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# A note on Strumilin's model of optimal saving

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## 1. INTRODUCTION

The question of optimal saving is never to be solved once and for all. How should we decide on problems of time horizon or preference of future against present consumption? And what do we really know about production possibilities in the course of time? These questions provide no reason for complaint. Different models seem to serve different purposes. Beautiful generalisations in mathematical language are available to show what can be said on basis of a few traditional propositions about utility and production. Other constructions have a more narrow scope. They intend to supply some information and specific solutions based on crude observations of reality. For instance, there may be some evidence to regard the Cobb-Douglas production function as a good approximation of the input–output structure. The planning period may be specified within some range and people could be questioned about their ideas about desirable accumulation.

The numerical model of S. G. Strumilin [5], first published in 1962, belongs to the class of simple models. Because the ideas of this author are based on his Russian experience, it could be of interest to compare his contribution with the work of western scholars. As an example of the latter Haavelmo's [2] analyses of optimal accumulation would be an excellent candidate. However, we intend to do more. Strumilin's numerical model is formalised by Nemchinov in [4]. The mathematical model of Nemchinov does not represent the period analysis of Strumilin in an accurate manner and no solution is given. In section two of this note we shall formulate the correct symbolical version of Strumilin's model and solve it in analytical terms as far as conventional mathematics allows. Finally an extension of the model is introduced and discussed in the third section of our note.

[1] We are indebted to Mr. H. P. J. van de Kerkhof for helpful comments.

## 2. THE STRUMILIN MODEL

The dynamic system starts from a given value of labor and capital in period 1. There is no population growth. It is assumed that $\lambda$ units of labor, where $\lambda$ is a given number, can produce $\lambda$ units of output measured in labor terms without the aid of capital. The increase of output depends on what is called "Nutseffekt der Investitionen". Writing $Y$ for output and $K$ for capital the latter is defined as:

$$E = \sum_{1}^{t} \triangle Y_t \Big/ K_t \tag{1}$$

It has to be observed that the rise in output is taken in relation to labour-produced output. From (1) and the assumed input–output relation for labour ($L$) it then follows:

$$Y_t = Y_1 + \sum_{1}^{t} \triangle Y_t = L_1 + EK_t \tag{2}$$

We therefore may conclude that the assumption about production comes down to a linear production function. One may wonder whether in the light of western research a substitution elasticity of infinity is acceptable.[1] Moreover, technical change is absent. The implied production function should perhaps merely be regarded as an illustration. It certainly is a remarkable coincidence that Haavelmo in his study mentioned above uses the same function for reasons of convenience.

A constant part ($s_k$) of the increase in output caused by the accumulation of capital goods is saved and invested. So we can write:

$$s_k \sum_{1}^{t} \triangle Y_t = \triangle K_t \tag{3}$$

Equations (2) and (3) supplemented by the marginal productivity theory of distribution imply that a constant proportion of capital income is saved, while all income from labour is consumed. The model reduces to a linear production function and a Cambridge saving function.

Marginal productivity may be the wrong term in case the state owns all capital and redistributes part of the revenues from this source. Nevertheless, seen from the perspective of efficient production an explanation in terms of imputed factor incomes is more straightforward. We should of course grant the Soviet economists the right to define their own concepts. However, if we talk about economics, we must have the same thing in mind.

[1] See for instance David and van de Klundert [1].

There may be more room for a divergence of opinions with respect to the objective function. In Strumilin's model total consumption over a period of 40 years (the average working time of a generation) is maximised. We therefore have to choose $s_k$ in such a way that

$$\check{C} \equiv \sum_{1}^{40} C_t = \sum_{1}^{40} [C_0 + (1-s_k) \sum_{1}^{t} \triangle Y_t] \tag{4}$$

is as high as possible. It is interesting to compare this more or less paternalistic rule with Haavelmo's carefully, but introspective analysis of people's preferences. If men are allowed to follow their own desires only a few tentative proportions like the following seem to be possible. People do want a rising level of consumption. Fixing a planning period of about 5 years there should be a positive amount of investment at the end of this period, while it is held that the stock of capital should never be run down. More specific results are obtained by working through simple *excercises* bounded by the rules just described.

