Shareholder-efficient production plans in a multi-period economy

J.H. Drèze, O. Lachiri and E. Minelli

Discussion Paper 2007-42

Département des Sciences Économiques
de l'Université catholique de Louvain

UCL

Shareholder-efficient production plans in a multi-period economy[1].

J. H. Drèze[2], O. Lachiri[3] and E. Minelli[4]

October 2007

Abstract

We propose an objective for the firm in a general model of production economies extending over time under uncertainty and with incomplete markets. Trading in commodities and shares of stock occurs sequentially on spot markets at all date-events. We derive the objective of the firm from the assumption of initial-shareholders efficiency. Each shareholder is assumed to communicate to the firm her marginal valuation of profits at all date events (expressed in terms of initial resources). In defining her own marginal valuation of the firm's profits, a shareholder will take two elements into consideration. To evaluate the direct impact of a change in dividends the shareholder uses her own vector of marginal rates of substitution for revenue across date-events. In addition, the shareholder will take into account the impact of future dividends on the firm's stock price when she trades shares. To predict the effect on the stock price, she uses a (possibly different) state price process, her price theory. The only restriction that we impose on consumers' price theories is that they should be compatible with the observed equilibrium: given the equilibrium prices and production plans, a price theory must satisfy a no-arbitrage condition. The firm computes its own shadow prices for profits at all date-events by simply adding up the marginal valuations of all its initial shareholders. We prove existence of an equilibrium.

[1]We would like to thank Martine Quinzii and Jean-Marc Bonnisseau for useful comments. The usual disclaimer applies.

[2]CORE, Université Catholique de Louvain, Belgium.

[3]Centre d'Economie de la Sorbonne, Université Paris 1 Panthéon-Sorbonne, France.

[4]Dipartimento di Scienze Economiche, Università di Brescia, Italia.

This paper consists of: (i) an extensive non-technical introduction, reviewing the problem and the previous literature (sections 1.1 to 1.4) and summarising our contribution (1.5 and 1.6); (ii) a compact technical descritption of our model (section 2) and a statement of our main result (section 3); (iii) the proof of our main result (section 4); (iv) an Appendix collecting the proofs of ancillary lemma's. Sections 1 and 2−3 are self-contained and can be read independently. However, the logic behind sections $2-3$ is explained with more detail in section 1.5, to which readers are referred back in section 2.

1 Subject-matter and overview

1.1 General equilibrium and incomplete markets

The purpose of this paper is to contribute to the positive theory of general equilibrium in production economies extending over time under uncertainty and with incomplete markets.

The standard model fitting these specifications is known as GEI: general equilibrium with incomplete markets; see, e.g. Geanakoplos (1990) or Magill and Shafer (1991) for surveys. The basic specification rests on that in Chapter 7 of Debreu (1959). The economy consists of two kinds of agents: consumers and firms. Time and uncertainty are captured by an event tree that specifies, for each date up to a finite horizon, the set of possible date-events reflecting the (common) information of the agents at that date. There are L physical commodities at each date-event. With N date-events over the tree, the commodity space is the NL-dimensional Euclidean space. Each consumer h is defined by its consumption set in that space, by its initial endowment of commodities in the same space, and by its preferences among NL-dimensional consumption vectors. Each firm j is defined by its production set in the same space. In addition, all firms are initially owned by the consumers.

In Debreu's model, there exist markets at date 0 for trading all commodities (that is, for trading claims to each physical commodity contingent on each date-event). The resulting model is formally analogous to that of a production economy extending over a single period: consumers face a single budget constraint, over which they maximise preferences; firms maximise the present value of profits at market prices. Under this complete market system, trading in shares of ownership is redundant: each firm is analogously defined by a vector of event-dependent profits, with present value equal to the firm's market value. Trading in contingent commodities is a perfect substitute for trading in shares of the firms. Uncertainty makes no difference, due to the perfect insurance opportunities provided by the complete markets.

An alternative interpretation of the same model had appeared earlier in a seminal paper by Arrow (1953). Restrict trade in commodities to spot markets at each date event, and add markets at each date-event for elementary securities, each paying off in a specific successor date-event. The set of attainable allocations is the same as in Debreu.

The assumption of complete markets has long been recognised as unrealistic. In the real world, not all contingencies are amenable to perfect insurance. In the words of Magill and Quinzii (1996, p.4): "... the ideal structure of markets in which everything is traded out in advance would involve prohibitively large transactions costs". What we encounter in practice is a sequence of spot markets on which commodities are exchanged (as with Arrow), together with a limited set of asset markets through which *limited* reallocation of resources over time and across date-events is possible (at variance with Arrow). The resulting model is labelled GEI. Compared with the complete-markets model, two new features emerge: (i) consumers are faced with multiple budget constraints; and (ii) firms are not able to evaluate production plans in terms of market values.

A complete specification of the GEI model calls for defining the set of tradable assets. Ideally, one would like to see that set emerge endogenously from primitive considerations on transactions costs. In the words of Magill and Quinzii (1996): "only those contracts can survive for which the benefits from exchange outweigh the costs involved in their enforcement"[1]. Although there exists some investigation of that principle (e.g. Bisin 1998), most of the work on GEI (starting with the seminal paper by Radner 1972) includes the definition of tradable assets among the primitives of the model. Two notable specifications have been studied: (i) in models of exchange, a given set of assets paying off in units of account is given exogenously; (ii) in models of production, the basic assets are the shares of the firms – sometimes completed by default-free bonds.

In this introductory section, we refer successively to exchange economies, then to production economies extending over two periods, then finally to production economies extending over multiple periods.

[1]As an interesting illustration, Miller (1991, p. 18) attributes the short-lived fate of the Chicago market for CPI contracts (which he labels "the economist's dream contract") to "insufficient demand for immediacy" (i.e. for "speedy execution of trades") - meaning "insufficient" to cover the fixed cost of that market.

1.2 Plans, prices and price expectations

For an exchange economy extending over time under uncertainty, with an incomplete market structure exogenously given, the analysis of Radner (1972) yields existence of an equilibrium. The basic premise is that, although spot markets for commodities and assets open sequentially, all the agents hold *common, correct, point expectations* about market clearing prices. The equilibrium is defined by the property that, at the prices prevailing at time 0 and at the associated expectations for prices on spot markets at later date-events, the quantities demanded and supplied by the utility-maximising consumers (faced with multiple budget constraints) clear all markets.

The crucial assumption about price expectations is often labelled *perfect foresight*. It is of course a very strong assumption, in the absence of any mechanism apt to coordinate consumer expectations. That is also the reason why an alternative approach has been developed under the name TGE: temporary general equilibrium. Under that approach, it is not assumed that expectations about future prices are common, or correct, or single-valued. Only that consumer expectations are continuously related to market observations at 0, and have overlapping ranges across agents. We come back to TGE at the end of this section. Suffice it to mention here that it has been criticised by GEI theorists as placing insufficient constraints on the consumer price expectations; see, e.g. Drèze (1999 sect. 4). Thus, GEI and TGE are polar cases regarding expectations.

For the GEI exchange economy, it is easy to define the set of allocations that are *feasible*: namely, the allocations such that consumer consumptions are compatible (they add up to aggregate initial endowments), and susceptible of being attained through trade of the existing assets. In other words, feasibility recognises the limits to transfers of resources across date-events (the limits to insurance opportunities) inherent in the incomplete asset structure. On that basis, one can define the concept of Constrained Pareto Optimality (CPO): an allocation is *constrained Pareto optimal* if and only if it is feasible, and there does not exist another feasible allocation that Pareto-dominates it. It is shown in Geanakoplos and Polemarchakis (1986) that, generically in initial endowments and utility functions, equilibria as defined above are *not* CPO.

1.3 Investment under private ownership: the two period case

The standard model of a two-period production economy raises the new issue of defining the decision criteria of the firms. Production plans have a time 0 component, and $N-1$ time 1 components (one per terminal date-event). There is a stock market open a time 0, where shares of the firms are traded. A shareholder at time 0 after stock-market clearing receives a dividend equal to the value of the time 0 component of the production plan; in case of net investment by the firm, that dividend is negative (it operates like an addition to the stock price)[2] . At time 1, the value of the production plan at a date-event accrues to the time 0 shareholders as a dividend (assumed non-negative under limited liability). The stock market does not reopen (it would be redundant). Consumers know the production plans of all firms when choosing their portfolios; and the stock market at time 0 clears through prices.

The new difficulty is that, under genuinely incomplete markets, the profits at time 1 under date-event ξ ($\xi = 1, .., N-1$) do not have a well-defined market value at time 0, when production plans are chosen. Indeed, with J firms, $J < N-1$, only the J vectors of date 1 profits (namely $(N-1)$ - vectors) are priced by the market. Maximising the present value of profits (through choice of the production plan) is not well defined. Thus, one needs to specify a decision criterion for the firm. And the assumption of common, correct point expectations applies only to commodity prices at time 1 date-events. It does not apply to present values of profits at date 0, since there are no market prices for these.

The concept of Constrained Pareto Optimality is still well defined, for this model. An allocation is constrained feasible if and only if it is physically feasible, and susceptible of being attained through asset trading at given production plans. For normative purposes, it is easy to formulate the necessary first-order conditions (FOC) on production plans required by CPO. To that end, remember that each consumer optimises its consumption subject to N distinct budget constraints. Denote by λ the N-vector of Lagrange multipliers associated with these constraints, and by $\bar{\lambda}$ the N-vector of ratios $\frac{\lambda_\xi}{\lambda_0}$. These define marginal rates of substitution between "income" at date 1 in date-event ξ and income at date 0. (Under the assumption of perfect foresight, these marginal rates of substitution are defined at common market-clearing spot prices for commodities at all date-events.) Consider a firm owned by a single consumer h, deciding simultaneously about its consumption and about the production plan of its firm. Then, jointly optimal consumption and production plans would entail that firm profits are maximal (over the production set) at the shadow prices $\bar{\lambda}^h$. For the the general

[2]That is, firms do not engage in financial transactions of their own, a straightforward application of the Modigliani-Miller theorem.

case in which the firm may have multiple shareholders, it is shown in Drèze (1974) that necessary FOC for CPO impose that profits of each firm should be maximal at shadow prices defined as weighted averages of the marginal rates of substitution of the firm's shareholders, with weights given by respective shareholdings. That is also a necessary FOC condition for Pareto efficiency of the production plan from the viewpoint of that firm's final shareholders. Pending that condition, there would exist changes in production and zero-sum transfers among shareholders making all of them better off. That result is an important clarification of the normative issues raised by market incompleteness in a production economy: it defines unambiguously a desirable decision criterion for the firms, in a general model under standard assumptions. (It is often referred-to in the literature as "the Drèze criterion".)

Drèze (1974) also brings out the important feature that, under incomplete markets, the set of feasible allocations is not convex. Indeed, the dividends received by a shareholder, which enter as parameters of their budget constraints, are defined as a product of two endogenous variables, namely the shareholding and the firm profits. This bilinearity results in a non-convex feasible set for the economy, the very set over which CPO is defined. Thus, necessary FOC are in general not sufficient. If equilibria are defined by the Drèze criterion, equilibria exist, but they need not be CPO. It is shown in Geanakoplos et al. (1990) that, generically in initial endowments, they are *not* CPO.

The relevance of this analysis for positive economics is open to debate. The notion of shareholder (Pareto) efficiency of production decisions is clearly appealing for privately owned firms or small partnerships. For large corporations listed on stock exchanges, the appeal is much less compelling: shareholder preferences (their $\bar{\lambda}$'s) have no natural channel of expression; the role of shareholders is limited to approval voting at general assemblies. It would take a lot of faith in the Coase theorem to claim realism for the Drèze criterion. However, an important step in the direction of realism is provided by later work of Drèze (1985, 1989 chapters 2 and 3)) on "equilibria of production and exchange", then "equilibria of production, exchange and labour contracts". The new ingredient is the so-called control principle: for each firm j, given shareholdings θ^{hj}, there exists a uniquely (endogenously) defined subset of shareholders, say the Board of Directors; decisions about production plans must be endorsed by a majority of shareholders *including all the directors*. Thus, directors are veto players, a feature that also circumvents the Condorcet paradox of voting. And it is reasonable to assume that production decisions will be Pareto efficient for the small set of directors (subject to majority approval by all shareholders). Since the set of directors is endogenous (related to shareholdings), the general specification has undeniable realism. The only special assumption is that the correspondence defining the set of directors (as a function of shareholdings) is upper hemi-continuous for the discrete topology (see appendix 2 of Drèze 1989). An interesting by-product of this extension is that it covers as a special case delegation of the production decisions to a manager, as in Radner (1972), with the mild requirement that the manager be also a shareholder. (That mild requirement is also satisfied if the manager receives a profit share.) Indeed, a predetermined shareholder-manager is a particular, upper hemi-continuous (constant) selection of directors.

