nash (1987)). We are able to state the following duality results.

Property 9 (Weak duality)
Under no arbitrage, we have : $\inf_{\lambda \in \Lambda} \sum_{j \in J} \lambda_j P_j \geq \sup_{f_{U_i} \in \mathcal{F}^{q,e}_{U_i}} U_i(0) \int g_{U_i} f_{U_i} d\mu.$

Proof : see Appendix.

This guarantees that the two optimization problems have finite solutions. Furthermore the super-replication price can be associated with a super-replicating portfolio since the greatest lower bound is attained.

Property 10 (existence of a minimum price super-replicating portfolio)
Under non redundancy and no arbitrage, there exists a solution λ^* to the primal problem :

$$\min_{\lambda} \sum_{j \in J} \lambda_j P_j \quad s.t. \quad \sum_{j \in J} \lambda_j g_{U_i,j} \geq g_{U_i}, \ \mu \ a.s.$$

Proof : see Appendix.

Another interesting property of the super-replication problem is that there is no duality gap i.e. the super-replication problem and its dual have the same finite value.

Property 11 (absence of a duality gap)
Under no arbitrage, the super-replication price is equal to :

$$\sup_{f_{U_i} \in \mathcal{F}^{q,e}_{U_i}} U_i(0) \int g_{U_i} f_{U_i} d\mu,$$

i.e. the supremum of the expectation of payoff g_{U_i} taken w.r.t. the set of all equivalent risk-neutral measures.

Proof : see Appendix.

Hence, the functional which associates the super-replication price to a given payoff is sublinear (since it is the supremum of linear functionals). We can also characterize the super-replication problems where strong duality is achieved.

Property 12 (strong duality)
The set of payoffs g_{U_i} where strong duality is achieved, i.e. where there exists $f^*_{U_i}$ maximizing the expected value of the payoff over all equivalent risk-neutral probability density measures $f_{U_i} \in \mathcal{F}^{q,e}_{U_i}$ is the static investment opportunity set.

Proof : see Appendix.

This result corresponds to the characterization of attainable payoffs stated by Jacka (1992) in the dynamic case. In order to compute the optimal value, extensions of the simplex algorithm for solving either the primal or dual problems are given in Section 4.5 of Anderson and Nash (1987).

Where observed prices correspond to caplet payoffs and when we consider a caplet payoff of arbitrary exercice rate, the super-replication price is obtained as the linear interpolation of observed prices.

Let us take $c \in [c_i, c_{i+1}[$. The super-replicating portfolio of $(x-c)^+\delta$ has the form :

$$\delta \left[\frac{c_{i+1}-c}{c_{i+1}-c_i}(x-c_i)^+ + \frac{c-c_i}{c_{i+1}-c_i}(x-c_{i+1})^+ \right]. \tag{6.9}$$

Indeed it is easy to show that any portfolio $\sum \lambda_j (x-c_j)^+\delta$ dominating $(x-c)^+\delta$ also dominates the payoff (6.9). By the positivity of the price operator, we get that the price asssociated to (6.9) is the super-replication price :

$$\frac{c_{i+1}-c}{c_{i+1}-c_i}P_i + \frac{c-c_i}{c_{i+1}-c_i}P_{i+1}. \tag{6.10}$$

7 Numerical example and empirical application

In this last section, we begin by checking the practical relevance of our approach on a numerical example. The example is designed to obtain prices similar to those currently traded on the market. We then proceed further on real caplet data.[2]

[2]We thank Paribas Capital Markets for kindly providing the market data. The Gauss programs developped for this section are available on request.

For the numerical example, ten caplet prices have been generated with the Gaussian model (eq. (3.5)). These prices are one year caplet prices on three month Libor ($\tau_0 = \tau_1 = 1$, $\tau_2 = 1.25$, $\delta = 0.25$) with equally spaced exercice rates from 2.5% to 7%. The volatility parameter of the Gaussian model is set equal to 0.242% (eq. (3.6) : $\bar{\sigma} = 0.242\%$, $\sigma = 1\%$, $\lambda = 5\%$). The yield curve is taken 4% flat ($U_1(0) = 0.9608$, $U_2(0) = 0.9512$, $U_3(0) = 0.0096$, $x(0) = 4.02\%$). These data are used as input data for a calibration procedure based on the market model (eq. (3.4), (5.3), (5.5)). Hence we take the market model as our a priori model in this example. The volatility parameter of the market model corresponds to the implied volatility of the observed at-the-money caplet price ($\sigma = 24.478\%$). In Table 6 the price forecasts are compared with the (unobserved) true prices and the super-replication prices (eq. (6.10)). The strike rates of the caplet prices to be infered are the intermediate rates from 2.75% to 6.75%.

