Optimal real consumption and asset allocation for a HARA investor with labour income

Francesco Menoncin[a,*]

[a]Dipartimento di Scienze Economiche, Università di Brescia, Italy, and IRES, Université catholique de Louvain, Louvain-La-Neuve, Belgium

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Abstract

In this paper we take into account a very general setting with: (i) a set of stochastic investment opportunities, (ii) a set of risky assets, (iii) a riskless asset paying a stochastic interest rate, (iv) a stochastic inflation risk, (v) stochastic labor income, and (vi) HARA preferences. We compute a quasi-explicit solution for both the optimal consumption and asset allocation. This solution is based on two changes in the probability measure. We also show that the investor behaves as if he could rely on his wealth augmented by the expected value of all his "forward real labor incomes".

JEL classification: G11, C61.

Key words: Asset allocation; Inflation risk; Stochastic labour income; Stochastic investment opportunities; Feynman-Kač theorem.

*Dipartimento di Scienze Economiche, Università di Brescia, Via S. Faustino, 74/B, 25122 - Brescia - Italy. Tel: 0039-030-2988806; fax: 0039-030-2988837; e-mail: menoncin@eco.unibs.it; and IRES, Université catholique de Louvain. We are grateful for the financial support from the Belgian French Community's program "Action de Recherches Concertée' 99/04-235".

1 Introduction

This paper deals with the problem of finding the optimal consumption and asset allocation for an investor having a fixed financial horizon and HARA (Hyperbolic Absolute Risk Aversion) preferences. The market structure we study is very general since we do not specify any functional form for the drift and diffusion coefficients of the stochastic variables that enter the model. In particular, we take into account: (i) a set of stochastic investment opportunities, (ii) a set of risky assets, (iii) a riskless asset paying a stochastic interest rate. Furthermore, the investor must face an inflation risk and he is endowed with a stochastic labour income (or expenses). We will refer to this kind fo risk outside the financial market also as back ground risk.

A huge literature about asset allocation has been developed in different frameworks. The main characteristics of some papers that present a closed form (or quasi-explicit) solution are summarized in Table 1. These papers are classified by taking into account their preference specification (Utility), the number of state variables (S. V.) in the framework and whether or not they deal with consumption (Cons.), background risk (B. R.), market incompletness (Inc.), and inflation (Inf.).

Table 1: Papers presenting a closed form solution

Authors	**Utility**	**S.V.**	**Cons.**	**B.R.**	**Inc.**	**Inf.**
Bodie *et al.* (1992)	CRRA	1	no	yes	no	no
Kim and Omberg (1996)	HARA	1	no	no	yes	no
Wachter (1998)	CRRA	1	yes	no	no	no
Chacko and Viceira (1999)	CRRA	1	yes	no	yes	no
Deelstra *et al.* (2000)	CRRA	1	no	yes	no	no
Boulier *et al.* (2001)	CRRA	1	no	yes	no	no
Lioui and Poncet (2001)	CRRA	s	no	yes	no	no
Brennan and Xia (2002)	CRRA	2	yes	no	yes	yes
Menoncin (2002)	CARA	s	no	yes	no	yes
This paper	HARA	s	yes	yes	no	yes

After the seminal paper of Cox *et al.* (1985), the interest towards the inflation risk has risen only in recent period (see Brennan and Xia, 2002, and Menoncin, 2002). Nevertheless, when a long (and even medium) period of time is considered, the inflation risk and its hedging cannot be neglected. In particular, Brennan and Xia (2002) use the same framework as in Cox *et al.* (1985), while Menoncin (2002) does not specify any particular functional form for the drift and diffusion term of the consumption price process. Here, we use the same framework after Menoncin (2002) but we generalize to the case of consumption and HARA preferences.

Another brench of the literature has developped some existence and uniqueness result for optimal consumption and investment, without supplying any closed form sultion. In a very general setting, El Karoui and Jeanblanc-Picqué

(1998) analyse the case of a constrained investor who cannot borrow against the future and whose wealth cannot therefore be negative. They show that the optimal constrained solution consists in investing a part of the wealth in the unconstrained strategy and spending the remainder for financing an American Put written on the free wealth. This option provides an insurance against the constraint. Their analysis is carried out with a wider class of utility function than the HARA class.

Also Cuoco (1997) offers an existence result for the optimal portfolio for a constrained investor who is endowed with a stochastic labour income flow. The type of constraint he analyzes is sufficiently general for describing the case: (i) of nontradeable assets (i.e. incomplete markets), (ii) of short-sale constraints, (iii) of buying constraints, (iv) of portfolio-mix constraints, and (v) of minimal capital requirements. The last three constraints are relevant for banks and other financial institutions whose portfolios are affected by regulation of an authority (like a central bank).

Nevertheless, the general frameworks cited above are not able to supply the reader with an easy rule to implement for creating an optimal portfolio. In fact, a closed form solution is computed only in very particular cases. In this work, instead, we present a quasi-explicit solution for a quite general setting. Thus, we bridge the gap between the very theoretical framework where existence and uniqueness results are obtained and the very particular cases where an exact asset allocation is derived.

In this work we deal with an unconstrained problem (i.e., we do not check for the positivity of investor's consumption and wealth) because (as stated in Merton, 1990, Chapter 6.1) the positivity of optimal consumption is guaranteed by the use of a (strictly) HARA utility function. In fact, a HARA function with strictly positive parameters and with a subsistence level of consumption (wealth), exhibits a marginal utility tending to infinity for a given positive amount of consumption (wealth). This means that the optimal consumption (wealth) can never reach this value and will always stay above the subsistance level. If this was not the case, the investor would have an infinite increase in his utility by marginally increasing the consumption (wealth) level.

Bodie *et al.* (1992) offer an interesting analysis of the asset allocation problem when a labour income is present (but consumption is not). Nevertheless, they deal with a nominal market, where there is no more than one asset following a geometric Brownian motion. Their main result states that the investor who has a labour income behaves as if he could rely, at each instant, on the expected present value of all his future income flows. In this work we confirm this kind of result with the suitable modifications due to the presence of the inflation risk. In particular, we show that the consumer behaves as if he owned not only his present wealth but also the present expected value of what we will call "forward real labour income" (which is a measure of the real labour income).

A similar analysis to those presented in this paper is carried out by Lioui and Poncet (2001) but in the pure investing case. They take into account the problem of an investor who is endowed with a portfolio of discount bonds that he chooses (is obliged) not to trade until a deterministic time horizon. In our work, instead, the investor is supposed to be endowed with a non-financial income flow. We will present what this difference implies with respect to the optimal asset allocation, and we will also show that the qualitative solution after Lioui and Poncet is maintained. In fact, the two models are very similar since they are both interested in characterizing a very general solution where the involved random variables follow general stochastic processes. We maintain the completeness hypothesis made by Lioui and Poncet, but we generalize to consumption. Furthermore, the investor's preferences belong to the HARA family while these authors restrict the analysis to the CRRA case.

In this work we are able to find a quasi-explicit solution for both the optimal consumption and asset allocation. Furthermore, we show that the computation of the optimal solutions can be done through two suitable changes in probability. Thus, the usual result according to which the optimal portfolio can be reached simply thanks to the use of the risk neutral probability does not apply. For HARA investors, in fact, two new probability measures must be used. One "real " risk neutral probability making the asset prices behave as martingales when discounted by the consumption price process, and a subjective probability depending on investor's preferences.

The paper is organised as follows. Section 2 shows the structure of the problem we are dealing with. In particular we present: (i) the market structure, (ii) the stochastic labour income, (iii) the consumption price process, (iv) a first change in the probability measure, (v) the behaviour of the investor's wealth under the self-financing condition, (vi) the definition of the "real forward labor income", and (vii) the investor's preferences. In Section 3 the solution for the optimal consumption and asset allocation is computed. Furthermore, we analyse: (i) the role of the labor income, (ii) the portfolio hedging component based on the so-called elasticity approach by presenting a second change in the probability measure, (iii) the behaviour of the log-consumer, and (iv) the difference between the portfolio maximizing the utility of terminal real wealth and the portfolio maximizing the utility of both consumption and terminal real wealth. Section 4 concludes while some technical computations about probability changes are left to the appendix.

2 The economy

In this paper we take into account a very general framework where the asset prices depend on a set of s stochastic investment opportunities following the differential equation

$$\underset{s\times 1}{dX} = \underset{s\times 1}{f}(X,t)dt + \underset{s\times k}{g}(X,t)'\underset{k\times 1}{dW}, \quad X(t_0) = X_0, \tag{1}$$

where dW is the differential of a $k-$dimensional Wiener process. The drift and diffusion terms $f(X,t)$ and $g(X,t)$ are supposed to satisfy the usual Lipschitzian conditions guaranteeing that Equation (1) has a unique strong solution (see

Karatzas and Shreve, 1991). Furthermore, f and g are $\mathcal{F}_t$ measurable, where $\mathcal{F}_t$ is the $\sigma-$algebra through which the Wiener processes are measured on the complete probability space $(\Theta, \mathcal{F}, \mathbb{P})$. All the processes that will be introduced in what follows will be supposed to satisfy the same properties as those here stated for Equation (1). Hereafter, the prime denotes transposition. The values of all the state variables are known in t_0 and so X_0 is a deterministic vector of real variables.