1.4 Stochastic production economies: the T-period case

The additional difficulties arising in T-period production economies are substantial, and the state of the literature on that topic is much less advanced (which provides the motivation for the present paper). In the two-period model, the stock market is operative only at period 0; stock prices are thus known to all agents when they finalise consumption, portfolio or production decisions. With more than two periods, the stock market reopens at all future (non-terminal) date-events. Perfect foresight may be invoked for future stock prices. But future stock market transactions will modify the shareholdings of the firms. If production plans are subject to revisions, these will be decided by shareholders at the time of revisions. The identity as well as preferences of these future shareholders are not observable at the initial date. Accordingly, the future revisions of the initial production decisions are difficult to fathom. Perfect foresight does not apply to these future revisions, and it would stretch the imagination to postulate that it does.

A further difficulty comes from conflicts of interest among initial shareholders with divergent portfolio plans. If consumer h plans *selling* shares of firm j at a future date-event ξ, then h will benefit from a *high* market price for j at ξ (say q^j_ξ). If consumer h' plans *buying* additional shares at ξ, then h' will benefit from a *low* market price for j at ξ. When assessing the desirability of alternative production plans, shareholders like h or h' will be concerned about the impact of profits at distant date-events on the market price at ξ; this raises a new issue in forcasting: not only future prices matter, but also "derivatives" of such market prices with respect to future profits. The specification of the model, and significantly the definition of a decision criterion for the firm, have to cope with these new complications. It is thus not surprising that the relevant theory is still in infancy. We know of only two significant contributions, one normative and one positive.

To start with the normative side, there is an extremely useful recent contribution by Bonnisseau and Lachiri ((2002), BL from now on). They consider a production economy extending over T periods, with a single physical commodity ($L = 1$), and with shares of the firms as the only assets. At each non-terminal date-event, shares of the firms are traded on the stock market. BL rely on a permissive concept of Constrained Pareto Optimality. An allocation is defined by consumptions x^h, portfolios θ^h and production plans y^j. An allocation (x, θ, y) is constrained feasible if and only if: (i) it is physically feasible (consumption = endowments + production, $\sum_h \theta_\xi^{hj} = 1$ at each ξ); (ii) it is financially decentralisable, i.e. there exist stock prices q such that the budget constraints of consumers are satisfied at each ξ, given their x's, their θ's and these prices. An allocation is constrained Pareto optimal if (i) and (ii) hold and there does not exist another constrained feasible allocation that Pareto-dominates it. This is a "permissive" concept of CPO, because it does not impose any constraints on the stock prices, not even the standard mild requirement of "no arbitrage". (See also the example at the end of section 3 in BL.)

Under that definition, BL are able to provide necessary FOC for CPO: the profits of all firms should be maximal at shadow prices obtained as weighted sums of the marginal rates of substitution (our $\bar{\lambda}$'s) of shareholders, *with the weights at each date-event corresponding to shareholdings at that date-event.* This is a natural extension to multiple periods of the Drèze criterion. But it is much more demanding here than in the two-period model: the identity, shareholdings and preferences of all future shareholders of a firm must be known! The BL result, in its simplicity, is thus extremely instructive from a normative viewpoint, but it is definitely unrealistic from a positive viewpoint. Indeed, the informational requirement is simply prohibitive.

Turning to the positive side, we are taken back to the seminal paper by Grossman-Hart ((1979), GH). These authors were the first to identify the specific difficulties associated with extending the horizon from 2 to T periods. They deal with a general production economy extending over T periods with L commodities. The only assets are shares of the firms, traded on the stock market at each non-terminal date-event. Three special assumptions are added, each of which requires interpretation.

(GH1): competitive price perceptions; we discuss that assumption extensively below;

(GH2): no revisions of production plans; that assumption stipulates that a production plan y^j is chosen once and for all at date 0 by shareholders before the stock market opens; that is, by the consumers inheriting positive shareholdings from the past, as part of the primitives; this is an assumption about the sequence of decisions (production plans first, trading of shares thereafter); it is in the nature of a technological assumption, clearly introduced for simplicity and clarity: there is no concern about potential revisions of the production plans by future shareholders, since these are ruled out by the technology;

(GH3): possibility of closure; at each date event ξ, irrespective of the production plan implemented through the predecessors of ξ, 0 belongs to the production set over the subtree starting with ξ; this may be interpreted as an extreme form of limited liability: a firm may close down and disappear at any point, reneging on any obligations it might have incurred previously; actually, the assumption is less severe than it looks; it amounts to stating that the production set of firm j allows for the possibility of inaction from ξ onward, without consequences for the feasibility of plans before ξ; at equilibrium, there will be no unmet obligations after closing down.

Assumption **(GH1)** deserves clarification. It bears on the perceptions by consumers of the impact of additional profits at some date-event ξ' on the market values of the firm at earlier date-events, say ξ. Let us represent these perceptions by a "state price process", to be denoted α^h for consumer h and called "h's price theory". What does it mean to entertain "competitive price perceptions"? Magill and Quinzii (1996, p.382) mention two properties:

- "the price of a bundle of goods is the sum of the prices of its components;

- the unit price of each component is indepenent of the number of units of the good purchased or sold."

Applied to the pricing of firms on the stock market, these principles imply that the state price process α^h must satisfy, for all j's and ξ:

$$\alpha_\xi q_\xi^j = \sum_{\xi' \geq \xi} \alpha_{\xi'} p_{\xi'} \cdot y_{\xi'}^j \text{ for all } \xi \in \cup_{t=0}^{T-1} \Xi_t \text{ with, for any } \xi \in \Xi_T, q_\xi^j = 0, \tag{1.1}$$

where q_ξ^j is the market price of j at ξ and $p_{\xi'} y_{\xi'}^j$ are the profits of j at ξ'. Indeed, shares at ξ are the "bundle of goods" defined by vectors of future profits. Provided the α^h's are independent of j's profits, this formula satisfies the two desired properties. If all the α_ξ^h are strictly positive it also implies that the stock prices are arbitrage-free. This will be our definition of competitive price perceptions – neither more nor less.

Assumption **(GH1)** in GH satisfies this definition, but goes far beyond: these authors assume that h's price theory is given by h's own marginal rates of substitution $\bar{\lambda}^h$. A trivial application reveals the shortcomings of that restrictive specification. If h is a consumer expecting to die (from a terminal illness) before ξ leaving no heirs behind, $\lambda^h_{\xi'}$ could be zero. It would be preposterous for h to assume on that ground that additional profits at $\xi' > \xi$ will not be valued by the market at ξ! Accordingly, we do not follow GH on that path (the justification by MQ on p.386 notwithstanding). It is unfortunate that two distinct assumptions, namely "competitive price perceptions", which is fine, and "price perceptions reflecting own consumption preferences" ("egocentric price perceptions"?) have been lumped - initially by GH, but subsequently by the whole literature - under a single heading. As explained below (see equation (2.8)), "egocentric price perceptions" have the momentous implication of cancelling from the shareholders' evaluations of production plans all the terms involving future portfolio transactions.

1.5 Price theories and equilibrium

We are now (at long last!) ready to introduce the contribution of this paper to the positive theory of equilibria in production economies extending over T periods under uncertainty and incomplete markets. We propose an equilibrium concept based on initial-shareholders efficiency, for a general model of a real economy with L commodities and with assets consisting exclusively of shares of stock. (The introduction of other assets, in particular bonds, should not be problematic.) Trading in commodities and shares of stock occurs sequentially on spot markets at all date-events. We retain the assumption of perfect foresight, that is common, correct (at equilibrium), single-valued price expectations. Importantly, we do not introduce any information about quantities traded. We allow for future revisions of the production plans (as ownership of the firms evolves through share-trading). But we assume non-strategic behaviour with respect to revisions of the production plans – an assumption of bounded rationality. We endow each household h with a price theory α^h satisfying competitive price perceptions as defined by (1.1) above. And we define the decision criteria of firms by the principle of initial shareholders efficiency, as presently to be defined.

The idea of initial shareholders efficiency is straightforward: the production plan chosen by firm j at date zero is such that there do not exist an alternative production plan and transfers of initial resources among the shareholders making all of them better off. This principle leads at once to the property that the value of the chosen production plan is maximal at shadow prices reflecting the marginal valuations of the shareholders (Lemma 2.2).

Each shareholder is assumed to communicate to firm j its marginal valuation of profits at all date events (a valuation expressed in terms of initial resources). In defining its own marginal valuation of firm j's profits, a shareholder h will take two elements into consideration:

- profits (dividends) at date-event ξ are first valued at h's shadow price ($\bar{\lambda}^h$) for resources at ξ, multiplied by the shareholding of j by h at ξ, θ_ξ^{jh};

- next, h takes into account the impact of profits at ξ on the market value of j at every node $\xi' < \xi$ where h is trading j's shares; this impact is assessed according to h's price theory, and applied to the volume of h's trade at ξ' – thus with a positive sign in case of a sale, and a negative sign in case of a purchase; the resulting impact is multiplied by h's shadow price for resources at ξ', the date-event where the trade occurs[3].

The sum of the terms in the second evaluation is added to the term in the first evaluation, and this sum defines h's marginal valuation of firm j's profits at date-event ξ; call this sum β_ξ^{hj}.

Note that these calculations rely on the consumption and portfolio plans of h. By themselves the shadow prices for resources $\bar{\lambda}^h$ (the only ingredient of the BL FOC) do not convey the relevant information, if not combined with the planned portfolio trading plans and the consumer price theories.

Now, firm j computes its own shadow prices for profits at all date-events, β^j, by simply adding up the marginal valuations of all its initial shareholders. There is no weighting involved, because shareholdings have been taken into account by the shareholders themselves in computing their own marginal valuations (see eq. (2.7) for the formal expression).

An equilibrium is a feasible allocation (x, θ, y) and prices (p for commodities, q for shares) such that all markets clear and all agents optmise: (x^h, θ^h) is best for h's preferences subject to h's budget constraints; y^j has maximal present value at the shadow prices β^j .

This equilibrium concept borrows from GEI the perfect foresight assumption; but it is in TGE spirit because: (*i*) future revisions of production plans are ignored; and (*ii*) price perceptions are allowed to be idiosyncratic. It is, of course, a sophisticated concept through the treatment of initial shareholders' efficiency. We regard it as a first step, calling for the same extension that Drèze (1989) adds to Drèze (1974) , namely boards of directors, the control principle, ...

[3]This formulation does not take into account a possible impact of investment (negative profits) at node ξ on the market value of $j$at later nodes $\xi' > \xi$. The reason is that investments have no "value" of their own, beyond the profits which they generate later on; and these profits are duly taken into account under our formulation.

Our main result, Theorem 3.1, demonstrates the existence of an equilibrium under standard assumptions (about preferences, endowments, production sets, ...). That is, the Grossman-Hart theorem is here extended to more general price perceptions (competitive, but not "egocentric")[4].

1.6 Extensions

The reason for basing here the production decisions of the firms on the marginal valuations of initial shareholders rather then node 0 final shareholders is technical: the shadow prices of firm j are not continuous when some θ^{hj} goes to zero. At any positive value of θ^{hj}, h's marginal valuations of j's profits enter the calculation of β^j; that marginal valuation includes a "trading" component that is not proportional to θ^{hj} and remains finite when θ^{hj} goes to zero[5]. Hence, there is a discontinuity of β^j at $\theta^{hj}_{\xi_0} = 0$. When β^j is calculated on the basis of initial shareholdings, as given by the primitives, this potential discontinuity is immaterial: the set of relevant θ's is fixed, and only positive theta's matter. If instead β^j were calculated from after-trade shareholdings, there is a discontinuity of β^j when some $\theta^{hj}_{\xi_0}$ goes through zero. Hence the option, in this paper, to retain the simpler specification: initial shareholders.

Of course, there exist other ways around the potential discontinuity. Following Drèze (1985, 1989), just let the marginal valuations of after-trade shareholder h enter the calculation of β^j if and only if $\theta^{hj} \geq \epsilon > 0$. Because the proof of our existence theorem is already quite involved, we have postponed that ancillary refinement.

A realistic formulation of the sequence of production decisions and share trading would recognise that stock markets operate more or less continuously (daily). New information about the state of the economy also arrives more or less continuously. In contrast, production decisions are revised at discrete intervals, for instance when shareholders meet. The formulation in GH and here assumes that the economy "starts" at a point in time where a meeting of shareholders is held.

In closing this introduction, it is appropriate to mention one final issue

[4] As an informative side-comment, the proof uses a detour to cope with excess supply of shares at a zero price. That case has been problematic ever since Radner's (1972) seminal paper. It is handled here by verifying that, in such a case, rescaling production downward keeps it on the efficient boundary of the production set.

[5] Indeed, the dividend part of β^{hj}_{ξ} is proportional to θ^{hj}_{ξ} and the trading part of β^{hj}_{ξ} is a sum of terms at $\xi' < \xi$ proportional $\theta^{hj}_{\xi'}$; neither θ^{hj}_{ξ} nor $\theta^{hj}_{\xi'}$ needs go to zero with $\theta^{hj}_{\xi_0}$.

raised by the calculation of β^j as a sum of shareholder marginal valuations, namely truthful revelation of these valuations. A shareholder inflating systematically her marginal valuations (or some of them) would thereby increase her influence on production decisions in the firm. But the increased influence comes in at the cost of a distortion: because the marginal valuations are expressed in terms of current resources, inflating them entails an implicit discount rate lower than desired by the shareholder. This distortion is already present in the two-period model. The revelation issue is the same as in a pure public good problem: for shareholders, the production plans are comparable to public goods (with exclusion). That issue is discussed in Drèze (1974, section 6.2). Ignoring the issue is congruent with our general assumption of non-strategic behaviour. But the issue still belongs on the research agenda.

