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On Gale and Shapley *'College admissions and stability of marriage'*

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Abstract

In this note, we start to claim that established marriages can be heavily destabilized when the population of existing couples is enriched by the arrival of new candidates to marriage. Afterwards, we discuss briefly how stability concepts can be extended to account for entry and exit phenomena affecting the composition of the marriage market.

Keywords: matching, stability, marriage model, divorce cascades.

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Gale and Shapley (1962) investigate *marriage stability* within a hetero-sexual and monogamic community formed by one set of men and one set of women. Stability (henceforth G−S stability) in the marriage market is satisfied when women and men are matched so that there are no woman and no man who are not married to each other though they would prefer each other to their actual mates. Gale and Shapley (1962) show that, in such a market, there always exists a stable matching.[5]

Yet, divorces are a factual evidence and get more and more frequent! In this note, we start to claim that established marriages can be heavily destabilized when the population of existing couples is enriched by the arrival of new candidates to marriage. Using a striking example with n men and n women, we show that the entrance of a new couple in the marriage market completely disrupts, however large n may be, the unique stable matching prevailing before. Afterwards, we discuss briefly how stability concepts can be extended to account for entry and exit phenomena affecting the composition of the marriage market.

. DIVORCE CASCADES: AN EXAMPLE

Consider a marriage market with the same number (n) of women and men. In this market, both women (as well as men) have homogeneous preferences over the set of potential mates. More precisely, both men and women can be ranked in such a way that if a man is higher in the ranking of men, then each woman strictly prefers him to all others ranked below him, and similarly for the preferences of men over the set of women. Clearly, the unique G−S stable matching for this market consists in matching the i^{th}-ranked man with the i^{th}-ranked woman, $i = 1, .., n$.

Consider now that a new single woman and a new single man arrive in this market. Moreover, assume that this new single woman - the *Beauty* - is now ranked by all existing men at the top of the set of women, while the new single man - the *Beast* - is ranked by every women at the bottom of the set of men. In other words, the entry of this couple is as if "*The Beauty and the Beast*" had entered the market! Of course now the "top man" starts to be strongly interested in the *Beauty* and wishes to divorce from the

[5] This result illustrates a well-known French proverb, which says "Chaque casserole a son couvercle"!

past "top woman" to marry the *Beauty.* Thus, the past "top woman" now becomes available to the 2^{nd}-ranked man, who prefers her to his current mate. Consequently, a new divorce occurs as he wishes to marry the past "top woman" who is now willing to accept him. And so on and so forth, until the past "bottom man" decides to divorce from the "bottom woman," who now, poor woman, is left with the *Beast*! Accordingly, and whatever large the number n, the entry of a single pair of persons of opposite sex entails a cascade of divorces and fully disrupts all the n previous G$-$S stable matches: this entry generates a *domino effect,* where the disruption of one couple automatically provokes the disruption of the next one in the ranking, until all previous matches are destroyed.

. EXTERNAL STABILITY

The above example evokes the possibility of defining a concept of *external stability* related to a matching which is G$-$S stable. In this example, we start with a situation that is G$-$S stable and we slightly modify the market by allowing one further man and one further woman to enter the market. For the particular set-up of preferences introduced above, this marginal change entails a completely different new G$-$S stable matching in the resulting market. For this reason, we argue that the initial matching is *externally unstable.* For different specifications of preferences, the degree of external instability could be of course different, to the extent that the number of disrupted couples due to the entry can vary between 0 and n. This justifies the following definition.

DEFINITION 1. *Given a marriage market and a G$-$S stable matching in this market, this matching is said to be $k-$externally stable whenever, at any G$-$S stable matching in the marriage market consisting of one further man and one further woman, at least k existing past matches are not disrupted.*

In the example above, the initial matching is clearly $0-$*externally stable* since all existing initial matches are disrupted at the stable matching in the new marriage market. Notice that it can be the case that no couple is disrupted by the entry of a new pair: assume, for instance, that the new man puts the new woman at the top of his ranking and vice versa, while each existing mate ranks the entrant of the opposite sex in the very bottom of his/her previous ordering. In this case, no couple will be disrupted by the entrants; however, according to the definition, the initial matching is 0-externally stable, as there exists an entering pair that disrupts all couples. [6]

A further observation is that the maximal level of $k-$ external stability that can be achieved is $n-2$. In fact, for any preference profile and for any matching that is stable for that profile, there always exists at least one pair that disrupts two couples, whilst all others may remain matched with the same partner.

The degree of external stability of G$-$S stable matchings clearly depends on the individual rankings of existing and potential marriage mates. Given the richness of possible preference rankings, one can hardly make very general statements about the degree of external stability. Nevertheless, when restricting to more particular types of preference rankings, one might get some insights into this question. To illustrate this point, we consider in the following section, two specific categories of preference set-ups, namely, *vertical* and *horizontal* set-ups.