Table 6 : True prices, price forecasts and super-replication prices (in percent)

strike	true	forecast	super-repl.
2.75	0.31263	0.31234	0.31394
3.25	0.21172	0.21181	0.21395
3.75	0.12848	0.12845	0.13131
4.25	0.06813	0.06814	0.07105
4.75	0.03085	0.03084	0.03313
5.25	0.01169	0.01170	0.01307
5.75	0.00365	0.00365	0.00429
6.25	0.00093	0.00093	0.00116
6.75	0.00019	0.00019	0.00025

The results show that our simple procedure is very successful in rebuilding the unobserved data while matching exactly (by construction) the available market prices. The difference is not visible to the naked eye if the true prices are plotted on a graph together with their forecasts for each exercice rate. The absolute errors are of orders 10^{-6} to 10^{-8} while the relative errors are of orders 10^{-2} to 10^{-4}. Reversing the role of the Gaussian model and the market model in such an example leads to similar results.

Let us now apply the calibration approach to real market data. The collected data are one year caplet prices for the three month DEM Libor. The quotes (Tue. 06/10/1998 around 4 pm) take the form of a lognormal volatility smile which can be translated into caplet prices. The data (implied volatilities and caplet prices) are presented in Table 7. The discount bond prices are equal to 0.9665 and 0.9582 for the one year and fifteen month maturities, respectively.

Table 7 : Implied volatilities and observed caplet prices (in percent)

strike	volatility	price
2.5	24.841	0.23747
3	24.321	0.14219
3.5	24.249	0.07558
4	24.464	0.03695
4.5	24.857	0.01739
5	25.331	0.00814
5.5	25.799	0.00384
6	26.197	0.00182
6.5	26.500	0.00086
7	26.712	0.00040

From these observed data, we get the following price forecasts taking as a priori either the market model or the Gaussian model (Table 8). The volatility parameters are taken equal to their respective at-the-money implied volatility ($\sigma = 24.464\%$, $\bar{\sigma} = 0.225\%$). The price forecasts made by the two models do not differ very much from each other. For the strike rates : 2.75%, 3.75%, 4.75%, 5.75%, 6.75%, the Gaussian model forecasts are slightly higher while the reverse holds for the other exercice rates. However we do not see a particular reason for this special alternate ordering.

Table 8 : Price forecasts with the market model and the Gaussian model (in percent)

strike	market	Gaussian	super-repl.
2.75	0.18671	0.18707	0.18983
3.25	0.10506	0.10496	0.10889
3.75	0.05325	0.05327	0.05627
4.25	0.02541	0.02539	0.02717
4.75	0.01189	0.01189	0.01276
5.25	0.00558	0.00558	0.00599
5.75	0.00264	0.00265	0.00283
6.25	0.00125	0.00122	0.00134
6.75	0.00059	0.00062	0.00063

8 Concluding remarks

We have presented a general approach for the valuation of a book of interest rate derivative products, such as caplets. The proposed valuation is slightly different from the usual approach based on continuously adjusted self-financing portfolios. In particular, we do not rely on particular time evolutions of the state variables and institutional assumptions such as frictionless trading. The usual dynamic approach makes it difficult to take into account observed prices, and is mainly aimed at providing a structural model that will be modified according to the information contained in observed prices. The quality of the cross price prediction will depend on the one hand on the number of currently traded assets and observed prices and on the other hand on the quality of the structural model.

Our main focus was interest rate products such as caplets. This is a rich framework to introduce different numéraires and provide simple general formulas for interest rate options. Our framework ought to be extended in order to take into account different exercise dates. Such an extension has been made in exchange rate option pricing models and for special dynamics (Avellaneda et al. (1997) and Laurent and Leisen (1998)). The work is harder when considering HJM type models since the standard state variable is the forward curve whose dynamics are complex even for standard models such as the market model. Another extension would be to include other kinds of asset payoffs such as swaptions.