2 The Economy

We study a private ownership economy, populated by consumers and privately owned firms in finite numbers, that we index $H = \{1, \ldots, h, \ldots, H\}$ and $J = \{1, \ldots, j, \ldots, J\}$ respectively.

The economy evolves over a discrete and finite number of time periods. Indexing today $t = 0$ and the future from tomorrow to the final day $t = 1, \ldots, T$, we can describe aggregate uncertainty by an event tree. Let the set of date-events be $\Xi = \cup_{t=0}^{T} \Xi_t$. We write Ξ_t for the set of the events that may occur at day t for $t = 0, \ldots, T$ and define $\Xi^- = \Xi \backslash \Xi_T$ and $\Xi_+ = \Xi \backslash \Xi_0$. The root of the tree is denoted $\xi_0 = \Xi_0$, the unique state at time 0. For each date-event $\xi \in \Xi$ other than those belonging to Ξ_T, ξ^+ represents the set of the date-events that immediately succeed date-event ξ and $\Xi^+[\xi]$ the sub-tree having ξ as the root. For a given $\xi \in \Xi$ other that the root state ξ_0, we write $\xi' < \xi$ to mean all date-events ξ' that belong to the backward walk along the path from ξ to ξ_0, ξ excluded, and ξ^- to mean the unique immediate predecessor of ξ.

There are L physical commodities ($\ell = 1, 2, \ldots, L$) available for consumption and production at each date-event. The commodity space across the event tree can be written as $\mathbb{R}_+^{L\Xi}$. We write $x^h = (x_\xi^h)_{\xi \in \Xi} \in \mathbb{R}_+^{L\Xi}$ for the bundle of commodities consumed by consumer h and $y^j = (y_\xi^j)_{\xi \in \Xi} \in \mathbb{R}^{L\Xi}$ for the production plan of firm j. Commodities are traded only on spot markets at prices $p = (p_\xi)_{\xi \in \Xi} \in \mathbb{R}_+^{L\xi}$, where $p_\xi \in \mathbb{R}_+^L$ represents spot prices at date-event ξ.

There are J security markets where ownership shares of the J firms are traded sequentially between consumers, in every date-event other than the terminal ones, at the stock prices $q = (\ldots q_\xi^j, \ldots)$, with q_ξ^j the price of stock j at date-event ξ. We write $\theta^h = (\theta_\xi^h)_{\xi \in \Xi}$ for the sequence of portfolios of shares, with $\theta_\xi^h \in R_+^j$ the portfolio of shares held by consumer h at $\xi \in \Xi$.

2.1 Consumers

Each consumer $h \in \mathcal{H}$ is described by a real-valued intertemporal utility function u^h, defined on the set of possible consumption bundles $\mathbb{X}^h := \mathbb{R}_+^{L\Xi}$, and by an initial endowment $(w^h, \theta_{\xi_0^-}^h) := ((w_\xi^h)_{\xi \in \Xi}; (\theta_{\xi_0^-}^{hj})_{j \in J}) \in \mathbb{R}_+^{L\Xi} \times \mathbb{R}_+^J$ of commodities across date-events and of shares at ξ_0^-.

In any date-event $\xi \in \Xi$ other than the terminal ones, a consumer h may buy a portfolio of ownership shares $\theta_\xi^h \geq 0$, at the competitive stock prices q_ξ. Consumer h holding the portfolio of shares θ_ξ^h at date-event ξ, has an obligation to invest, or the right to receive dividends, in the amount $\sum_{j \in \mathcal{J}} \theta_\xi^{hj}(p_\xi \cdot y_\xi^j)$. That is, dividends at each node accrue to the new (after trade) shareholders. In addition to the exchange on the stock markets, a consumer h has to select a bundle of commodities $x^h \in \mathbb{R}_+^{L\Xi}$ for consumption at the cost of $p_\xi \cdot x_\xi^h$ in each state $\xi \in \Xi$.

Consumers are constrained in their choices on consumption and shares by their state-dependent budget constraints. At each date-event, the expense to purchase consumption and ownership shares should not exceed incomes from net dividends and from sales of of commodities and shares.

Consumer h takes spot prices p, stock prices q and production plans y as given, and chooses a consumption plan $\tilde{x}^h$ and a portfolio of shares $\tilde{\theta}^h$ which maximize her utility over the budget set $\mathbb{B}^h(p, q, y)$. The budget set $\mathbb{B}^h(p, q, y)$ collects all feasible consumption plans x^h and portfolios of shares θ^h that satisfy, at given prices (p, q) and production plans y, her state-dependent budget constraints:

$$\begin{gathered} p_\xi \cdot x_\xi^h + \sum_{j \in \mathcal{J}} q_\xi^j \theta_\xi^{hj} \leq p_\xi \cdot w_\xi^h + \sum_{j \in \mathcal{J}} q_\xi^j \theta_{\xi^-}^{hj} + \sum_{j \in \mathcal{J}} \theta_\xi^{hj}(p_\xi \cdot y_\xi^j) \text{ for all } \xi \in \Xi^-, \\ p_\xi \cdot x_\xi^h \leq p_\xi \cdot w_\xi^h + \sum_{j \in \mathcal{J}} \theta_{\xi^-}^{hj}(p_\xi \cdot y_\xi^j) \text{ for all } \xi \in \Xi_T. \end{gathered} \tag{2.1}$$

Formally, $(\tilde{x}^h, \tilde{\theta}^h)$ solves the problem:

$$\text{Max }_{(x^h, \theta^h)} \Big\{ u^h(x^h) \text{ s.t. } (x^h, \theta^h) \in \mathbb{B}^h(\tilde{p}, \tilde{q}, \tilde{y}) \Big\}$$

for given spot prices $\tilde{p}$, stock prices $\tilde{q}$ and production plans $\tilde{y}$.

We assume that for each consumer h, the individual characteristics (u^h, w^h, θ^h) satisfy the following properties: u^h is **(A.1)** continuously differentiable, **(A.2)** quasi-concave, **(A.3)** weakly monotone and **(A.4)** strictly monotone in good $l = 1$ in every date-event ξ; and **(A.5)** $w^h \in \mathbb{R}^{L\Xi}_{++}$ and $\sum_{h\in\mathcal{H}} \theta^{hj}_{\xi_0^-} = 1$, $\forall j \in \mathcal{J}$.

The consumer's decision problem is a standard mathematical programming problem. Assumptions (A.1), (A.2) ensure a first order characterization by the Kuhn-Tucker (KT) conditions since $(\tilde{x}^h, \tilde{\theta}^h)$ is a regular point for the budget constraints. This will give rise to h's shadow prices $\tilde{\lambda}^h = (\tilde{\lambda}^h_\xi)_\xi$ for resources across date-events (Lagrange multipliers associated to the constraints), measured in term of "utils". Assumption $(A.4)$ delivers positive λ's. This helps towards getting some basic ideas across without too many technical complications.

Lemma 2.1 summarizes this result, by stating the first-order necessary and sufficient conditions for optimality of the solutions of the consumer h decision problem. For notational convenience, for any two arbitrary vectors $p \in \mathbb{R}^{L\Xi}$ and $\lambda \in \mathbb{R}^{\Xi}$, we define the vector $\lambda^h \,\square\, p$ as a matrix whose Ξ rows are $\lambda_\xi p_\xi = (\lambda_\xi p_{\ell\xi})_{\ell\in\mathcal{L}} \in \mathbb{R}^L$.

Lemma 2.1 *Assume that* $(A.1)$*,* $(A.2)$ *and* $(A.3)$ *hold. Then,* $(\tilde{x}^h, \tilde{\theta}^h) \in \mathbb{R}^{\Xi}_+ \times \mathbb{R}^{J\Xi^-}_+$ *is optimal for the decision problem of consumer* h *and* $\tilde{\lambda}^h \in \mathbb{R}^{\Xi}_+ \setminus \{0\}$ *(respectively,* $\tilde{\lambda}^h \in \mathbb{R}^{\Xi}_{++}$ *if Assumption* $(A.4)$ *holds) are Lagrange multipliers for the inequality constraints if and only if the Kuhn-Tucker conditions stated below hold:*

$$\text{the consumer } h \text{ budget inequalities (2.1) are binding at } (\tilde{x}^h, \tilde{\theta}^h) \tag{2.2}$$

$$\tilde{\lambda}^h \,\square\, p = \nabla u^h(\tilde{x}^h) \text{ (the gradient of } u^h \text{ at } \tilde{x}^h\text{);} \tag{2.3}$$

for all $(j, \xi) \in \mathcal{J} \times \Xi^- \setminus \Xi_{T-1}$*,*

$$\tilde{\lambda}^h_\xi \left(q^j_\xi - p_\xi \cdot y^j_\xi\right) - \sum_{\xi'\in\xi^+} \tilde{\lambda}^h_{\xi'} q^j_{\xi'} \geq 0 \tag{2.4a}$$

$$0 = \tilde{\theta}^{hj}_\xi \left(\tilde{\lambda}^h_\xi \left(q^j_\xi - p_\xi \cdot y^j_\xi\right) - \sum_{\xi'\in\xi^+} \tilde{\lambda}^h_{\xi'} q^j_{\xi'}\right); \tag{2.4b}$$

and for all $(j,\xi) \in \mathcal{J} \times \Xi_{T-1}$,

$$\tilde{\lambda}^h_\xi \left(q^j_\xi - p_\xi \cdot y^j_\xi\right) - \sum_{\xi' \in \xi^+} \tilde{\lambda}^h_{\xi'} p_{\xi'} \cdot y^j_{\xi'} \geq 0 \tag{2.5a}$$

$$0 = \tilde{\theta}^{hj}_\xi \left(\tilde{\lambda}^h_\xi \left(q^j_\xi - p_\xi \cdot y^j_\xi\right) - \sum_{\xi' \in \xi^+} \tilde{\lambda}^h_{\xi'} p_{\xi'} \cdot y^j_{\xi'}\right); \tag{2.5b}$$

The proof uses standard techniques, and we omit it.

2.2 Firms

Each firm $j \in \mathcal{J}$ is assumed to have a production set $\mathbb{Y}^j \subseteq \mathbb{R}^{L\Xi}$ that satisfies standard assumptions: $\mathbb{Y}^j$ is **(*B*.1)** convex, **(*B*.2)** closed, **(*B*.3)** satisfies free disposal: $-\mathbb{R}^{L\Xi}_+ \subset \mathbb{Y}^j$; moreover, **(*B*.4)** $\left(\sum_{h\in\mathcal{H}} w^h + \sum_{j\in\mathcal{J}} \mathbb{Y}^j\right) \cap \mathbb{R}^{L\Xi}_+$ is compact.

The production plan of firm j is chosen by the initial shareholders at date $t = 0$; that is, we assume that the shareholders' meeting takes place *before* they trade on the stock market at $t = 0$, yet with full knowledge of the prices q_0. When contemplating a change in the production plan of the firm, shareholders take as given their consumption and portfolio plans as well as the production plans of all other firms, but they anticipate the effect of the contemplated production change on future dividends and on the current and future prices of firm j's stock. As explained in section 1.5, we assume that each consumer h calculates the direct impact of a change in dividends by using her own vector of marginal utilities for revenue across date-events, $\lambda^h \in \mathbb{R}^\Xi_{++}$. To predict the effect on the stock price, she uses a (possibly different) state price process $\alpha^h \in \mathbb{R}^\Xi_{++}$, which we call h's *price theory*. The only restriction that we impose on consumers' price theories is that they should be compatible with the observed equilibrium: given the equilibrium prices and production plans, the processes α^{hj} must satisfy a no-arbitrage condition for each firm j, that we now define.

Given spot prices and production plans, (p, y), the stock prices q satisfy no-arbitrage, written $q \in N(p, y)$, if and only if there exists $\alpha \in \mathbb{R}^\Xi_{++}$ such that, for all $j \in \mathcal{J}$,

$$\alpha_\xi q^j_\xi = \sum_{\xi' \geq \xi} \alpha_{\xi'} p_{\xi'} \cdot y^j_{\xi'} \text{ for all } \xi \in \Xi^- \text{ with, for any } \xi \in \Xi_T, q^j_\xi = 0. \tag{2.6}$$

For given (p, q, y), let $\Omega(p, q, y)$ be the set of processes $\alpha \in \mathbb{R}^\Xi_{++}$ satisfying 2.6. To any $\alpha \in \mathbb{R}^\Xi_{++}$ we can associate an element of the simplex

$\Delta_{\Xi-1} := \{\alpha \in \mathbb{R}_+^{\Xi} \mid \sum_{\xi\in\Xi} \alpha_\xi = 1\}$. We assume that the state prices α^h that consumers use in their theories are uniformly bounded below [6]. Let $0 < b < \frac{1}{\Xi}$ and $\Delta^b_{\Xi-1} := \{\alpha^h \in \mathbb{R}_+^{\Xi} \mid \sum_{\xi\in\Xi} \alpha^h_\xi = 1, \alpha_\xi \geq b \,\forall \xi \in \Xi\}$. Then we can state the following Assumption:

> ***(A.6)***: Every consumer h is characterized by a continuous map $A^h : \mathbb{R}_+^{\mathcal{L}\Xi} \times \mathbb{R}_+^{\mathcal{J}\Xi} \times \bigotimes_j \mathbb{Y}^j \to \Delta^b_{\Xi-1}$ which associates a price theory $\alpha^h \in \Omega(p,q,y) \cap \Delta_{\Xi-1}$ to any (p,q,y) such that $q \in N(p,y)$.