. VERTICAL AND HORIZONTAL SET-UPS

In real life, individuals' preferences regarding potential marriage mates are highly subjective and heterogeneous. In this section, we illustrate that some preference set-ups engender marriage markets with higher degree of $k-$*external stability* than others. We analyse two particular preference set-ups: a horizontal set-up, where each individual has its own preferred match (peak load preference set-up), and a vertical set-up, where all individuals have the same preference ordering over the potential marriage partners.

Definition 2. *A vertical set-up corresponds to a profile of preference relations over the potential marriage partners, such that, for every agent* k, $k=1,...,n$, we have $i \succ_k j \Leftrightarrow i > j$ *for every* $i,j=1,...,n$, $i \neq j$.

Hence, the vertical set-up case corresponds to the set of preference relations adopted in our example, with both men and women being ranked in such a way that if a woman (man) is higher in the ranking of women

[6] Also notice that the concept of k - external stability applies as well to the case in which the new marriage market would consist of the previous market with one woman and one man *less,* which corresponds to a situation of exit.

(men), then each man (woman) strictly prefers her (him) to all others ranked below her (him).

Definition 3. *A peak load preference set-up corresponds to a profile of preferences over the potential marriage partners such that, for every agent* k*,* $k = 1, ..., n$*, we have* $i \succ_k j \Leftrightarrow |k-i| < |k-j|$ *and, whenever* $|k-i| = |k-j|$*,* $i \succ_k j \Leftrightarrow i > j$*, for every* $i, j = 1, ..., n$*,* $i \neq j$*.*

This definition simply transposes the usual notion of distance used in continuous models *à la* Hotelling to define preference relations characterized by the fact that the further distant an option is from the ideal option of an individual, the least preferred it is. In this case, the ranking of potential marriage mates is no longer unanimous and the top men and top women will differ from individual to individual. For example, this is the case when women (as well as men) are differentiated with respect to some characteristic and, consequently, they rank potential marriage partners according to the similarity with their own characteristic. This implies that each woman (resp. man) has her (his) specific individual ranking over marriage partners, with marriage mates whose characteristic is closer to her (his) own being better positioned in her (his) individual ranking. The following representation of preferences illustrates the characteristics of peak load preferences set-up, considering a marriage market with three men and three women:

$$\begin{array}{ccc} \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_2 \\ m_3 & m_1 & m_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \succ_{m_1} & \succ_{m_2} & \succ_{m_3} \\ w_1 & w_2 & w_3 \\ w_2 & w_3 & w_2 \\ w_3 & w_1 & w_1 \end{array}$$

Transposing this preference set-up to a marriage market with n women and n men and ranking men and women according to the characteristic with respect to which they differ, it can be easily seen that the unique G$-$S stable matching for this market consists in matching the i^{th}-ranked man with the i^{th}-ranked woman, $i = 1, .., n$.

3.1. Vertical and horizontal rankings: external stability

In the previous section, we argued that the degree of external stability in marriage markets is determined by the nature of individuals' preference set-ups over potential marriage partners. In this section, we illustrate this point by contrasting the cases of vertical and peak load preference set-ups, which exhibit substantially different properties in terms of external stability. In fact, preference structures corresponding to peak load preference set-ups are more externally-stable than the ones corresponding to vertical preference set-ups in the sense that the former produce stable matchings that have a higher degree of of external-stability than the latter.

Let us consider a marriage market composed of a same number of men and women. Assume that preference relations over potential marriage mates are described by the vertical preference set-up. Finally, assume that an additional man and an additional woman enter this market. If all existing women rank this additional man between the existing men j and $j+1$, while all existing men rank this additional woman between the existing women $j+h$ and $j+h+1$, then it is easy to see that all the $|h|$ marriages corresponding to the existing matches of individuals ranked in-between the positions of the new members of the market are now disrupted. In the limit, when we consider the largest possible gap (n), as in the example of "*The Beauty and the Beast,*" all the existing matches at the initial G-S stable matching are disrupted at the new G-S stable matching. Hence, vertical preference set-ups produce stable matchings that are 0-externally stable.

In contrast, peak-load preferences are substantially more stable. Independently of the relative positioning of the new members of the marriage market, the entrance of a new woman and a new man in the market disrupts, at most, two matches of the initial G-S stable matching (all the remaining couples stay together at the new G-S stable matching). It follows that, in the case of peak-load preferences, the degree of external stability is equal to $n-2$.

. CONCLUSION

In this note, we have suggested that the notion of G-S stability in a marriage market could be advantageously complemented with a concept of stability related to the possibility of entry in this market. Starting from an illustrative example, we show how entry can heavily destabilize established marriages. We propose a tentative concept which could capture the extent of external stability of a marriage market.

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