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Appendix A

Proof of Proposition 2

We define the linear functional $\pi^*_{U_1} : g_{U_1} \in G_{U_1} \longrightarrow \sum_{j\in J} \lambda_j P_j$, with $\lambda_j\ \forall j$ s.t. $g_{U_1} = \sum_{j\in J} \lambda_j g_{U_1,j}$. $(G_{U_1}, \pi^*_{U_1})$ is a price system in the terminology of HARRISON and KREPS (1979), and $\pi^*_{U_1}$ provides the prices of static portfolios.

The absence of arbitrage opportunities is equivalent to the assumption that $\pi^*_{U_1}$ is strictly positive. In the terminology of HARRISON and KREPS (1979), $(G_{U_1}, \pi^*_{U_1})$ is an arbitrage free price system. Since strictly positive linear functionals on a subspace can be extended as strictly positive linear functionals on the whole space $L^p(\mu)$ (DUFFIE (1988) p. 72) and since positive linear functionals on $L^p(\mu)$ are continuous (DUFFIE (1988) p. 63), $\pi^*_{U_1}$ can be extended to any payoff in $L^p(\mu)$: there exists a strictly positive linear functional on $L^p(\mu)$, π_{U_1}, such that the restriction of π_{U_1} to G_{U_1}, $\pi_{U_1}|G_{U_1}$, is equal to $\pi^*_{U_1}$.

Once π_{U_1} has been defined on $L^p(\mu)$, we know from Section 2.1.1 that corresponding π_{U_2} and π_{U_3} can be defined on $L^p(\mu)$.

The converse is straightforward.

Proof of Property 1

Applying successively Proposition 1 and the state price measure invariance property (equation (2.5)), we get :

$$\begin{aligned} Cap(c) &= (B(0,\tau_1) - B(0,\tau_2)) \int_a^b \frac{(x-c)^+}{x} f_{U_3}(x) d\mu(x) \\ &= (B(0,\tau_1) - B(0,\tau_2))(1 - F_{U_3}(c)) - c\delta B(0,\tau_2)(1 - F_{U_2}(c)). \end{aligned}$$

It proves (a) and (b). (c) is obtained from the definition of the forward Libor.

Proof of Property 2

From Proposition 1, equation (2.8) and the transfer theorem, we get :

$$\begin{aligned} Cap(c) &= \delta U_2(0) \int_a^b (x-c)^+ f_{U_2}(x) d\mu(x) \\ &= \delta U_2(0) \int_a^b (x-c)^+ \bar{f}_{U_2}(\delta x + 1) d\mu(x) \\ &= \delta U_2(0) \int_{1+a\delta}^{1+\delta b} (\frac{z-1}{\delta} - c)^+ \bar{f}_{U_2}(z) d\bar{\mu}(z) \\ &= U_2(0) \int_{1+c\delta}^{1+\delta b} z \bar{f}_{U_2}(z) d\bar{\mu}(z) \\ &\quad - U_2(0)(1+c\delta)(1 - \bar{F}_{U_2}(1+c\delta)). \end{aligned}$$

Since :

$$\frac{\bar{f}_{U_2}(z)}{\bar{f}_{U_1}(z)} = \frac{f_{U_2}(\frac{z-1}{\delta})}{f_{U_1}(\frac{z-1}{\delta})} = \frac{U_1(0)}{U_2(0)}\,\frac{1}{z},$$

we have :

$$\begin{aligned} U_2(0)\int_{1+c\delta}^{1+\delta b} z\bar{f}_{U_2}(z)d\bar{\mu}(z) &= U_1(0)\int_{1+c\delta}^{1+\delta b} \bar{f}_{U_1}(z)d\bar{\mu}(z) \\ &= U_1(0)(1-\bar{F}_{U_1}(1+c\delta)), \end{aligned}$$

and the stated result follows immediately.