Following Drèze (1974) we define a criterion for the firm by postulating that the chosen production plan must be efficient from the point of view of the initial shareholders. The production plan chosen by firm j at date zero must be such that there do not exist an alternative production plan and transfers of initial resources among the shareholders making all of them better off. Starting from a given production plan y^j there must not exists a feasible change dy^j such that the induced change in utilities satisfies[7]:

$$\sum_{h\in\mathcal{H}_{\xi_0^-}} \frac{du^h}{\lambda^h_{\xi_0}} > 0$$

where $\mathcal{H}^j_{\xi_0^-} := \{h \in \mathcal{H} : \theta^{hj}_{\xi_0^-} > 0\}$ is the set of initial shareholders of firm j. In our multi-period setting, the effect on consumers' utilities of a change in the production plan is defined with respect to given price theories (α^h), which initial shareholders use to evaluate the impact of the proposed change on the stock's price.

Lemma 2.2 *Given portfolios, consumer state prices, consumer price theories and commodity prices* $(\tilde\theta, \tilde\lambda, \tilde\alpha, \tilde p)$, *a production plan* $\tilde y^j$ *is efficient from the point of view of firm* j*'s initial shareholders if*

$$\tilde y^j \textit{ solves } Max_{\,y^j} \left\{ V^j(\beta^j(\tilde\theta, \tilde\lambda, \tilde\alpha); \tilde p, y^j) \textit{ s.t. } y^j \in \mathbb{Y}^j \right\}$$

[6] Assumption $(A.6)$ places an arbitrarily small but strictly positive lower bound on the values α^h_ξ of the price processes. The motivation is purely technical. Of course, given finitely many states and agents, and strictly monotonic preferences, α^h_ξ should be positive for all h and ξ. In particular, it has a positive minimum (over h and ξ) at equilibrium. We did not feel that dispensing with $(A.6)$ passed a cost-benefit test...

[7] The quotients $\frac{du^h}{\lambda^h_{\xi_0}}$ have the dimension of units of account at ξ_0; a positive sum entails the possibility of Pareto domination through transfers of initial resources.

where for all $y^j \in \mathbb{Y}^j$,

$$V^j(\beta^j(\tilde{\theta},\tilde{\lambda},\tilde{\alpha});\tilde{p},y^j) = \sum_{\xi\in\Xi} \beta^j_\xi(\tilde{\theta},\tilde{\lambda},\tilde{\alpha}) \, \left(\tilde{p}_\xi \cdot y^j_\xi\right)$$

and for all $\xi \in \Xi$

$$\beta^j_\xi(\tilde{\theta},\tilde{\lambda},\tilde{\alpha}) := \sum_{h\in\mathcal{H}_{\xi_0^-}} \beta^{hj}_\xi(\tilde{\theta},\tilde{\lambda},\tilde{\alpha}) \tag{2.7}$$

$$\textit{with } \beta^{hj}_\xi(\tilde{\theta},\tilde{\lambda},\tilde{\alpha}) := \frac{\tilde{\lambda}^h_\xi}{\tilde{\lambda}^h_{\xi_0}}\tilde{\theta}^{hj}_{\xi^-} + \sum_{\xi'<\xi} \frac{\tilde{\lambda}^h_{\xi'}}{\tilde{\lambda}^h_{\xi_0}} \left(\tilde{\theta}^{hj}_{\xi'^-} - \tilde{\theta}^{hj}_{\xi'}\right) \frac{\tilde{\alpha}^h_\xi}{\tilde{\alpha}^h_{\xi'}}.$$

<u>Proof</u>: Starting from given portfolios, consumer state prices, consumer price theories, commodity prices and a production plan for firm j, $(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{p}, \tilde{y}^j)$, consider the effect of a marginal change of production at node ξ, $d\tilde{y}^j_\xi$, on the utility of individual $h \in \mathcal{H}^j_{\xi_0^-}$. To simplify notation we omit the tildes. Using the individual budget constraint and first order conditions we obtain:

$$\frac{du^h}{\lambda^h_{\xi_0}} = \frac{\lambda^h_\xi}{\lambda^h_{\xi_0}}\theta^{hj}_\xi(p_\xi dy^j_\xi) + \sum_{\xi'\leq\xi} \frac{\lambda^h_{\xi'}}{\lambda^h_{\xi_0}} \left(\theta^{hj}_{\xi'^-} - \theta^{hj}_{\xi'}\right) dq^j_{\xi'}.$$

To evaluate the price effect, the consumer uses her own price theory α^h: $dq^j_{\xi'} = \frac{\alpha^h_\xi}{\alpha^h_{\xi'}}(p_\xi dy^j_\xi)$ for all $\xi' \leq \xi$. Using this expression and rearranging terms we obtain:

$$\frac{du^h}{\lambda^h_{\xi_0}} = [\frac{\lambda^h_\xi}{\lambda^h_{\xi_0}}\theta^{hj}_{\xi^-} + \sum_{\xi'<\xi} \frac{\lambda^h_{\xi'}}{\lambda^h_{\xi_0}} \left(\theta^{hj}_{\xi'^-} - \theta^{hj}_{\xi'}\right) \frac{\alpha^h_\xi}{\alpha_{\xi'}}](p_\xi dy^j_\xi) = \beta^{hj}_\xi(p_\xi dy^j_\xi).$$

Thus the coefficient β^{hj}_ξ in the Lemma is exactly consumer h's marginal valuation, at the proposed allocation, of an additional unit of firm j's profit at node ξ. Initial shareholders' efficiency can thus be expressed by saying that there should not exist a change of production plan dy^j such that:

$$\sum_{h\in\mathcal{H}_{\xi_0^-}} \frac{du^h}{\lambda^h_{\xi_0}} = \sum_{\xi\in\Xi}\sum_{h\in\mathcal{H}_{\xi_0^-}} \beta^{hj}_\xi(p_\xi dy^j_\xi) = \sum_{\xi\in\Xi} \beta^j_\xi(p_\xi dy^j_\xi) > 0.$$

This will be the case if the firm uses the criterion V^j. $\square$

We can interpret the firm's criterion as follows. Each shareholder communicates to the firm her marginal valuation of profits at all date events (expressed in terms of initial resources). In defining her own marginal valuation of firm j's profits, a shareholder h evaluates profits at date-event ξ using her own shadow price for resources at ξ, multiplied by the shares held at that node, θ_{ξ}^{jh}. To evaluate the impact of profits at ξ on the market value of j at every node $\xi' < \xi$, h uses her price theory α^h, and applies the result to her trade at ξ' – thus with a positive sign in case of a sale, and a negative price in case of a purchase; the resulting impact is multiplied by h's shadow prices for resources at ξ', the date-event where the trade occurs.

The two terms are then added, and this sum defines h's marginal valuation of firm j's profits at date-event ξ, β_{ξ}^{hj}.

Firm j then computes its own vector of shadow prices for profits at all date-events, β^j, by simply adding up the marginal valuations of all its initial shareholders. There is no weighting involved, because shareholdings have been taken into account by the shareholders themselves in computing their own marginal valuations.

If consumers did not consider the effect of a change in profits on market value, $dq_{\xi'}^j = 0$ for all $\xi' \leq \xi$, the effect of dy_{ξ}^j on a consumer utility would simply be $\frac{du^h}{\lambda_{\xi_0}^h} = \frac{\lambda_{\xi}^h}{\lambda_{\xi_0}^h}\theta_{\xi}^{hj}(p_{\xi} dy_{\xi}^j)$, leading to:

$$\beta_{\xi}^j = \sum_{\mathcal{H}_{\xi_0^-}} \frac{\lambda_{\xi}^h}{\lambda_{\xi_0}^h}\theta_{\xi}^{hj},$$

where, as in the original Drèze criterion, a shareholder's weight at a given node is her equilibrium asset holding at that node [8].

If on the other hand , as in the paper by Grossman and Hart (1979), each consumer were to evaluate the effect on market value using her vector of shadow prices for resources, i.e. if for all h $\alpha^h = \lambda^h$, all terms in β^j involving changes in portfolios would drop out and we would obtain the Grossman-Hart criterion:

$$\beta_{\xi}^j = \sum_{\mathcal{H}_{\xi_0^-}} \frac{\lambda_{\xi}^h}{\lambda_{\xi_0}^h}\theta_{\xi_0^-}^{hj}. \tag{2.8}$$

[8] BL (2002) obtain a related formula, but with a crucial difference: on the right-hand-side the summation is over all h, not just over intial shareholders.

3 Competitive stock market equilibrium

The above data define the economy:

$$\mathcal{E} = \left((\mathbb{X}^h, u^h, w^h, A^h, (\theta^{hj}_{\xi_0})_{j\in\mathcal{J}})_{h\in\mathcal{H}}, (\mathbb{Y}^j)_{j\in\mathcal{J}} \right).$$

A *competitive stock market equilibrium* consists of spot prices $\tilde{p}$, stock prices $(\tilde{q}^j)_j$, consumer price theories $(\tilde{\alpha}^h)_h$, consumer state prices $(\tilde{\lambda}^h)_h$, consumption plans $(\tilde{x}^h)_h$, portfolio plans of ownership shares $(\tilde{\theta}^h)_h$, production plans $(\tilde{y}^j)_j$, such that,

(1) for each $\xi \in \Xi^-$ and $j \in \mathcal{J}$: $\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_\xi = 1$;

(2) for each $\xi \in \Xi$: $\sum_{h\in\mathcal{H}} (\tilde{x}^h_\xi - w^h_\xi) = \sum_{j\in\mathcal{J}} \tilde{y}^j_\xi$;

(3) for each $h \in \mathcal{H}$, $\tilde{\alpha}^h = A^h(\tilde{p}, \tilde{q}, \tilde{y})$;

(4) for each $h \in \mathcal{H}$:
$(\tilde{x}^h, \tilde{\theta}^h)$ solves $\text{Max}_{(x^h,\theta^h)} \left\{ u^h(x^h) \text{ s.t. } (x^h, \theta^h) \in \mathbb{B}^h(\tilde{p}, \tilde{q}, \tilde{y}) \right\}$, and $\frac{\tilde{\lambda}^h}{\tilde{\lambda}^h_0}$ is the vector of marginal rates of substitution between revenue at all date-events and at ξ_0;

(5) for each $j \in \mathcal{J}$:
$\tilde{y}^j$ solves $\text{Max}_{y^j} \left\{ V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}); \tilde{p}, y^j) \text{ s.t. } y^j \in \mathbb{Y}^j \right\}$ where:

(5a) for all $y^j \in \mathbb{Y}^j$, $V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}); \tilde{p}, y^j) = \sum_{\xi\in\Xi} \beta^j_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha})(\tilde{p}_\xi \cdot y^j_\xi)$;

(5b) for all $\xi \in \Xi$,

$$\beta^j_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}) := \sum_{h\in\mathcal{H}_{\xi_0^-}} \beta^{hj}_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha})$$

$$\text{with } \beta^{hj}_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}) := \frac{\tilde{\lambda}^h_\xi}{\tilde{\lambda}^h_{\xi_0}} \tilde{\theta}^{hj}_{\xi^-} + \sum_{\xi'<\xi} \frac{\tilde{\lambda}^h_{\xi'}}{\tilde{\lambda}^h_{\xi_0}} \left(\tilde{\theta}^{hj}_{\xi'^-} - \tilde{\theta}^{hj}_{\xi'} \right) \frac{\tilde{\alpha}^h_\xi}{\tilde{\alpha}^h_{\xi'}}$$

Conditions (1) and (2) are market clearing equations. Condition (4) and (5) are the consumers' and firms' optimization conditions. Finally, condition (3) expresses the no-arbitrage restriction on consumer price theories.

To prove the existence of an equilibrium, we use the following additional assumption, which, as in Grossman and Hart (1979) (cfr. our discussion in section 1.4), says that production can be stopped at any moment:

(*B*.5) At an arbitrary date-event $\xi \in \Xi$, and for any feasible production plan $y^j \in \mathbb{Y}^j$, there exists another feasible production plan $z^j \in \mathbb{Y}^j$ with $z^j(\xi') = 0$ for all $\xi' \in \Xi^+[\xi]$ and $z^j(\xi') = y^j(\xi')$ otherwise.

Theorem 3.1 *Under assumptions* $(A.1) - (A.6)$ *and* $(B.1) - (B.5)$ *there exists an equilibrium for* $\mathcal{E}$*.*

The proof is in the next section.

4 Proof of Theorem 3.1

To prove the existence of a competitive equilibrium we introduce an auxiliary concept, which we call pseudo equilibrium. A pseudo-equilibrium differs from an equilibrium only because, at a pseudo-equilibrium, we allow for the possible free disposal of shares when the price of a firm is zero.