But let us return to the model of Strumilin. It may be solved in a conventional manner. For simplicity we change over to continuous time. From (1) and (3) it can be seen that capital is growing at a constant rate:

$$K_t = K_0 e^{s_k E t} \tag{5}$$

Writing (4) in continuous time, taking account of (1) and substituting (5) we get:

$$\check{C} = TC_0 + (1-s_k) EK_0 \int_0^T e^{s_k E t} \, dt = TC_0 + \frac{1-s_k}{s_k} K_0 [e^{s_k E T} - 1] \tag{6}$$

where $T$ stands for the now unspecified length of the planning period.

The first derivative of $\check{C}$ with respect to $s_k$ is easily computed:

$$\frac{d\check{C}}{ds_k} = \frac{K_0}{{s_k}^2} \{[(1-s_k)As_k - 1]e^{As_k} + 1\}, \tag{7}$$

where $A \equiv ET$. Putting the *RHS* of (7) equal to zero we get as a condition for maximal total consumption:[1]

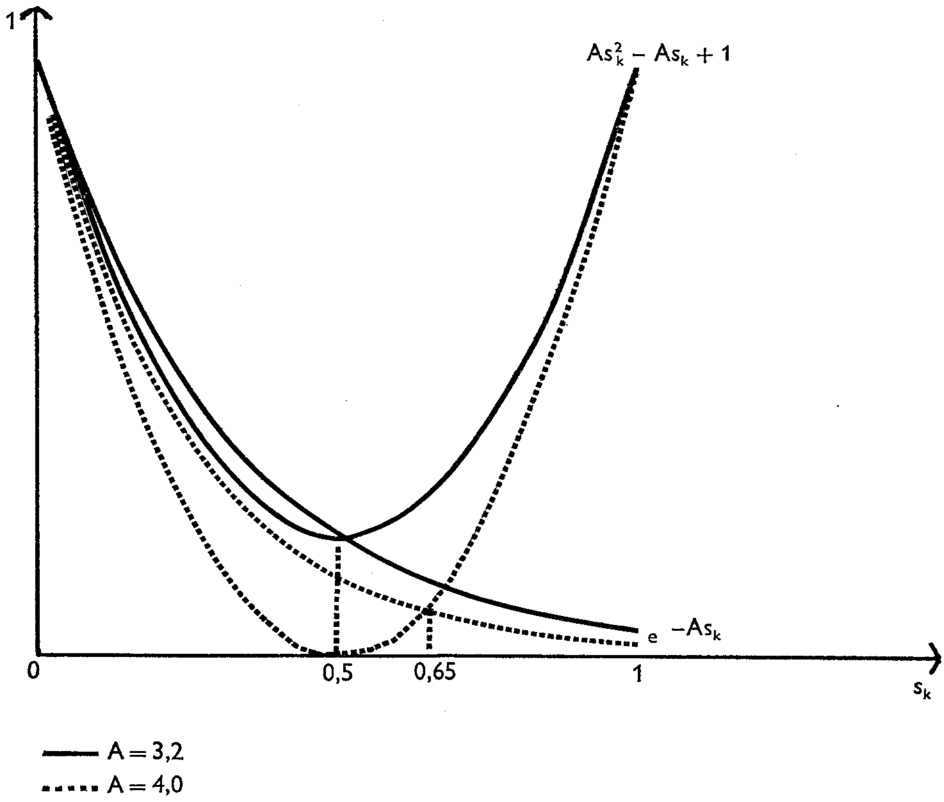
$$As_k^2 - As_k + 1 = e^{-As_k} \tag{8}$$

It is not possible to solve (8) for $s_k$ explicitely. As inspection of (8) shows the parabola on the *LHS* and the exponential function on the *RHS* both go through the point with coordinates (0, 1). A somewhat elaborate

[1] The expression for the second derivative is not easily to handle. The reader should verify by numerical experimentation that a maximum value is involved.

mathematical analysis. not to be presented here, reveals that a second point of intersection exists for $A > 2$. This is shown in Figure 1 for two different values of $A$.

Figure 1.




The second point of intersection gives us the optimal $s_k$. To derive the direction of change for an increase in $A$ we transform (8) to:

$$e^{-x} + x - 1 = \frac{1}{A} x^2, \text{ where } x \equiv As_k \tag{8a}$$

Dividing both sides of (8a) by $x$ we get:

$$s_k = \frac{e^{-x} + x - 1}{x} \tag{8b}$$

Differentiation of (8b) leads to:

$$\frac{ds_k}{dx} = \frac{x(-e^{-x}+1)-e^{-x}-x+1}{x^2} \cong \frac{\frac{1}{2!}+\frac{x}{3!}+\ldots\ldots}{e^x} > 0$$

The optimal value of $s_k$ increases from $0 \to 1$ as $A$ goes from $2 \to \infty$. A rise in the marginal productivity of capital or in the length of the planning period causes an increase in the optimal savings out of capital income.

