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TIME-FRACTIONAL DERIVATIVES IN RELAXATION PROCESSES: A TUTORIAL SURVEY

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*Dedicated to Professor Michele Caputo (Accademico dei Lincei)
on the occasion of his 80th birthday (5 May 2007)*

Abstract

The aim of this tutorial survey is to revisit the basic theory of relaxation processes governed by linear differential equations of fractional order. The fractional derivatives are intended both in the Rieamann-Liouville sense and in the Caputo sense. After giving a necessary outline of the classical theory of linear viscoelasticity, we contrast these two types of fractional derivatives in their ability to take into account initial conditions in the constitutive equations of fractional order. We also provide historical notes on the origins of the Caputo derivative and on the use of fractional calculus in viscoelasticity.

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Introduction

In recent decades the field of fractional calculus has attracted interest of researchers in several areas including mathematics, physics, chemistry, engineering and even finance and social sciences. In this survey we revisit the fundamentals of fractional calculus in the framework of the most simple time-dependent processes like those concerning relaxation phenomena. We devote particular attention to the technique of Laplace transforms for treating the operators of differentiation of non integer order (the term "fractional" is kept only for historical reasons), nowadays known as Riemann-Liouville and Caputo derivatives. We shall point out the fundamental role of the Mittag-Leffler function (the Queen function of the fractional calculus), whose main properties are reported in an *ad hoc* Appendix. The topics discussed here will be: (i) essentials of fractional calculus with basic formulas for Laplace transforms (Section 1); (ii) relaxation type differential equations of fractional order (Section 2); (iii) constitutive equations of fractional order in viscoelasticity (Section 3). The last topic is treated in detail since, as a matter of fact, the linear theory of viscoelasticity is the field where we find the most extensive applications of fractional calculus already since a long time, even if often only in an implicit way. Finally, we devote Section 4 to historical notes concerning the origins of the Caputo derivative and the use of fractional calculus in viscoelasticity in the past century.

1. Definitions and properties

For a sufficiently well-behaved function $f(t)$ (with $t \in \mathbf{R}^+$) we may define the derivative of a positive non-integer order in two different senses, that we refer here as to *Riemann-Liouville* (R-L) derivative and *Caputo* (C) derivative, respectively.

Both derivatives are related to the so-called Riemann-Liouville fractional integral. For any $\alpha > 0$ this fractional integral is defined as

$$J_t^\alpha \, f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, f(\tau) \, d\tau \,, \tag{1.1}$$

where $\Gamma(\alpha) := \displaystyle\int_0^\infty \mathrm{e}^{-u} \, u^{\alpha-1} \, du$ denotes the Gamma function. For existence of the integral (1) it is sufficient that the function $f(t)$ is locally integrable in $\mathbf{R}^+$ and for $t \to 0$ behaves like $O(t^{-\nu})$ with a number $\nu < \alpha$.

For completion we define $J_t^0 = I$ (Identity operator).

We recall the semigroup property

$$J_t^\alpha \, J_t^\beta = \, J_t^\beta \, J_t^\alpha = J_t^{\alpha+\beta} \,, \quad \alpha, \beta \geq 0 \,. \tag{1.2}$$

Furthermore we note that for $\alpha \geq 0$

$$J_t^\alpha \, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} \, t^{\gamma+\alpha} \,, \quad \gamma > -1 \,. \tag{1.3}$$

The fractional derivative of order $\mu > 0$ in the *Riemann-Liouville* sense is defined as the operator D_t^μ which is the left inverse of the Riemann-Liouville integral of order μ (in analogy with the ordinary derivative), that is

$$D_t^\mu \, J_t^\mu = I \,, \quad \mu > 0 \,. \tag{1.4}$$

If m denotes the positive integer such that $m-1 < \mu \leq m$, we recognize from Eqs. (1.2) and (1.4):

$$D_t^\mu \, f(t) := D_t^m \, J_t^{m-\mu} \, f(t) \,, \quad m-1 < \mu \leq m \,. \tag{1.5}$$

In fact, using the semigroup property (1.2), we have

$$D_t^\mu \, J_t^\mu = D_t^m \, J_t^{m-\mu} \, J_t^\mu = D_t^m \, J_t^m = I \,.$$

Thus (1.5) implies

$$D_t^\mu \, f(t) = \begin{cases} \dfrac{d^m}{dt^m} \left[\dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t \dfrac{f(\tau)\, d\tau}{(t-\tau)^{\mu+1-m}} \right], & m-1 < \mu < m; \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \tag{1.5'}$$

For completion we define $D_t^0 = I$.

