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Abstract. Motivated by problems from dynamic economic models, we consider the problem of defining a uniform measure on inverse limit spaces. Let $f : X \to X$ where X is a compact metric space and f is continuous, onto and piecewise one-to-one and $Y := \varprojlim(X, f)$. Then starting with a measure μ_1 on the Borel sets $\mathcal{B}(X)$, we recursively construct a sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$ on $\mathcal{B}(X)$ satisfying $\mu_n(A) = \mu_{n+1}[f^{-1}(A)]$ for each $A \in \mathcal{B}(X)$ and $n \in \mathbb{N}$. This sequence of probability measures is then uniquely extended to a probability measure on the inverse limit space Y. If μ_1 is a uniform measure, we argue that the measure induced on the inverse limit space by the recursively constructed sequence of measures is a uniform measure. As such, the measure has uses in economic theory for policy evaluation and in dynamical systems in providing an ambient measure (when Lebesgue measure is not available) with which to define an SRB measure or a metric attractor for the shift map on the inverse limit space.

1. Introduction

In dynamic economic models, an equilibrium is a sequence $\{x_1, x_2, \ldots\}$ where each x_j is in some compact metric space X. This sequence is typically generated by a dynamical system $x_{t+1} = f(x_t)$ where $f : X \to X$ where f is continuous. However, there are dynamic economic models where these equilibrium sequences are backward orbits of a non-invertible continuous map $f : X \to X$, i.e., $x_t = f(x_{t+1})$. In such instances, we say the model has *backward dynamics* and we call f the *backward map*. Two such models that can have backward dynamics are the overlapping generations (OG) model ([3]) and the cash-in-advance (CIA) model ([11]).

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The inverse limit of $f : X \to X$ is denoted by $Y := \varprojlim(X, f)$ and consists of $(x_1, x_2, \ldots) \in X^\infty$ with $x_i = f(x_{i+1})$ for $i \in \mathbb{N}$, i.e., all backward orbits of f. If f is a backward map from a model with backward dynamics, then the set of equilibria in the model is the inverse limit space $\varprojlim(X, f)$. Inverse limits is a relatively new approach to analyzing dynamic economic models with backward dynamics. [8, 9] use inverse limits to analyze the long-run behavior of an OG model with backward dynamics. [5, 6] investigate the topological structure of the inverse limit space associated with the CIA model of [7]. [4] utilize the inverse limit space to show that a multi-valued dynamical system with backward dynamics is chaotic going forward in time if and only if it is chaotic going backward in time.

Let $Y := \varprojlim(X, f)$ be the set of equilibria in a dynamic economic model with backward dynamics. There are numerous circumstances in which one would like to have Lebesgue measure (a scaled uniform measure) on the space Y. However, when the bonding map f is chaotic, the inverse limit space is topologically complicated and does *not* have such a measure. This necessitates an ambient measure on the inverse limit space that can play the role of Lebesgue measure. We would like to construct a probability measure on Y that does two things.

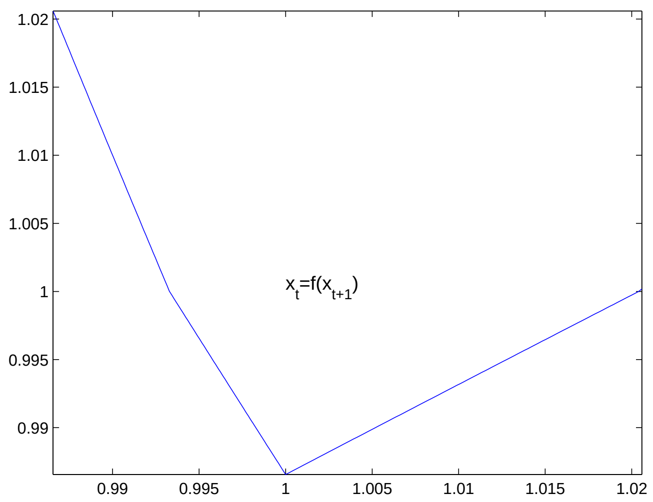
(1) If X has (finite) Lebesgue measure λ, then the probability of seeing tower sets $\pi_1^{-1}(B) \subseteq Y$ where $B \in \mathcal{B}(X)$ is given by $\lambda(B)/\lambda(X)$.
(2) Given x_t, the points $x_{t+1} \in f^{-1}(x_t)$ are equally likely.

