Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 13, NUMBER 5 (2010) ISSN 1311-0454

SOME DISTORTION THEOREMS FOR STARLIKE HARMONIC FUNCTIONS

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Dedicated to Professor Yaşar Polatoğlu
the occasion of his 60th birthday

Abstract

In this paper, we consider harmonic univalent mappings of the form $f = h + \bar{g}$ defined on the unit disk $\mathbb{D}$ which are starlike. Distortion and growth theorems are obtained.

MSC 2010: 30C55, 30C45
Key Words and Phrases: harmonic function, starlike harmonic function, growth theorem, distortion theorem

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathfrak{D}$ is said to be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$, that is, u, v satisfy, respectively the Laplace equations

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0.$$

There is a well-known relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v which are defined on a simply connected domain $\mathfrak{D}$ there exist analytic functions U and V so that

$$u = \mathfrak{Re}(U) \text{ and } v = \mathfrak{Im}(V).$$

Therefore, it has a canonical decomposition

$$f = h + \overline{g} \tag{1}$$

where h and g are, respectively, the analytic functions

$$h = \frac{U+V}{2} \text{ and } g = \frac{U-V}{2}.$$

We call h the analytic part and g the co-analytic part of f. It is fact that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function. However, not every complex-valued harmonic function is analytic.

The Jacobian J_f of a function $f = u + iv$ has a very important place in the theory of harmonic mappings, defined by

$$J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

Or, in terms of f_z and $f_{\bar{z}}$, we have

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'(z)|^2 - |g'(z)|^2,$$

where $f = h + \overline{g}$ is the harmonic function in $\mathfrak{D}$.

If $f = h + \overline{g}$ is a harmonic function on $\mathfrak{D}$ with $J_f > 0$, then we say that f is a sense-preserving (or orientation preserving) harmonic function on $\mathfrak{D}$. In this case we have

$$|g'(z)| < |h'(z)|$$

for all $z \in \mathfrak{D}$. If f has $J_f < 0$, then $\overline{f}$ is sense preserving. For convenience, we will only examine sense preserving harmonic functions.

The mapping $z \to f(z)$ is sense preserving and locally univalent in $\mathfrak{D}$ if and only if $J_f > 0$ in $\mathfrak{D}$. The function $f = h + \bar{g}$ is said to be harmonic univalent in $\mathfrak{D}$ if the mapping $z \to f(z)$ is sense preserving harmonic and univalent in $\mathfrak{D}$.

The second complex dilatation of a harmonic function $f = h + \overline{g}$ is the quantity

$$\omega(z) = \frac{\overline{f_{\bar{z}}}}{f_z} = \frac{g'(z)}{h'(z)} \quad (z \in \mathfrak{D}). \tag{2}$$

Let $\mathcal{S}_{\mathcal{H}}$ denote the family of functions $f = h + \overline{g}$ that are harmonic, sense preserving, and univalent in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{3}$$

It follows from the sense-preserving property if $f \in \mathcal{S}_{\mathcal{H}}$, then we have $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Thus, it is easy to see that $|b_1| < 1$. Since the second complex dilatation ω of a sense preserving harmonic mapping f is always an analytic function of modulus less than one, then this function ω will be called the analytic dilatation of f. Also $f \in \mathcal{S}_{\mathcal{H}}$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its members is zero. In 1984 Clunie and Sheil-Small [1] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Many studies have been done on this class and its subclasses, and continued taking place.

A sense-preserving harmonic mapping $f \in \mathcal{S}_{\mathcal{H}}$ is in the class $\mathcal{S}^*_{\mathcal{H}}$ if the range $f(\mathbb{D})$ is starlike with respect to the origin. A function $f \in \mathcal{S}^*_{\mathcal{H}}$ is a called harmonic starlike mapping in $\mathbb{D}$. A function $f = h + \overline{g}$ with such a property must satisfy the condition

$$\mathfrak{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) > 0$$

for all $z \in \mathbb{D}$.

In our proofs we use the following lemma:

LEMMA 1.1[2]. *If $f = h + \overline{g} \in \mathcal{S}^*_{\mathcal{H}}$, then there exist angles α and β such that*

$$\mathfrak{Re}\left\{\left(e^{i\alpha}\frac{h(z)}{z} + e^{-i\alpha}\frac{g(z)}{z}\right)\left(e^{i\beta} - e^{-i\beta}z^2\right)\right\} > 0 \tag{4}$$

for all $z \in \mathbb{D}$.

