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EULER EQUATION BRANCHING

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Euler Equation Branching

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Abstract

Some macroeconomic models exhibit a type of global indeterminacy known as Euler equation branching (e.g., the one-sector growth model with a production externality). The dynamics in such models are governed by a differential inclusion $\dot{x} \in F(x)$, where F is a set-valued function. In this paper, we show that in models with Euler equation branching there are multiple equilibria and that the dynamics are chaotic. In particular, we provide sufficient conditions for a dynamical system on the plane with Euler equation branching to be chaotic and show analytically that in a neighborhood of a steady state, these sufficient conditions will typically be satisfied. We also extend the results of Christiano and Harrison (1999) for the one-sector growth model with a production externality. In a more general setting, we provide necessary and sufficient conditions for Euler equation branching in this model. We show that chaotic and cyclic equilibria are possible and that this behavior is not dependent on the steady state being "locally" a saddle, sink or source.

Keywords: global indeterminacy, Euler equation branching, multiple equilibria, cycles, chaos, increasing returns to scale, externality, regime switching.

JEL: E13, E32, E62.

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1 Introduction

Some macroeconomic models exhibit a type of global indeterminacy known as Euler equation branching. In typical dynamic models, the dynamics are described by a differential equation $\dot{x} = f(x)$. However, in models with Euler equation branching the dynamics are governed by a *differential inclusion* $\dot{x} \in F(x)$, where F is a set-valued function. Models that exhibit Euler equation branching include the one-sector growth model with a production externality of Christiano and Harrison (1999). One also finds Euler equation branching in the two-sector model of Benhabib and Farmer (1996) and the RBC model with a balanced-budget rule of Schmitt-Grohé and Uribe (1997).[1] Two fundamental questions for such models are

- Does Euler equation branching imply the existence of multiple equilibria?
- If there are multiple equilibria, are the dynamics simple or chaotic?

In this paper, we show that in models with Euler equation branching there are multiple equilibria and that the dynamics are chaotic. Given that a dynamical system generated by a differential inclusion is *non-standard*, there is not a set of readily available results for defining chaos and establishing the (non)existence of chaos. In this paper, we use a definition of chaos for a dynamical system generated by a differential inclusion that is in the spirit of one of the more commonly used definitions of chaos, Devaney (2003). We provide sufficient conditions for a dynamical system on the plane with Euler equation branching to be chaotic and show analytically that in a neighborhood of a steady state, these sufficient conditions will typically be satisfied. These results hold even if there is a unique steady state equilibrium *and* this unique steady state is locally determinate.

We also extend the results of Christiano and Harrison (1999) for the one-sector growth model with a production externality. By using a particular parameterization, they obtain a clean, closed-form expression for the global set of competitive equilibria illustrating the existence of Euler equation branching. They illustrate the possibility of deterministic and stochastic regime switching equilibria along with equilibria that appear chaotic. We consider a more general setting and provide necessary and sufficient conditions for Euler equation branching. Moreover, we show that chaotic and cyclic equilibria are possible in the model and that this behavior is not dependent on the steady state being locally determinate or indeterminate.

[1] See Stockman (2007a,b).

This paper joins the literature that has stressed the importance of global analysis in exploring possible equilibria in dynamic general equilibrium models.[2] Benhabib and Perli (1994) analyze the endogenous growth model of Lucas and illustrate the possibility of global indeterminacy with multiple balanced-growth paths. They extend the model to include a labor-leisure choice and illustrate that there are two balanced-growth paths for a given a level of physical and human capital, and the choice of labor can put the economy on either of these two paths. Boldrin et al. (2002) develop a method for characterizing the global dynamics in the two-sector growth model. They find that global indeterminacy can arise and that the growth rate along an equilibrium trajectory can fluctuate chaotically. Stockman (2007a) extends the work of Schmitt-Grohé and Uribe (1997) and illustrates that a balanced-budget rule induces aggregate instability regardless of the determinacy of the steady state. In particular, he provides sufficient conditions for Euler equation branching to be a generic quality under a balanced-budget rule. Consequently, regime switching equilibria involving cycles and chaotic behavior are possible.

In the next section we briefly describe the one-sector model. In section 3, we provide necessary and sufficient conditions under which Euler equation branching will exist in this model. In addition, we provide sufficient conditions for a more general one-sector model. Our main methodological contribution for establishing the existence of chaos for a model on the plane with Euler equation branching is in section 4 along with our application to the one-sector model. We conclude in section 5.

2 Model

The model of Christiano and Harrison (1999, 1996) is a standard real business cycle model with a production externality. We briefly describe a continuous-time version of the model here. Preferences are given by

$$\int_0^\infty e^{-\rho t}[\log(c_t) + \sigma \log(1 - n_t)]dt, \tag{1}$$

with $\rho < 0$ and $\sigma > 0$. Output is produced at many locations using capital K and labor N, but also depends on the average level of production across these location y:

$$Y = f(y, K, N) := y^\gamma K^\alpha N^{1-\alpha},$$

[2]In addition to the papers discussed here, see also Hommes and de Vilder (1995), Michener and Ravikumar (1998), Benhabib et al. (2001), Guo and Lansing (2002), Coury and Wen (2002), Medio and Raines (2007) and Stockman (2007b).

with $0 < \alpha < 1$ and $0 \leq \gamma < 1$. Solving for the economy-wide average one gets

$$y = k^{\alpha_k} n^{\alpha_n}, \tag{2}$$

where $\alpha_k := \alpha/(1-\gamma)$ and $\alpha_n := (1-\alpha)/(1-\gamma)$. The equilibrium rental rate of capital and wage are given by

$$r_t = \alpha \frac{y_t}{k_t} \text{ and } w_t = (1-\alpha)\frac{y_t}{n_t}.$$

The household's optimality conditions are

$$\begin{aligned}
1/c_t &= \lambda_t, & (3)\\
\sigma/(1-n_t) &= \lambda_t w_t, & (4)\\
\dot{\lambda}_t &= \lambda_t(\rho + \delta - r_t), & (5)\\
\dot{k}_t &= w_t n_t + r_t k_t - c_t - \delta k_t, & (6)\\
0 &= \lim_{t\to\infty} e^{-\rho t}\lambda_t k_t. & (7)
\end{aligned}$$

Substituting for the input factor prices, an equilibrium in the model must satisfy

$$\begin{aligned}
\sigma/(1-n_t) &= \lambda_t(1-\alpha)k_t^{\alpha_k} n_t^{\alpha_n - 1}, & (8)\\
\dot{\lambda}_t &= \lambda_t(\rho + \delta - \alpha k_t^{\alpha_k - 1} n_t^{\alpha_n}), & (9)\\
\dot{k}_t &= k_t^{\alpha_k} n_t^{\alpha_n} - c_t - \delta k_t, & (10)
\end{aligned}$$

along with (3) and (7). With k_t as the state and λ_t as the co-state, the equilibria in the model must satisfy the following

$$\begin{aligned}
\dot{k}_t &= k_t^{\alpha_k} n_t^{\alpha_n} - 1/\lambda_t - \delta k_t, & (11)\\
\dot{\lambda}_t &= \lambda_t(\rho + \delta - \alpha k_t^{\alpha_k - 1} n_t^{\alpha_n}), & (12)\\
\sigma/(1-n_t) &= \lambda_t(1-\alpha)k_t^{\alpha_k} n_t^{\alpha_n - 1}, & (13)
\end{aligned}$$

along with (7). Benchmark parameter values are reported in Table 1.