On the financial market there are n risky assets and one riskless asset whose prices follow the differential equations

$$\underset{n\times 1}{dS} = \underset{n\times 1}{\mu (S, X, t)}dt + \underset{n\times k}{\Sigma (S, X, t)'}\underset{k\times 1}{dW}, \quad S(t_0) = S_0, \tag{2}$$

$$dG = Gr(X,t)\,dt, \quad G(t_0) = G_0, \tag{3}$$

where $r(X,t)$ is the instantaneous riskless interest rate. The values of S and G in t_0 are supposed to be deterministic (positive) variables. The set of risk sources for the risky assets is the same we have used for the state variables (dW). This assumption is not restrictive because of potential handling of various situations via the matrices g and Σ.

The set of risky assets S may contain stocks, bonds, and also derivatives. Thus, our model is able to describe the more particular structure generally taken into account in the literature and containing one stock and one bond.

We recall the main result concerning completeness and arbitrage in this kind of market (for the proof of the following theorem see Øksendal, 2000).

Theorem 1 *A financial market as in (2) and (3) is arbitrage free (complete) if and only if there exists a (unique) $k-$dimensional vector $\xi(t,X)$ such that*

$$\Sigma(t,X)'\,\xi(t,X) = \mu(t,X) - r(t,X)\,S(t,X),$$

and such that

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\|\xi(t,X)\|^2 dt}\right] < \infty.$$

If on the market there are less assets than risk sources ($n < k$), then the market cannot be complete even if it is arbitrage free. In this work we assume that $n \leq k$ and that the rank of matrix Σ is maximum (i.e. it equals n). We underline that if the market is arbitrage free and $n > k$, then $n - k$ assets must be redundant (i.e. linearly dependent). Accordingly, in an arbitrage free market where the redundant assets have been eliminated, only the case $n \leq k$ can arise.

2.1 The stochastic labour income

The investor is endowed with a set of non-financial money flows he cannot buy nor sell in the financial market. These flows can be positive (corresponding to stochastic labour incomes) or negative (corresponding to stochastic expenses).

The cumulated non-financial flows (L) follow the stochastic differential equation

$$\underset{l\times 1}{dL} = \underset{l\times 1}{\mu_L (L, X, t)}dt + \underset{l\times k}{\Lambda (L, X, t)'}\underset{k\times 1}{dW}, \quad L(t_0) = L_0, \tag{4}$$

where we have taken into account a set of l possible income flows for the sake of generality.

Remark 1 *We stress that we indicate with L the cumulated income process. This means that $L(t)$ is the sum of all non-financial flows received (or paid) until time t. Accordingly, the investor's revenue between time t_1 and time t_2 ($t_2 > t_1$) can be represented as $L(t_2) - L(t_1) = \int_{t_1}^{t_2} dL(t)$. Analogously, the instantaneous revenue between time t and time $t + dt$ is given by $dL(t)$.*

The risk sources for L are represented by dW which is the same set of risk we have for both asset prices and state variables. This hypothesis is not restrictive because we can model a lot of different cases by choosing the suitable elements for the matrices Λ, Σ, and g. Nevertheless, we underline that, in this framework, the notion of completeness for the financial market must be clarified. In particular, we want to widen it in order to include also the labour income risk.

Since we consider the same risk set (dW) for both the financial market and the labour income, in order to check the completeness of the market we should disentangle the two sets of risk sources. In particular, Equations (2) and (4) should be written in the following way:

$$\begin{cases} \underset{n\times 1}{dS} = \underset{n\times 1}{\mu} dt + \underset{n\times k_S}{\Sigma'_S}\underset{k_S\times 1}{dW_S}, \\ \underset{l\times 1}{dL} = \underset{l\times 1}{\mu_L} dt + \underset{l\times k_S}{\Lambda'_S}\underset{k_S\times 1}{dW_S} + \underset{l\times(k-k_S)}{\Lambda'_L}\underset{(k-k_S)\times 1}{dW_L}, \end{cases} \tag{5}$$

provided that the background variables can be affected by a risk set (dW_L) which does not affect the asset prices. In this case, when matrix Σ_S is invertible we have a complete market (as stated in Theorem 1), but we also have a set of risks (dW_L) which cannot be hedged through a suitable combination of assets. Blake *et al.* (2000) consider a market having the same structure as in System (5). Thus, they define the risk contained in vector dW_L as a "non-hedgeable" risk.

When we want to write System (5) with the same risk sources for both processes:

$$\underset{k\times 1}{dW} = \begin{bmatrix} dW_S & dW_L \end{bmatrix}',$$

as in Equations (2) and (4), then we must create two new matrices:

$$\underset{n\times k}{\Sigma'} = \begin{bmatrix} \underset{n\times k_S}{\Sigma'_S} & \underset{n\times(k-k_S)}{\mathbf{0}} \end{bmatrix}, \quad \underset{l\times k}{\Lambda'} = \begin{bmatrix} \underset{l\times k_S}{\Lambda'_S} & \underset{l\times(k-k_S)}{\Lambda'_L} \end{bmatrix},$$

where $\mathbf{0}$ is a matrix of zeros. We underline that even if Σ_S is invertible, the new matrix Σ cannot be (unless $\Lambda_L = \mathbf{0}$). This means that even if the financial market is complete, this "property" may not hold if we include in the "market" notion also the labour income.

Accordingly, in our analysis, we will use the usual definition of completeness ($\exists \Sigma^{-1}$) and we will consider a wider concept of "market" for including also the labour income risk. Thus, what Blake *et al.* (2000) define as a "non-hedgeable" risk in a complete market, for us is just an incomplete market, where the "non-hedgeable" component is not disentangled.

Nevertheless, we outline that our analysis does not loose generality because it is always possible to write the matrices g, Σ, and Λ as block-matrices in order to disentangle the hedgeable and the non-hedgeable risks. Finally, as an example, the reader is referred to Battocchio and Menoncin (2002) where an incomplete "wide" market is considered, even if the financial market *stricto sensu* is complete.

We underline that in our framework the stochastic labour income plays the same role as the non-tradeable position in a discount bond does for the investor considered by Lioui and Poncet (2001). So, while in Lioui and Poncet the investor is interested in hedging (by the mean of a future contract) this non-tradeable position, in our model he is interested in hedging the risk linked with his labour income.

2.2 The inflation and the real market

The consumer-investor can also freely buy and sold any quantity of a representative consumption good whose price P behaves according to the following stochastic differential equation:

$$dP = P\mu_\pi (P, X, t)\, dt + P \underset{1\times k}{\sigma_\pi (P, X, t)'} \underset{k\times 1}{dW}, \quad P(t_0) = 1, \tag{6}$$

where P can also be interpreted as the consumption price process. The initial value of P is conventionally put equal to 1 without loss of generality because prices can always be normalized. For the sake of generality we do not specify any particular form for the drift and the diffusion coefficients of this process.

We recall that Cox *et al.* (1985) propose the following stochastic equation for the price level

$$dP = P\pi dt + P\sigma_P \sqrt{\pi} dW_P,$$

where σ_P is a constant and π is the inflation rate which is supposed to behave according to one of the two following differential equations:

$$\begin{aligned} d\pi &= k_1 \pi (\theta_1 - \pi)\, dt + \sigma_1 \pi^{\frac{3}{2}} dW_\pi, \\ d\pi &= k_2 (\theta_2 - \pi)\, dt + \sigma_2 \sqrt{\pi} dW_\pi, \end{aligned}$$

where k_i, θ_i, and σ_i, $i \in \{1, 2\}$ are all positive constant.

Brennan and Xia (2002) use a simpler framework where

$$
\begin{aligned}
dP &= P\pi dt + P\sigma_P dW_P, \\
d\pi &= k(\theta - \pi)\, dt + \sigma_\pi dW\pi.
\end{aligned}
$$

In all these models W_P and W_π are correlated. Nevertheless, we recall that a set of correlated Wiener processes can always be transformed into a set of uncorrelated Wiener precesses by means of the Cholesky matrix. Accordingly, we always deal with uncorrelated risks, without loss of generality.

Our general model is able to account for all these particular specifications. In fact, the drift of the price level is assumed to depend on a set of state variables that may contain also the inflation rate.

In our framework the inflation risk plays the same role as the interest risk in the model after Lioui and Poncet (2001). In fact, it is the only risk which is explicitly disentangled from the process of the state variables X.

Here, the consumption price process is exogenous. This means that we do not care about the price determination process. This would be the subject of an extension of our approach to the case of the general equilibrium, like in Cox *et al.* (1985) where prices are endogenously determined.

We introduce, now, a variable upcoming in the following work: the inverse of the consumption price level. This variable ($m \equiv P^{-1}$) is known as "deflator" and it represents the purchasing power of a nominal monetary unit. Furthermore, if we identify the value of a monetary unit with the number of goods that can be purchased against it, then m can also be interpreted as the "value of money". By Itô's lemma, the variable m follows the stochastic differential equation

$$
dm = -m\left(\mu_\pi - \sigma_\pi' \sigma_\pi\right) dt - m\sigma_\pi' dW, \quad m(t_0) = 1. \tag{7}
$$

Accordingly, the real asset values can be computed from Equations (2) and (3) by applying Itô's differential as follows

$$
\begin{aligned}
d(mS) &= dm \cdot S + m \cdot dS + dm \cdot dS \\
&= m\left(\mu - \Sigma' \sigma_\pi - S\left(\mu_\pi - \sigma_\pi' \sigma_\pi\right)\right) dt + m\left(\Sigma' - S\sigma_\pi'\right) dW, \\
d(mG) &= dm \cdot G + m \cdot dG \\
&= m\left(Gr - G\left(\mu_\pi - \sigma_\pi' \sigma_\pi\right)\right) dt - mG\sigma_\pi' dW.
\end{aligned}
$$

Thus, after defining $\hat{S} \equiv \begin{bmatrix} mS' & mG \end{bmatrix}'$, we can write the real market structure as

$$
\underset{(n+1)\times 1}{d\hat{S}} = \underset{(n+1)\times 1}{M} dt + \underset{(n+1)\times k}{\Gamma'} \underset{k\times 1}{dW}, \tag{8}
$$

where

$$
M \equiv m \begin{bmatrix} \mu - S\mu_\pi + S\sigma_\pi' \sigma_\pi - \Sigma' \sigma_\pi \\ Gr - G\mu_\pi + G\sigma_\pi' \sigma_\pi \end{bmatrix}, \quad \Gamma' \equiv m \begin{bmatrix} \Sigma' - S\sigma_\pi' \\ -G\sigma_\pi' \end{bmatrix}.
$$

As widely explained in Menoncin (2002), whose framework is identical to this one, we underline that in the real market the riskless asset looses its characteristic for becoming like a risky asset. In fact, the ex-riskless asset acquires a diffusion coefficient corresponding to the opposite of the price diffusion term, and the diffusion matrix of the "real market" (Γ) has one more column with respect to the nominal one (Σ). Thus, when the inflation has a positive shock the real value of the riskless asset has a negative shock, and *vice versa.* While in Brennan and Xia (2002) there is no real riskless asset in the market, in our framework the role of the riskless assset is plaid by the consumption good.