[10] construct a similar measure when f is a unimodal map on an interval $I \subset \mathbb{R}$. Our method of construction is quite different and framework is more general (we obtain their measure as a special case). Here are three possible uses for such a measure.

First, dynamic economic models often have associated with them various real-valued functions defined on the state space, X, that measure economic features such as the amount of utility inherent in a given state (a so-called *utility function*). Each of these functions induces a real-valued function on the inverse limit space which measures the same feature for a sequence of allowed states in the model (equilibria). By integrating these functions we can describe the average amount of, say, utility inherent in the model. Then by analyzing how this integral depends on the parameters of the model we can see how to make policy suggestions in order to change, for example, the parameters and maximize utility or minimize the variance in utility.

To be more concrete, consider the standard CIA model of [7] as analyzed in [4] [5, 6] and [11]. As shown in [4], an equilibrium in this model corresponds to a sequence $\{x_1, x_2, \ldots\}$ satisfying $f(x_{t+1}) = x_t$ where $f : [x_l, x_h] \to [x_l, x_h]$, $0 < x_l < x_h < +\infty$, f is continuous, onto and non-invertible. See Figure 1 for one backward map from the CIA model.

FIGURE 1. Backward map $f : [x_l, x_h] \to [x_l, x_h]$ from the cash-in-advance model.



The inherent *utility* associated with an equilibrium $\mathbf{x} := \{x_1, x_2, \ldots\} \in Y$ is given by

$$W(\mathbf{x}) := \sum_{t=1}^{\infty} \beta^t w(x_t), \tag{1}$$

where $0 < \beta < 1$ is called the *discount factor* and $w(\cdot)$ takes the following form:

$$w(x) := \frac{\min\{1, x\}^{1-\sigma}}{1-\sigma} + \frac{(2 - \min\{1, x\})^{1-\gamma}}{1-\gamma},$$

with $\sigma > 0$ and $\gamma > 0$. See [11] or [4] for more details. Let M_t be the money supply at time t. The government controls the money supply using a money growth rule: $M_{t+1} = (1 + \theta)M_t$, where $\theta \geq 0$ is the money growth rate. For each θ, there is different interval $X(\theta) := [x_l(\theta), x_h(\theta)]$, backward map $f_\theta : X(\theta) \to X(\theta)$ and inverse limit space

$Y_\theta := \varprojlim(X(\theta), f_\theta)$. Economists are interested in

$$V(\theta) := \int_{Y(\theta)} W \, dm,$$

where m is an *appropriately* chosen measure on $Y(\theta)$. The function $V(\theta)$ is an *indirect* utility function that depends on the chosen monetary policy θ. One possible use for such a function would be to identify an optimal policy θ^* from some considered set of policies Θ:

$$\theta^* := \arg\max_{\theta \in \Theta} V(\theta).$$

Second, an SRB measure for σ on the inverse limit space is of interest in economics, because it captures the dynamics (in a frequency sense) of the model. From a modeling perspective, economists are concerned with how a dynamic model conforms to the data in terms of certain moments (mean, variance, covariance) calculated from *time averages*. An SRB measure allows one to determine the time averages of these moments in the model by calculating the space average of certain continuous functions. Let $Y := \varprojlim(X, f)$ and $\sigma : Y \to Y$ be the shift map given by $\sigma((x_1, x_2, \ldots)) = (x_2, x_3, \ldots)$. Let μ be an σ-invariant measure. A set $B \subset Y$ is a *basin* of μ if for all continuous $g : Y \to \mathbb{R}$ and $\mathbf{x} \in B$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g(\sigma^{i-1}(\mathbf{x})) = \int g \, d\mu.$$

The standard way of defining an SRB measure is to require the basin to have positive Lebesgue measure. This necessitates an ambient measure on the inverse limit space that can play the role of Lebesgue measure.