3 Euler Equation Branching

Given k_t and λ_t, one uses equation (13) to solve for n_t. If there is more than one solution for n_t, the model exhibits Euler equation branching. The next proposition gives necessary and sufficient conditions for there to be either 2 or 0 solutions for n.

Table 1: Benchmark parameter values.

$\alpha = 0.30$	(capital's share)
$\delta = 0.10$	(depreciation)
$\rho = 0.05$	(discount factor)
$\sigma = 2.00$	(leisure preference parameter)

Proposition 1. *In this model with preferences given by (1) and technology given by (2), there is Euler equation branching if and only if* $\alpha_n := (1-\alpha)/(1-\gamma) > 1$.

Proof. Suppose $\alpha_n > 1$. Equilibrium in the labor market implies

$$\frac{\sigma}{1-n} = \lambda(1-\alpha)k^{\alpha_k}n^{\alpha_n - 1}.$$

Rearranging one gets

$$\left(\frac{1-\alpha}{\sigma}\right)\lambda k^{\alpha_k} = B(n) := \frac{1}{(1-n)n^{\alpha_n - 1}}.$$

Note that $\lim_{n\to 0} B(n) = \lim_{n\to 1} B(n) = +\infty$ and $B(n) > 0$ for all $n \in (0,1)$. The derivative is given by

$$B'(n) = (1-n)^{-2}n^{1-\alpha_n} + (1-\alpha_n)(1-n)^{-1}n^{-\alpha_n} = B(n)\left[\frac{1}{1-n} + \frac{1-\alpha_n}{n}\right].$$

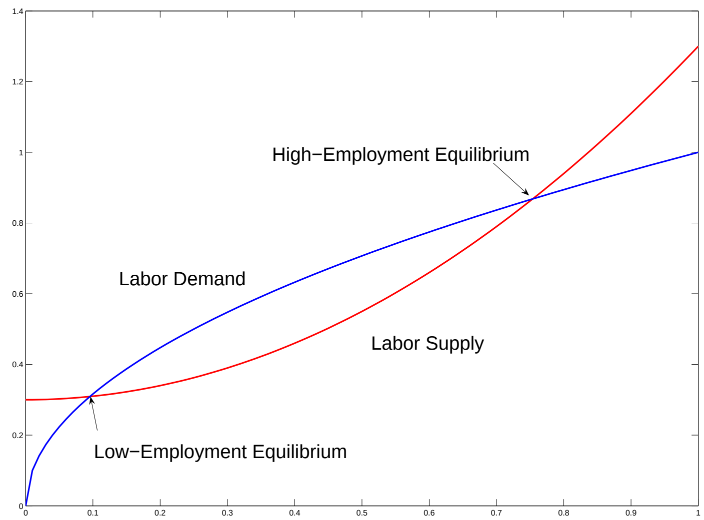
This implies $B'(n) = 0$ for $n \in (0,1)$ iff $n^* = (\alpha_n - 1)/\alpha_n$. Moreover, for $n \in (0, n^*)$ one has $B'(n) < 0$ and for $n \in (n^*, 1)$ one has $B'(n) > 0$. Therefore the minimum value taken on by B occurs at $n = n^*$ and this value is $B^* := B(n^*)$. Any equilibrium requires

$$\left(\frac{1-\alpha}{\sigma}\right)\lambda k^{\alpha_k} \geq B^*.$$

In general, for a given k and λ there are either 2 or 0 solutions for n implying the existence of Euler equation branching. Note that for $\alpha_n \leq 1$, $B'(n) > 0$ implying that there exists at most one equilibrium in the labor market. □

We see the required size of the externality γ is such that the equilibrium labor demand curve is upward sloping. By the *equilibrium labor demand* we mean the relationship $w = (1-\alpha)k^{\alpha_k}n^{\alpha_n - 1}$. Though each individual firm's technology exhibits a diminishing marginal product of labor, the *equilibrium* or aggregate marginal product of labor need not

Figure 1: Labor Market with Euler Equation Branching.



be diminishing if the externality is sufficiently large. It is important to note that the existence of Euler equation branching does not depend on the "local" determinacy properties of the (unique) steady state in the model. Note that in Christiano and Harrison (1999), $\gamma = 1 - \alpha$ and $\alpha = 1/3$ so $\alpha_n = 2 > 1$, which confirms their finding of Euler equation branching.

The intuition for Euler equation branching is simple. In the labor market, the externality causes the equilibrium labor demand curve to be upward sloping. The marginal product of labor goes to zero as labor goes to zero, as opposed to infinity in the standard neoclassical model. This curve starts out below the Frisch labor supply curve (this curve gives the amount of labor the household would like to supply for a given wage level holding the marginal utility of income fixed). It also ends up below the Frisch labor supply curve as labor approaches the time endowment. Hence, if the two curves cross at all, they will typically cross multiple times. See Figure 1 for an illustration of the labor market with Euler equation branching.

The existence of Euler equation branching can be established for a broader set of preferences and technology. The next proposition provides sufficient conditions for Euler equation branching to be a *generic* property.

Proposition 2. *Consider the model described in section 2. Let $\bar{H}$ be the time endowment.*

Let preferences over consumption and work $U(C,H)$ be represented by a $\mathcal{C}^2$ function $U : \mathbb{R}_{++} \times [0,\bar{H}) \to \mathbb{R}$ satisfying

(i) $U_C > 0$, $U_H < 0$ and negative definite Hessian,

(ii) $\lim_{H\to\bar{H}} U_H(C,H) = -\infty$ for all $C > 0$,

(iii) $U_H(C,0) := \lim_{H\to 0} U_H(C,H) < 0$ for all $C > 0$,

(iv) $U_C(C,0) := \lim_{H\to 0} U_C(C,H) > 0$ for all $C > 0$.,

Let the production function $F : \mathbb{R}_+^2 \to \mathbb{R}$ be $\mathcal{C}^2$ with $F_K > 0$, $F_H > 0$ and $F_H(K,0) := \lim_{H\to 0} F_H(K,H) = 0$ for all $K > 0$. If the labor supply and demand curves intersect (transversally), there will be an even number of such intersections, i.e., generically, transversal crossings come in pairs.

Proof. These conditions are sufficient for the Frisch labor supply curve to be upward sloping and the labor demand curve to be initially beneath and ultimately below the labor supply curve. In this proposition, we are considering only transversal intersections (the generic case) and do not consider non-transversal intersections.