Finally, we outline that the matrix M containing the risk premium does not measure the difference between the asset returns and the riskless interest rate as in the usual "nominal" analysis. Instead, in our framework, it contains the difference between the nominal asset return and the inflation drift term. Furthermore, this difference is adjusted for the diffusion terms of assets and inflation.

2.3 A new probability measure

When a nominal financial market is taken into account, one of the most known result is that under the so-called risk neutral probability measure the discounted[1] asset prices are martingales. It is easy to show that this property does not hold for the real financial market we have defined in (8). In fact, the introduction of the inflation risk makes it necessary to use another change in probability measure (and another *numéraire*) in order to secure the discounted asset prices to be martingales. In the real market the consumption price process P is the suitable *numéraire* since all asset prices are divided by it. This means that the discount factor we are implicitly taking into account is the money value m. Accordingly, we must find the corresponding change in probability which guarantees that the discounted asset values (mS and mG) are martingales. In algebraic terms we must find a probability measure $\mathbb{Q}$ such that

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\hat{S}\left(t\right)\right]=\hat{S}\left(t_0\right),$$

or, equivalently, we want to check for the existence of $\mathbb{Q}$ such that[2]

$$d\hat{S}=\Gamma' dW^{\mathbb{Q}}.$$

The Girsanov Theorem[3] states that this probability exists if there exists a vector ξ satisfying the equality[4]

$$\Gamma'\xi = M.$$

[1] The discount factor is G^{-1}. This means that the riskless asset is taken as the *numéraire* of the economy.

[2] We recall that a diffusion process having zero drift is a stochastic integral and thus, a martingale.

[3] For a complete exposition of the Girsanov Theorem the reader is referred to Duffie (1996), Björk (1998), and Øksendal (2000).

[4] Actually, it is also necessary that the following condition holds:

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\xi'\xi dt}\right]<\infty.$$

It is evident that, in a market where all the redundant assets have been eliminated, the new probability measure exists and it is unique if and only if the matrix Γ is invertible. We underline that this property implies the completeness of the real financial market, as stated in Theorem 1. Thus, in what follows we will always suppose the following assumption holds.

Assumption 1 *The market is complete for $n+1$ risky assets (i.e. $\exists \Gamma^{-1}$).*

Furthermore, the vector ξ has a suitable economic interpretation. Actually, it is given by the ratio between the real risk premium[5] and the volatility of the real asset values. Accordingly, it measures the real market price of risk.

Definition 1 *Given the real market structure in Equation (8), the "real market price of risk" is*

$$\xi = \Gamma'^{-1} M.$$

Now, by applying the Girsanov Theorem, we can define the new probability measure as follows.

Definition 2 *Given the market structure (9) and the historical probability $\mathbb{P}$, a "real risk neutral probability" $\mathbb{Q}$ satisfies*

$$d\mathbb{Q} = \exp\left(-\int_{t_0}^{H} M'\Gamma^{-1} dW_t - \frac{1}{2}\int_{t_0}^{H} \left\|\Gamma'^{-1}M\right\|^2 dt\right) d\mathbb{P},$$

if

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{H}\left\|\Gamma'^{-1}M\right\|^2 dt}\right] < \infty.$$

Then

$$dW^{\mathbb{Q}} = \Gamma'^{-1} M dt + dW,$$

is a Wiener process with respect to $\mathbb{Q}$.

The new probability $\mathbb{Q}$ transforms the dynamic equations of nominal asset values and money value according to the following proposition.

[5] We have already argued above that the drift term M can be interpreted as the real risk premium since it contains the difference between the asset nominal returns and the inflation rate.

Proposition 1 *Given the nominal asset prices in (2) and the money value in (7), under the "real risk neutral probability"* $\mathbb{Q}$ *(as in Definition 2), the nominal asset values follow the stochastic process*

$$dS = (Sr + \Sigma' \sigma_\pi)\, dt + \Sigma' dW^{\mathbb{Q}},$$

and the money value follows the stochastic process

$$dm = -mrdt - m\sigma_\pi' dW^{\mathbb{Q}}.$$

Proof. See Appendix A. ■

Thus, under the real risk neutral probability, the money value (m) turns out to be just a stochastic discount factor. In fact, its drift coincides with the (opposite of the) riskless interest rate.

2.4 The investor's wealth

After what we have presented in the previous subsections, the market structure can be summarized as follows:

$$\begin{cases} \underset{s\times 1}{dX} = \underset{s\times 1}{f}(X,t)dt + \underset{s\times k}{g}(X,t)' \underset{k\times 1}{dW}, \\ \underset{n\times 1}{dS} = \underset{n\times 1}{\mu}(S,X,t)dt + \underset{n\times k}{\Sigma}(S,X,t)' \underset{k\times 1}{dW}, \\ dG = r(X,t)\, Gdt, \\ dP = P\mu_\pi(P,X,t)\, dt + P\underset{1\times k}{\sigma_\pi}(P,X,t)' \underset{k\times 1}{dW}, \\ \underset{l\times 1}{dL} = \underset{l\times 1}{\mu_L}(L,X,t)dt + \underset{l\times k}{\Lambda}(L,X,t)' \underset{k\times 1}{dW}, \end{cases} \tag{9}$$

where we stress that the variables contained in L are potentially different for each investor while the variables contained in X, S, and the value of G are common for all economic agents.

If we indicate with $\theta(t) \in \mathbb{R}^{n\times 1}$ and $\theta_G(t) \in \mathbb{R}$ the number of risky assets held and the quantity of riskless asset held respectively, then the nominal wealth R_N, at each time t, can be written as

$$R_N(t) = \theta(t)' S + \theta_G(t)\, G + \theta_P(t)\, P, \tag{10}$$

where θ_P is the quantity of the representative consumption good held in the portfolio (see, for the same approach, Damgaard *et al.*, 2003).

The Itô differential of (10) is[6]

$$dR_N = \theta' dS + \theta_G dG + d\theta' (S + dS) + d\theta_G \cdot G + d\theta_P (P + dP).$$

[6] We recall that θ, θ_G, and S are stochastic variables, while G is a deterministic function. Accordingly, the term $d\theta_G \cdot dG$ disappears.

Now, in order to have a self-financing portfolio, the changes in the portfolio composition ($d\theta$, $d\theta_G$, and $d\theta_P$) must finance the nominal consumption (C) during the period dt and must be financed by the instantaneous stochastic revenue (dL). Accordingly, the self-financing condition can be written as

$$d\theta' (S + dS) + d\theta_G \cdot G + d\theta_P (P + dP) = u'dL - Cdt,$$

where $u \in \mathbb{R}^{l \times 1}$ is a parameter vector indicating in which proportion the elements of vector L affect the wealth level. For instance, we could suppose that an investor must cope with both a labour income L_1 and a flow of uncertain expenses L_2. Then, in this case, u is a two dimensional vector whose elements are 1 and -1 because the income increases the nominal wealth while the expenses decrease it.

Finally, the evolution of the investor's nominal wealth is

$$dR_N = \theta' dS + \theta_G dG + \theta_P P + u'dL - Cdt,$$

and after substituting the differentials from System (9), we can write the dynamic budget constraint as

$$dR_N = \left(\theta'\mu + \theta_G Gr + \theta_P P\mu_\pi + u'\mu_L - C\right) dt + \left(\theta'\Sigma' + \theta_P P\sigma'_\pi + u'\Lambda'\right) dW.$$

Now, the investor's goal is supposed to be the maximization of the expected utility of his real consumption and real wealth. The real wealth R (real consumption c) is defined as the ratio between the nominal wealth R_N (nominal consumption C) and the price level (or, alternatively, the product between the nominal wealth or consumption and the money value). Accordingly, in order to find the dynamic behaviour of the investor's real wealth, we have to differentiate the following formula:

$$R = \frac{R_N}{P} = m \cdot R_N.$$

By applying Itô differentiation we obtain

$$\begin{aligned} dR &= dm \cdot R_N + m \cdot dR_N + dm \cdot dR_N \\ &= m\left(\theta'\mu + \theta_G Gr + \theta_P P\mu_\pi + u'\mu_L - C - R_N\left(\mu_\pi - \sigma'_\pi\sigma_\pi\right)\right) dt \\ &\quad -m\left(\theta'\Sigma'\sigma_\pi + \theta_P P\sigma'_\pi\sigma_\pi + u'\Lambda'\sigma_\pi\right) dt \\ &\quad +m\left(-R_N\sigma'_\pi + \theta'\Sigma' + \theta_P P\sigma'_\pi + u'\Lambda'\right) dW \end{aligned}$$

and, after substituting for the value of θ_P given in Equation (10):

$$dR = (w'M + k - c)\, dt + (w'\Gamma' + K')\, dW, \tag{11}$$

where M and Γ are defined as in (8) while

$$\underset{(n+1)\times 1}{w} \equiv \left[\begin{array}{cc} \theta' & \theta_G \end{array}\right]', \quad k \equiv mu'\left(\mu_L - \Lambda'\sigma_\pi\right), \quad \underset{k\times 1}{K} \equiv m\Lambda u,$$

and c is the real consumption rate given by C/P.