Third, [8, 9] analyze the overlapping generations model when backward dynamics is present (see [1], [2] and [3] for more on this model). Under certain specifications for functions and model parameters, the dynamics in the overlapping generations model are given by the logistic map: $x_t = 4x_{t+1}(1 - x_{t+1}) =: F(x_{t+1})$ where $F : [0,1] \to [0,1]$. Attractors of the shift map on the inverse limit space corresponds to "long-run" equilibria in the model, i.e., these trajectories are where the economy is heading as $t \to \infty$. [8, 9] discuss two types of attractors: *metric* and *topological*. A metric attractor requires the basin of attraction to be large in a measure sense (positive measure) using an appropriate ambient measure on the inverse limit space. Again, Lebesgue or uniform measure on the inverse limit space is the natural choice for an ambient measure. See [8, 9] for more detail.

2. Uniform Measures on Inverse Limit Spaces

Let $\{X, \mathcal{B}(X), \lambda\}$ be a probability space where X is a compact metric space and $\mathcal{B}(X)$ the σ-algebra of Borel sets. Suppose $f : X \to X$ is continuous, onto and *piecewise one-to-one*, i.e.,

> there exists an $M \in \mathbb{N}$ (finite) so that the domain space of f can be partitioned by $\{D_1, D_2, \ldots D_M\}$ with $D_j \in \mathcal{B}(X)$ where $f|_{D_j}$ is one-to-one. Without loss of generality, assume M is selected minimally.

Note that a function $f : [a, b] \to [a, b]$ that is continuous, onto and piecewise monotone would be piecewise one-to-one.

Theorem 1. *Suppose $f : X \to X$ is continuous, onto and piecewise one-to-one. If $A \in \mathcal{B}(X)$, then $f(A) \in \mathcal{B}(X)$, i.e., f is bi-measurable.*

Proof. This follows from [13] since for each $y \in X$, $\#f^{-1}(y)$ is finite. □

Let $I_j := f(D_j)$ for $j = 1, 2, \ldots, M$. Since f is bi-measurable, we have $I_j \in \mathcal{B}(X)$. For $j = 1, 2, \ldots, M$, let $M_j := M!/[(M - j)!j!]$ be the number of ways of selecting j sets from a collection of M sets. For $j = 1, 2, \ldots, M$, let $\{(a^p_{j1}, a^p_{j2}, \ldots, a^p_{jj}) \mid p = 1, 2, \ldots, M_j\}$ denote the M_j possible indices in selecting j sets from a collection of M sets.

For $j \in \mathbb{N}$, let

$$R_j := \{y \in X | \#f^{-1}(y) = j\}.$$

Since y has at most one preimage in each D_i, we have $R_j = \varnothing$ for $j > M$. Let $1 \leq N \leq M$ be the largest N with $R_N \neq \varnothing$. The sets $\{R_1, R_2, \ldots, R_N\}$ partitions X.

Lemma 1. *For $j = 1, 2, \ldots, N$, $R_j \in \mathcal{B}(X)$.*

Proof. We have

$$R_N = \cup_{p=1}^{M_N} \left(\cap_{i=1}^{N} I_{a^p_{Ni}} \right).$$

Since R_N is the finite union and intersection of $\mathcal{B}(X)$ sets, $R_N \in \mathcal{B}(X)$. We proceed recursively. Suppose $R_{k+1}, R_{k+2}, \ldots R_N \in \mathcal{B}(X)$. For $p = 1, 2, \ldots M_k$, let $R^p_k = [\cap_{i=1}^{k} I_{a^p_{ki}}] \setminus [R_{k+1} \cup R_{k+1} \cup \cdots \cup R_N]$. Again, we see $R^p_k \in \mathcal{B}(X)$ and $R_k = \cup_{p=1}^{M_k} R^p_k \in \mathcal{B}(X)$. Since $R_N \in \mathcal{B}(X)$, by induction $R_j \in \mathcal{B}(X)$, for $j = 1, 2, \ldots, N - 1$ as well. □

Lemma 2. *Let $R_{ij} := I_i \cap R_j$ and $D_{ij} := D_i \cap f^{-1}(R_j)$ for $i = 1, 2, \ldots M$ and $j = 1, 2, \ldots, N$. Then*

(1) $R_{ij}, D_{ij} \in \mathcal{B}(X)$;
(2) $f(D_{ij}) = R_{ij}$;
(3) for each $i = 1, 2, \ldots, M$, $\{R_{i1}, \ldots, R_{iN}\}$ partitions I_i;

(4) for each $i = 1, 2, \dots, M$, $\{D_{i1}, \dots, D_{iN}\}$ partitions D_i; and
(5) $\{D_{ij}\}$ is a partition of X.