The Frisch labor supply curve $H(w,\lambda)$ and demand for consumption $C(w,\lambda)$ are defined implicitly from the following first-order conditions:

$$\begin{aligned} U_C(C,H) &= \lambda, \\ -U_H(C,H) &= \lambda w. \end{aligned}$$

By use of the implicit function theorem, we have

$$\frac{\partial H(w,\lambda)}{\partial w} = \frac{-U_{CC}\lambda}{U_{CC}U_{HH} - U_{CH}^2}.$$

The numerator is positive and the denominator is positive since U has a negative definite Hessian. Thus this partial derivative is positive and hence the Frisch labor supply curve is upward sloping (and continuous since U is $\mathcal{C}^2$). Note that at $H = 0$ with $C > 0$, the implied w is strictly positive since $U_C(C,0) > 0$ and $-U_H(C,0) > 0$. This means the w-intercept for the labor supply curve is strictly positive. Moreover, as $H \to \bar{H}$ the implied w from the labor supply curve is $+\infty$.

Since $F_H(K,0) = 0$, for low H the labor supply curve is above the labor demand curve (the first transversal crossing of the labor supply curve by the labor demand curve is from below). We have $F_H(K, \bar{H}) < \infty$ for all K, so the labor demand curve is eventually below the labor supply curve. Consequently, given a crossing of the labor supply curve by the labor demand curve from below, it must be that (1) there is at least one more crossing and (2) the last transversal crossing of the labor supply curve by the labor demand curve is from above.

□

4 Chaos and Cycles

Given the existence of Euler equation branching, the dynamics in this model when $\left(\frac{1-\alpha}{\sigma}\right)\lambda k^{\alpha_k} > B(n^*)$ are given by a differential inclusion or multi-valued dynamical system (MVDS):

$$\begin{bmatrix} \dot{k} \\ \dot{\lambda} \end{bmatrix} \in \{G(k, \lambda, n_1), G(k, \lambda, n_2)\},$$

where $0 < n_1 < n^* < n_2 < 1$ are the two equilibrium values in the labor market and

$$G(k, \lambda, n) := \begin{bmatrix} k^{\alpha_k} n^{\alpha_n} - 1/\lambda - \delta k \\ \lambda(\rho + \delta - \alpha k^{\alpha_k - 1} n^{\alpha_n}) \end{bmatrix}.$$

The existence of Euler equation branching will typically imply the existence of cycles and chaotic behavior. Here, we follow Stockman (2007a) closely. First, we give some definitions for a MVDS. Let the state space X be a metric space with metric d and $T := [0, \infty)$ our time index. The space of all possible orbits on X is denoted by $W := \{\gamma \mid \gamma : T \to X\}$. Let $F : X \to 2^X$ be a set-valued function. A dynamical system on X generated by F is a subset of W given by

$$D := \{\gamma \in W \mid \dot{\gamma}(t) \in F(\gamma(t))\}.$$

Definition 1. D has a *periodic orbit of length* $m > 0$ if there exists an orbit $\gamma \in D$ with $\gamma(t) = \gamma(t + m)$ for all $t \in T$ and there does not exist an $n \in (0, m)$ with $\gamma(t) = \gamma(t + n)$ for all $t \in T$. D has a *periodic orbit of length* $m = 0$ if there exists an orbit $\gamma \in D$ with $\gamma(t) = \bar{\gamma}$ for all $t \in T$.

Definition 2. D has *sensitive dependence on initial conditions* if there exists a sensitivity constant $\delta > 0$ such that for any given $x \in X$ and neighborhood $N(x)$, there exists orbits $\gamma, \sigma \in D$ and $m \geq 0$ such that $\gamma(0) = x$, $\sigma(0) \in N(x)$ and $d(\gamma(m), \sigma(m)) > \delta$.

Definition 3. D has a *dense set of periodic points* if for any given $x \in X$ and neighborhood $N(x)$, there exists a periodic orbit $\gamma \in D$ with $\gamma(0) \in N(x)$.

Definition 4. D is *topologically transitive* if for any (non-empty) open sets $U, V \subset X$, there exists an orbit $\gamma \in D$ and $N \in T$ with $\gamma(0) \in U$ and $\gamma(N) \in V$.

Definition 5. D is *chaotic* in the sense of Devaney (2003) if D is topologically transitive, has a dense set of periodic points and has sensitive dependence on initial conditions.

4.1 Linear MVDSs

Stockman (2007a,b) considers the following example of a MVDS generated by a linear function and a constant function. He shows that such simple building blocks for a multi-valued dynamical system can generate rich dynamical behavior.

Example 1. Let $X := \mathbb{R}^2$ and $H(x) := \{Ax, b\}$, where A is a 2×2 matrix with no purely imaginary eigenvalues and $b \in X$.

He considers three cases for $x^* = 0 \in X$ under A: (1) sink, (2) source and (3) saddle. In all of these cases, there will typically exist an invariant closed set with a non-empty interior on which H will be chaotic.

Theorem 1 (sink or source – real root). *Let $X := \mathbb{R}^2$ and $H(x) := \{Ax, b\}$ where $b \in X$ and A is a 2×2 matrix with real eigenvalues $\lambda_1 < \lambda_2 < 0$ or $0 < \lambda_1 < \lambda_2$ and eigenvectors e_1 and e_2. Without loss of generality, assume that A is diagonal and e_1 and e_2 are the canonical basis vectors. Then provided $b \neq \alpha e_1$ and $b \neq \beta e_2$, the dynamical system generated by H restricted to a cone in $\mathbb{R}^2$ is chaotic. If $b = \alpha e_1$ or $b = \beta e_2$ or $\lambda_1 = \lambda_2 \neq 0$, then the dynamical will be chaotic on a ray emanating from the origin.*

Proof. See Stockman (2007b). □

Theorem 2 (sink or source – complex root). *Let $X := \mathbb{R}^2$ and $H(x) := \{Ax, b\}$ where $b(\neq \vec{0}) \in X$ and A is a 2×2 matrix with complex eigenvalues λ_1 and λ_2 satisfying Re $(\lambda_i) \neq 0$. Then the dynamical system generated by H is chaotic on $\mathbb{R}^2$.*

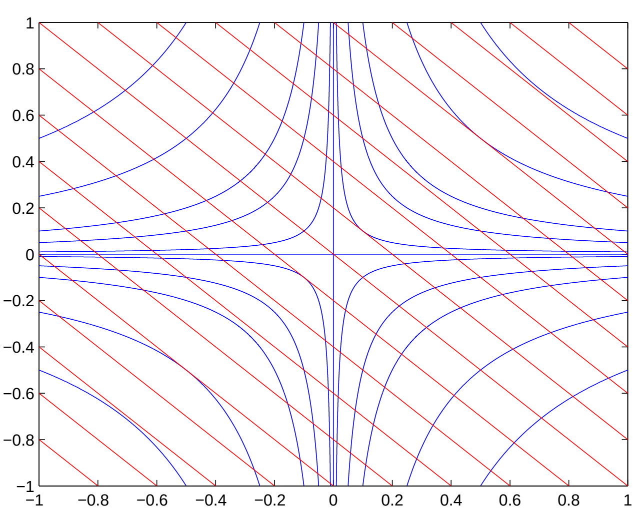
Proof. See Stockman (2007b). □

Theorem 3 (saddle). *Let $X := \mathbb{R}^2$ and $H(x) := \{Ax, b\}$ where $b \in X$ and A is a 2×2 matrix with real eigenvalues $\lambda_2 < 0 < \lambda_1$ and eigenvectors e_1 and e_2. Without loss of generality, assume that A is diagonal and e_1 and e_2 are the canonical basis vectors. Then provided $b \neq \alpha e_1$ and $b \neq \beta e_2$, the dynamical system generated by H restricted to one of the orthants of $\mathbb{R}^2$ is chaotic.*