The solution to the optimisation problem for the consumer-investor will show the optimal value of vector w while the optimal quantity of the consumption good held (θ_P) will be determined through the budget constraint (10).

2.5 The forward real labor income

We underline that the diffusion process $kdt + K'dW$ appearing in (11) does not correspond to $d(mL)$. In order to have this correspondence, the self-financing condition should have been defined on the real market. Nevertheless, this would not be correct. In fact, only the nominal wealth can be actually invested since the real wealth is just a fictitious measure. Accordingly, it seems useful to better describe the role of matrix K and scalar k. The differential of the cumulated real income is given by

$$d(m \cdot u'L) = dm \cdot u'L + m \cdot u'dL + dm \cdot u'dL,$$

which is

$$\begin{aligned} d(m \cdot u'L) &= mu'(\mu_L - \Lambda'\sigma_\pi)\,dt + mu'\Lambda'dW \\ &\quad -mu'L(\mu_\pi - \sigma'_\pi\sigma_\pi)\,dt - mu'L\sigma'_\pi dW. \end{aligned}$$

By using the notation K and k this differential can be written as

$$d(m \cdot u'L) = kdt + K'dW + u'L \cdot dm,$$

and finally, we can observe that the process $kdt + K'dW$ in the real wealth corresponds to[7]

$$kdt + K'dW = d(m \cdot u'L) - u'L \cdot dm,$$

which is the real revenue received between t and $t+dt$ diminished by the change in the money value on the total cumulated revenue.

This is consistent with what we have argued above. Actually, since the self-financing condition can be valid only for the nominal market, then we must take into account the changes in the real revenues $d(m \cdot u'L)$ corrected by the changes in the money value. In fact, these last changes must not be taken into account when computing the self-financing condition.

Furthermore, since the equality

$$d(m \cdot u'L) - u'L \cdot dm = u'dL\,(m + dm)\,,$$

holds and $u'dL$ is the instantaneous labour income while $m + dm$ is the forward money value,[8] then we are able to define the "forward real labour income" (Y) which will be useful in the following sections.

[7] Of course, we recall that also

$$kdt + K'dW = u'dL\,(m + dm)\,,$$

is valid. Nevertheless, we prefer to use the other notation for avoiding the product of two differentials ($dL \cdot dm$) in what follows.

[8] We recall that the Itô differential is expressed as a forward difference. Thus, the term Δm (finite difference) can be written as

$$\Delta m = m(t + \Delta t) - m(t)\,,$$

and accordingly, the term $m + \Delta m$ is

$$m + \Delta m = m(t + \Delta t)\,,$$

which is the forward money value.

Definition 3 *Given the stochastic labour income (4) and the money value (7), the "forward real labour income" (Y) satisfies*

$$\begin{aligned} dY &= mu'\left(\mu_L - \Lambda'\sigma_\pi\right)dt + mu'\Lambda' dW \\ &= d\left(m \cdot u'L\right) - u'L \cdot dm \\ &= kdt + K'dW. \end{aligned} \tag{12}$$

When the money value m has a deterministic behaviour, the differential of the real cumulated labour is

$$d\left(m \cdot u'L\right) = u'L \cdot dm + m \cdot u'dL,$$

and consequently, from (12) we can write

$$dY = m \cdot u'dL,$$

which is just the real value of the differential of L (since there is no risk inside the money value process).

2.6 The consumer's preferences

The most widely used utility function belongs to the CRRA family (see, for instance the papers listed in Table 1). In Menoncin (2002) a set of results is derived for an investor whose preferences are described by a CARA (Constant Absolute Risk Aversion) utility function. Instead, in this work, even if we maintain the same market structure as in Menoncin (2002), we generalize his results by taking into account a HARA (Hyperbolic Absolute Risk Aversion) utility function. In particular, we suppose that the real consumption c gives a consumer the following utility

$$U\left(c,t\right) = \delta\left(t\right)\left(\gamma c - \alpha\right)^{1-\frac{\beta}{\gamma}},$$

where $\delta\left(t\right)$ is a discount factor and whose Arrow-Pratt absolute risk aversion index ($\mathcal{R}$) is an hyperbolic function of c:

$$\mathcal{R} \equiv -\frac{\partial^2 U}{\partial c^2}\left(\frac{\partial U}{\partial c}\right)^{-1} = \frac{\beta}{\gamma c - \alpha}. \tag{13}$$

Furthermore, for having a well defined maximization problem, we need the utility function to be increasing and concave in its argument. These conditions imply the following restrictions on the preference parameters:

$$\begin{aligned} \frac{\partial U}{\partial c} &> 0 \Longrightarrow \delta\left(t\right)\left(\gamma - \beta\right) > 0, \\ \frac{\partial^2 U}{\partial c^2} &< 0 \Longrightarrow -\beta\delta\left(t\right)\left(\gamma - \beta\right) < 0, \end{aligned}$$

from which we can immediately obtain

$$\beta > 0, \quad \delta(t)(\gamma - \beta) > 0. \tag{14}$$

The parameter α plays a crucial role for what concerns the relevance of the non-negativity constraint on consumption. In fact, if α is non negative then it can be interpreted as a measure of the subsistence consumption level (as exposed in Karatzas and Shreve, 1998). This result can be easily shown after computing the marginal utility of consumption

$$\frac{\partial U}{\partial c} = \delta(t)(\gamma - \beta)(\gamma c - \alpha)^{-\frac{\beta}{\gamma}}.$$

When $\beta/\gamma > 0$ there exists a level of consumption (equal to α/γ) giving an infinite marginal utility. Consequently, if the consumption reached the value α/γ the consumer would have an infinite increase in his utility by increasing his consumption (even by a very little amount). This means that the optimal consumption will never reach the value α/γ which can be interpreted as the subsistence consumption level. Accordingly, during our work we will neglect the non-negativity constraint on consumption since we suppose $\alpha/\gamma \geq 0$.

The HARA utility function can be thought of as the "mother" of all the utility functions commonly used in the economic literature. In fact, we can distinguish the following sub-cases:

1. when $\alpha = 0$ and $\gamma = 1$ we have the CRRA (Constant Relative Risk Aversion) utility function in the form $U(R) = \delta(t) c^{1-\beta}$; in this case the subsistence consumption level equals zero;

2. when $\alpha = -1$ and γ tends to zero we have the CARA (Constant Absolute Risk Aversion) utility function in the form $U(R) = \delta(t) e^{-\beta c}$;[9] in this case there does not exist any finite and non-negative level of consumption giving an infinite marginal utility; thus, in the case of CARA preferences the non-negativity constraint on consumption should be explicitly imposed;

3. when $\alpha = 0$, $\gamma = 1$, $\delta = (1-\beta)^{-1}$ and β tends to 1 we have the same results as for the log utility function;[10]

4. when $1 - \frac{\beta}{\gamma} = 2$ we have the quadratic utility function.

[9] In this case, according to Conditions (14), the function $\delta(t)$ must be negative for all t.

[10] The result of the optimization problem does not change if we add to the objective function another function independent of the control variables. In our case (with $\alpha = 0$, $\gamma = 1$, $\delta(t) \equiv \delta_1(t)(1-\beta)^{-1}$) we add $-\delta_1(t)(1-\beta)^{-1}$ and the objective function becomes

$$U(c,t) = \delta_1(t) \frac{1}{1-\beta} c^{1-\beta} - \delta_1(t) \frac{1}{1-\beta}.$$

The limit of this utility, for β tending to 1, is the log utility:

$$\lim_{\beta \to 1} \delta_1(t) \frac{c^{1-\beta} - 1}{1-\beta} = \delta_1(t) \ln c.$$

Accordingly, any result stated in the case of a HARA utility function remains valid for a wide range of preferences. We underline that very few works deal with the general HARA family for the investor's preferences and are also able to find a closed form solution for the optimal asset allocation. At least to our knowledge Kim and Omberg (1996) are the only authors who do this.

3 The optimal portfolio and consumption

After defining the model structure in the previous section, here we compute the optimal portfolio and we check whether the results presented in Bodie *et al.* (1992) for a particular case of our model, are valid also in our more general framework. We recall that these authors show that an investor behaves as if his wealth were augmented by the present value of his future income flows.