At a pseudo equilibrium, consumers are aware of the possibility of free disposal of shares, they fully anticipate its occurrence, and revise their marginal valuation of an additional unit of revenue in a given state accordingly. The concept is not meant to have any descriptive appeal. We will use it only in an intermediate step, and prove that there always exists a pseudo-equilibrium in which no disposal of shares takes place, i.e. an equilibrium for our economy.

A *Pseudo-Equilibrium* of a competitive stock market consists of spot prices $\tilde{p}$, stock prices $(\tilde{q}^j)_j$, consumer price theories $(\tilde{\alpha}^h)_h$, consumer state prices $(\tilde{\lambda}^h)_h$, consumption plans $(\tilde{x}^h)_h$, portfolio plans of ownership shares $(\tilde{\theta}^h)_h$, production plans $(\tilde{y}^j)_j$, and anticipated "re-scaling factors" $(\tilde{\tau}^j)_j$ such that,

(1) for each $\xi \in \Xi^-$ and $j \in \mathcal{J}$: $\tilde{q}^j_\xi(\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_\xi - \sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi^-}) = 0$ and $(\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_\xi - \sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi^-}) \leq 0$;

(2) for each $\xi \in \Xi^-$: $\sum_{h\in\mathcal{H}}(\tilde{x}^h_\xi - w^h_\xi) = \sum_{j\in\mathcal{J}}(\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_\xi)\tilde{y}^j_\xi$; and for each $\xi \in \Xi_T$: $\sum_{h\in\mathcal{H}}(\tilde{x}^h_\xi - w^h_\xi) = \sum_{j\in\mathcal{J}}(\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi^-})\tilde{y}^j_\xi$;

(3) for each $h \in \mathcal{H}$, $\tilde{\alpha}^h = A^h(\tilde{p}, \tilde{q}, \tilde{y})$;

(4) for each $h \in \mathcal{H}$:

$(\tilde{x}^h, \tilde{\theta}^h)$ solves Max $_{(x^h,\theta^h)}$ $\{u^h(x^h)$ s.t. $(x^h, a^h) \in \mathbb{B}^h(\tilde{p}, \tilde{q}, \tilde{y})\}$, and $\frac{\tilde{\lambda}^h}{\tilde{\lambda}^h_0}$ is the vector of marginal rates of substitution between revenue at all date-events and at ξ_0;;

(5) for each $j \in \mathcal{J}$:

$\tilde{y}^j$ solves Max $_{y^j}$ $\left\{V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}); \tilde{p}, y^j) \text{ s.t. } y^j \in \mathbb{Y}^j\right\}$ where:

(5a) for all $y^j \in \mathbb{Y}^j$,

$V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}); \tilde{p}, y^j) = \sum_{\xi\in\Xi} \beta^j_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau})(\tilde{p}_\xi \cdot y^j_\xi)$;

(5b) for all $\xi \in \Xi$,

$$\beta^j_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) := \sum_{h\in\mathcal{H}_{\xi_0^-}} \beta^{hj}_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) \tag{4.1}$$

$$\text{with } \beta^{hj}_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) := \frac{\tilde{\lambda}^h_\xi}{\tilde{\lambda}^h_{\xi_0}} \tilde{\tau}^j_{\xi^-} \tilde{\theta}^{hj}_{\xi^-} + \sum_{\xi'<\xi} \frac{\tilde{\lambda}^h_{\xi'}}{\tilde{\lambda}^h_{\xi_0}} \left(\tilde{\tau}^j_{\xi'^-} \tilde{\theta}^{hj}_{\xi'^-} - \tilde{\tau}^j_{\xi'} \tilde{\theta}^{hj}_{\xi'}\right) \frac{\tilde{\alpha}^h_\xi}{\tilde{\alpha}^h_{\xi'}}$$

(6) for each $j \in \mathcal{J}$ and $\xi \in \Xi^-$:

$$\tilde{\tau}^j_\xi = \frac{1}{\sum_h \tilde{\theta}^j_\xi}.$$

Condition (1) says that excess supply of shares might be possible at some date-events if the price is zero. The consequences on the commodity markets are expressed in condition (2): production is scaled down by a factor $\sum_h \tilde{\theta}^{hj}_\xi \leq \sum_h \tilde{\theta}^{hj}_{\xi^-} \leq 1$. Condition (3) expresses the no-arbitrage restriction on consumer price theories. Conditions (4) and (5) are the consumers' and firms' optimization conditions. In the definition of the objective of the firm we have taken into account that free disposal of shares may alter the actual share of a firm held by a consumer at the pseudo equilibrium, namely $\tilde{\tau}^j_\xi \tilde{\theta}^{hj}_\xi$. Finally, condition (6) defines the rescaling factors τ^j_ξ in terms of the actual number of shares held by consumers at each node.

It is of central importance for us to notice here that, if at a pseudo equilibrium $\tilde{\tau}^j_\xi = 1$ for all j and all ξ, the corresponding prices, price theories and allocations $(\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y})$ constitute an equilibrium for the economy.

The proof of Theorem 3.1 will be divided in three parts. In Part I we show that a pseudo equilibrium exists for an economy compactified by imposing some artificial bounds. In Part II we show that, starting from a pseudo-equilibrium, we can construct an equilibrium. Finally, in PART III we show that the artificial bounds can be removed.

PART I.

We first introduce some notation. Let $\text{proj}_N(K)$ be the orthogonal projection of the subset $K \subset \mathbb{R}^n$ on the subset $N \subset \mathbb{R}^m$, where n and m are non-null integers. Such a mapping is continuous on its domain. Thus the image of any compact subset of its domain is a compact subset as well. Endow the n dimensional Euclidean space $\mathbb{R}^n$ with the l^1-norm: for a generic vector $x := (x_d)_{d=1}^n \in \mathbb{R}^n$, $\|x\| = \sum_{d=1}^n |x_d|$.

We normalize prices as follows. Let

$$\mathbb{Q} := \left\{ (p,q) \in \mathbb{R}_+^{L\Xi} \times \mathbb{R}_+^{\mathcal{J}(\Xi^-)} \;\middle|\; \begin{array}{ll} \|p_\xi\| + \|q_\xi\| = 1 & \forall \xi \in \Xi^-, \\ \|p_\xi\| = 1 & \forall \xi \in \Xi_T, \end{array} \right\},$$

be the space of the spot and stock prices p, q; the usual simplex $\Delta_{\Xi-1} := \{\lambda^h \in \mathbb{R}_+^\Xi \mid \sum_{\xi\in\Xi} \lambda_\xi^h = 1\}$ be the space of all λ^h, for each $h \in H$; and $\Delta_{\Xi-1}^b := \{\alpha^h \in \mathbb{R}_+^\Xi \mid \sum_{\xi\in\Xi} \alpha_\xi^h = 1 \text{ and } b \le \alpha_\xi^h \forall \xi \in \Xi\}$ be the space of the price theories for each $h \in H$.

Next, we impose some artificial bounds on choice spaces.

Define

$$\mathbb{F} := \left\{ (x,y) \in (\mathbb{R}^{L\Xi})^H \times (\bigotimes_{j\in\mathcal{J}} \mathbb{Y}^j) \mid \sum_{h\in\mathcal{H}} (x^h - w^h) - \sum_{j\in\mathcal{J}} y^j \le 0 \right\}$$

as the set of feasible actions for commodities. Then, according to Assumption $(B.4)$, there exists a positive real number $\bar{m}$, such that, for any $m \ge \bar{m}$, $\mathbb{X}_m^h := \mathbb{R}_+^{L\Xi} \cap M \subset \text{proj}_{\mathbb{R}_+^{L\Xi}} \mathbb{F}$ for all $h \in \mathcal{H}$, and $\mathbb{Y}_m^j := \mathbb{Y}^j \cap M \subset \text{proj}_{\mathbb{Y}^j} \mathbb{F}$ for all $j \in \mathcal{J}$, where M defines the n-dimensional hyper-cube $[-m, m]^n$ with $n := (\Xi H + \Xi J)$.

We impose an artificial lower bound on portfolios $b^{hj}(\epsilon) := \epsilon \theta_{\xi_0^-}^{hj}$ to avoid $\sum_h \theta_\xi^{hj} = 0$ at any ξ. In Lemma **(4.1)** we will show that for any m big enough we can always choose $\epsilon(m)$ small enough that $\mathbb{B}^h(p,q,y) \ne \emptyset$ on the relevant domain. Let $\Theta_m^h = ([b^{hj}(\epsilon(m)), m]^{\mathcal{J}})^{\Xi^-}$ denote the set of constrained portfolios.

Lemma 4.1 *Let* $\mu = min\ \{w_{\ell\xi}^h \mid \ell \in \mathcal{L}, h \in \mathcal{H} \text{ and } \xi \in \Xi\}$ *and* $Jm > \mu$. *If* $\epsilon(m) = (\frac{\mu}{Jm})^{T+1}$, *then for any arbitrary* $(p,q) \in$

$\mathbb{Q}$ *and* $y \in \bigotimes_{j\in\mathcal{J}} \mathbb{Y}^j_m$, *consumer* h*'s budget set is non - empty, i.e.* $\mathbb{B}^h(p,q,y) \neq \emptyset$.

Now, having bounded portfolio holdings away from zero and from above, we can limit the range of the scaling factors $\tau^j_\xi = \frac{1}{\sum_h \theta^{hj}_\xi}$ $(\xi \in \Xi^-)$ to a compact set $\mathbb{T} \subset \mathbb{R}^{\Xi^-}_+$.

Call the truncated economy $\mathcal{E}_m$.

Proposition 4.1 *Assume* $(A.1) - (A.6)$ *and* $(B.1) - (B.4)$. *Then there exists a pseudo-equilibrium for* $\mathcal{E}_m$.

Proof of Proposition 4.1: The pseudo-equilibrium existence proof parallels that in Radner (1972). We provide a sketch, divided in two main steps. The proofs of some intermediary lemmas are collected in Appendix **A**.

Step 1: Defining the fixed point correspondence.

Let $\mathbb{Z} = \mathbb{Q} \times \bigotimes_{\mathcal{H}} \Delta^b_{\Xi-1} \times \bigotimes_{\mathcal{H}} \Delta_{\Xi-1} \times \bigotimes_{h\in\mathcal{H}} \mathbb{X}^h_m \times \bigotimes_{h\in\mathcal{H}} \Theta^h_m \times \bigotimes_{j\in\mathcal{J}} \mathbb{Y}^j_m \times \bigotimes_{\mathcal{J}} \mathbb{T}$ and $z = (p,q,\alpha,\lambda,x,\theta,y,\tau)$. The pseudo-equilibria correspondence is defined as

$$\begin{aligned} \mathcal{G} &: \mathbb{Z} \rightrightarrows \mathbb{Z} \\ z &\mapsto ((\mathcal{C}^h(z))_h, (\mathcal{P}^j(z))_j, \mathcal{M}(z), (A^h(z))_h), (\Lambda^h(z))_h, (\mathcal{T}^j(z))_j) \end{aligned} \tag{4.2}$$

where $A^h : \mathbb{Q} \times \bigotimes_{j\in\mathcal{J}} \mathbb{Y}^j_m \to \Delta^b_{\Xi-1}$ is the continuous function describing h's price theory, $\mathcal{T}^j : \bigotimes_{h\in\mathcal{H}} \Theta^h \to \mathbb{T}$ is the continuous function defining firm j's scaling factors, as in point (6) of the definition of pseudo-equilibrium; and the correspondences $\{(\mathcal{C}^h)_h, (\mathcal{P}^j)_j, \mathcal{M}, (\Lambda^h)_h\}$ are respectively, consumers' demand correspondences, firms' choice correspondences, and the market auctioneer's correspondences, that we define as follows.

For any $h \in H$,

$$\begin{aligned} \mathcal{C}^h &: \mathbb{Z} \rightrightarrows \mathbb{X}^h_m \times \Theta^h_m \\ z &\mapsto \ \text{Argmax} \left\{ u^h(x^h) \mid (x^h, \theta^h) \in \underline{\mathbb{B}}^h(p,q,\lambda^h,y) \right\} \end{aligned} \tag{4.3}$$

where $\underline{\mathbb{B}}^h(p,q,\lambda^h,y) = \{(x^h,\theta^h) \in \mathbb{X}^h_m \times \Theta^h_m \mid \lambda^h \cdot G^h(p,q,y,x,\theta) \leq 0\}$ and G^h is the column vector whose Ξ elements are $(p_\xi \cdot (x^h_\xi - w^h_\xi) - \sum_j (\theta^h_{\xi^-} - \theta^h_\xi) q^j_\xi - \sum_j \theta^h_\xi (p_\xi \cdot y^j_\xi))$ for the first Ξ^- components and $(p_\xi \cdot (x^h_\xi - w^h_\xi) - \sum_j \theta^h_{\xi^-}(p_\xi \cdot y^j_\xi))$ for the Ξ_T terminal ones.

