CONJUGATE COMPOSITIONS IN EVEN-DIMENSIONAL AFFINELY CONNECTED SPACES WITHOUT A TORSION

Georgi Zlatanov, Bistra Tsareva

Abstract. Let in even-dimensional affinely connected space without a torsion \( A_{2m} \) be given a composition \( X_m \times X_m \) by the affinor \( a^\beta_\alpha \). The affinor \( b^\beta_\alpha \), determined with the help of the eigen-vectors of the matrix \( (a^\beta_\alpha) \), defines the second composition \( Y_m \times Y_m \). Conjugate compositions are introduced by the condition: the affinors of any of both compositions transform the vectors from the one position of the composition, generated by the other affinor, in the vectors from the another its position. It is proved that the compositions define by affinors \( a^\beta_\alpha \) and \( b^\beta_\alpha \) are conjugate. It is proved also that if the composition \( X_m \times X_m \) is Cartesian and composition \( Y_m \times Y_m \) is Cartesian or chebyshevian, or geodesic than the space \( A_{2m} \) is affine.

Keywords: affinely connected space, net, composition, conjugate, chebyshevian, Cartesian, geodesic compositions

2010 Mathematics Subject Classification: 53Bxx, 53B05

1. Preliminary

Let \( A_N \) be an affinely connected space without a torsion, i.e. with a symmetric affinely connectedness, define by the coefficients \( \Gamma^\gamma_{\sigma\beta} \). According to \cite{2} the space \( A_N \) assumes a composition \( X_n \times X_m \) of two base manyfolds \( X_n \) and \( X_m \) \((n + m = N)\) if and only if there exists an affinor \( a^\beta_\alpha \), such that

\[
\begin{align*}
 a^\beta_\sigma a^\sigma_\alpha &= \delta^\beta_\alpha.
\end{align*}
\]

This space will be denoted \( A_N(X_n \times X_m) \) and \( a^\beta_\alpha \) will be called the affinor of the composition \( A_N(X_n \times X_m) \) \cite{2}. Two positions \( P(X_n), P(X_m) \) of the base manyfolds pass through any point of \( A_N(X_n \times X_m) \).

We shall consider an affinely connected spaces \( A_N(X_n \times X_m) \) with integrable structure of the compositions. According to \cite{3}, \cite{5} the integrability condition of the structure is characterized with the equality

\[
\begin{align*}
 a^\sigma_\beta \nabla_{[\alpha} a^\nu_{\sigma]} - a^\sigma_\beta \nabla_{[\beta} a^\nu_{\sigma]} &= 0.
\end{align*}
\]

For the projecting affinors \( \frac{n}{\alpha} \frac{\beta}{\alpha} \), \( \frac{m}{\alpha} \frac{\beta}{\alpha} \), define by the conditions \( \frac{n}{\alpha} \frac{\beta}{\alpha} = \frac{1}{2}(\delta^\beta_\alpha + a^\beta_\alpha) \), \( \frac{m}{\alpha} \frac{\beta}{\alpha} = \frac{1}{2}(\delta^\beta_\alpha - a^\beta_\alpha) \), the following equalities are fulfilled: \( \frac{n}{\alpha} \frac{\beta}{\alpha} \frac{n}{\alpha} \frac{\sigma}{\alpha} = \frac{m}{\alpha} \frac{\beta}{\alpha} \frac{m}{\alpha} \frac{\sigma}{\alpha} = \frac{n}{\alpha} \frac{\sigma}{\alpha} \frac{m}{\alpha} \frac{\sigma}{\alpha} \frac{n}{\alpha} \frac{\beta}{\alpha} = 0 \) \cite{3}, \cite{4}.