The maximization problem can be written as

$$\begin{cases} \max_{w,c} \mathbb{E}_{t_0} \left[\int_{t_0}^{H} \chi \delta(t) \left(\gamma c(t) - \alpha \right)^{1-\frac{\beta}{\gamma}} dt + \delta(H) \left(\gamma R(H) - \alpha \right)^{1-\frac{\beta}{\gamma}} \right] \\ \begin{bmatrix} dz \\ dR \end{bmatrix} = \begin{bmatrix} \mu_z \\ w'M + k - \chi c \end{bmatrix} dt + \begin{bmatrix} \Omega' \\ w'\Gamma' + K' \end{bmatrix} dW, \\ R(t_0) = R_0, \quad z(t_0) = z_0, \quad \forall t_0 < t < H \end{cases} \tag{15}$$

assuming Conditions (14) hold. H is the time horizon of the investor and

$$\begin{aligned} \underset{(s+n+1+l)\times 1}{z} &\equiv \begin{bmatrix} X' & S' & G & L' \end{bmatrix}', \\ \underset{(s+n+1+l)\times 1}{\mu_z} &\equiv \begin{bmatrix} f' & \mu' & Gr & \mu_L' \end{bmatrix}', \\ \underset{k\times(s+n+1+l)}{\Omega} &\equiv \begin{bmatrix} g & \Sigma & \mathbf{0} & \Lambda \end{bmatrix}. \end{aligned}$$

The variable χ takes value in $\{0, 1\}$ and it is zero when the consumption problem is not taken into account. The Hamiltonian of Problem (15) is

$$\begin{aligned} \mathcal{H} = {} & \chi \delta(t) \left(\gamma c(t) - \alpha \right)^{1-\frac{\beta}{\gamma}} \\ & + \mu_z' J_z + J_R \left(w'M + k - \chi c \right) + \frac{1}{2} \operatorname{tr} \left(\Omega' \Omega J_{zz} \right) \\ & + \left(w'\Gamma' + K' \right) \Omega J_{zR} + \frac{1}{2} J_{RR} \left(w'\Gamma' + K' \right) \left(\Gamma w + K \right), \end{aligned} \tag{16}$$

where $J(R, z, t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation and the subscripts on J indicate the partial derivatives.

The first order condition on c is[11]

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial c} &= \delta(t)(\gamma-\beta)(\gamma c(t)-\alpha)^{-\frac{\beta}{\gamma}} - J_R = 0 \\ \Rightarrow & \quad c^* = \frac{1}{\gamma}\left(\frac{1}{\delta(t)(\gamma-\beta)}J_R\right)^{-\frac{\gamma}{\beta}} + \frac{\alpha}{\gamma},\end{aligned} \tag{17}$$

while the first order condition on w is

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial w} &= J_R M + \Gamma'\Omega J_{zR} + J_{RR}\Gamma'\Gamma w + J_{RR}\Gamma' K = 0 \\ \Rightarrow & \quad w^* = -\Gamma^{-1}K - \frac{J_R}{J_{RR}}(\Gamma'\Gamma)^{-1}M - \frac{1}{J_{RR}}\Gamma^{-1}\Omega J_{zR}.\end{aligned} \tag{18}$$

After substituting w^* and c^* into the Hamiltonian we can write the Hamilton-Jacobi-Bellman (hereafter HJB) partial differential equation as

$$\begin{aligned}0 &= J_t + \chi\delta^{\frac{\gamma}{\beta}}\frac{\beta}{\gamma}(\gamma-\beta)^{\frac{\gamma}{\beta}-1}J_R^{1-\frac{\gamma}{\beta}} \\ &\quad + \mu_z' J_z + J_R\left(k - K'\Gamma'^{-1}M - \chi\frac{\alpha}{\gamma}\right) - \frac{1}{2}\frac{J_R^2}{J_{RR}}M'(\Gamma'\Gamma)^{-1}M \\ &\quad - \frac{J_R}{J_{RR}}M'\Gamma^{-1}\Omega J_{zR} + \frac{1}{2}\operatorname{tr}(\Omega'\Omega J_{zz}) - \frac{1}{2}\frac{1}{J_{RR}}J_{zR}'\Omega'\Omega J_{zR}.\end{aligned}$$

Now, since the value function often inherits its functional form from the objective function, we try the following guess-function:

$$J(z,R,t) = F(z,t)(V(t,z)+\gamma R)^{1-\frac{\beta}{\gamma}}, \tag{19}$$

where $F(z,t)$ and $V(z,t)$ are two functions that must be determined. The boundary conditions on F and V are

$$\begin{aligned}F(z,H) &= \delta(H), \\ V(z,H) &= -\alpha.\end{aligned}$$

When the guess-function is substituted into the HJB equation, it is easy to check that there exist only two kinds of terms containing the wealth level R: $(V+\gamma R)^{1-\frac{\beta}{\gamma}}$ and $(V+\gamma R)^{-\frac{\beta}{\gamma}}$. Accordingly, since the HJB must equate zero for each value of R, it can be split into two partial differential equations (one for each kind of term containing R). Thus, the functions $F(z,t)$ and $V(z,t)$ must solve

$$\left\{\begin{aligned}&0 = V_t + \left(\mu_z' - M'\Gamma^{-1}\Omega\right)V_z + \left(\gamma k - \gamma M'\Gamma^{-1}K - \chi\alpha\right) + \tfrac{1}{2}\operatorname{tr}(\Omega'\Omega V_{zz}), \\ &0 = F_t + \left(\mu_z' + \tfrac{\gamma-\beta}{\beta}M'\Gamma^{-1}\Omega\right)F_z + \tfrac{1}{2}\tfrac{\gamma-\beta}{\beta}FM'(\Gamma'\Gamma)^{-1}M + \tfrac{1}{2}\operatorname{tr}(\Omega'\Omega F_{zz}) \\ &\qquad + \chi\delta^{\frac{\gamma}{\beta}}\tfrac{\beta}{\gamma}F^{1-\frac{\gamma}{\beta}} + \tfrac{1}{2}\tfrac{\gamma-\beta}{\beta}\tfrac{1}{F}F_z'\Omega'\Omega F_z.\end{aligned}\right. \tag{20}$$

[11] The derivative of $\mathcal{H}$ with respect to c has been divided by χ since it has value 1 when consumption is taken into account. In fact, when $\chi = 0$ the consumption problem does not arise at all.

The solution to the first partial differential equation in System (20) gives the functional form of $V(z,t)$ while the solution to the second equation gives the functional form of $F(z,t)$. Once these two functions have been found, we can plug the guess function (19) into Equations (17) and (18) for writing

$$\begin{aligned} c^* &= \frac{1}{\gamma}\delta(t)^{\frac{\gamma}{\beta}} F(z,t)^{-\frac{\gamma}{\beta}} (V(z,t)+\gamma R) - \frac{\alpha}{\gamma}, & (21) \\ w^* &= -\Gamma^{-1}K + \frac{1}{\beta}(V(z,t)+\gamma R)(\Gamma'\Gamma)^{-1} M - \frac{1}{\gamma}\Gamma^{-1}\Omega\frac{\partial V(z,t)}{\partial z} & (22) \\ &\quad + \frac{1}{\beta}(V(z,t)+\gamma R)\Gamma^{-1}\Omega\frac{1}{F(z,t)}\frac{\partial F(z,t)}{\partial z}. \end{aligned}$$

This solution confirms the so called "elasticity approach" for the optimal asset allocation. In fact, the term $\frac{1}{F}\frac{\partial F}{\partial z}$ is just the elasticity of function F with respect to the state variables z. In the following subsections we expose the computations leading to the quasi-explicit solutions for the equations in System (20) and we will be more precise about the optimal consumption and asset allocation.

3.1 The role of the forward real labor income

The first equation in System (20) can be solved through the Feynman-Kač representation theorem.[12] This quasi-explicit solution has the following form:

$$V(z,t) = \mathbb{E}_t^{Z_V}\left[\int_t^H \left(\gamma k - \gamma M'\Gamma^{-1}K - \chi\alpha\right) ds - \alpha\right],$$

where the variables Z_V follow the modified stochastic equation[13]

$$dZ_V = \left(\mu_z - \Omega'\Gamma'^{-1}M\right) ds + \Omega' dW, \quad Z_V(t) = z.$$

We can immediately check that the stochastic variables Z_V coincide with the stochastic variables z once one switches from the historical to the real risk neutral probability measure. In fact, after recalling what we have presented in Definition 2, we can write

$$\begin{aligned} dz &= \mu_z ds + \Omega' dW, \\ &= \mu_z ds + \Omega'\left(dW^{\mathbb{Q}} - \Gamma'^{-1}M ds\right), \\ &= \left(\mu_z - \Omega'\Gamma'^{-1}M\right) ds + \Omega' dW^{\mathbb{Q}}. \end{aligned}$$

Consequently, the function $V(z,t)$ can alternatively be written as

$$V(z,t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^H \left(\gamma k - \gamma M'\Gamma^{-1}K - \chi\alpha\right) ds - \alpha\right],$$

[12] For a complete exposition of the Feynman-Kač Theorem the reader is referred to Duffie (1996), Björk (1998) and Øksendal (2000).

[13] We recall that matrices k, K, M, and Γ depend on the state variables z. Nevertheless, all the dependences have been omitted in order to make the presentation less heavy.

without specifying any new stochastic variable. Furthermore, since χ, α, and γ are deterministic parameters, another simplification can be done

$$V(z,t) = \gamma \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^H \left(k - M'\Gamma^{-1}K \right) ds \right] - \chi\alpha (H-t) - \alpha.$$

Now, we recall what we have presented in Definition 3 as the forward real labour income. Under the new probability $\mathbb{Q}$, the forward real labour income Y follows the process

$$dY = \left(k - K'\Gamma'^{-1}M \right) dt + K'dW^{\mathbb{Q}}.$$

Thus, we see that the argument of the expected value in V coincides with the drift term of Y under $\mathbb{Q}$. This means that we can carry out the following simplifications on V:

$$\begin{aligned} V(z,t) &= \gamma \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^H dY \right] - \chi\alpha (H-t) - \alpha \\ &= \gamma \left(\mathbb{E}_t^{\mathbb{Q}} [Y(H)] - Y(t) \right) - \chi\alpha (H-t) - \alpha. \end{aligned} \tag{23}$$

We recall that Y is the cumulated flow of forward labour incomes (as in Definition 3). Thus the difference $Y(H) - Y(t)$ measures all the forward labour incomes received between t and H. This revenue flow must be added to the investor's wealth. More precisely, we can define another measure of wealth as Bodie *et al.* (1992) argue. In fact, the investor behaves *as if* he could borrow against his future income that can be accordingly viewed as an asset. Like any other financial asset it must be priced. For this purpose, it must be discounted with a suitable *numéraire* (in this framework the consumption price process) and evaluated under a risk neutral probability (in this framework the real risk neutral probability $\mathbb{Q}$).