For any $j \in \mathcal{J}$, let $\underline{\beta}^j$ the mapping defined from $\bigotimes_{h\in\mathcal{H}} \Theta^h_m \times \bigotimes_{\mathcal{H}} \Delta_{\Xi-1} \times \bigotimes_{\mathcal{H}\mathcal{J}} \Delta^b_{\Xi-1} \times \bigotimes_j \mathbb{T}^j$ to $\mathbb{R}^{\Xi}$ by

$\underline{\beta}^j_{\xi_0}(\theta, \lambda, \alpha, \tau) = \prod_h \lambda^h_{\xi_0}$ if $\xi = \xi_0$ and

$$\underline{\beta}^j_{\xi_0}(\theta, \lambda, \alpha, \tau) = \sum_{h\in\mathcal{H}^j_{\xi_0^-}} \Big(\lambda^h_\xi \tau^j_{\xi^-}\theta^{hj}_{\xi^-} + \sum_{\xi'<\xi} \lambda^h_{\xi'}(\tau^j_{\xi'^-}\theta^{hj}_{\xi'^-} - \tau^j_{\xi'}\theta^{hj}_{\xi'})\frac{\alpha^h_\xi}{\alpha^h_{\xi'}}\Big) \prod_{h'\neq h} \lambda^{h'}_{\xi_0}$$

otherwise. Define the set $\mathbb{S}^j_m = \underline{\beta}^j(\bigotimes_{h\in\mathcal{H}} \Theta^h_m \times \bigotimes_{\mathcal{H}} \Delta_{\Xi-1} \times \bigotimes_{\mathcal{H}} \Delta^b_{\Xi-1} \times \bigotimes_{\mathcal{J}} \mathbb{T})$. Then, let V^j be the real valued mapping defined on $\mathbb{S}^j_m \times \mathrm{proj}_{\mathbb{R}^{L\Xi}_+}(\mathbb{Q}) \times \mathbb{Y}^j_m$ by $V^j(\underline{\beta}^j; p, y^j) = \sum_{\xi\in\Xi} \underline{\beta}^j_\xi(\theta, \lambda, \alpha, \tau)(p_\xi \cdot y^j_\xi)$.

Then, for any $j \in J$,

$$\begin{aligned} \mathcal{P}^j : \mathbb{Z} &\rightrightarrows \mathbb{Y}^j_m \\ z &\mapsto \operatorname{Argmax}\left\{V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau); p, y^j) \mid y^j \in \mathbb{Y}^j_m\right\}. \end{aligned} \tag{4.4}$$

Let

$$\begin{aligned} \mathcal{M} : \mathbb{Z} &\rightrightarrows \mathbb{Q} \\ z &\mapsto \mathcal{M}(z) \end{aligned} \tag{4.5}$$

where

$$\mathcal{M}(z) := \left\{ (p,q) \in \mathbb{Q} \;\middle|\; \begin{array}{l} \forall (p',q') \in \mathbb{Q}, \\ (p_\xi - p'_\xi) \cdot \big(\sum_{h\in\mathcal{H}}(x^h_\xi - w^h_\xi) - \sum_{j\in\mathcal{J}}(\sum_{h\in\mathcal{H}} \theta^{hj}_\xi) y^j_\xi)\big) + \\ + \sum_{j\in\mathcal{J}}(q^j_\xi - q'^j_\xi)\big(\sum_{h\in\mathcal{H}} \theta^{hj}_\xi - \sum_{h\in\mathcal{H}} \theta^{hj}_{\xi^-}\big) \geq 0, \xi \in \Xi^-; \\ (p_\xi - p'_\xi) \cdot (\sum_{h\in\mathcal{H}}(x^h_\xi - w^h_\xi) - \sum_{j\in\mathcal{J}}(\sum_{h\in\mathcal{H}} \theta^{hj}_{\xi^-}) y^j_\xi) \geq 0, \xi \in \Xi_T \end{array} \right\}$$

Finally, for any $h \in H$, let

$$\begin{aligned} \Lambda^h : \mathbb{Z} &\rightrightarrows \Delta_{\Xi-1} \\ z &\mapsto \operatorname{Argmax}\left\{\lambda^h \cdot G^h(p, q, x, \theta, y) \mid \lambda^h \in \Delta_{\Xi-1}\right\}. \end{aligned} \tag{4.6}$$

The correspondence $\mathcal{G}$ embodies all equilibrium correspondences for consumers, producers, and the auctioneer, respectively. The set $\mathbb{Z}$ is compact and convex by construction and Assumptions $(B.1)$ and $(B.2)$. The set $\mathbb{S}$ is compact as well. We need to show that the correspondence $\mathcal{G}$ satisfies the hypotheses of Kakutani's Fixed-Point Theorem, so that a fixed point $\tilde{z}_m$ (we omit the subscript m from variables, unless specified otherwise) exists.

Lemma 4.2 *The demand correspondences* $(\mathcal{C}^h)_h$*, the supply correspondences* $(\mathcal{P}^j)_j$ *and the auctioneer's correspondences* $\mathcal{M}$ *and* $(\Lambda^h)_h$ *are non empty and convex valued, and upper hemi-continuous.*

Lemma 4.3 *The pseudo-equilibrium correspondence* $\mathcal{G}$ *is non empty and convex valued, and upper hemi-continuous.*

By Kakutani's Fixed-Point Theorem, there exists a fixed point $\tilde{z}_m := (\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{a}, \tilde{y}, \tilde{\tau}) \in \mathbb{Z}$.

<u>*Step 2:*</u> The fixed point $\tilde{z}_m$ is a pseudo-equilibrium for the truncated economy $\mathcal{E}_m$.

Lemma 4.4 *For all* $h \in \mathcal{H}$*,*
$(\tilde{x}^h, \tilde{\theta}^h)$ *solves Max* $_{(x^h,\theta^h)}\{u^h(x^h)$ *s.t.* $(x^h, \theta^h) \in \mathbb{B}^h(\tilde{p}, \tilde{q}, \tilde{y}) \cap [\mathbb{X}^h_m \times \Theta^h_m]\}$*, and the components of* $\tilde{\lambda}^h \in \mathbb{R}^{\Xi}_{++}$ *are the Lagrange multipliers of the above maximization problem.*

Lemma 4.5 *For all* $j \in \mathcal{J}$*,*
$\tilde{y}^j$ *solves Max* $_{y^j}$ $\left\{V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tau); \tilde{p}, y^j) \text{ s.t. } y^j \in \mathbb{Y}^j_m\right\}$ *where for all* $y^j \in \mathbb{Y}^j_m$*,* $V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}); \tilde{p}, y^j) = \sum_{\xi \in \Xi} \beta^j_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau})(\tilde{p}_\xi \cdot y^j_\xi)$*, and for all* $\xi \in \Xi$

$$\beta^j_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) := \sum_{h \in \mathcal{H}_{\xi_0^-}} \beta^{hj}_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau})$$

$$\text{with } \beta^{hj}_\xi(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) := \frac{\tilde{\lambda}^h_\xi}{\tilde{\lambda}^h_{\xi_0}} \tilde{\tau}^j_{\xi^-} \tilde{\theta}^{hj}_{\xi^-} + \sum_{\xi' < \xi} \frac{\tilde{\lambda}^h_{\xi'}}{\tilde{\lambda}^h_{\xi_0}} \left(\tilde{\tau}^j_{\xi'^-} \tilde{\theta}^{hj}_{\xi'^-} - \tilde{\tau}^j_{\xi'} \tilde{\theta}^{hj}_{\xi'}\right) \frac{\tilde{\alpha}^h_\xi}{\tilde{\alpha}^h_{\xi'}}.$$

Lemma 4.6 *For all* $\xi \in \Xi^-$*,*

$$\sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_\xi - \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_{\xi^-} \leq 0 \quad \textit{and} \quad \sum_{h \in \mathcal{H}} (\tilde{x}^h_\xi - w^h_\xi) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_\xi \tilde{y}^j_\xi \leq 0.$$

For all $\xi \in \Xi_T$*,*

$$\sum_{h \in \mathcal{H}} (\tilde{x}^h_\xi - w^h_\xi) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_{\xi^-} \tilde{y}^j_\xi \leq 0.$$

Lemma 4.7 *For all* $h \in \mathcal{H}$*,*

$$\tilde{p}_\xi \cdot \tilde{x}^h_\xi + \sum_{j \in \mathcal{J}} \tilde{q}^j_\xi \tilde{\theta}^{hj}_\xi = \tilde{p}_\xi \cdot w^h_\xi + \sum_{j \in \mathcal{J}} \tilde{q}^j_\xi \tilde{\theta}^{hj}_{\xi^-} + \sum_{j \in \mathcal{J}} \tilde{\theta}^{hj}_\xi (\tilde{p}_\xi \cdot \tilde{y}^j_\xi) \textit{ for all } \xi \in \Xi^-,$$

$$\tilde{p}_\xi \cdot \tilde{x}^h_\xi = \tilde{p}_\xi \cdot w^h_\xi + \sum_{j \in \mathcal{J}} \tilde{\theta}^{hj}_{\xi^-}(\tilde{p}_\xi \cdot \tilde{y}^j_\xi) \textit{ for all } \xi \in \Xi_T.$$

Lemma 4.8 *For all* $\xi \in \Xi^-$,

$$\sum_{h \in \mathcal{H}} (\tilde{x}^h_\xi - w^h_\xi) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_\xi \tilde{y}^j_\xi = 0$$

$$\textit{and for all } j \in \mathcal{J}, \tilde{q}^j_\xi \left(\sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_\xi - \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_{\xi^-} \right) = 0. \quad (4.7)$$

For all $\xi \in \Xi_T$,

$$\sum_{h \in \mathcal{H}} (\tilde{x}^h_\xi - w^h_\xi) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj}_{\xi^-} \tilde{y}^j_\xi = 0.$$

This ends the proof of Proposition 4.1.

Part II.

Proposition 4.2 *Assume* $(A.1) - (A.6)$ *and* $(B.1) - (B.5)$. *Then there exists an equilibrium for* $\mathcal{E}_m$.

Proof of Proposition 4.2: We start with the following Lemma showing that, at a pseudo-equilibrium, the stream of future dividends at any given node, evaluated using $\tilde{\beta}$, does not exceed the observed price of the firm at that node.

Lemma 4.9 *At a pseudo equilibrium, for all* j *and all* ξ,

$$\tilde{\beta}^j_\xi \tilde{q}^j_\xi \geq \sum_{\xi' \geq \xi} \tilde{\beta}^j_{\xi'}(\tilde{p}_{\xi'} \cdot \tilde{y}_{\xi'})$$

Proof of Lemma 4.9: To simplify notation, consider the case of a single initial owner, and omit the indexes h and j, and the tildes. We also use the notation $\bar{\lambda}_\xi = \frac{\lambda_\xi}{\lambda_{\xi_0}}$. At any given ξ:

$$\sum_{\xi' \in \xi_+} \beta_{\xi'} q_{\xi'} = \sum_{\xi' \in \xi_+} \left(\bar{\lambda}_{\xi'} \tau_\xi \theta_\xi + \sum_{\xi_0 \leq \xi" \leq \xi} \bar{\lambda}_{\xi"} (\tau_{\xi"_-} \theta_{\xi"_-} - \tau_{\xi"} \theta_{\xi"}) \frac{\alpha_{\xi'}}{\alpha_{\xi"}} \right) q_{\xi'}.$$

We can rewrite the term on the right as:

$$\tau_\xi\theta_\xi \sum_{\xi'\in\xi_+} \bar\lambda_{\xi'} q_{\xi'} + \Big(\sum_{\xi_0\le\xi"<\xi} \bar\lambda_{\xi"} (\tau_{\xi"_-}\theta_{\xi"_-} - \tau_{\xi"}\theta_{\xi"}) \frac{\alpha_\xi}{\alpha_{\xi"}} \Big) \Big(\sum_{\xi'\in\xi_+} \frac{\alpha_{\xi'}}{\alpha_\xi} q_{\xi'} \Big)$$
$$+ \bar\lambda_\xi (\tau_{\xi_-}\theta_{\xi_-} - \tau_\xi\theta_\xi) \sum_{\xi'\in\xi_+} \frac{\alpha_{\xi'}}{\alpha_\xi} q_{\xi'}.$$

Using the KT conditions of the decision problems of consumers (2.4)-(2.5), and the no-arbitrage condition for α (2.6), we obtain:

$$\sum_{\xi'\in\xi_+} \beta_{\xi'} q_{\xi'} \le \Big(\bar\lambda_\xi \tau_\xi \theta_\xi + \sum_{\xi_0\le\xi"<\xi} \bar\lambda_{\xi"} (\tau_{\xi"_-}\theta_{\xi"_-} - \tau_{\xi"}\theta_{\xi"}) \frac{\alpha_\xi}{\alpha_{\xi"}}$$
$$+ \bar\lambda_\xi (\tau_{\xi_-}\theta_{\xi_-} - \tau_\xi\theta_\xi) \Big) (q_\xi - p_\xi \cdot y_\xi)$$

that is:

$$\sum_{\xi'\in\xi_+} \beta_{\xi'} q_{\xi'} \le \beta_\xi \left(q_\xi - p_\xi \cdot y_\xi \right).$$

A simple recursion argument using the fact that at the terminal nodes stock prices are null concludes the proof. □

We now argue that if at a pseudo-equilibrium, for some firm j and some node ξ, $\tilde q^j_\xi = 0$ and $\sum_h \theta^{jh}_\xi < \sum_h \theta^{jh}_{\xi-}$, we can always re-scale the production plan, the shareholdings and the price and obtain a full equilibrium.