According to \cite{4} for an arbitrary vector \( v^\alpha \in A_N \) we have \( v^\alpha = \frac{n}{\alpha} \frac{\sigma}{\sigma} v^\sigma + \frac{m}{\alpha} \frac{\sigma}{\sigma} v^\sigma = \tilde{V}^\alpha + \hat{V}^\alpha \), where \( \tilde{V}^\alpha = \frac{n}{\alpha} \frac{\sigma}{\sigma} v^\sigma \in P(X_n) \), \( \hat{V}^\alpha = \frac{m}{\alpha} \frac{\sigma}{\sigma} v^\sigma \in P(X_m) \).
Following [3] we will write the known characteristics for the affinor of some special compositions $X_n \times X_m$ :

**Proposition 1.** The positions $P(X_n)$ and $P(X_m)$ of the c, c-composition (Cartesian) $X_n \times X_m$ are parallelly translated along any line in the space if and only if $\nabla_\alpha a^\alpha_\beta = 0$.

**Proposition 2.** The positions $P(X_n)$ and $P(X_m)$ of the ch, ch-composition (Chebyshevian) $X_n \times X_m$ are parallelly translated along the lines of $X_m$ and $X_n$, respectively if and only if $\nabla_\alpha a^\alpha_\beta = 0$.

**Proposition 3.** The positions $P(X_n)$ and $P(X_m)$ of the g, g-composition (geodesic) $X_n \times X_m$ are parallelly translated along the lines of $X_n$ and $X_m$, respectively if and only if $\nabla_\alpha a^\alpha_\beta + a^\alpha_\beta \nabla_\sigma a^\sigma_\beta = 0$.

2. **Conjugate compositions in affinely connected spaces without a torsion $A_{2m}$**

Let the affinor $a^\alpha_\beta$ defines a composition $X_m \times X_m$ in affinely connected spaces without a torsion $A_{2m}$.

Let accept:

$$(3) \quad \alpha, \beta, \gamma, \sigma, \nu \in \{1, 2, \ldots, 2m\}; \quad i, j, k, p, q, r, s \in \{1, 2, \ldots, m\}; \quad \overline{i}, \overline{j}, \overline{k}, \overline{p}, \overline{q}, \overline{r}, \overline{s} \in \{m+1, m+2, \ldots, 2m\}.$$

Let $v^\alpha_1, v^\alpha_2, \ldots, v^\alpha_m, \ldots, v^\alpha_{2m}$ be the eigen-vectors of the matrix $(a^\alpha_\beta)$, as

$$(4) \quad a^\alpha_\beta v^\alpha_s = v^\beta_s, \quad a^\alpha_\beta v^\alpha_\sigma = -v^\beta_\sigma.$$

They define the net $\left(v_1, v_2, \ldots, v_{2m}\right)$.

The reciprocal covectors $v^\alpha_\sigma (\alpha = 1, 2, \ldots, 2m)$ are defined by the equalities

$$(5) \quad v^\alpha_\sigma v^\beta_\sigma = \delta^\beta_\alpha \iff v^\beta_\sigma v^\alpha_\sigma = \delta^\alpha_\beta.$$

Following the paper [6], we can consider the affinor $a^\alpha_\beta$ of the composition $X_m \times X_m$ as an affinor, associated with the net $\left(v_1, v_2, \ldots, v_{2m}\right)$. Therefore $a^\alpha_\beta$ has the presentation

$$(6) \quad a^\alpha_\beta = v^\alpha_1 v_1 + \cdots + v^\alpha_m v_m - v^\alpha_{m+1} v_{m+1} - \cdots - v^\alpha_{2m} v_{2m} = v^\alpha_i v_i - v^\alpha_\overline{i} v_\overline{i}.$$

Now according to [6] for the projecting affinors we have $\overline{m}^\beta_\alpha = v^\beta_i v_\overline{i}$. Let net $\left(v_1, v_2, \ldots, v_{2m}\right)$ be chosen as a coordinate one. Then we have

$$(7) \quad v^\sigma_1 (1, 0, \ldots, 0), \quad v^\sigma_2 (0, 1, \ldots, 0), \ldots, \quad v^\sigma_{2m} (0, 0, \ldots, 0, 1).$$
Let consider the vectors
\[ w_i^\alpha = v_i^\alpha + v_{m+i}^\alpha, \quad w_i^\alpha = v_i^\alpha - v_{m+i}^\alpha. \]