Proof of Property 3

Form (a) is immediately deduced from Proposition 1 :

$$\begin{aligned} Dig(c) &= \pi_{U_2}[\mathbf{I}_E(x)\delta] \\ &= \delta U_2(0)Q_{U_2}(E). \end{aligned}$$

We get form (b) noting that the payoff $\delta\mathbf{I}_E(x)$ is the a.s. limit of the sequence of payoffs $\delta((x-c+1/p)^+ - (x-c)^+)/(1/p)$ obtained from caplet spread payoffs. These payoffs are also bounded by δ (whose expectation is finite and equal to δ). The corresponding sequence of prices is $(Cap(c-1/p)-Cap(c))/(1/p)$ and from Lebesgue convergence theorem, we can state that :

$$Dig(c) = -\lim_{p\longrightarrow+\infty}\frac{Cap(c-1/p)-Cap(c)}{-1/p}.$$

Proof of Property 4

From Property 3, the digital caplet price may be written :

$$Dig(c,U_2(0),U_3(0)) = c\delta U_2(0)Q_{U_2}(x\geq c \mid U_2(0),U_3(0)). \tag{8.11}$$

From the homogeneity property of the option price, we have :

$$eDig(c,eU_2(0),eU_3(0)) = c\delta eU_2(0)Q_{U_2}(x\geq c \mid eU_2(0),eU_3(0)). \tag{8.12}$$

Comparing (8.11) and (8.12), we see that Q_{U_2} is homogeneous of degree zero in $(U_2(0),U_3(0))$. Starting from the caplet price given in property 1, we similarly get :

$$\begin{aligned} Cap(c,U_2(0),U_3(0)) &= U_3(0)Q_{U_3}(x\geq c \mid U_2(0),U_3(0)) \\ &\quad -c\delta U_2(0)Q_{U_2}(x\geq c \mid U_2(0),U_3(0)), && (8.13) \\ eCap(c,eU_2(0),eU_3(0)) &= eU_3(0)Q_{U_3}(x\geq c \mid eU_2(0),eU_3(0)) \\ &\quad -c\delta eU_2(0)Q_{U_2}(x\geq c \mid eU_2(0),eU_3(0)). && (8.14) \end{aligned}$$

Since Q_{U_2} is homogeneous of degree zero, it is clear from (8.13) and (8.14) that Q_{U_3} is also homogeneous of degree zero in $(U_2(0), U_3(0))$, and hence in $(U_2(0), U_1(0))$ since $U_3(0)/U_2(0) = U_1(0)/U_2(0) - 1$, and we note with a slight abuse of notation :

$$Q_{U_3}(x \geq c \mid 1, U_2(0)/U_3(0)) = Q_{U_3}(x \geq c \mid U_2(0)/U_3(0)).$$

Using the expression of Property 2, we may prove along the same lines that Q_{U_1} is homogeneous of degree zero in $(U_2(0), U_1(0))$.

Proof of Property 6

Let us assume that $f_{U_1}^{1\sigma} = \left(f_{U_1}^{0\sigma} + \sum_{j\in J} \lambda_{U_1,j}^{\sigma} g_{U_1,j}\right)^+ > 0$, μ a.s.. Then, $f_{U_1}^{0\sigma} + \sum_{j\in J} \lambda_{U_1,j}^{\sigma} g_{U_1,j} > 0$, μ a.s., and $f_{U_1}^{1\sigma} = f_{U_1}^{0\sigma} + \sum_{j\in J} \lambda_{U_1,j}^{\sigma} g_{U_1,j}$. Since $f_{U_1}^{1\sigma} \in \mathcal{F}_{U_1}^2 \subset \mathcal{F}_{U_1}^{2,s}$, $f_{U_1}^{1\sigma}$ satisfies the first order conditions of the variance optimal signed measure density problem and is equal to $\tilde{f}_{U_1}^{1\sigma}$ by unicity.

Proof of Property 7

From Riesz-Fréchet Representation Theorem, there exists a unique function $f_{U_i}^* \in L^2(\mu)$ such that $P_{U_i}^*[g_{U_i}] = U_i(0) \int g_{U_i} f_{U_i}^* d\mu$, $\forall g_{U_i} \in L^2(\mu)$. We also have $\int f_{U_i}^* d\mu = 1$. Indeed, the numéraire U_i is a traded asset and its approximation price is equal to $U_i(0)$. Thus, $f_{U_i}^*$ belongs to the set $\mathcal{F}_{U_i}^{2,s}$ of risk-neutral signed density measures.