Consider a pseudo equilibrium $(\tilde p, \tilde q, \tilde\alpha, \tilde\lambda, \tilde x, \tilde\theta, \tilde y, \tilde\tau)$ with $\tilde q^j_\xi = 0$ for some firm j and some node ξ.

First notice that:

$$\sum_{\xi'\ge\xi} \tilde\beta^j_{\xi'} (\tilde p_{\xi'} \cdot \tilde y^j_{\xi'}) = 0. \tag{4.8}$$

Indeed, from Lemma 4.9, $\sum_{\xi'\ge\xi} \tilde\beta^j_{\xi'} (\tilde p_{\xi'} \cdot \tilde y^j_{\xi'}) \le 0$. Using $(B.5)$ we can conclude that the inequality is in fact an equality because otherwise the firm would have done better by stopping activity at node ξ.

We now re-scale production plans, portfolios and prices as follows: $\hat\theta^{jh}_\xi = \tilde\tau^j_\xi \tilde\theta^{jh}_\xi$, $\hat y^j_\xi = \frac{1}{\tilde\tau^j_\xi} \tilde y^j_\xi$, $\hat q^j_\xi = \frac{1}{\tilde\tau^j_\xi} \tilde q^j_\xi$.

Clearly, $\sum_h \hat\theta^{jh}_\xi = 1$ for all ξ, and we may set $\hat\tau_\xi = 1$ for all ξ. Then, for all j and all ξ, $\beta^j_\xi(\tilde\theta, \tilde\lambda, \tilde\alpha, \tilde\tau) = \beta^j_\xi(\hat\theta, \tilde\lambda, \tilde\alpha, \hat\tau) = \hat\beta^j_\xi$ (see eq. (4.1)).

The rescaled portfolios are feasible: $\epsilon\theta^{hj}_{\xi_0^-} \leq \hat{\theta}^{hj}_{\xi_0} \leq 1$ for all $\xi \in \Xi^-$. Indeed, the right side inequality is easely obtained, since by the positivity of all $\tilde{\theta}^{hj}_{\xi}$ we have $\tilde{\theta}^{hj}_{\xi} \leq \sum_h \tilde{\theta}^{hj}_{\xi} = 1/\tilde{\tau}^j_{\xi}$ for all $\xi \in \Xi^-$. The left side inequality is obtained proceeding as follows. Considering the first relation in Lemma 4.6, we obtain $\sum_h \tilde{\theta}^{hj}_{\xi} \leq \sum_h \tilde{\theta}^{hj}_{\xi_0} = 1$ for all $\xi \in \Xi^-$, that is $1 \leq \tilde{\tau}^j_{\xi}$ for all $\xi \in \Xi^-$. Then, if in addition we use the feasibility of the optimal portfolios $\tilde{\theta}^{hj}_{\xi}$ with $\epsilon\theta^{hj}_{\xi_0^-} \geq 0$, we get $\epsilon\theta^{hj}_{\xi_0^-} \leq \epsilon\tilde{\theta}^{hj}_{\xi_0^-}\tau^j_{\xi} \leq \tilde{\theta}^{hj}_{\xi}\tau^j_{\xi} = \hat{\theta}^{hj}_{\xi}$ for all $\xi \in \Xi^-$.

The re-scaling does not affect the individual budget, nor the consumers' first order conditions (cfr. (2.1), keeping in mind that $\tilde{q}^j_{\xi} = 0$ whenever $\tilde{\tau}_{\xi} > \tilde{\tau}_{\xi^-}$).

Finally, we must show that the rescaled production plan $\hat{y}^j$ is feasible and optimal for firm j at the (unchanged) state prices $\hat{\beta}^j = \tilde{\beta}^j$.

Proceeding backwards, let t_1 be the largest t such that for some $\xi_1 \in \Xi_1$ and some j we have $\tilde{\tau}^j_{\xi_1} > \tilde{\tau}^j_{\xi_1^-}$. Then $\tilde{q}^j_{\xi_1} = 0$ and, by (4.8),

$$\sum_{\xi' \geq \xi_1} \tilde{\beta}^j_{\xi'}(\tilde{p}_{\xi'} \cdot \tilde{y}^j_{\xi'}) = 0. \tag{4.9}$$

For every such ξ_1, proceed as follows. Re-scale $\tilde{y}^j$ in the sub-tree starting at ξ_1 by a factor $(\tilde{\tau}^j_{\xi_1^-})/(\tilde{\tau}^j_{\xi_1}) < 1$. By assumption (B.5) the re-scaled production plan is feasible. Moreover, by equation ((4.9)), its value at state prices $\hat{\beta}^j = \tilde{\beta}^j$ is zero. That is, the rescaled plan is maximizing firm j profit at the given state prices in the sub-tree starting at ξ_1.

Proceed backwards and repeat the foregoing operation as many times as needed if there exists another date t_2 at which for some $\xi_2 \in \Xi_2$ and some j we have $\tilde{\tau}^j_{\xi_2} > \tilde{\tau}^j_{\xi_2^-}$.

Notice that these successive re-scalings generate exactly the production plan $\hat{y}^j$. Thus, we conclude that this re-scaled plan belongs to the efficient boundary of the production set Y^j at the (unchanged) state prices used by the firm.

To summarize, we have shown that: for the truncated economy $\mathcal{E}_m$, a quasi-equilibrium $\tilde{z}_m = (\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{\tau})$ leads to an equilibrium $\hat{z}_m = (\tilde{p}, \hat{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \hat{\theta}, \hat{y})$ after re-scaling.

Part III.

Proposition 4.3 *Assume Assumptions* $(A.1)-(A.6)$ *and* $(B.1)-(B.4)$*. If* $m \to \infty$ *then* $\hat{z}_m \to \hat{z}$*, an equilibrium of* $\mathcal{E}$*.*

Proof of Proposition 4.3: Assumption $(B.4)$ ensures that one can choose m big enough so that all endogenous variables of the sequence of pseudo-equilibria $\hat{z}_m$ other than ownership shares are in the interior of the hyper-cube M, moreover $\sum_h \hat{\theta}^{hj} = 1$ for all j and all ξ, so that the the upper bound on ownership shares is not binding. By Assumptions $(A.2)$ and $(A.4)$, $(\tilde{x}^h, \hat{\theta}^h)$ is a solution of consumer h unconstrained decision problem. Using Assumption $(B.1)$ and the convexity of the objective function V^j of firm j on $\mathbb{Y}^j$, $\hat{y}^j$ is a solution of firm j decision problem.

This concludes the proof of Theorem 3.1.

References

[1] Arrow K., (1953): "Le rôle des valeurs boursières pour la repartition la meillure des risques," Econometrie, Colloq. Internat. Centre National de la Recherche Scientifique 40 (Paris 1952), 41-47; discussion, 47-48, C.N.R.S. (Paris 1953); English translation in *Review of Economic Studies* 31 (1964), 91-96.

[2] Bisin A., (1998): "General Equilibrium and Endogenously Incomplete Financial Markets", *Journal of Economic Theory*, vol. 82, 19-45.

[3] Bonnisseau J .M. O. Lachiri, (2002): "Drèze's criterion in a multi-periods economy with stock markets", *Cahiers de la MSE Série bleue*, no. 2002.98.

[4] ———, (2004): "On the Objective of Firms under Uncertainty with Stock Markets", *Journal of Mathematical Economics*, vol. 40, no. 5.

[5] Debreu G., (1959): *The Theory of Value: An axiomatic analysis of economic equilibrium*, New Haven and London, Yale University Press.

[6] Drèze J. H., (1974): "Investment under Private ownership: Optimality, Equilibrium and Stability", in *Allocation Under Uncertainty; Equilibrium and Optimality*, J.H. Drèze (ed.), London: Macmillan, pp. 129–66.

[7] ———, (1985): "(Uncertainty and) The Firm in General Equilibrium Theory", *Economic Journal*, 95 (Sup- plement: Conference Papers), pp. 1–20.

[8] ______, (1989): *Labour Management, Contracts and Capital Markets: A General Equilibrium Approach*, Oxford: Basil Blackwell Inc.

[9] ______, (1999): "The formulation of Uncertainty: Prices and States", Chap. 3 in G. Chichilnisky, Ed., *Markets, Information, and Uncertainty, Essays in Economic Theory in Honor of K.J. Arrow*, Cambridge University Press Cambridge, UK.

[10] Duffie D. and W. Shafer, (1986): "Equilibrium and the Role of the Firm in Incomplete Markets", *Research Paper* no. 915, Graduate School of Business, Stanford University.

[11] Geanakoplos J., (1990): "Introduction to General Equilibrium with Incomplete Asset Markets," *Journal of Mathematical Economics*, vol. 19, issue (1-2), pp. 1–38.

[12] Geanakoplos J., M. Magill, M. Quinzii and J. H. Drèze, (1990): "Generic Inefficiency of Stock Market Equilibium When Markets Are Incomplete", *Journal of Mathematical Economics*, vol. 19, issue (1), pp. 113–151.

[13] Geanakoplos J. and H.M. Polemarchakis, (1986): "Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete," in W. Heller, R. Starr and D. Starrett (eds.), *Uncertainty, Information and Communication: Essays in Honor of K. J. Arrow*, vol. III, Cambridge University Press.

[14] Grossman S. J. and O. Hart, (1979): "A Theory of Competitive Equilibrium in Stock Market Economies", *Econometrica*, vol. 47, pp. 293-330.

[15] Magill, M. and M. Quinzii, (1996): *Theory of Incomplete Markets*, Cambridge (Mass.): MIT Press.

[16] Magill, M. and W. Shafer, (1991): "Incomplete Markets", p. 1523-1614 in *Handbook of Mathetical Economics*, vol. IV, North-Holland, Amsterdam.

[17] Miller, M. , (1991): *Financial Innovations and Market Volatility*, Blackwell, Oxford.

[18] Radner, R. , (1972): "Existence of Equilibrium of Plans, Prices, and Price Expectations in Sequence of Markets", *Econometrica*, 40, pp. 289-303.

Appendix: Proofs of Lemmas

Proof of Lemma 4.1

We prove the result by defining an ϵ and a portfolio such that constraints **(2.1)** are satisfied for an arbitrary event ξ that may occur at time t. Since the stock market does not exist in terminal date-events, let for any $j \in \mathcal{J}$ and any $\xi \in \Xi_T$, $q^j_\xi = 0$ and $\theta^{hj}_\xi = \theta^{hj}_{\xi^-} \forall h \in \mathcal{H}$.

Let $0 < \eta < 1$ and, for an arbitrary event $\xi \in \Xi$ that may occur at time t $(0 \leq t \leq T)$, let $\underline{\theta}^h_\xi = \eta^{t+1}\theta^h_{\xi_0{}^-}$.

At this portfolio, the income of consumer h at ξ is

$$\sum_{j \in \mathcal{J}} \eta^t(1-\eta)\theta^{hj}_{\xi_0{}^-} q^j_\xi + \sum_{\ell \in \mathcal{L}} p_{\ell\xi} \left(w^h_{\ell\xi} + \sum_{j \in \mathcal{J}} \eta^{t+1}\theta^{hj}_{\xi_0{}^-} y^j_{\ell\xi} \right).$$

The first term is positive since $0 < \eta^{t+1} < 1$, while the second term is positive for all η such that $0 < \eta \leq \frac{\mu}{Jm} < 1$, where $\mu = min\{w^h_{\ell\xi} : \ell \in \mathcal{L}, h \in \mathcal{H}, \xi \in \Xi\}$ and m big enough so that $\frac{\mu}{Jm} < 1$.

Indeed, each of the terms inside the parentheses is bounded below by $\mu - Jm\eta^{t+1}$ because $-m \leq y^j_{\ell\xi} \leq 0$ for all $(\ell, j) \in \mathcal{L} \times \mathcal{J}$, $\theta^{hj}_{\xi_0^-} \geq 0$ for all $j \in \mathcal{J}$ and $\sum_{j \in \mathcal{J}} \theta^{hj}_{\xi_0^-} = 1$. If $\eta^{t+1} \leq \frac{w^h_{\ell\xi}}{Jm}$ for all $\ell \in \mathcal{L}$ at date-event (t, ξ), the lower bound is positive.

Feasibility of $\underline{\theta}^h$ is guaranteed by taking $\epsilon = \eta^{T+1}$, since $0 < \eta < 1$.

Proof of Lemma 4.2

We shall proceed in three parts.