Let $\mathcal{A}$ denote the class of all functions s_1 analytic in the open unit disk $\mathbb{D}$ with the usual normalization $s_1(0) = s_1'(0) - 1 = 0$. If s_1 and s_2 are analytic in $\mathbb{D}$, we say that s_1 is subordinate to s_2, written $s_1 \prec s_2$ or $s_1(z) \prec s_2(z)$, if s_2 is univalent, then we have $s_1(0) = s_2(0)$ and $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in the open unit disk $\mathbb{D}$. If p in $\mathcal{P}$ satisfies $\mathfrak{Re}p(z) > 0$ for $z \in \mathbb{D}$, then we say that p is the Carathéodory function. It has been shown that for a function $p(z) \in \mathcal{P}$, the following inequalities are satisfied ([3]):

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r}, \tag{5}$$

and

$$\left| z\frac{p'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2} \tag{6}$$

for all $|z| = r < 1$.

2. Results

Lemma 2.1. *Let $f = h + \overline{g}$ be an element of $\mathcal{S}^*_{\mathcal{H}}$, then we have*

$$\begin{aligned} \frac{(A+|B|)r-(A-|B|)r^2}{(1+r)(1+r^2)} &\leq |h(z) - e^{-2i\alpha} g(z)| \\ &\leq \frac{(A+|B|)r+(A-|B|)r^2}{(1+r)(1+r^2)}, \end{aligned} \tag{7}$$

for $|z| = r < 1$ where $A = \cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha) > 0$, $B = \sin(\beta+\alpha) - b\cos(\beta-\alpha) - a\sin(\beta-\alpha)$, $g'(0) = b_1 = a + ib$ for some choice of angles α and β.

P r o o f. Since $f = h + \overline{g}$ is element of $\mathcal{S}^*_{\mathcal{H}}$, then we have

$$\frac{h(z)}{z}\Big|_{z=0} = 1, \quad \frac{g(z)}{z}\Big|_{z=0} = b_1 = a + ib,$$

and if we consider (4) as a function with positive real part

$$p(z) = \left(e^{i\alpha}\frac{h(z)}{z} + e^{-i\alpha}\frac{g(z)}{z} \right)\left(e^{i\beta} - e^{-i\beta}z^2 \right) \tag{8}$$

has the properties $\mathfrak{Re}p(z) > 0$ and $p(0) = [\cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha)]i[\sin(\beta+\alpha) - b\cos(\beta-\alpha) - a\sin(\beta-\alpha)]$ where $b_1 = a + ib$ and α, β are angles.

On the other hand, the assumption $p(0) = 1$ is not restriction for the Carathédory class. Indeed, let $p(z)$ be element of the Carathédory class with $p(0) = A + iB$, $A > 0$, then the function

$$p_1(z) = \frac{1}{A}(p(z) - iB)$$

satisfies the condition $p_1(0) = 1$ and $\mathfrak{Re}p_1(z) > 0$. This shows that $p_1(z)$ is the element of the Carathédory class. Therefore, the function

$$p_1(z) = \frac{1}{A}(p(z) - iB) = \frac{1}{\cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha)} \times \left[\left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z} \right) - i(\sin(\beta+\alpha) - b\cos(\beta-\alpha) - a\sin(\beta-\alpha)) \right] \tag{9}$$

is the Carathédory function under the condition $A = \cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha) > 0$. Then we have

$$\frac{1-r}{1+r} \leq |p_1(z)| \leq \frac{1+r}{1-r} \tag{10}$$

for $p_1(z) \in \mathcal{P}$ and $|z| = r < 1$. If we substitute (9) into (10) and after simple calculations we get

$$\frac{(A+|B|) - (A-|B|)r}{1+r} \leq |p(z)| \leq \frac{(A+|B|) + (A-|B|)r}{1-r}. \tag{11}$$

Using (8) and (11) we obtain

$$\frac{(A+|B|)r - (A-|B|)r^2}{(1+r)|e^{i\beta} - e^{-i\beta}z^2|} \leq |e^{i\alpha}h(z) - e^{-i\alpha}g(z)| \leq \frac{(A+|B|)r + (A-|B|)r^2}{(1-r)|e^{i\beta} - e^{-i\beta}z^2|}. \tag{12}$$

On the other hand, we have

$$\frac{1}{1+r^2} \leq \frac{1}{|e^{i\beta} - e^{-i\beta}z^2|} \leq \frac{1}{1-r^2}. \tag{13}$$