Let us now take some g_{U_i} orthogonal to the investement opportunity set $G_{U_i} = \{g_{U_i,j}\}$. Its approximation price is zero and thus we have the implication :

$$\forall j \in J, \int g_{U_i} g_{U_i,j} d\mu = 0 \Longrightarrow \int g_{U_i} f_{U_i}^* d\mu = 0.$$

Therefrom we deduce that $f_{U_i}^*$ belongs to the investment opportunity set G_{U_i} which means that $f_{U_i}^*$ has the interpretation of a portfolio. Let us denote A_{U_i}, the subspace of G_{U_i} spanned by the zero price portfolios :

$$A_{U_i} = \{\sum_{j\in J} \lambda_j g_{U_i,j} \mid \sum_{j\in J} \lambda_j P_j = 0\}.$$

Since the approximation price of any arbitrage portfolio is equal to zero, we deduce that the portfolio $f_{U_i}^*$ is orthogonal to A_{U_i}.

We now have to show that $\tilde{f}_{U_i}^i = f_{U_i}^*$. Since $\tilde{f}_{U_i}^i$ is the unique $L^2(\mu)$-norm minimum element of $\mathcal{F}_{U_i}^{2,s}$, we simply have to show that :

$$\int (\tilde{f}_{U_i}^i)^2 d\mu = \int (f_{U_i}^*)^2 d\mu.$$

or equivalently that :

$$\int \tilde{f}^i_{U_i}(\tilde{f}^i_{U_i} + f^*_{U_i})d\mu = \int f^*_{U_i}(\tilde{f}^i_{U_i} + f^*_{U_i})d\mu.$$

$\tilde{f}^i_{U_i} + f^*_{U_i}$ is the payoff of a portfolio since we know from our previous results that $\tilde{f}^i_{U_i}$ and $f^*_{U_i}$ are two portfolios. Since the expectations of a given portfolio under any measure consistent with observed prices are the same, the last equality is true.

Proof of Property 9

From the following implication : $\forall \lambda_j,\ \forall f_{U_i} \in \mathcal{F}^{q,e}_{U_i}$

$$\sum_{j\in J} \lambda_j g_{U_i,j} \geq g_{U_i} \ \mu \text{ a.s.} \Longrightarrow \int (\sum_{j\in J} \lambda_j g_{U_i,j}) f_{U_i} d\mu \geq \int g_{U_i} f_{U_i} d\mu,$$

Since the probability measure $f_{U_i}d\mu$ is compatible with observed prices :

$$\sum_{j\in J} \lambda_j P_j \geq U_i(0) \int g_{U_i} f_{U_i} d\mu,$$

which ends the proof.

Proof of Property 10

The proof uses standard functional analysis arguments. Let us denote by $\Lambda \subset \mathbb{R}^J$, the set $\{\lambda \in \mathbb{R}^J : \sum_{j\in J} \lambda_j g_{U_i,j} \geq g_{U_i}\ \mu \text{ a.s.}\}$ assumed to be non empty. This set is closed and convex. Let us choose a given $f_{U_i} \in \mathcal{F}^{q,e}_{U_i}$. The primal problem is equivalent to :

$$\inf_{\lambda\in\Lambda} \sum_{j\in J} \lambda_j P_j - U_i(0) \int g_{U_i} f_{U_i} d\mu = \|U_i(0)(\sum_{j\in J} \lambda_j g_{U_i,j} - g_{U_i})\|_{L^1(f_{U_i}d\mu)}.$$

Under non redundancy, it may be easily checked that the mapping :

$$\lambda \longrightarrow \|U_i(0) \sum_{j\in J} \lambda_j g_{U_i,j}\|_{L^1(f_{U_i}d\mu)}$$

is a norm in $\mathbb{R}^J$. Let us now consider $\lambda^0 \in \Lambda$ and the set :

$$\Lambda^0 = \{\lambda \in \Lambda : \|U_i(0)(\sum_{j\in J} \lambda_j g_{U_i,j} - g_{U_i})\|_{L^1(f_{U_i}d\mu)} \leq \|U_i(0)(\sum_{j\in J} \lambda^0_j g_{U_i,j} - g_{U_i})\|_{L^1(f_{U_i}d\mu)}\}.$$