The reciprocal covectors \( \alpha w_\sigma (\alpha = 1, 2, \ldots, 2m) \) are defined by the equalities
\[ w_\alpha^\beta \sigma = \delta_\alpha^\beta \iff w_\alpha^\beta \beta = \delta_\sigma^\alpha. \]

Let introduce the affinor
\[ b_\beta^\alpha = w_i^\beta w_\alpha - w_i^\beta w_\alpha. \]

From (9), (10) we obtain \( b_\beta^\alpha b_\alpha^\beta = \delta_\sigma^\alpha. \) Hence the affinor \( b_\alpha^\beta \) defines a composition \( Y_m \times \overline{Y}_m. \) We denote by \( P(Y_m) \) and \( P(\overline{Y}_m) \) the positions of this composition. Using (9), (10) we establish
\[ b_\alpha^\beta w_\sigma^\alpha = w_\sigma^\beta, \quad b_\beta^\alpha w_\sigma^\alpha = -w_\beta^\sigma, \]
from where it follows that \( w_\sigma^\alpha \) and \( w_\beta^\alpha \) are the eigen-vectors of the matrix \( \left( b_\alpha^\beta \right). \)

According to [6] the projecting affinors of the composition \( Y_m \times \overline{Y}_m \) have the following form
\[ b_\alpha^\beta = w_i^\beta w_\alpha, \quad \overline{b}_\alpha^\beta = w_i^\beta \overline{w}_\alpha. \]

**Definition 1.** The compositions \( X_m \times \overline{X}_m \) and \( Y_m \times \overline{Y}_m \) be called conjugate if
1) for arbitrary vectors \( v^\alpha \in P(X_m) \) and \( \overline{v}^\alpha \in P(\overline{X}_m) \) are fulfilled
\[ b_\alpha^\beta v^\alpha \in P(X_m) \quad \text{and} \quad b_\alpha^\beta \overline{v}^\alpha \in P(\overline{X}_m); \]
2) for arbitrary vectors \( u^\alpha \in P(Y_m) \) and \( \overline{u}^\alpha \in P(\overline{Y}_m) \) are fulfilled
\[ a_\alpha^\beta u^\alpha \in P(Y_m) \quad \text{and} \quad a_\alpha^\beta \overline{u}^\alpha \in P(\overline{Y}_m). \]

**Theorem 1.** The compositions \( X_m \times \overline{X}_m \), define by the affinor (6) and associated with the net \( \left( v, v, \ldots, v \right) \) and the composition \( Y_m \times \overline{Y}_m \), define by the affinors (10) are conjugate.

**Proof:** With the help of (4), (6) and (8) we find
\[ a_\alpha^\beta w^\alpha = a_\alpha^\beta \left( v^\alpha + v^{\alpha}_{s+m} \right) = v^\alpha - v^{\alpha}_{s+m} = w_\alpha^\beta, \]
\[ a_\alpha^\beta \overline{w}^\alpha = a_\alpha^\beta \left( v^\alpha - v^{\alpha}_{s} \right) = v^\alpha + v^{\alpha}_{s-m} = \overline{w}_\alpha^\beta. \]

Now if an arbitrary vector \( v^\alpha \in P(Y_m) \), then \( v^\alpha = \lambda^i w_i^\alpha \), where \( \lambda^i \) are functions of the point and \( a_\alpha^\beta v^\alpha = \lambda^i a_\alpha^\beta w_i^\alpha \). Taking into account (12) we can
write \( a^\beta_\alpha v^\alpha = \frac{1}{m+1} w^\beta + \frac{2}{m+2} w^\beta + \ldots \), which means that \( a^\beta_\alpha v^\alpha \in P(\overline{Y}_m) \).