Accordingly, we can define the potential investor's wealth (R_Y) as

$$R_Y \equiv R + \mathbb{E}_t^{\mathbb{Q}} [Y(H) - Y(t)]. \tag{24}$$

3.2 The elasticity component of optimal portfolio

The second equation in System (20) can be solved through the following transformation

$$F(z,t) = h(z,t)^{\frac{\beta}{\gamma}},$$

shown in Zariphopoulou (2001). The suitable boundary condition becomes

$$h(z,H) = \delta(H)^{\frac{\gamma}{\beta}}.$$

Thanks to this transformation the non linear partial differential equation for F becomes a parabolic equation in h as follows:

$$\begin{aligned} 0 &= h_t + \left(\mu_z' + \frac{\gamma-\beta}{\beta} M'\Gamma^{-1}\Omega \right) h_z + \frac{1}{2}\operatorname{tr}\left(\Omega'\Omega h_{zz}\right) \\ &\quad + \chi\delta^{\frac{\gamma}{\beta}} + \frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta} M' \left(\Gamma'\Gamma\right)^{-1} Mh. \end{aligned}$$

The solution of this equation can be represented through the Feynman-Kač theorem in the following way:

$$\begin{aligned} h(z,t) &= \mathbb{E}_t^{Z_F}\left[\int_t^H \chi\delta(s)^{\frac{\gamma}{\beta}} e^{\frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta}\int_t^s M'\left(\Gamma'\Gamma\right)^{-1}Md\tau} ds \right. \\ &\quad \left. + \delta(H)^{\frac{\gamma}{\beta}} e^{\frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta}\int_t^H M'\left(\Gamma'\Gamma\right)^{-1}Md\tau} \right], \end{aligned}$$

where the modified stochastic variables Z_F follow the diffusion process

$$dZ_F = \left(\mu_z + \frac{\gamma-\beta}{\beta}\Omega'\Gamma'^{-1}M \right) ds + \Omega' dW, \quad Z_F(t) = z.$$

Also in this case, as shown in the previous subsection, the variables Z_F can be led back to the original variables z by means of a suitable change in the probability measure. Nevertheless, this time, the new probability measure will depend on the preference parameters γ and β. This means that the new probability measure we are going to present is not the same for all the investors (like $\mathbb{Q}$). For this reason we will call it "subjective probability". After applying Girsanov Theorem as in Definition 2, we can refer to a new probability measure as in the following definition.

Definition 4 *Given the market structure (9) and the historical probability $\mathbb{P}$, a "subjective probability" $\mathbb{Q}_\gamma$ satisfies*

$$d\mathbb{Q}_\gamma = \exp\left(-\frac{\beta-\gamma}{\beta}\int_{t_0}^H M'\Gamma^{-1}dW_t - \frac{1}{2}\left(\frac{\beta-\gamma}{\beta}\right)^2 \int_{t_0}^H \left\| \Gamma'^{-1}M \right\|^2 dt \right) d\mathbb{P},$$

if

$$\mathbb{E}\left[e^{\frac{1}{2}\left(\frac{\beta-\gamma}{\beta}\right)^2 \int_{t_0}^H \left\|\Gamma'^{-1}M\right\|^2 dt} \right] < \infty.$$

Then

$$dW^{\mathbb{Q}_\gamma} = \frac{\beta-\gamma}{\beta}\Gamma'^{-1}Mdt + dW,$$

is a Wiener process with respect to $\mathbb{Q}_\gamma$.

The new probability $\mathbb{Q}_\gamma$ transforms the dynamic equation of asset values according to the following proposition.

Proposition 2 *Given the market structure (9), under the "subjective probability" $\mathbb{Q}_\gamma$, the asset values follow the stochastic process*

$$dS = \left((Sr + \Sigma'\sigma_\pi) + \frac{\gamma}{\beta} (\mu - Sr - \Sigma'\sigma_\pi) \right) dt + \Sigma' dW^{\mathbb{Q}_\gamma}.$$

Proof. The proof is analogous to those presented in Appendix A, where the matrix Γ must be substituted by the matrix $(\beta - \gamma)\beta^{-1}\Gamma$. ■

The risk aversion index computed in (13) can be written as $\mathcal{R} = \left(\frac{\gamma}{\beta}R - \frac{\alpha}{\beta}\right)^{-1}$ and, in this way, it is easy to check that the ratio γ/β measures how strong the reaction of $\mathcal{R}$ is with respect to changes in the real wealth. In particular, for the case of a CARA utility function (when $\alpha = -1$ and γ tends to zero), the index $\mathcal{R}$ does not depend at all on R. So, we can conclude what follows.

Proposition 3 *When the utility function belongs to the CARA family (i.e. $\alpha = -1$ and $\gamma \to 0$), the "subjective probability" $\mathbb{Q}_\gamma$ and the "real risk neutral probability" $\mathbb{Q}$ coincide.*

This proposition explains why Menoncin (2002) meeds only one change in probability even if he considers a set of stochastic labour incomes.

We underline that the "subjective probability" cannot be considered as a "risk neutral probability" since, under it, the asset returns do not coincide with the riskless interest rate (nor with the riskless interest rate corrected by the inflation risk).

Finally, the function $h(z,t)$ can be written as

$$\begin{aligned} h(z,t) &= \mathbb{E}_t^{\mathbb{Q}_\gamma}\left[\int_t^H \chi\delta(s)^{\frac{\gamma}{\beta}} e^{\frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta}\int_t^s M'(\Gamma'\Gamma)^{-1}M d\tau} ds \right. \\ &\quad \left. + \delta(H)^{\frac{\gamma}{\beta}} e^{\frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta}\int_t^H M'(\Gamma'\Gamma)^{-1}M d\tau}\right]. \end{aligned} \tag{25}$$

3.3 A new (subjective) riskless asset

Now, in order to understand better the role of the function $h(z,t)$ we can define a new asset on the financial market. It is a particular kind of riskless asset whose price depends on investor's preference parameters. Let us call the price of this new asset $G_\gamma(z,s)$ and suppose that its instantaneous revenue is given by

$$\frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta}M'(\Gamma'\Gamma)^{-1}M,$$

then we can define this new asset as follows.

Definition 5 *The value of a subjective riskless asset (G_γ) follows the differential equation*

$$\frac{dG_\gamma(z,s)}{G_\gamma(z,s)} = \frac{1}{2}\frac{\gamma}{\beta}\frac{\gamma-\beta}{\beta}M'(\Gamma'\Gamma)^{-1}Mds, \quad G_\gamma(z,t)=1. \tag{26}$$

The return on this particular asset is given by the square of the real market price of risk (or real Sharpe ratio), weighted by a combination of preference parameters. This is a new riskless asset because its value follows a deterministic differential equation even if its revenue is allowed to be stochastic. In fact, we recall that both matrices M and Γ do depend on the stochastic variables z.

The boundary condition in (26) comes from the need to equate the value of G_γ to the value of the exponential that can be found in $h(z,t)$ in 25. Furthermore, it means that G_γ can be thought of as the *numéraire* of the economy.

After defining G_γ the function $h(z,t)$ can be written as

$$h(z,t) = \int_t^H \chi\delta(s)^{\frac{\gamma}{\beta}}\mathbb{E}_t^{\mathbb{Q}_\gamma}[G_\gamma(z,s)]\,ds + \delta(H)^{\frac{\gamma}{\beta}}\mathbb{E}_t^{\mathbb{Q}_\gamma}[G_\gamma(z,H)],$$

and its value can be interpreted as the present value of a coupon bond paying $G_\gamma(z,s)$ at each instant and $G_\gamma(z,H)$ at its expiration date. The discount factor is given by $\delta(s)^{\gamma/\beta}$. It also has a "subjective" component given by a ratio between two preference parameters γ/β.

When the investor's preferences belong to the CARA family (i.e. γ tends to zero) then the subjective riskless asset has a constant value (equal to 1) and the function $h(z,t)$ has the simplified form

$$h(z,t) = \chi(H-t)+1.$$

In fact, the subjective discount rate is 1 and the corresponding coupon bond pays 1 monetary unit each period from t to H and 1 monetary unit at the expiration date.

3.4 A quasi-explicit solution

After substituting the values of V and F into Equations (17) and (18) the optimal consumption and portfolio can be written as in the following proposition.[14]

[14] In the optimal value of consumption the value of χ has been set to 1. In fact, χ is needed for distinguish between the portfolio compositions.