Proof. See Stockman (2007a). □

For the intuition on the saddle result, see Figure 2. Assume that the vertical axis and the horizontal axis are the unstable and stable manifolds of A (respectively) and the flow from b is running from "northwest" to "southeast." The importance of the vector b not being a scalar multiple of either eigenvector is so that an integral curve generated by b will intersect those generated by A typically twice (or not at all). This system is chaotic on the "northeast" quadrant. Sensitive dependence of initial conditions is easy to see since every open set in this quadrant has a point with an orbit that diverges to $(0, +\infty)$ along with an orbit that converges to $(0, 0)$. To get to $(0, +\infty)$ simply follow the integral curves generated by A. To get to $(0, 0)$ simply follow the integral curve generated by b to the stable manifold of A and then follow the stable manifold to $(0, 0)$. A dense set of periodic points follows since every point in the interior of this quadrant is part of a cyclic orbit. To see this, note that one can construct cyclic orbits that look like "half moons" using integral curves of A and the integral curves of b. In fact, there is an orbit connecting any two points in the interior of this quadrant. From this, topological transitivity follows.

Figure 2: Saddle: integral curves from the MVDS given by $\dot{x} \in \{Ax, b\}$.



Stockman (2007a) conjectures that this simple family of MVDSs is useful as a way of understanding the behavior of non-linear MVDSs. To see this, suppose one has a non-linear system $\dot{x} \in \{M(x), N(x)\}$ with $M(x^*) = 0$ and $N(x^*) \neq 0$. Suppose further that x^* is a hyperbolic point of M, i.e., $DM(x^*)$ has no purely imaginary eigenvalues. Then in a neighborhood of x^*, $M(x)$ behaves like Ax where $A = DM(x^*)$ and $N(x)$ behaves like b where $b = N(x^*)$. For this reason, one can expect that most non-linear MVDSs near a steady state will be chaotic as well. We prove this conjecture in the next subsection. Note this is not a trivial result that follows directly from the Hartman-Grobman theorem. Even though Ax is conjugate to $M(x)$ and b is conjugate to $N(x)$ in a neighborhood of x^*, we cannot conclude that $\{Ax, b\}$ is conjugate to $\{M(x), N(x)\}$ in a neighborhood of x^* since in all likelihood the two single conjugacies would require *two* different coordinate changes.

4.2 Nonlinear MVDSs

We now turn to analyzing non-linear MVDSs on the plane generated by two functions. Our approach is two-fold: (1) we establish sufficient conditions for chaos and (2) we show that these sufficient conditions will typically be satisfied near a steady state equilibrium.

Let $\dot{x} \in \{f(x), g(x)\}$ and ϕ and ψ be the flows generated by continuous functions f and g (mapping $\mathbb{R}^2 \to \mathbb{R}^2$). Let $T := [0, \infty)$ and define

$$\Gamma := \{\gamma : T \to \mathbb{R}^2 \mid \dot{\gamma}(t) \in \{f(\gamma(t)), g(\gamma(t))\}.$$

Definition 6. Let $a, b \in \mathbb{R}^2$. We say there is a *path from a to b* provided there exists $\gamma \in \Gamma$ and $t_0, t_1 \in T$ such that $t_0 < t_1$ with $\gamma(t_0) = a$ and $\gamma(t_1) = b$. The *path* is $P := \{\gamma(t) : t_0 \leq t \leq t_1\}$.

Definition 7. Let $a, b \in \mathbb{R}^2$. We say there is a *simple path from a to b* provided there exists a path γ from a to b such that $\dot{\gamma}$ has finitely many discontinuities on $[t_0, t_1]$ and $a \neq \gamma(s) \neq b$ for all $t_0 < s < t_1$. The *simple path* is $P := \{\gamma(s) : t_0 \leq s \leq t_1\}$.

Next, we define a *chaotic set* and provide sufficient conditions for establishing its existence. Then we show that the existence of a chaotic set will imply chaos for the MVDS as we have defined it above.

Definition 8. A set $R \subset \mathbb{R}^2$ is a *chaotic set* provided

1. for all $a, b \in R$, there exists a path from a to b in R,

2. there exists nonempty $U \subset R$ open (relative to R) and a $\gamma \in D$ such that $\gamma(t) \in R \setminus U$ for all $t \in T$.

Let $K \subset \mathbb{R}^2$ be (nonempty) closed such that g has no bounded solutions in K, i.e., there are no trajectories (forward or backward) generated by g that stay in K forever. Let $a \in \mathbb{R}^2$ and P a simple path from a to a such that $P \subset K$. In general, P is compact and is a finite union of arcs. We will need the following lemma which involves the topological concept of a *component* which we define now. Let $A \subset \mathbb{R}^2$ (nonempty). If $x \in A$, then the largest connected subset C_x of A containing x is called the *component* of x. The components of A form a partition of A where each component is a "maximal" connected piece. For example, let $P \subset \mathbb{R}^2$ be the unit circle and $A = \mathbb{R}^2 \setminus P$, then A has 2 components – the interior of the unit disc and the complement of the unit disc.

Lemma 1. *$\mathbb{R}^2 \setminus P$ has only one unbounded component.*

Proof. Note that since P is a compact subset of $\mathbb{R}^2$, P is closed and bounded. Thus there is some $q \in \mathbb{R}^2$ and $r > 0$ such that $P \subseteq B_r(q)$. Thus $\mathbb{R}^2 \setminus B_r(q) \subseteq \mathbb{R}^2 \setminus P$, and $\mathbb{R}^2 \setminus B_r(q)$ is one unbounded component. This implies that $\mathbb{R}^2 \setminus P$ has only one unbounded component. If not then let C_1 and C_2 be the two unbounded components. Both C_1 and C_2 must intersect $\mathbb{R}^2 \setminus B_r(q)$, in fact $\mathbb{R}^2 \setminus B_r(q) \subseteq C_1 \cup C_2$, a contradiction to $C_1 \cup C_2$ being disconnected. $\square$