So we proved that from \( v^\alpha \in P(Y_m) \) it follows \( a^\beta_\alpha v^\alpha \in P(\overline{Y}_m) \). The proof of the proposition - from \( v^\alpha \in P(\overline{Y}_m) \) it follows \( a^\beta_\alpha v^\alpha \in P(Y_m) \) - is similar.

From (5), (8) and (9) we obtain

\[
\begin{align*}
\frac{s}{s} w^\alpha_s v^\alpha &= \frac{1}{2}, \\
\frac{s}{s} w^\alpha_s v^\alpha &= \frac{1}{2}, \\
\frac{s}{s} w^\alpha_k v^\alpha &= 0, \\
\frac{s}{s} k^m v^\alpha &= 0, & s \neq k \;
\end{align*}
\]

Now if an arbitrary vector \( v^\alpha \in P(X_m) \), then \( v^\alpha = \mu^i v^\alpha \), where \( \mu^i \) are functions of the point and \( b^\beta_\alpha v^\alpha = \mu^i b^\beta_\alpha v^\alpha \). Taking into account (14) we can write

\[
\begin{align*}
b^\beta_\alpha v^\alpha &= \mu^i v^\alpha, & \text{which means that } b^\beta_\alpha v^\alpha \in P(X_m). \]
\]

The proof of the proposition - from \( v^\alpha \in P(X_m) \) it follows \( b^\beta_\alpha v^\alpha \in P(X_m) \) - is similar.

Let consider the affinor \( c^\beta_\alpha = -a^\beta_\alpha b^\alpha_\sigma \). From (5), (6), (8) and (10) we obtain

\[
\begin{align*}
c^\beta_\alpha &= -a^\beta_\sigma b^\alpha_\sigma = w^\beta w^\alpha m^\alpha w^\alpha + w^\beta w^\alpha m^\alpha w^\alpha. \\
\end{align*}
\]

Since according to (9) and (15) \( c^\alpha_\beta c^\sigma_\alpha = -w^\beta_\sigma w^\alpha_\sigma = -\delta^\beta_\sigma \), the affinor \( c^\alpha_\beta \) defines an elliptic composition as, while the affinors \( a^\alpha_\beta \) and \( b^\alpha_\beta \) define hyperbolic compositions. If \( z^\alpha \) is an eigen-vector of the matrix \( (c^\alpha_\beta) \), then \( c^\alpha_\beta z^\alpha = \pm i z^\beta \), where \( i^2 = -1 \).

From the equalities \( a^\alpha_\beta a_\sigma^\alpha = \delta^\beta_\sigma \), \( b^\alpha_\beta b^\alpha_\sigma = \delta^\beta_\sigma \), \( c^\alpha_\beta c^\alpha_\sigma = -\delta^\beta_\sigma \), \( a^\alpha_\beta b^\alpha_\sigma = -c^\beta_\sigma \) easily follow

\[
\begin{align*}
a^\alpha_\beta a^\beta_\sigma &= -b^\beta_\alpha a^\alpha_\sigma = -c^\beta_\sigma, & b^\beta_\alpha c^\alpha_\sigma = -c^\beta_\sigma b^\alpha_\sigma = a^\beta_\sigma, \\
\end{align*}
\]

Because of (5), (6), (9), (10) and (15) we have \( a^\alpha_\beta = b^\alpha_\beta = c^\alpha_\beta = 0 \), from

where we obtain \( a^\alpha_\beta a^\beta_\alpha = b^\alpha_\beta b^\alpha_\beta = c^\alpha_\beta c^\alpha_\beta = 0 \). Then from (4), (5), (6), (7), (9), (10) and (15) it follows that in the parameters of the chosen coordinate
system the matrix \((a_{\alpha}^\beta), (b_{\alpha}^\beta), (c_{\alpha}^\beta)\), have the following form

\[
(a_{\alpha}^\beta) = \begin{pmatrix}
\delta_{i} & 0 \\
0 & -\delta_{i}^r
\end{pmatrix}, \quad (b_{\alpha}^\beta) = \begin{pmatrix}
1 & \circ \\
1 & \circ \\
1 & \circ
\end{pmatrix},
\]