Part 1: A continuous function defined on a compact set attains a maximum. Thus the demand correspondence $\mathcal{C}^h$ is non empty valued thanks to the continuity of u^h. The upper hemi-continuity of $\mathcal{C}^h$ follows from its definition, the continuity of u^h and Berge's maximum principle. To apply Berge's theorem we must show that the correspondence $\underline{\mathbb{B}}^h$ defined from the non empty convex compact subset $\mathbb{Q} \times \Delta_{\Xi-1} \times \bigotimes_{j \in \mathcal{J}} \mathbb{Y}^j_m$ to the non empty convex compact subset $\mathbb{X}^h_m \times \Theta^h_m$ is convex, non empty valued and continuous. Non empty convex valuedness and upper hemi-continuity are immediate. To prove the lower hemi-continuity of the correspondence $\underline{\mathbb{B}}^h$ we use its scaling property, also an immediate fact given $(A.5)$, $w \gg 0$. But recall first the meaning of the scaling property of $\underline{\mathbb{B}}^h(p, q, \lambda^h, y)$. It says that, for a fixed and arbitrary $v \in]0, 1[$, and for any $(x,\theta) \in \underline{\mathbb{B}}^h(p,q,\lambda^h,y)$, there exists a neighborhood V of (p,q,λ,y) such that for all $(p',q',\lambda'^h,y') \in V$, $(\upsilon x,\upsilon\theta) \in \underline{\mathbb{B}}(p',q',\lambda'^h,y')$. Let now $(p_n,q_n,\lambda_n^h,y_n)_n \in \bigotimes_n[\mathbb{Q}\times\Delta_{\Xi-1}\times\bigotimes_{j\in\mathcal{J}}\mathbb{Y}_m^j]$ be a sequence that converges to (p,q,λ,y) and let $(x,\theta) \in \underline{\mathbb{B}}^h(p,q,\lambda^h,y)$. Because of the scaling property of $\underline{\mathbb{B}}^h(p,q,\lambda^h,y)$, $(\upsilon x,\upsilon\theta)$ still belongs to $\underline{\mathbb{B}}^h(p_n,q_n,\lambda_n,y_n)$ for n sufficiently large and for an arbitrary $\upsilon \in]0,1[$. Taking $\upsilon := 1-1/n$, $n \geq n_0$ and n_0 sufficiently large, we get the following: for $(p_n,q_n,\lambda_n^h,y_n) \to (p,q,\lambda^h,y)$ when $n \to +\infty$ and for $(x,\theta) \in \underline{\mathbb{B}}^h(p,q,\lambda^h,y)$, there exists a sequence $(x_n,\theta_n)_n$ that converge to (x,θ) (defined by, for all integer n, $x_n := (1-1/n)x$ and $\theta_n := (1-1/n)\theta$) and for all $n \geq n_0$, $(x_n,\theta_n) \in \mathbb{B}^h(p_n,q_n,\lambda_n^h,y_n)$.

To show the convexity of the values of the demand correspondence $\mathcal{C}^h$, let $(x^h,x'^h) \in \bigotimes_2 \mathcal{C}^h(z)$ where $z \in \mathbb{Z}$ and let $\mu \in [0,1]$. By the definition of $\mathcal{C}^h$, there exists $(\theta^h,\theta'^h) \in \bigotimes_2 \Theta_m^h$ so that $((x,\theta),(x'^h,\theta'^h)) \in \bigotimes_2 \underline{\mathbb{B}}^h(y,p,q,\lambda^h)$. Then, let ${x_\mu}^h$ be $\mu x^h + (1-\mu)x'^h$ and ${\theta_\mu}^h$ be $\mu\theta^h + (1-\mu)\theta'^h$. Clearly $({x_\mu}^h,{\theta_\mu}^h)$ belongs to $\underline{\mathbb{B}}^h(p,q,\lambda^h,y)$. Consequently, if ${x_\mu}^h \notin \mathcal{C}^h(z)$, then there exists $x''^h \in \mathbb{X}_m^h$ and $\theta''^h \in \Theta_m^h$ such that, $u(x''^h) > u({x_\mu}^h)$ together with $\lambda^h \cdot G(p,q,\lambda^h,x''^h,\theta''^h,y) \leq 0$. From Assumption $(A.2)$ and $u(x''^h) > u(x_\mu^h)$, we easily deduce that either $u(x''^h) > u(x^h)$ or $u(x''^h) > u(x'^h)$. Considering that fact, together with $\lambda^h \cdot G(p,q,\lambda^h,x''^h,\theta''^h,y) \leq 0$, we contradict $(x^h,x'^h) \in \bigotimes_2 \mathcal{C}^h(z)$.

Part 2: (i) Clearly $\underline{\beta}^j$ are well defined and continuous on their respective domains, since $\mathcal{T}^j$ is continuous too. Then, $\mathbb{S}_m^j$ is compact since its is the image of the product of compact subsets. Clearly V^j is continuous, since it results from the composition of continuous maps. Therefore, the maximum of $V^j(\underline{\beta}^j(\theta,\lambda,\alpha,\tau);p,y^j)$ over the non empty compact convex subset $\mathbb{Y}_m^j$ exists for any given $(\theta,\lambda,\alpha,p)$ in the domain of V^j. This guarantees the non emptiness of the values of the correspondence $\mathcal{P}^j$. (ii)The compactness of the values of $\mathcal{P}^j$ can be easily proved since $\mathbb{Y}_m^j$ is compact too (by Assumption $(B.2)$ and by construction). (iii) The convexity of the values of $\mathcal{P}^j$ follows from Assumption $(B.1)$ and the definition of $\mathcal{P}^j$. (iv) Finally, showing that the graph of $\mathcal{P}^j$ is a closed subset is sufficient for the upper hemi-continuity of $\mathcal{P}^j$ since the correspondence takes its values in a compact subset. Let $(z_n)_n = (p_n,q_n,\alpha_n,\lambda_n,x_n,\theta_n,y_n,\tau_n)_n \in \bigotimes_n \mathbb{Z}$ be a sequence of elements that converges to $z = (p,q,\alpha,\lambda,x,\theta,y,\tau)$ and $y_n^j \in \mathcal{P}^j(z_n)$ for all n. Recall that $\mathbb{Z}$ results from the cartesian product of a family of compacts, hence closed subsets. So, each limit point belongs to each relative set. Using the continuity of $\underline{\beta}^j$ and V^j on their domains, then one obtain by passing to the limit that: for all $y'^j \in \mathbb{Y}_m^j$, $V^j(\underline{\beta}^j(\theta,\lambda,\alpha,\tau)\ p,y^j) \geq$

$V^j(\underline{\beta}^j(\theta,\lambda,\alpha,\tau)\ p, y'^j)$ which means that $y^j \in \mathcal{P}^j(z)$. Hence, the graph of $\mathcal{P}^j$ is closed.

Part 3: It is easy to see that the correspondence $\mathcal{M}$ is non empty compact valued and upper hemi-continuous. The non emptiness of the values of Λ^h is ensured by the fact that we are maximizing a continuous function over a compact set. The proof of the convexity and the compactness of the values of the correspondence Λ^h is routine. The upper-hemi continuity of Λ^h is obtained by proving that its graph is closed.

Proof of Lemma 4.3

By Lemma **4.2** and the continuity of the function $\mathcal{T}^j$ for each $j \in \mathcal{J}$, it follows that the correspondence $\mathcal{G}$ is non-empty and convex valued as well as upper hemi-continous.

Proof of Lemma 4.4

The proof follows from the definition of $\mathcal{C}^h$ and Λ^h, Lemma **2.1** and $\tilde{\lambda}^h >> 0$ by the monotonicity of u^h, (A.4) .

Proof of Lemma 4.5

The proof follows from the definition of $\mathcal{P}^j$ by noticing that $\underline{\beta}^j_\xi = \Pi_h \lambda^h_{\xi_0} \beta^j_\xi$ for all $\xi \in \Xi$, and $\Pi_h \tilde{\lambda}^h_{\xi_0} > 0$ by (A.4) .

Proof of Lemma 4.6

We consider only the initial date-event ξ_0, since clearly from that same reasoning one can get the result for every subsequent date-event $\xi \in \Xi$. Combining the definition of $\mathcal{M}$ with aggregation over all consumers ($h \in \mathcal{H}$) of their budget constraints in (2.1) shows that,
for all $(p_{\xi_0}, q_{\xi_0}) \in \text{proj}_{\mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}}_+}(\mathbb{Q})$,

$$p_{\xi_0} \cdot \sum_{h\in\mathcal{H}} \left(\tilde{x}^h_{\xi_0} - w^h_{\xi_0} - \sum_{j\in\mathcal{J}} \tilde{\theta}^{hj}_{\xi_0} \tilde{y}^j_{\xi_0} \right) + \sum_{j\in\mathcal{J}} q^j_{\xi_0} \left(\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi_0} - \sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi_0^-} \right) \leq$$

$$\tilde{p}_{\xi_0} \cdot \sum_{h\in\mathcal{H}} \left(\tilde{x}^h_{\xi_0} - w^h_{\xi_0} - \sum_{j\in\mathcal{J}} \tilde{\theta}^{hj}_{\xi_0} \tilde{y}^j_{\xi_0} \right) + \sum_{j\in\mathcal{J}} \tilde{q}^j_{\xi_0} \left(\sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi_0} - \sum_{h\in\mathcal{H}} \tilde{\theta}^{hj}_{\xi_0^-} \right) \leq 0,$$

From the continuity of the orthogonal projection mapping, we know that $\text{proj}^{\perp}_{\mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}}_+}(\mathbb{Q})$ is a compact set since $\mathbb{Q}$ is compact as well. So, it follows that

$$\left(\left(\sum_{h\in\mathcal{H}}(\tilde{x}^h_{\xi_0} - w^h_{\xi_0}) - \sum_{j\in\mathcal{J}}\left(\sum_{h\in\mathcal{H}}\tilde{\theta}^{hj}_{\xi_0}\right)\tilde{y}^j_{\xi_0}\right), \left(\sum_{h\in\mathcal{H}}\tilde{\theta}^h_{\xi_0} - \sum_{h\in\mathcal{H}}\tilde{\theta}^h_{\xi_0^-}\right)\right)$$

belongs to $(\text{proj}^{\perp}_{\mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}}_+}(\mathbb{Q}))^{\circ}$ i.e. to the negative polar cone of $\text{proj}^{\perp}_{\mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}}_+}(\mathbb{Q})$, that is the set of vectors $x \in \mathbb{R}^{L+\mathcal{J}}$ such that the natural scalar product between x and all vectors of $\text{proj}^{\perp}_{\mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}}_+}(\mathbb{Q})$ is less or equal to 0. Then, the results hold.

Proof of Lemma 4.7

Suppose to the contrary that there exists at least a date-event $\xi \in \Xi$ and a consumer $h \in \mathcal{H}$ such that

$$\tilde{p}_\xi \cdot \tilde{x}^h_\xi + \sum_{j\in\mathcal{J}}\tilde{q}^j_\xi\tilde{\theta}^{hj}_\xi < \tilde{p}_\xi \cdot w^h_\xi + \sum_{j\in\mathcal{J}}\tilde{q}^j_\xi\tilde{\theta}^{hj}_{\xi^-} + \sum_{j\in\mathcal{J}}\tilde{\theta}^{hj}_\xi(\tilde{p}_\xi \cdot \tilde{y}^j_\xi).$$

(with $\tilde{q}^j_\xi = 0$ and $\tilde{\theta}^{hj}_\xi = \tilde{\theta}^{hj}_{\xi^-}$ if $\xi \in \Xi_T$).

Because of Lemma **4.2**, we can assume that m is big enough so that the consumption of each consumer at the pseudo-equilibrium is in the interior of the the hyper-cube $M = [-m, m]^{\Xi\mathcal{H}+\Xi\mathcal{J}}$. Therefore, by monotonicity of u^h, $(A.4)$, we could find a consumption plan $u^h(x^h) > u^h(\tilde{x}^h)$ such that $(x^h, \tilde{\theta}^h) \in \underline{\mathbb{B}}^h(\tilde{p}, \tilde{q}, \tilde{y})$, a contradiction.

Proof of Lemma 4.8

Aggregating over all consumers ($h \in \mathcal{H}$) the binding budget constraints (thanks to Lemma **4.7**) and considering Lemma **4.6** and the non-negativity of the prices $\tilde{p}$ and $\tilde{q}$, one can check easily that, for all $\xi \in \Xi^-$,

$$\tilde{q}^j_\xi\Big(\sum_{h\in\mathcal{H}}\tilde{\theta}^{hj}_\xi - \sum_{h\in\mathcal{H}}\tilde{\theta}^{hj}_{\xi^-}\Big) = 0,\ j \in \mathcal{J} \text{ and } p_\xi \cdot \Big(\sum_{h\in\mathcal{H}}(\tilde{x}^h_\xi - w^h_\xi) - \sum_{j\in\mathcal{J}}\sum_{h\in\mathcal{H}}\tilde{\theta}^{hj}_\xi\tilde{y}^j_\xi\Big) = 0,$$

together with, for all $\xi \in \Xi_T$,

$$p_\xi \cdot \Big(\sum_{h\in\mathcal{H}}(\tilde{x}^h_\xi - w^h_\xi) - \sum_{j\in\mathcal{J}}\sum_{h\in\mathcal{H}}\tilde{\theta}^{hj}_{\xi^-}\tilde{y}^j_\xi\Big) = 0.$$

Therefore, by free-disposal $(B.3)$, we obtain equalities in the market clearing conditions for commodities.

Département des Sciences Économiques
de l'Université catholique de Louvain
Institut de Recherches Économiques et Sociales

Place Montesquieu, 3
1348 Louvain-la-Neuve, Belgique

ISSN 1379-244X D/2007/3082/042

Département des Sciences Économiques
de l'Université catholique de Louvain
Institut de Recherches Économiques et Sociales

Place Montesquieu, 3
1348 Louvain-la-Neuve, Belgique

ISSN 1379-244X D/2007/3082/042