This set is a closed bounded set of $\mathbb{R}^J$:

$$\|U_i(0) \sum_{j\in J} \lambda_j g_{U_i,j}\|_{L^1(f_{U_i}d\mu)} \leq \|U_i(0)(\sum_{j\in J} \lambda_j g_{U_i,j} - g_{U_i})\|_{L^1(f_{U_i}d\mu)} + \|U_i(0) g_{U_i}\|_{L^1(f_{U_i}d\mu)}.$$

Since the mapping $\lambda \longrightarrow \|U_i(0)(\sum_{j\in J} \lambda_j g_{U_i,j} - g_{U_i})\|_{L^1(f_{U_i}d\mu)}$ is continuous, there exists a minimum point λ^* in Λ^0 and the super replication problem is solvable.

Proof of Property 11

Let us take P^* the super-replication price and assume that :

$$\varepsilon = P^* - \sup_{f_{U_i} \in \mathcal{F}_{U_i}^{q,e}} U_i(0) \int g_{U_i} f_{U_i} d\mu > 0.$$

Let us denote by $\pi^*_{U_i,g}$ the linear functional defined by :

$$\pi^*_{U_i,g}[\sum_{j \in J} \lambda_j g_{U_i,j} + \lambda g_{U_i}] = \sum_{j \in J} \lambda_j P_j + \lambda(P^* - \varepsilon/2),$$

where λ_j, λ are arbitrary real numbers. This functional is strictly positive, i.e.

$$\sum_{j \in J} \lambda_j g_{U_i,j} + \lambda g_{U_i} > 0 \Longrightarrow \sum_{j \in J} \lambda_j P_j + \lambda(P^* - \varepsilon/2) > 0.$$

When $\lambda \geq 0$, the strict positivity of the right-hand side is deduced by taking the expectation of the left-hand side under some arbitrary equivalent risk-neutral probability measure and using the definitions of P^* and ε (which imply : $P^* - \varepsilon/2 > U_i(0) \int g_{U_i} f_{U_i} d\mu$).

When $\lambda < 0$, the right-hand side can be written as :

$$\sum_{j \in J} -\frac{\lambda_j}{\lambda} g_{U_i,j} - g_{U_i} > 0.$$

By definition of P^*, we get :

$$\sum_{j \in J} -\frac{\lambda_j}{\lambda} P_j \geq P^* > P^* - \varepsilon/2.$$

Multiplying by $-\lambda$ provides the right-hand side inequality.

Since $\pi^*_{U_i,g}$ is strictly positive, it can be extended on $L^p(\mu)$ and represented by an equivalent risk-neutral probability measure $\tilde{f}_{U_i} d\mu$ (say). We then have :

$$U_i(0) \int \tilde{f}_{U_i} g_{U_i} d\mu = P^* - \varepsilon/2 > \sup_{f_{U_i} \in \mathcal{F}_{U_i}^{q,e}} U_i(0) \int g_{U_i} f_{U_i} d\mu,$$

which leads to a contradiction.

Proof of Property 12

Let us first consider an arbitrary payoff $\sum \lambda_j g_{U_i,j}$ in the static investment opportunity set. Due to the price compatibility constraints, we get that for any $f_{U_i} \in \mathcal{F}_{U_i}^{q,e}$, $U_i(0) \int g_{U_i} f_{U_i} d\mu$ is equal to $\sum \lambda_j P_j$. Thus the maximum is attained for every equivalent risk-neutral probability measure.

Conversely, let us take a payoff g_{U_i} that does not belong to the investment opportunity set and consider the minimum price surreplicating portfolio $\sum \lambda_j^* g_{U_i,j}$. If we assume the existence $f_{U_i}^* d\mu$ maximizing the expected value of the payoff, we get (from the absence of duality gap) :

$$\int \left(\sum \lambda_j^* g_{U_i,j} - g_{U_i} \right) f_{U_i}^* d\mu = 0.$$

Noticing that the integrand is non negative and that $f_{U_i}^* d\mu$ is equivalent to μ, we deduce that the integrand must be equal to zero μ a.s.