Since $\mathbb{R}^2 \setminus P$ only has one unbounded component, we can write $\mathbb{R}^2 \setminus P$ as

$$\left(\bigcup_{\alpha \in A} C_\alpha \right) \cup C_0$$

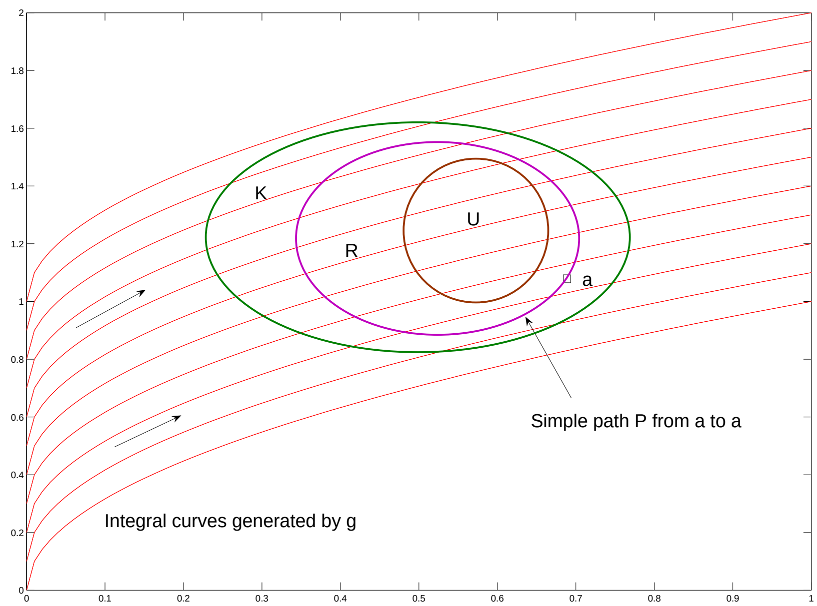
where each C_α and C_0 are components of $\mathbb{R}^2 \setminus P$ and C_0 is the unique unbounded component. Let $R := \bigcup_{\alpha \in A} C_\alpha \cup P$. Note that this is the same as $\mathbb{R}^2 \setminus C_0$. Since C_0 is open, we know that R is closed. See Figure 3 for a possible configuration of these sets. Next, we show that R is a chaotic closed set.

Lemma 2. *R is a chaotic closed set.*

Proof. Let $x, y \in R$. Then there exists a simple path from x to some point $z \in P$ and simple path from some $w \in P$ to y. Thus there is a simple path from x to z, from z to a, from a to w and from w to y.

Suppose $R^\circ \neq \varnothing$. Since P is a finite union of arcs, $P^\circ = \varnothing$. Thus there exists $U \subset R$ open in $\mathbb{R}^2$ such that $P \cap U = \varnothing$. In this case we take P to be our orbit to satisfy part (b) of the definition of chaotic set.

Figure 3:



Suppose $R^\circ = \varnothing$. Then P contains no simple closed curves. Then there exists $< 0 < t_1, t_2$ such that $\psi^t(a) \in P$ for all $0 \leq t \leq t_1$ and $\phi^t(\psi^{t_1}(a)) \in P$ for $0 \leq t \leq t_2$ with $\phi^{t_2}(\psi^{t_1}(a)) = a$ and $P_2 := \{\phi^t(\psi^{t_1}(a)) : 0 \leq t \leq t_2\} = \{\psi^s(a)) : 0 \leq s \leq t_1\} =: P_1$. Let $b \in P_1$ such that $b \neq a$ and $b \neq \psi^{t_1}(a)$. Let $t_3 \in (0, t_1)$ such that $\psi^{t_3}(a) = b$ and $t_4 \in (0, t_2)$ such that that $\phi^{t_4}\psi^{t_1}(a) = b$. Let $\lambda \in \Gamma$ be the periodic trajectory (with period $t_3 + t_2 - t_4$) such that $\lambda(0) = a$, $\lambda(s) = \psi^s(a)$ for $0 \leq s \leq t_3$ and $\lambda(s) = \phi^{t_4+(s-t_3)}\psi^{t_3}(a)$ for $t_3 \leq s \leq t_3 + (t_2 - t_4)$. This shorter trajectory then satisfies part (b) of the definition of chaotic set. □

Lemma 3. *Let R be a chaotic set. Then $\Gamma|_R$ has a dense set of periodic points.*

Proof. Let $x \in R$. Then there exists a periodic orbit in R from x to x. □

Lemma 4. *Let R be a chaotic set. Then $\Gamma|_R$ is topologically transitive.*

Proof. Let $U, V \subset R$ be nonempty and open in the relative topology on R. Let $x \in U$ and $y \in V$. Then there exists a path from x to y so there exists an orbit that starts in U and passes through V. □

Lemma 5. *Let R be a chaotic set. Then $\Gamma|_R$ has sensitive dependence on initial conditions.*

Proof. Let $U \subset R$ be open and γ a trajectory such that $\gamma(t) \in R \setminus U$ for all $t \in T$. Let $z \in U$ and $\delta > 0$ such that $B_\delta(z) \subset U$. Let $x \in R$. We establish sensitive dependence on initial conditions by showing that there are two different trajectories in our system that each start at x but end up being more than δ apart.

Let $w \in R$ such that $\gamma(t) = w$ for some $t \in T$. Then there exists a trajectory λ from x to w that then follows γ. There is also another trajectory λ' from x to w to z reaching w at the same time λ reaches w. Then when λ' reaches z, λ is on γ so $d(\lambda(t), \lambda'(t)) > \delta$. □

The next theorem follows from Lemmas 3–5.

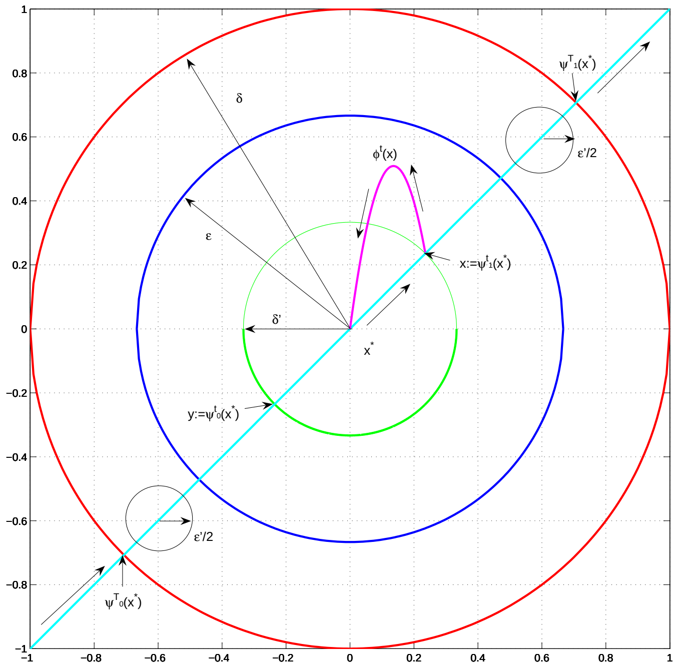
Theorem 4. *If R is a chaotic set, then $\Gamma|_R$ is chaotic.*

We now turn to establishing that our non-linear MVDS will typically be chaotic on a region near a steady state. Let $x^* \in \mathbb{R}^2$ with $f(x^*) = 0$ and $g(x^*) \neq 0$. Let $A = Df(x^*)$ with eigenvalues λ_1, λ_2 and corresponding eigenvectors e_1, e_2. Choose $\delta > 0$ such that $g(x) \neq 0$ for all $x \in \overline{B}_\delta(x^*)$.