On the other hand, the fractional derivative of order μ in the *Caputo* sense is defined as the operator ${}_*D_t^\mu$ such that

$${}_*D_t^\mu \, f(t) := J_t^{m-\mu} \, D_t^m \, f(t) \,, \quad m-1 < \mu \leq m \,. \tag{1.6}$$

This implies

$${}_*D_t^\mu \, f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\mu+1-m}} \,, & m-1 < \mu < m \,; \\ \dfrac{d^m}{dt^m} f(t) \,, & \mu = m \,. \end{cases} \tag{1.6'}$$

Thus, when the order is not integer the two fractional derivatives differ in that the standard derivative of order m does not generally commute with the fractional integral. Of course the Caputo derivative (1.6′) needs higher regularity conditions of $f(t)$ than the Riemann-Liouville derivative (1.5′).

We point out that the *Caputo* fractional derivative satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than m, if its order μ is such that $m-1<\mu\le m$. Furthermore we note for $\mu\ge 0$:

$$D_t^\mu\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)}\, t^{\gamma-\mu}\,, \quad \gamma > -1\,. \tag{1.7}$$

It is instructive to compare Eqs. (1.3), (1.7).

In [57] we have shown the essential relationships between the two fractional derivatives for the same non-integer order

$$ {}_*D_t^\mu\, f(t) = \begin{cases} D_t^\mu \left[f(t) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!} \right], \\ D_t^\mu\, f(t) - \displaystyle\sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)\, t^{k-\mu}}{\Gamma(k-\mu+1)}, \end{cases} \quad m-1<\mu<m\,. \tag{1.8}$$

In particular we have from (1.6′) and (1.8)

$$\begin{aligned} {}_*D_t^\mu f(t) &= \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\mu}\, d\tau \\ &= D_t^\mu\, \left[f(t) - f(0^+)\right] = D_t^\mu\, f(t) - f(0^+)\, \frac{t^{-\mu}}{\Gamma(1-\mu)}\,, \end{aligned} \quad 0<\mu<1\,. \tag{1.9}$$

The *Caputo* fractional derivative represents a sort of regularization in the time origin for the *Riemann-Liouville* fractional derivative. We note that for its existence all the limiting values $f^{(k)}(0^+) := \lim_{t\to 0^+} D_t^k f(t)$ are required to be finite for $k=0,1,2,\dots,m-1$. In the special case $f^{(k)}(0^+)=0$ for $k=0,1,m-1$, the two fractional derivatives coincide.

We observe the different behaviour of the two fractional derivatives at the end points of the interval $(m-1,m)$ namely when the order is any positive integer, as it can be noted from their definitions (1.5), (1.6). In fact, whereas for $\mu\to m^-$ both derivatives reduce to D_t^m, as stated in Eqs. (1.5′), (1.6′), due to the fact that the operator $J_t^0 = I$ commutes with D_t^m, for $\mu\to(m-1)^+$ we have

$$\mu \to (m{-}1)^+ : \begin{cases} D_t^\mu f(t) \to \; D_t^m \, J_t^1 \, f(t) = D_t^{(m-1)} \, f(t) = f^{(m-1}(t), \\ \\ {}_*D_t^\mu f(t) \to J_t^1 \, D_t^m \, f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+). \end{cases} \tag{1.10}$$

As a consequence, roughly speaking, we can say that D_t^μ is, with respect to its order μ, an operator continuous at any positive integer, whereas ${}_*D_t^\mu$ is an operator only left-continuous.