Therefore, if we use (13) in (12) we obtain the desired result. ■

THEOREM 2.2. *Let $f = h + \overline{g}$ be element of $\mathcal{S}^*_{\mathcal{H}}$, then we have*

$$|h'(z) - e^{-2i\alpha}g'(z)| \leq \frac{(A+|B|) + (A-|B|)r}{(1-r)^3}$$

for $|z| = r < 1$ where $A = \cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha) > 0$, $B = \sin(\beta+\alpha) - b\cos(\beta-\alpha) - a\sin(\beta-\alpha)$, $g'(0) = b_1 = a + ib$ for some choice of angles α and β.

P r o o f. Using Lemma 2.1, we obtain that

$$z\frac{p'(z)}{p(z)} = z\frac{Ap_1'(z)}{Ap_1(z)+iB}$$

for all z in the open unit disc. Also, we know that the following inequality satisfies for functions which in the Carathédory class:

$$\left|z\frac{p'(z)}{p(z)}\right| = \left|z\frac{p_1'(z)}{p_1(z)+i\frac{B}{A}}\right| \leq \frac{2r}{1-r^2}. \tag{14}$$

On the other hand, from the equation (8) we have

$$\frac{zh'(z)-e^{-2i\alpha}zg'(z)}{h(z)-e^{-2i\alpha}g(z)} = \frac{1+e^{-2i\beta}z^2}{1-e^{-2i\beta}z^2} + z\frac{p'(z)}{p(z)}. \tag{15}$$

Considering (14) and (15) together, we obtain

$$\begin{aligned}\left|\frac{zh'(z)-e^{-2i\alpha}zg'(z)}{h(z)-e^{-2i\alpha}g(z)}\right| &= \left|\frac{1+e^{-2i\beta}z^2}{1-e^{-2i\beta}z^2} + z\frac{p'(z)}{p(z)}\right| \\ &\leq \left|\frac{1+e^{-2i\beta}z^2}{1-e^{-2i\beta}z^2}\right| + \left|z\frac{p'(z)}{p(z)}\right|.\end{aligned} \tag{16}$$

Also we know that

$$\frac{1-r^2}{1+r^2} \leq \left|\frac{1+e^{-2i\beta}z^2}{1-e^{-i2\beta}}\right| \leq \frac{1+r^2}{1-r^2}. \tag{17}$$

Using (17) and (14) in (16), we get

$$\left|\frac{zh'(z)-e^{-2i\alpha}zg'(z)}{h(z)-e^{-2i\alpha}g(z)}\right| \leq \left|\frac{1+e^{-2i\beta}z^2}{1-e^{-2i\beta}z^2}\right| + \left|z\frac{p'(z)}{p(z)}\right| = \frac{1+r}{1-r}. \tag{18}$$

Using Lemma 2.1 in (18), we obtain that

$$|h'(z)-e^{-2i\alpha}g'(z)| \leq \frac{(A+|B|)+(A-|B|)r}{(1-r)^3}.$$

■

LEMMA 2.3. *Let $\omega(z)$ be the analytic dilatation of $f = h+\overline{g} \in \mathcal{S}_{\mathcal{H}}$ defined by $\omega(z) = g'(z)/h'(z)$ for all $z \in \mathbb{D}$, then we have*

$$\frac{(1-r)(1-|b_1|)}{1+|b_1|r} \leq |1-e^{-2i\alpha}\omega(z)| \leq \frac{(1+r)(1+|b_1|)}{1+|b_1|r} \tag{19}$$

($|z| = r < 1$) where $g'(0) = b_1 \neq 0$ and $|b_1| < 1$.

P r o o f. Let we define the function

$$\phi(z) = \frac{\omega(z)-b_1}{1-\overline{b_1}\omega(z)}$$

where $g'(0) = b_1$ for all $z \in \mathbb{D}$. Since $\phi(z)$ is a transformation which maps $\mathbb{D}$ onto itself we have $|\phi(z)| < 1$ and $\phi(0) = 1$. Thus we can write

$$\omega(z) \prec \frac{b_1 + z}{1 + \overline{b_1} z}.$$

On the other hand, the function $\omega(z) = \frac{b_1+z}{1+\overline{b_1}z}$ maps $|z| = r$ into the circle centered at

$$C(r) = \left\{ \frac{\mathfrak{Re} b_1(1-r^2)}{1-|b_1|^2 r^2}, \frac{\mathfrak{Im} b_1(1-r^2)}{1-|b_1|^2 r^2} \right\},$$

having the radius

$$\rho(r) = \frac{(1-|b_1|^2)r}{1-|b_1|^2 r^2}.$$

So we have

$$\left| \omega(z) - \frac{b_1(1-r^2)}{1-|b_1|^2 r^2} \right| \leq \frac{(1-|b_1|^2)r}{1-|b_1|^2 r^2}.$$

Therefore, we get the result after some simple calculations.