Theorem 5 (sink or source). *Suppose there exists $\epsilon > 0$ such that x^* is a sink (asymptotically stable) or a source (asymptotically unstable) for f on $B_\epsilon(x^*)$. Then Γ has a chaotic set.*

Proof. Figure 4 contains a schematic for this proof. Suppose x^* is a sink. One can pick $\epsilon < \delta/2$. Let $\delta' > 0$ such that if $z \in B_{\delta'}(x^*)$ then $\phi^t(z) \in B_\epsilon(x^*)$ for all $t \in T$ and $\phi^t(z) \to x^*$. We can pick $t_0 < 0$ (maximal) and $0 < t_1$ (minimal) such that $y := \psi^{t_0}(x^*) \in \partial B'_\delta(x^*)$ and $x := \psi^{t_1}(x^*) \in \partial B'_\delta(x^*)$, i.e., t_0 is the first time (going forward in time) that the flow $\psi^t(x^*)$ intersects $\partial B'_\delta(x^*)$ and t_1 is the first time (going backward in time) that the flow $\psi^t(x^*)$ intersects $\partial B'_\delta(x^*)$. Since x^* is asymptotically stable under f, then as $t \to \infty$, $\phi^t(x) \to x^*$ and $\phi^t(x) \in B_\epsilon(x^*)$ for all $t \in T$. Let $L := \{\psi^s(x^*) : t_0 \leq s \leq t_1\}$. If there exists $t > 0$ such that $\phi^t(x) \in L$, then we have a simple path from x to x. If not $\phi^t(x) \notin L$ for all $t > 0$. Let $\epsilon' := \sup_{t>0} \inf_{w \in L}\{d(\phi^t(x), w)\} \leq \epsilon$. Let $T_0 < 0$ be minimal and $0 < T_1$ be maximal such that $\psi^{T_0}(x^*), \psi^{T_1}(x^*) \in \partial B_\delta(x^*)$. So choose $t'_0 < 0 < t'_1$ such that $B_{\epsilon'/2}(\psi^{t'_0}(x^*)) \cap B_\epsilon(x^*) = \varnothing$ and $B_{\epsilon'/2}(\psi^{t'_1}(x^*)) \cap B_\epsilon(x^*) = \varnothing$. Let $0 < \lambda < \epsilon'/2$ such that if $z \in B_\lambda(\psi^{t'_0}(x^*))$ then $d(\psi^s(z), \psi^{t'_0+s}(x^*)) < \epsilon'/2$ for all $0 \leq s \leq t'_1 - t'_0$. Then there exists z and $0 < t_3 < t_4$ such that $\phi^{t_3}(x), \phi^{t_4}(x) \in \{\psi^s(z) : 0 \leq s \leq t'_1 - t'_0$. Thus there is a simple path from $\phi^{t_3}(x)$ to $\phi^{t_3}(x)$.

Figure 4: Integral curves ψ and ϕ from Theorem 5.



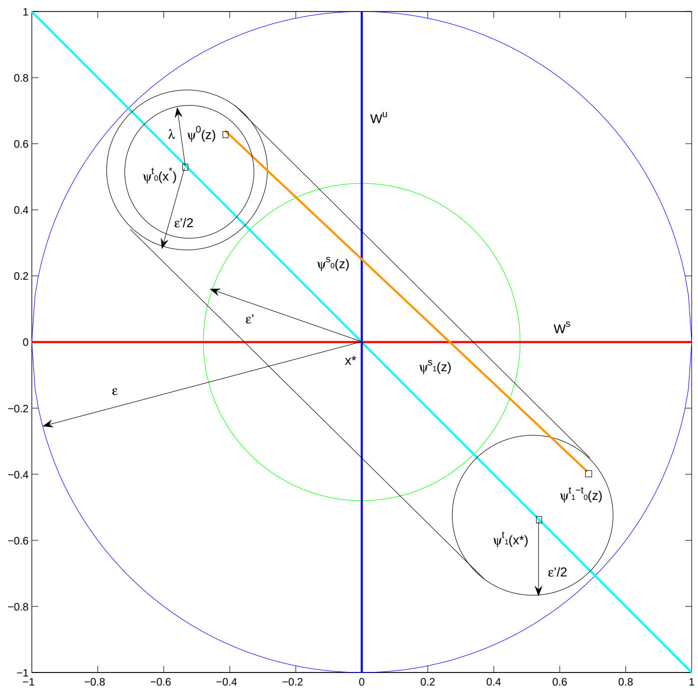
The proof for a source is essentially the same with the direction of time reversed.

□

Theorem 6 (saddle). *Suppose $\lambda_1 < 0$ and $\lambda_2 > 0$ with $g(x^*) \neq \alpha e_1$ and $g(x^*) \neq \beta e_2$. Then Γ has a chaotic set.*

Proof. Figure 5 contains a schematic for this part of the proof. Let $\epsilon < \delta/2$ be such that (1) W^s and W^u divide $B_\epsilon(x^*)$ into four quadrants which we number moving counterclockwise from 1 to 4 and (2) without loss of generality, $\phi^t(x^*)$ is in quadrant 4 for sufficiently small $t > 0$ and in quadrant 2 for sufficiently small $t < 0$ (in Figure 5, quadrant 1 is the top-right quadrant). The restriction on $g(x^*)$ implies that in a sufficiently small neighborhood of x^*, we will have $\psi^t(x^*)$ in quadrant j for $t < 0$ sufficiently close to 0 and $\psi^t(x^*)$ in quadrant $j+2$ mod 4 for $t > 0$ sufficiently close to 0. Let $T_0 < 0$ be minimal and $0 < T_1$ be maximal such that $\psi^{T_0}(x^*), \psi^{T_1}(x^*) \in \partial B_\epsilon(x^*)$. Let $\epsilon' < \epsilon/2$ and $t_0 < 0$ (maximal) and $0 < t_1$ (minimal) such that $B_{\epsilon'/2}(\psi^{t_0}(x^*)) \cap B_{\epsilon'}(x^*) = \varnothing$ and $B_{\epsilon'/2}(\psi^{t_1}(x^*)) \cap B_{\epsilon'}(x^*) = \varnothing$. Let $\lambda > 0$ such that

Figure 5: Schematic for first part of proof for Theorem 6.



if $z \in B_\lambda(\psi^{t_0}(x^*))$ then $d(\psi^s(z), \psi^{t_0+s}(x^*)) < \epsilon/2$ for all $0 \le s \le t_1 - t_0$. Thus there exists $z \in B_\lambda(\psi^{t_0}(x^*))$ and $0 < s_0 < s_1$ such that $\psi^{s_0}(z) \in W^u$ and $\psi^{s_1}(z) \in W^s$.