In Table 1 we present some numerical results. As may be expected they conform the above results. The second row is in complete accordance with the results of Strumilin's numerical example.

Table 1. *Optimal savings for alternative values of A.*

| $A$ | $s_k$ | $\bar{s}$ |
|---|---|---|
| 2,5 | 0,3 | 0,066 |
| 3,2 | 0,5 | 0,186 |
| 4,0 | 0,65 | 0,512 |
| 5,0 | 0,73 | 0,579 |

Capital grows exponentially at a constant rate. The aggregate saving ratio and the capital coefficient both increase over time. The former approaches $s_k$ for $t \to \infty$, while the asymtotic value of the latter is $1/E$. Expressions for the behaviour of both variables over time are obtained from (2), (3) and (5):

$$s \equiv \frac{\dot{K}}{Y_t} = \frac{s_k}{1+\xi\, e^{-s_k Et}}, \quad \# \left[\dot{K}_t \equiv \frac{dK_t}{dt}\right] \tag{9}$$

$$\varkappa \equiv \frac{K_t}{Y_t} = \frac{1/E}{1+\xi\, e^{-s_k Et}} \tag{10}$$

where $\xi \equiv \dfrac{L_0}{EK_0}$. From the time-paths of income and investment we further may compute the average saving ratio out of total income ($\bar{s}$) corresponding to an optimal value of $s_k$. The results with respect to the chosen numerical examples are given in column three of Table 1.

**3. AN EXTENSION**

A rise of the saving ratio as well as of the capital coefficient does not agree with the "stylized" facts, to borrow from Kaldor [3], of western economics. Moreover the idea of balanced growth suggests that both ratios are more or less constant in the long run. As will be shown in this section it is easy to extend the Strumilin model in such a way that $s$ and $\varkappa$ are constant for certain values of the parameters involved.

First we may introduce population growth at a constant rate. However, if output is measured in efficiency units (with respect to labour) population growth and Harrod neutral technical change are identical from the production point of view. We therefore assume that labour *potential* is growing at a rate of 100 $\pi$ percent. In the light of the foregoing it seems a fair generalisation to state that all labour income, whether it comes from an increase in labour income per head or from pure expansion of numbers, is consumed. Secondly, to complete the story we should take account of depreciation of capital in real terms. The reasonable thing to assume is that a fixed proportion ($\delta$) of the capital stock is scheduled for replacement at every point of time.

Incorporating the extensions mentioned above the model becomes:

$$Y_t = L_t + EK_t \tag{11}$$

$$L_t = L_0\, e^{\pi t} \tag{12}$$

$$\dot{K}_t = s_k\,(Y_t - L_t) - \delta\, K_t \tag{13}$$

$$\max.\ \check{C} = \int_0^T C_t dt = (1 - s_k) \int_0^T (Y_t - L_T)\, dt + \int_0^T L_T\, dt. \tag{14}$$

A solution along the lines sketched in section two gives the following equation to solve for $s_k$:

$$A\,E\,s_k^2 - A(E+\delta)\,s_k + E - (1-A)\,\delta = (E-\delta)\,e^{-As_k + T\delta} \tag{15}$$

For the net savings gross output ratio and capital related to gross output we get respectively:

$$s = \frac{s_k - {}^{\delta}/_{E}}{1 + \xi\, e^{(\pi+\delta-s_k E)t}} \tag{16}$$

$$\varkappa = \frac{{}^{1}/_{E}}{1 + \xi\, e^{(\pi+\delta-s_k E)t}} \tag{17}$$

The increase in labour potential has no influence on the optimal value of $s_k$. Inspection of (15) reveals that the introduction of $\delta$ leads to a higher optimal value of savings out of capital income.[1] These results are to be expected. An increase in labour potential influences production and consumption in the same way. Physical depreciation of capital means more savings for the same future consumption. For a given $T$ this should increase the optimal $s_k$. Finally we conclude from (16) and (17) that balanced growth with constant $s$ and $k$ is possible if the parameters eventually obey the condition $\pi = s_k E - \delta$.

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[1] We can still discuss matters in terms of the intersection between a parabola and an exponential function.