THEOREM 2.4. *Let $f = h + \overline{g}$ be an element of $\mathcal{S}^*_{\mathcal{H}}$, then we have*

$$|h'(z)| \leq \frac{(1+|b_1|r)(A+|B|+(A-|B|r))}{(1-|b_1|r)(1-r)^4}, \tag{20}$$

$$|g'(z)| \leq \frac{(A+|B|)+(A-|B|)r}{(1-r)^3} \left(1 + \frac{1+|b_1|r}{(1-r)(1-|b_1|r)} \right) \tag{21}$$

for $|z| = r < 1$ where $A = \cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha) > 0$, $B = \sin(\beta+\alpha) - b\cos(\beta-\alpha) - a\sin(\beta-\alpha)$, $g'(0) = b_1 = a + ib$ for some choice of angles α and β.

P r o o f. Let consider the analytic dilatation function $\omega = g'/h'$ of $f = h + \overline{g}$. Then, we have

$$\begin{aligned} |h'(z) - e^{-2i\alpha} g'(z)| &= |h'(z) - e^{-2i\alpha}\omega(z)h'(z)| \\ &= |h'(z)||1 - e^{-2i\alpha}(z)|. \end{aligned} \tag{22}$$

Considering (19) and Theorem 2.2 in (22) we obtain,

$$|h'(z)| \leq \frac{(1+|b_1|r)(A+|B|+(A-|B|r))}{(1-|b_1|r)(1-r)^4},$$

and

$$|g'(z)| \leq \frac{(A+|B|)+(A-|B|)r}{(1-r)^3} \left(1 + \frac{1+|b_1|r}{(1-r)(1-|b_1|r)} \right)$$

for all $|z| = r < 1$. ■

COROLLARY 2.5. *Let* $f = h + \overline{g} \in \mathcal{S}^*_{\mathcal{H}}$, *then we have*

$$|f(z)| \leq \int_0^r \frac{(1+|b_1|\rho)((A+|B|)+(A-|B|)\rho)}{(1-|b_1|\rho)(1-\rho)^4} d\rho + \int_0^r \frac{(A+|B|)+(A-|B|)\rho}{(1-\rho)^3}\left(1+\frac{1+|b_1|\rho}{(1-\rho)(1-|b_1|\rho)}\right) d\rho$$

for $|z| = r < 1$ *where* $A = \cos(\beta+\alpha) - a\cos(\beta-\alpha) + b\sin(\beta-\alpha) > 0$, $B = \sin(\beta+\alpha) - b\cos(\beta-\alpha) - a\sin(\beta-\alpha)$, $g'(0) = b_1 = a + ib$ *for some choice of angles* α *and* β.

P r o o f. For $f = h + \overline{g}$, we have the following inequalities

$$f = h + \overline{g} = \int_0^r h'(\rho e^{i\theta})e^{i\theta} d\rho + \overline{\int_0^r g'(\rho e^{i\theta})e^{i\theta} d\rho}$$

$$= \int_0^r h'(\rho e^{i\theta})e^{i\theta} d\rho + \int_0^r \overline{g'(\rho e^{i\theta})}e^{-i\theta} d\rho = \int_0^r f_z(\rho e^{i\theta})e^{i\theta} d\rho + \int_0^r f_{\overline{z}}(\rho e^{i\theta})e^{-i\theta} d\rho.$$

Hence

$$|f| = |h + \overline{g}| \leq |h| + |g| \leq \int_0^r |f_z(\rho e^{i\theta})| d\rho + \int_0^r |f_{\overline{z}}(\rho e^{i\theta})| d\rho \Rightarrow$$

$$|f| \leq \int_0^r |h'(\rho e^{i\theta})| d\rho + \int_0^r |g'(\rho e^{i\theta})| d\rho \Rightarrow .$$

Applying inequalities (20) and (21) to the above, we obtain the result. ∎

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Received: GFTA, August 27-31, 2010