Figure 6 (a blown-up version of Figure 5) contains a schematic for this part of the proof. Choose $r_0, \lambda_0 > 0$ such that $B_{\lambda_0}(\phi^{r_0}(\psi^{s_1}(z))) \cap \{\psi^m(z) : s_0 \le m \le s_1\} = \varnothing$. Choose $\lambda_1 > 0$ such that if $w \in B_{\lambda_1}(\psi^{s_1}(z))$ then $d(\phi^m(w), \phi^m(\psi^{s_1}(z))) < \lambda_0$ for all $0 \le m \le r_0$. Let $w \in \{\psi^m(z) : s_0 \le m < s_1\} \cap B_{\lambda_1}(\psi^{s_1}(z))$. Then $w \notin W^s \cup W^u$ so $\phi^m(w) \notin W^u \cup W^s$ for all $m \ge 0$. Hence there exists M such that $\phi^M(w) \notin B_\epsilon(x^*)$. So there exists $M_0 > 0$ such that $\phi^{M_0}(w) \in \{\psi^m(z) : s_0 \le m \le s_1\}$. Thus there exists a simple path from w to w. □

4.3 One-Sector Model

We now turn to the non-linear MVDS from the model and consider two cases for the steady state: local determinacy and local indeterminacy. We will see that in both cases the model can exhibit chaotic behavior.

Figure 6: Schematic for second part of proof for Theorem 6. The simple path from w to w first follows ϕ to $\phi^{M_0}(w)$, then follows ψ back to w.

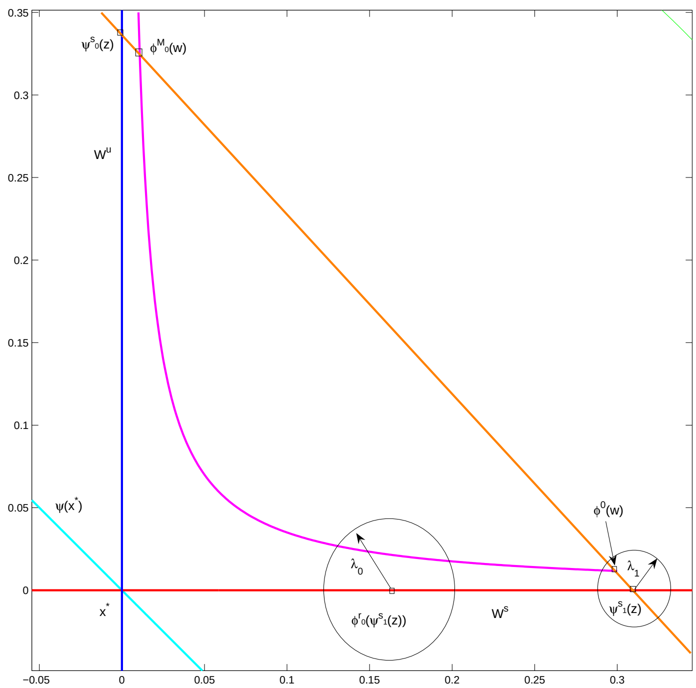
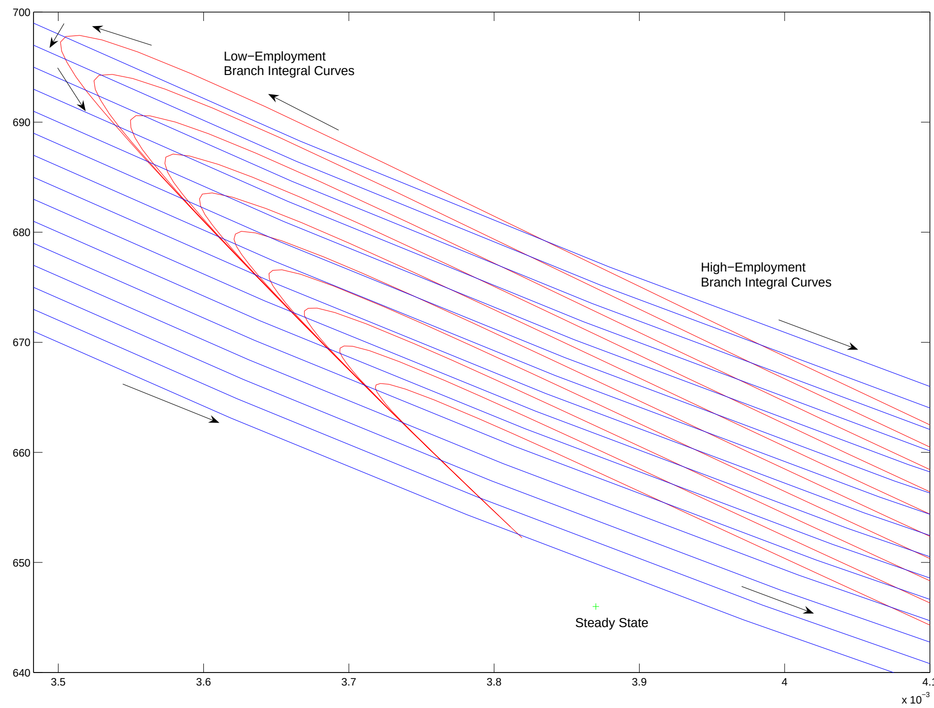


Figure 7: The steady state is locally a sink. Plotted are integral curves from both the low-employment and high-employment branches. The plotted integral curves from the high-employment branch flow from the top left to the bottom right. The plotted integral curves for the low-employment branch are flowing counter clockwise.



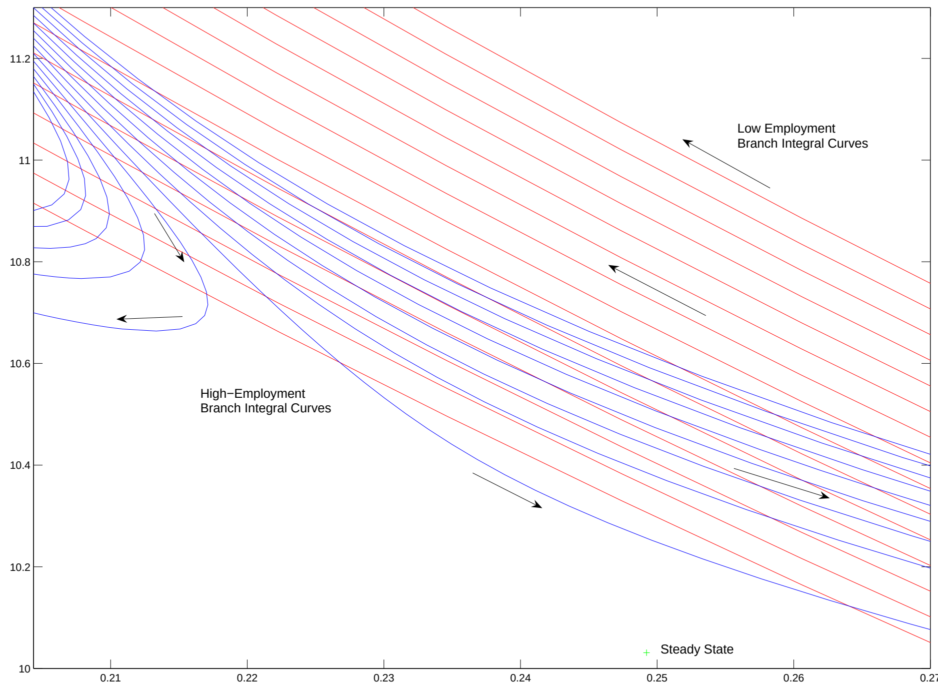
Example 2 (Local Indeterminacy). Let parameter values be set at baseline values (see Table 1) with $\gamma = 0.6$. The steady state and eigenvalues from the linearization around the steady state are given by $K = 0.0039$, $\lambda = 645.9942$, $N = 0.3043$, $C = 0.0015$, $Y = 0.0019$, eigenvalues $\mu_1 = -0.9091$, $\mu_2 = -0.0759$. We see the steady state is locally a sink.