Proof. (1) $R_{ij}, D_{ij} \in \mathcal{B}(X)$ since each is the intersection of Borel sets. (2) Let $x \in D_{ij}$. Then $x \in D_i$ so $f(x) \in I_i$. Since $x \in f^{-1}(R_j)$, we have $f(x) \in R_j$ so $f(x) \in I_i \cap R_j$. This implies $f(D_{ij}) \subseteq R_{ij}$. Let $y \in R_{ij}$. This implies there exists an $x \in I_i$ and $x \in f^{-1}(R_j)$ with $y = f(x)$. This implies $x \in D_{ij}$, so $y \in f(D_{ij})$ and $R_{ij} \subseteq f(D_{ij})$. (3) This follows since R_j is a partition of X and $I_i \subseteq X$. (4) This follows since $\{R_j\}$ is a partition of X, we have $\{f^{-1}(R_j)\}$ being a partition of X. (5) This follows since the $\{D_i\}$ are a partition of X. □

N.B. The set R_{ij} represents the points $y \in X$ that have exactly j preimages and one preimage in D_i. The points D_{ij} represents the points $x \in D_i$ with $f(x) \in R_j$, i.e., points in D_i whose image has exactly j preimages.

Lemma 3. *Suppose μ is a probability measure on $\mathcal{B}(X)$. Then*

$$\sum_{i=1}^{M} \mu(R_{ij}) = j\mu(R_j).$$

Proof. Note that $R_j = \cup_{i=1}^{M} R_{ij}$ and each $y \in R_j$ is an element of exactly j of the R_{ij}. Then

$$\mu(R_j) = \sum_{i=1}^{M} \mu(R_{ij}) - (j-1)\mu(\{y \in R_j : y \text{ is in exactly } j \text{ of the } R_{ij}\}),$$

or

$$\mu(R_j) = \sum_{i=1}^{M} \mu(R_{ij}) - (j-1)\mu(R_j).$$

Rearranging gives the desired result. □

Theorem 2. *Given a probability measure μ_n on $\mathcal{B}(X)$, define μ_{n+1} on $\mathcal{B}(X)$ as follows. For each $A \in \mathcal{B}(X)$, define*

$$\mu_{n+1}(A) := \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \mu_n[f(A \cap D_{ij})].$$

Then μ_{n+1} is a probability measure on $\mathcal{B}(X)$.

Proof. Suppose $A = X$. Then $f(A \cap D_{ij}) = f(D_{ij}) = R_{ij}$ and

$$\frac{1}{j} \sum_{i=1}^{M} \mu_n[f(D_{ij})] = \frac{1}{j} \sum_{i=1}^{M} \mu_n(R_{ij}) = \frac{j}{j}\mu_n(R_j) = \mu_n(R_j).$$

Since the R_j partition X, we have $\mu_{n+1}(X) = \sum_{j=1}^N \mu_n(R_j) = 1$.

Suppose $A \in \mathcal{B}(X)$. Then $f(A \cap D_{ij}) \subseteq R_{ij}$ and $0 \leq \mu_n[f(A \cap D_{ij})] \leq \mu_n[R_{ij}]$. Since

$$0 \leq \frac{1}{j} \sum_{i=1}^{M} \mu_n[f(A \cap D_{ij})] \leq \frac{1}{j} \sum_{i=1}^{M} \mu_n(R_{ij}) = \frac{j}{j} \mu_n(R_j) = \mu_n(R_j),$$

we have $0 \leq \mu_{n+1}(A) \leq \sum_{j=1}^N \mu_n(R_j)$. And since the R_j partition X, we have $\sum_{j=1}^N \mu_n(R_j) = 1$ implying $0 \leq \mu_{n+1}(A) \leq 1$ for all $A \in \mathcal{B}(X)$.