Proposition 4 *Under market structure (9) and Assumption 1, the optimal consumption and portfolio solving Problem (15) are*

$$c^* = \frac{\delta(t)^{\frac{\gamma}{\beta}}}{h(z,t)}\left(R_Y - \frac{\alpha}{\gamma} - \frac{\alpha}{\gamma}(H-t)\right) + \frac{\alpha}{\gamma}, \tag{27}$$

$$\begin{aligned} w^* = & -\Gamma^{-1}K + \frac{\gamma}{\beta}\left(R_Y - \frac{\alpha}{\gamma} - \chi\frac{\alpha}{\gamma}(H-t)\right)(\Gamma'\Gamma)^{-1}M \\ & -\Gamma^{-1}\Omega\frac{\partial}{\partial z}\left(\mathbb{E}_t^{\mathbb{Q}}[Y(H)] - Y(t)\right) \\ & +\left(R_Y - \frac{\alpha}{\gamma} - \chi\frac{\alpha}{\gamma}(H-t)\right)\Gamma^{-1}\Omega\frac{1}{h(z,t)}\frac{\partial h(z,t)}{\partial z}. \end{aligned} \tag{28}$$

where R_Y, h, Y, and $\mathbb{Q}$ are described in (24), (25), and Definitions 3 and 2, respectively.

The result presented in Proposition 4 allows us to argue that the optimal portfolio is formed by four components.

1. $w^*_{(1)} \equiv -\Gamma^{-1}K$. This component (as shown in Menoncin, 2002) minimizes the instantaneous variance of the investor's real wealth. Furthermore, it does not depend on any preference parameter and so it is suitable for every kind of investor-consumer.

2. $w^*_{(2)} \equiv \frac{\gamma}{\beta}\left(R_Y - \frac{\alpha}{\gamma} - \chi\frac{\alpha}{\gamma}(H-t)\right)(\Gamma'\Gamma)^{-1}M$. This is the typical Merton's speculative component where the investor's wealth (R) is increased by the discounted amount of his future incomes as in Bodie *et al.* (1992). Nevertheless, the suitable measure for the future incomes is given by the forward real income as shown in Definition 3. We recall here a very simple, naïve, and practical rule used by some practitioners for prescribing the "optimal" percentage of wealth that must be invested in risky assets. It would be sufficient to compute $70 - t$, where t is the age of the consumer, for having this "optimal" percentage. Such a rule suggests that young people should have riskier portfolios than old ones. Our model, instead, prescribes the opposite rule. We have already defined H as the consumer financial horizon. So, it can also be interpreted as the consumer expected life (for instance $H = 70$). In our framework, the younger the consumer the less risky his portfolio must be. In fact, young consumers still have to finance a long consumption path, while old people can afford to invest in riskier portfolios. The riskiest portfolio is held during the last period before H. In fact, in this case, there is no more consumption to finance.

3. $w^*_{(3)} \equiv -\Gamma^{-1}\Omega\frac{\partial}{\partial z}\left(\mathbb{E}_t^{\mathbb{Q}}[Y(H)] - Y(t)\right)$. This portfolio component hedges the investor's wealth against the fluctuations in his labor income due to the changes in the state variables. This component obviously disappears when no labor income enters the investor's wealth. Furthermore, it does

not depend on any preference parameter. From this point of view $w^*_{(3)}$ is similar to $w^*_{(1)}$. While the first component hedges the consumer portfolio against the risk of his labour income (measured by K), this third component hedges against the reactions of the labour drift to changes in the state variables.

4. $w^*_{(4)} \equiv \left(R_Y - \frac{\alpha}{\gamma} - \chi\frac{\alpha}{\gamma}(H-t)\right)\Gamma^{-1}\Omega\frac{1}{h(z,t)}\frac{\partial}{\partial z}h(z,t)$. This last part belongs to the so-called hedging-portfolio (like the previous one) and it contains the elasticity of the function $h(z,t)$ with respect to changes in the state variables z. In the previous subsection, we have already presented $h(z,t)$ as the value of a bond paying coupons that are proportional to the square of the Sharpe index. Thus, the portfolio component $w^*_{(4)}$ hedges against the changes in this bond value due to the changes in the state variables. When we consider the usual nominal setting, the only state variable is the riskless interest rate and there are some bonds, then the the elasticity of function h with respect to the interest rate r for the optimal bond allocation coincides with the duration.

Both the third and the fourth optimal portfolio components contain the matrix product $\Gamma^{-1}\Omega$ that represents the "weight" of the state variable risk (measured by matrix Ω) with respect to the asset risk (measured by matrix Γ).

We underline that when the forward real labour income exactly coincides with the subsistence consumption level (α/γ), i.e.

$$dY = \frac{\alpha}{\gamma}dt, \quad Y(0) = 0,$$

the total expected real labour income cumulated from t to H is

$$\mathbb{E}_t^{\mathbb{Q}}[Y(H)] - Y(t) = \frac{\alpha}{\gamma}(H-t),$$

and the portfolio component $w^*_{(1)}$ becomes

$$w^*_{(2)} = \frac{1}{\beta}(\gamma R - \alpha)(\Gamma'\Gamma)^{-1} M,$$

which is the classical Merton speculative component. Accordingly, if the forward real labour income is supposed to finance at least the subsistence consumption level, then what exceeds this level contributes to increase the portfolio riskiness.

Now, we turn back to the optimal consumption. From Proposition 4 it is evident that c^* is composed by two parts: (i) a constant part equal to the minimum consumption (α/γ), in fact we have already argued that the optimal consumption cannot be lower than α/γ, and (ii) a part proportional to the ratio between the real wealth augmented by the future labour incomes and the value of the bond payng G_γ.

3.5 The log-investor: he is not myopic, he is introvert

The optimal real consumption and portfolio allocation for a log-investor can easily be obtained from the result stated in Proposition 4 by putting $\alpha = 0$, $\gamma = 1$, and $\beta = 1$. After this substitution we obtain what follows.

Corollary 1 *Under market structure (9) and Assumption 1, the optimal portfolio and consumption solving Problem (15) for a log-utility function (i.e. $\alpha = 0$, $\gamma = 1$, and $\beta = 1$) are*[15]

$$\begin{aligned} c^*_{\ln} &= \frac{\delta(t)}{\int_t^H \delta(s)\,ds + \delta(H)} R_Y, \\ w^*_{\ln} &= -\Gamma^{-1}K + R_Y\left(\Gamma'\Gamma\right)^{-1}M - \Gamma^{-1}\Omega\frac{\partial}{\partial z}\left(\mathbb{E}^{\mathbb{Q}}_t\left[Y(H)\right] - Y(t)\right). \end{aligned}$$

This corollary is consistent with the usual result according to which the log-investor does not care about the changes in the market price of risk with respect to the state variables. In fact, the term that contained the derivative of $h(z,t)$ with respect to z (the whole set of the state variables) has disappeared. Accordingly, we can say that the log-investor is myopic because the future values of the Sharpe ratio (contained in the integral between t and H inside $h(z,t)$) does not affect his optimal portfolio. Nevertheless, his myopia is selective because he does care about his future incomes. When these money flows are not taken into account (i.e., $k = 0$ and $K = \mathbf{0}$) then we go back to the usual result of an optimal portfolio given just by the term $R\left(\Gamma'\Gamma\right)^{-1}M$.

From this point of view we can argue that the log-investor is not at all myopic because he does consider all his future income flows. Nevertheless, he can be said to be "introvert" since he just care about the income of his own, and not about the future behaviour of the "outside" market price of risk. In particular, he seems to be interested in what happens to his revenue when z changes, but not in what happens to the Sharpe index.

Remark 2 *When a (not necessarily) stochastic labour income is considered, the classical result of a log-optimal portfolio independent of the financial horizon does not apply.*

In the optimal asset allocation $w^*_{\ln}$ the variable χ does not play any role. This means that we can conclude what follows.

[15] We underline that in this case the value of the function $h(z,t)$ is

$$h(z,t) = \int_t^H \delta(s)\,ds + \delta(H).$$

Proposition 5 *The optimal portfolio maximizing the intertemporal expected utility of consumption is equal to the optimal portfolio maximizing the expected utility of termnal wealth when consumers have log utility functions.*

When the investment horizon tends to infinity, the optimal portfolio is still well defined only if

$$\lim_{H\to\infty} \mathbb{E}_t^{\mathbb{Q}}\left[Y\left(H\right)\right] < \infty, \quad \forall t \geq t_0.$$

If this condition holds, and the discount factor $\delta\left(t\right)$ has the form

$$\delta\left(t\right) = e^{-\rho\left(t-t_0\right)},$$

then the optimal consumption of the log-investor is given by

$$\lim_{H\to\infty} c_{\ln}^{*} = \rho\left(R - Y\left(t\right) + \lim_{H\to\infty} \mathbb{E}_t^{\mathbb{Q}}\left[Y\left(H\right)\right]\right),$$

that is the economic agent consumes, in each period, the discounted value of all his future income flows.

3.6 How to finance consumption

If we compute the difference between the optimal asset allocation with $\chi = 1$ and with $\chi = 0$ we obtain the investment strategy financing the consumption flow. Let us call w_1^* the asset allocation when consumption is taken into account and w_0^* the asset allocation maximizing the utility of investor's final wealth. It is easy to check that the difference between these two vectors has the form

$$\begin{aligned} w_0^* - w_1^* &= \frac{\alpha}{\beta}\left(H-t\right)\left(\Gamma'\Gamma\right)^{-1} M \\ &+ \frac{\alpha}{\gamma}\left(H-t\right)\Gamma^{-1}\Omega\frac{1}{h_1\left(z,t\right)}\frac{\partial h_1\left(z,t\right)}{\partial z} \\ &+ \frac{1}{\gamma}\left(\gamma R_Y - \alpha\right)\Gamma^{-1}\Omega\left(\frac{1}{h_0\left(z,t\right)}\frac{\partial h_0\left(z,t\right)}{\partial z} - \frac{1}{h_1\left(z,t\right)}\frac{\partial h_1\left(z,t\right)}{\partial z}\right), \end{aligned}$$

where

$$h_0\left(z,t\right) \equiv \left.h\left(z,t\right)\right|_{\chi=0}, \quad h_1\left(z,t\right) \equiv \left.h\left(z,t\right)\right|_{\chi=1}.$$

We can see that the portfolio financing the optimal consumption path heavily depends on the different behaviour of bond value $h\left(z,t\right)$ with respect to the state variables. Furthermore, this particular portfolio is formed by three components.