The integral curves from the non-linear system are plotted in Figure 7. From this figure it is clear that on a significant region near the steady state the dynamics are chaotic.

Example 3 (Local Determinacy). Let parameter values be set at baseline values (see Table 1) with $\gamma = 0.4$. The steady state and eigenvalues from the linearization around the steady state are given by $K = 0.2492$, $\lambda = 10.0311$, $N = 0.3043$, $C = 0.0997$, $Y = 0.1246$, eigenvalues $\mu_1 = 0.7839$, $\mu_2 = -0.2031$. We see the steady state is locally a saddle.

The integral curves from the non-linear system are plotted in Figure 8. We see a similarity

Figure 8: The steady state is locally a saddle. Plotted are integral curves from both the low-S and high-S branches. The plotted integral curves from the high-S branch (those associated with the local saddle) flow from the top left to the bottom right. The plotted integral curves for the low-S branch are flowing from the bottom right to the top left.

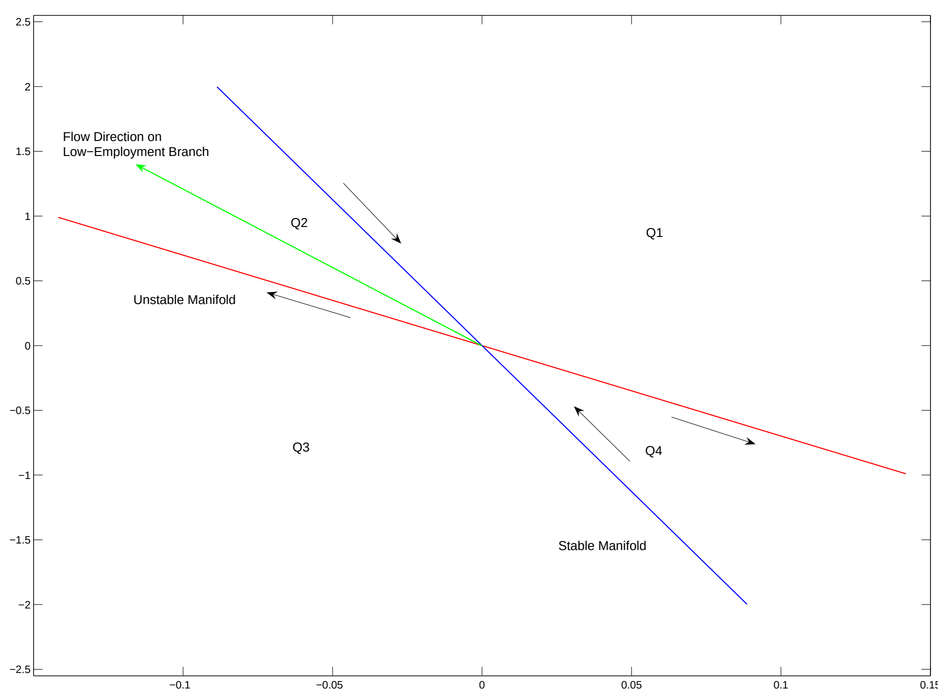


to the integral curves depicted in Figure 2 generated by a linear function and a constant function. From this figure it is clear that on a significant region near the steady state the dynamics are chaotic. Figure 9 contains the stable/unstable manifold of the linear approximation around the saddle steady state along with the direction of the vector field of the low-S branch at the steady state. The stable/unstable manifold divides the state space into four quadrants (Q1–Q4). From the linear approximation, chaos is expected in Q1 and this is exactly what one sees for the non-linear system in Figure 8.

5 Conclusion

In this paper, we make two contributions. First, we provide necessary and sufficient conditions for Euler equation branching in the one-sector model with a production externality of

Figure 9: Stable/unstable manifold from the linear approximation around the steady state. Included is the direction of the flow on the low-employment branch. The stable/unstable manifold divides the state space into four quadrants labeled Q1–Q4. Given the direction of the flow on the low-employment branch and the position of the stable/unstable manifolds, chaos is expected in Q1.



Christiano and Harrison (1999), though for a wider range of parameter values. We also provide sufficient conditions for Euler equation branching for a more general class of preferences and technologies. Second, in two-dimensional models with Euler equation branching, we provide sufficient conditions for establishing chaos and show analytically that these conditions are typically satisfied near a steady state equilibrium. Moreover, this chaotic behavior will occur if the steady state is "locally" a saddle, sink or source.

References

Benhabib, J., Farmer, R. E. A., 1996. Indeterminacy and sector specific externalities. Journal of Monetary Economics 37, 397–419.

Benhabib, J., Perli, R., 1994. Uniqueness and indeterminacy: on the dynamics of endogenous growth. Journal of Economic Theory 63, 113–142.

Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2001. The perils of Taylor rules. Journal of Economic Theory 96, 40–69.

Boldrin, M., Nishimura, K., Shigoka, T., Yano, M., 2002. Chaotic equilibrium dynamics in endogenous growth models. Journal of Economic Theory 96, 97–132.

Christiano, L. J., Harrison, S. G., 1996. Chaos, sunspots and automatic stabilizers. National Bureau of Economic Research Working Paper 5703.

Christiano, L. J., Harrison, S. G., August 1999. Chaos, sunspots and automatic stabilizers. Journal of Monetary Economics 44 (1), 3–31.

Coury, T., Wen, Y., 2002. Global indeterminacy and chaos in standard rbc models. Manuscript, Cornell University.

Devaney, R. L., 2003. An introduction to chaotic dynamical systems, 2nd Edition. Westview Press, Boulder, Colorado.

Guo, J.-T., Lansing, K. J., 2002. Fiscal policy, increasing returns, and endogenous fluctuations. Macroeconomic Dynamics 6, 633–664.

Hommes, C., de Vilder, R., 1995. Sunspot equilibria in an implicitly defined overlapping generations model. working paper, 1–20.

Medio, A., Raines, B., 2007. Backward dynamics in economics. the inverse limit approach. Journal of Economic Dynamics and Control, 31, 1633–1671.

Michener, R., Ravikumar, B., 1998. Chaotic dynamics in a cash-in-advance economy. Journal of Economic Dynamics and Control 22, 1117–1137.

Schmitt-Grohé, S., Uribe, M., 1997. Balanced-budget rules, distortionary taxes, and aggregate instability. Journal of Political Economy 105, 976–1000.

Stockman, D. R., 2007a. Balanced-budget rules: chaos and cycles. Manuscript, University of Delaware.

Stockman, D. R., 2007b. Chaos and sector-specific externalities. Manuscript, University of Delaware.