(17)

\[
(c_{\alpha}^\beta) = \begin{pmatrix}
\circ & 1 \\
-1 & 1 \\
-1 & \circ
\end{pmatrix}.
\]

Following [3] let introduce the notations \(A_{\alpha\beta}^\sigma = \nabla_{\alpha} a_{\beta}^\sigma, B_{\alpha\beta}^\sigma = \nabla_{\alpha} b_{\beta}^\sigma, C_{\alpha\beta}^\sigma = \nabla_{\alpha} c_{\beta}^\sigma\). In the chosen coordinate system, which is adapted with the space \((10-12 December 2010, Plovdiv, Bulgaria 229)\), let introduce the notations \((19)\)

\[
A_{ik}^s = 0, \quad A_{\tau i k}^s = 0, \quad A_{i k}^s = -2\Gamma_{ik}^s, \quad A_{i k}^s = -2\Gamma_{ik}^s,
\]

(18)

\[
A_{ik}^\tau = 0, \quad A_{\tau i k}^\tau = 0, \quad A_{ik}^\tau = 2\Gamma_{ik}^\tau, \quad A_{i k}^\tau = 2\Gamma_{ik}^\tau.
\]

According to (17) we establish in the chosen coordinate system the following equalities for \(B_{\alpha\beta}^\sigma\) and \(C_{\alpha\beta}^\sigma\)

\[
B_{ik}^\tau = \Gamma_{ik}^{s+\tau}, \quad B_{i k}^\tau = \Gamma_{i k}^{s+\tau} - \Gamma_{ik}^{\tau}, \quad B_{i k}^\tau = \Gamma_{i k}^{s+\tau} - \Gamma_{ik}^{\tau},
\]

(19)

\[
B_{\tau i k}^\tau = \Gamma_{\tau i k}^{s+\tau} - \Gamma_{\tau i k}^{\tau}, \quad B_{i k}^\tau = \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau}, \quad B_{i k}^\tau = \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau},
\]

\[
B_{i k}^\tau = \Gamma_{i k}^{s+\tau} - \Gamma_{i k}^{\tau}, \quad B_{i k}^\tau = \Gamma_{i k}^{s+\tau} - \Gamma_{i k}^{\tau},
\]

\[
C_{ik}^\tau = \Gamma_{ik}^{s+\tau} + \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau}, \quad C_{ik}^\tau = \Gamma_{ik}^{s+\tau} + \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau},
\]

(20)

\[
C_{ik}^\tau = \Gamma_{ik}^{s+\tau} + \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau}, \quad C_{ik}^\tau = \Gamma_{ik}^{s+\tau} + \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau},
\]

\[
C_{ik}^\tau = \Gamma_{ik}^{s+\tau} + \Gamma_{ik}^{s+\tau} - \Gamma_{ik}^{\tau}.
\]

**Theorem 2.** If the composition \(X_m \times \overline{X}_m\) is \(c, c\) - composition and its conjugate composition \(Y_m \times \overline{Y}_m\) is of the kind \((c, c)\) or \((g, g)\) or \((ch, ch)\), then the space \(A_{2m}\) is affine.