Suppose $\{A_1, A_2, \ldots\}$ is a collection of disjoint $\mathcal{B}(X)$ sets. Note that we have for each A_k,

$$\mu_{n+1}(A_k) = \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \mu_n[f(A_k \cap D_{ij})].$$

So we have

$$\mu_{n+1}(\cup_k A_k) = \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \mu_n[f(\cup_k A_k \cap D_{ij})] = \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \mu_n[\cup_k f(A_k \cap D_{ij})].$$

For each i, j, the sets $\{A_k \cap D_{ij}\}_{k=1}^\infty$ are disjoint. Since $f|_{D_{ij}}$ is one-to-one for each i, j, the sets $\{f(A_k \cap D_{ij})\}_{k=1}^\infty$ are disjoint as well. Since μ_n is a probability measure, we have

$$\mu_{n+1}[\cup_k A_k] = \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \sum_{k=1}^{\infty} \mu_n\left[f(A_k \cap D_{ij})\right] = \sum_{k=1}^{\infty} \mu_{n+1}(A_k).$$

□

The following lemma shows that f viewed as a map from $(X, \mathcal{B}(X), \mu_{n+1})$ to $(X, \mathcal{B}(X), \mu_n)$ is a measure preserving transformation.

Lemma 4. *Let $A \in \mathcal{B}(X)$, then $\mu_n(A) = \mu_{n+1}[f^{-1}(A)]$.*

Proof. Let $B = f^{-1}(A)$, $A_j = A \cap R_j$ and $B_{ij} = B \cap D_{ij}$. Then $\cup_{i=1}^M f(B_{ij}) = A_j$ for all $j = 1, 2, \ldots, N$. Since $B = f^{-1}(A)$, we have

$$\sum_{i=1}^{M} \mu_n\left[f(B_{ij})\right] = j\mu_n(A_j),$$

for $j = 1, 2, \ldots, N$. Then

$$\mu_{n+1}(B) = \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \mu_n[f(B_{ij})] = \sum_{j=1}^{N} \mu_n(A_j).$$

However, since the A_j are disjoint, we have $\sum_{j=1}^N \mu_n(A_j) = \mu_n(A)$. □

Suppose that μ_1 is a probability measure on $\mathcal{B}(X)$. Using Theorem 2, one can construct a sequence of probability measures $\{\mu_1, \mu_2, \ldots\}$. Note that in our construction of $\{\mu_n\}$, we have used μ_n to induce a measure μ_{n+1}. However, Lemma 4 illustrates that if one had μ_{n+1}, and induces a measure $\nu(A) := \mu_{n+1}[f^{-1}(A)]$, one would get $\nu = \mu_n$.

Theorem 3. *Let X be a compact metric space, $\mathcal{B}(X)$ the Borel sets of X, $f : X \to X$ be continuous, onto and piecewise one-to-one, $Y := \varprojlim(X, f)$, and $\mathcal{F}$ be the smallest σ-algebra such that all the projection maps $\pi_n : Y \to X$ are measurable. Let $\mu_1 = \lambda$ be a measure on $\mathcal{B}(X)$ and define μ_n for $n > 1$ recursively by*

$$\mu_{n+1}(A) := \sum_{j=1}^{N} \frac{1}{j} \sum_{i=1}^{M} \mu_n[f(A \cap D_{ij})].$$

Then μ_n is probability measure on $\mathcal{B}(X)$ and $\mu_{n+1}[f^{-1}(A)] = \mu_n(A)$ for all $A \in \mathcal{B}(X)$ and there exists a unique probability measure μ on $\mathcal{F}$ such that $\mu[\pi_n^{-1}(A)] = \mu_n(A)$ for all $A \in \mathcal{B}(X)$ and $n \in \mathbb{N}$.

Proof. The existence of a unique measure μ with the above properties follows from [12, Theorem 3.2, p. 139]. That μ is a probability measure follows since $\mu(Y) = \mu[\pi_1^{-1}(X)] = \mu_1(X) = 1$. □

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