1. A component proportional to the time to horizon. We have already presented a comment for this kind of term in the previous subsection.
2. A component proportional to both the time to horizon and the elasticity of the bond $h\left(z,t\right)$ with respect to z when consumption is present.

3. A component proportional to both the real wealth (augmented by the future labour income) and the difference between the elasticities of bond $h(z,t)$ with respect to z without and with consumption.

An important simplification can be obtained when the real market price of risk $\Gamma'^{-1}M$ does not depend on the state variables z. In this case, in fact, the derivative of h with respect to z equals zero (in both cases $\chi = 1$ and $\chi = 0$) and the difference $w_0^* - w_1^*$ shrinks to the formula shown in the following proposition.

Proposition 6 *If the market price of risk does not depend on the values of the state variables (i.e.* $\frac{\partial}{\partial z}\Gamma'^{-1}M = 0$*) then the portfolio financing the consumption flow for a HARA investor is given by*

$$w_0^* - w_1^* = \frac{\alpha}{\beta}(H-t)(\Gamma'\Gamma)^{-1}M.$$

An easy corollary follows.

Corollary 2 *If the market price of risk does not depend on the values of the state variables (i.e.* $\frac{\partial}{\partial z}\Gamma'^{-1}M = 0$*) then, for a CRRA investor (i.e.* $\alpha = 0$*) the portfolio maximizing the expected utility of intertemporal consumption is equal to the portfolio maximizing the utility of investor's final wealth.*

The result stated in Proposition 6 shows that in order to finance the consumption flow, it is necessary to undertake an investment strategy which is less risky than those maximizing the expected utility of final wealth (except for a CRRA investor as shown in Corollary 2). In particular, the less amount of risky assets due to the need of consumption is proportional to both the growth optimal (or log-optimal) portfolio (i.e. $(\Gamma'\Gamma)^{-1}M$) and the time to the financial horizon $(H-t)$. This means that the farther the financial horizon, the lower the amount that must be invested in the risky assets in order to finance the consumption during all the optimization period.

4 Conclusion

In this paper we have studied the problem of an investor maximizing the expected utility of his consumption and final wealth. The model takes into account a very general setting where: (i) there exists a set of stochastic investment opportunities, (ii) there exists a set of risky assets, (iii) there exists a riskless asset paying a stochastic interest rate, (iv) a stochastic inflation risk is explicitly considered, (v) investor is supposed to be endowed with a set of stochastic labour incomes (or expenses), and (vi) the behaviour of the investor is described by a

HARA utility function. All the drift and diffusion coefficients of the stochastic differential equations here considered does not have any particular functional form. Thus, the result we are able to find is very general and applies to any of the more specific financial market structures generally analysed in the literature where a closed form solution is found.

We obtain a quasi-explicit solution for both the optimal consumption and asset allocation. This result is no more based on a single change in probability as it is usual in the financial literature. Actually, we use two changes in probability. One is based on a "real" risk neutral probability making the asset prices behave as martingale when discounted by the consumption price process. The other is based on a subjective probability depending on investor's preferences.

We confirm the result according to which the investor endowed with a non-tradeable income behaves as if he could rely on the present value of his future incomes. Nevertheless, these incomes must be evaluated under the "real" risk neutral probability.

A comparison between the optimal asset allocation when consumption is present and when it isn't, show that the need to finance all the consumption path implies a less risky portfolio with respect to the pure financial model where only the expected utility of the final wealth is maximized. Consequently, a less risky investment strategy is prescribed for young people who still have to finance a long consumption path. Instead, older people can afford to invest in riskier portfolios.

A Appendix

Proof of Proposition 1. The stochastic differential equations (2) and (6) can be rewritten under the real risk neutral probability $\mathbb{Q}$ as:

$$\begin{aligned} dS &= (\mu - \Sigma'\xi)\,dt + \Sigma' dW^{\mathbb{Q}}, \\ dP &= P\,(\mu_\pi - \sigma_\pi'\xi)\,dt + P\sigma_\pi' dW^{\mathbb{Q}}, \end{aligned} \tag{29}$$

where

$$\xi \equiv \Gamma'^{-1} M,$$

is the real market price of risk. Application of Itô's lemma on $m \equiv P^{-1}$ gives

$$dm = -m\,(\mu_\pi - \sigma_\pi'\xi - \sigma_\pi'\sigma_\pi)\,dt - m\sigma_\pi' dW^{\mathbb{Q}}. \tag{30}$$

Now, it is necessary to compute the form of vector ξ. For making the computations easier we can write the matrices Σ and σ_π as block-matrices in the following way:

$$\underset{n\times(n+1)}{\Sigma'} = \left[\begin{array}{cc} \underset{n\times n}{\Sigma_n'} & \underset{n\times 1}{\sigma} \end{array}\right], \quad \underset{1\times(n+1)}{\sigma_\pi'} = \left[\begin{array}{cc} \underset{1\times n}{\sigma_{\pi n}'} & \sigma_{\pi 1} \end{array}\right].$$

Accordingly, we can the real market price of risk as

$$\begin{aligned}\xi &= \begin{bmatrix} \Sigma_n' - S\sigma_{\pi n}' & \sigma - S\sigma_{\pi 1} \\ -G\sigma_{\pi n}' & -G\sigma_{\pi 1} \end{bmatrix}^{-1} \\ &\quad\times \begin{bmatrix} \mu - S\left(\mu_\pi - \sigma_{\pi n}'\sigma_{\pi n} - \sigma_{\pi 1}^2\right) - \Sigma_n'\sigma_{\pi n} - \sigma\sigma_{\pi 1} \\ Gr - G\left(\mu_\pi - \sigma_{\pi n}'\sigma_{\pi n} - \sigma_{\pi 1}^2\right) \end{bmatrix}.\end{aligned}$$

The inverse matrix necessary for this computation has the following form:

$$\begin{bmatrix} \Sigma_n' - S\sigma_{\pi n}' & \sigma - S\sigma_{\pi 1} \\ -G\sigma_{\pi n}' & -G\sigma_{\pi 1} \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} \Phi & \Phi\left(\sigma - S\sigma_{\pi 1}\right)\frac{1}{G\sigma_{\pi 1}} \\ -\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi & -\frac{1}{G\sigma_{\pi 1}} - \frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\left(\sigma - S\sigma_{\pi 1}\right)\frac{1}{G\sigma_{\pi 1}} \end{bmatrix},$$

where

$$\Phi \equiv \left(\Sigma_n' - \frac{1}{\sigma_{\pi 1}}\sigma\sigma_{\pi n}'\right)^{-1},$$

and so

$$\xi = \begin{bmatrix} \Phi\left(\mu - Sr\right) + \Phi\sigma\frac{1}{\sigma_{\pi 1}}\left(r - \mu_\pi\right) - \sigma_{\pi n} \\ -\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\left(\mu - rS\right) - \left(\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\sigma + 1\right)\frac{1}{\sigma_{\pi 1}}\left(r - \mu_\pi\right) - \sigma_{\pi 1} \end{bmatrix}.$$

Now, we just need to compute the matrix products $\Sigma'\xi$ and $\sigma_\pi'\xi$:

$$\begin{aligned}\Sigma'\xi &= \begin{bmatrix} \Sigma_n' & \sigma \end{bmatrix} \\ &\quad\times \begin{bmatrix} \Phi\left(\mu - Sr\right) + \Phi\sigma\frac{1}{\sigma_{\pi 1}}\left(r - \mu_\pi\right) - \sigma_{\pi n} \\ -\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\left(\mu - rS\right) - \left(\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\sigma + 1\right)\frac{1}{\sigma_{\pi 1}}\left(r - \mu_\pi\right) - \sigma_{\pi 1} \end{bmatrix} \\ &= \mu - Sr - \Sigma_n'\sigma_{\pi n} - \sigma\sigma_{\pi 1} = \mu - Sr - \Sigma'\sigma_\pi, \\ \sigma_\pi'\xi &= \begin{bmatrix} \sigma_{\pi n}' & \sigma_{\pi 1} \end{bmatrix} \\ &\quad\times \begin{bmatrix} \Phi\left(\mu - Sr\right) + \Phi\sigma\frac{1}{\sigma_{\pi 1}}\left(r - \mu_\pi\right) - \sigma_{\pi n} \\ -\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\left(\mu - rS\right) - \left(\frac{1}{\sigma_{\pi 1}}\sigma_{\pi n}'\Phi\sigma + 1\right)\frac{1}{\sigma_{\pi 1}}\left(r - \mu_\pi\right) - \sigma_{\pi 1} \end{bmatrix} \\ &= \mu_\pi - r - \sigma_\pi'\sigma_\pi.\end{aligned}$$

After substituting these values into Equations (29) and (30) we have

$$\begin{aligned} dS &= \left(Sr + \Sigma'\sigma_\pi\right)dt + \Sigma' dW^{\mathbb{Q}}, \\ dm &= -mrdt - m\sigma_\pi' dW^{\mathbb{Q}}. \end{aligned}$$

QED. ∎

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