**Proof:** Let \(X_m \times \overline{X}_m\) be \(c, c\) - composition. According to Proposition 1 and [3] \(A_{\alpha\beta}^\sigma = \nabla_{\alpha} a_{\beta}^\sigma = 0\). In the chosen coordinate system these conditions...
accept the form [3]

\[ \Gamma^{s}_{ik} = \Gamma^{s}_{ik} = \Gamma^{s}_{ik} = 0. \]

Using (21) we can write (19) properly

\[ B^{s}_{ik} = B^{s}_{ik} = 0, \quad B^{s}_{ik} = \Gamma^{s}_{ik} - m, \quad B^{s}_{ik} = -\Gamma^{s}_{ik}, \]

(22)

\[ B^{s}_{ik} = B^{s}_{ik} = 0, \quad B^{s}_{ik} = -\Gamma^{s}_{ik}, \quad B^{s}_{ik} = \Gamma^{s}_{ik} + m. \]

1. Now let \( Y_{m} \times \overline{Y}_{m} \) be \( d, d \)-composition. From Proposition 1 it follows

\[ B^{s}_{\alpha\beta} = \nabla_{\alpha} b^{s}_{\beta} = 0. \]

Substituting in (22) we obtain \( \Gamma^{s}_{ik} - m = \Gamma^{s}_{ik} - k = 0. \) So the last results and (21) show us that \( \Gamma^{s}_{\alpha\beta} = 0 \) for any \( \alpha, \beta, \sigma. \)

2. Now let \( Y_{m} \times \overline{Y}_{m} \) be \( c h, c h \)-composition. From Proposition 2 it follows

\[ B^{s}_{\alpha\beta} = \nabla_{\alpha} b^{s}_{\beta} = 0. \]

Substituting in (22) we obtain \( \Gamma^{s}_{ik} - m = \Gamma^{s}_{ik} - k = 0. \) So the last results and (21) show us that \( \Gamma^{s}_{\alpha\beta} = 0 \) for any \( \alpha, \beta, \sigma. \)

3. Now let \( Y_{m} \times \overline{Y}_{m} \) be \( g, g \)-composition. Let consider the tensor \( M^{s}_{\alpha\beta} = b^{s}_{\alpha} B^{s}_{\beta} + b^{s}_{\beta} B^{s}_{\alpha}. \) Taking into account (17) and (22) for the components of the tensor \( M^{s}_{\alpha\beta} \) in the chosen coordinate system we have

\[ M^{s}_{ik} = \Gamma^{s}_{ik}, \quad M^{s}_{ik} = -\Gamma^{s}_{k+m \ i}, \quad M^{s}_{ik} = \Gamma^{s}_{k \ i+m}, \quad M^{s}_{ik} = -\Gamma^{s}_{k \ m}, \quad M^{s}_{ik} = \Gamma^{s}_{k+m \ i}. \]

But according to Proposition 3 \( Y_{m} \times \overline{Y}_{m} \) is an \( g, g \)-composition if and only if \( M^{s}_{\alpha\beta} = 0. \) Consequently \( Y_{m} \times \overline{Y}_{m} \) is an \( g, g \)-composition if and only if

\[ \Gamma^{s}_{ik} = \Gamma^{s}_{k+m \ i} = \Gamma^{s}_{k \ i} = \Gamma^{s}_{k \ i+m} = \Gamma^{s}_{k \ m} = \Gamma^{s}_{k+m \ i}. \]

So the last results and (21) show us that \( \Gamma^{s}_{\alpha\beta} = 0 \) for any \( \alpha, \beta, \sigma. \)

Obviously, in any of the above three cases the tensor of the curvature

\[ R^{s}_{\alpha\beta\gamma} = \partial_{\alpha} \Gamma^{s}_{\beta\gamma} - \partial_{\beta} \Gamma^{s}_{\alpha\gamma} + \Gamma^{s}_{\alpha\nu} \Gamma^{s}_{\beta\nu} - \Gamma^{s}_{\beta\nu} \Gamma^{s}_{\alpha\nu} = 0, \]

which means - the space \( A_{2m} \) is affine [1]. □

References


Georgi Zlatanov, Bistra Tsareva  
Faculty of Mathematics and Informatics  
University of Plovdiv “Paisii Hilendarski”  
24 Tsar Assen  
4000 Plovdiv, Bulgaria  
e-mail: zlatanov@uni-plovdiv.bg, btsareva@uni-plovdiv.bg