The above behaviours have induced us to keep for the Riemann-Liouville derivative the same symbolic notation as for the standard derivative of integer order, while for the Caputo derivative to decorate the corresponding symbol with subscript $*$.

We also note, with $m-1 < \mu \le m$, and c_j arbitrary constants,

$$D_t^\mu \, f(t) \; = \; D_t^\mu \, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j \, t^{\mu-j} \,, \tag{1.11}$$

$${}_*D_t^\mu \, f(t) \; = \; {}_*D_t^\mu \, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j \, t^{m-j} \,. \tag{1.12}$$

Furthermore, we observe that in case of a non-integer order for both fractional derivatives the semigroup property (of the standard derivative for integer order) does not hold for both fractional derivatives when the order is not integer.

We point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the Laplace transformation[1].

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[1]The Laplace transform of a well-behaved function $f(t)$ is defined as

$$\widetilde{f}(s) = \mathcal{L}\,\{f(t); s\} := \int_0^\infty \mathrm{e}^{\,-st} \, f(t) \, dt \,, \quad \Re\,(s) > a_f \,.$$

We recall that under suitable conditions the Laplace transform of the m-derivative of $f(t)$ is given by

$$\mathcal{L}\,\{D_t^m \, f(t); s\} = s^m \, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k} \, f^{(k)}(0^+) \,, \quad f^{(k)}(0^+) := \lim_{t \to 0^+} D_t^k f(t) \,.$$

For the Caputo derivative of order μ with $m-1 < \mu \le m$ we have

$$\mathcal{L}\{{}_*D_t^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k}\, f^{(k)}(0^+)\,,$$
$$f^{(k)}(0^+) := \lim_{t\to 0^+} D_t^k f(t)\,. \tag{1.13}$$

The corresponding rule for the Riemann-Liouville derivative of order μ is

$$\mathcal{L}\{D_t^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k}\, g^{(k)}(0^+)\,,$$
$$g^{(k)}(0^+) := \lim_{t\to 0^+} D_t^k g(t)\,, \quad \text{where} \quad g(t) := J_t^{(m-\mu)}\, f(t)\,. \tag{1.14}$$

Thus it is more cumbersome to use the rule (1.14) than (1.13). The rule (1.14) requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral. However, when all the limiting values $f^{(k)}(0^+)$ for $k = 0, 1, 2, \ldots$ are finite and the order is not integer, we can prove that the corresponding $g^{(k)}(0^+)$ vanish so that formula (1.14) simplifies into

$$\mathcal{L}\{D_t^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s)\,, \quad m-1 < \mu < m\,. \tag{1.15}$$

For this proof it is sufficient to apply the Laplace transform to the second equation (8), by recalling that $\mathcal{L}\{t^{\alpha}; s\} = \Gamma(\alpha+1)/s^{\alpha+1}$ for $\alpha > -1$, and then to compare (1.13) with (1.14).

For fractional differentiation on the positive semi-axis we recall another definition for the fractional derivative recently introduced by Hilfer, see [67] and [115], which interpolates the previous definitions (1.5) and (1.6). Like the two derivatives previously discussed, it is related to a Riemann-Liouville integral. In our notation it reads

$$D_t^{\mu,\nu} := J_t^{\nu(1-\mu)}\, D_t^1\, J_t^{(1-\nu)(1-\mu)}\,, \quad 0 < \mu < 1\,, \quad 0 \le \nu \le 1\,. \tag{1.16}$$

We can refer it to as the *Hilfer* (H) fractional derivative of order μ and type ν. The Riemann-Liouville derivative corresponds to the type $\nu = 0$ whereas that Caputo derivative to the type $\nu = 1$.

We have here not discussed the Beyer-Kempfle approach investigated and used in several papers by Beyer and Kempfle et al.: this approach is appropriate for causal processes not starting at a finite instant of time, see e.g. [8, 68]. They define the time-fractional derivative on the whole real line as a pseudo-differential operator via its Fourier symbol. The interested reader is referred to the above mentioned papers and references therein.

For further reading on the theory and applications of fractional integrals and derivatives (more generally of fractional calculus) we may recommend e.g. our CISM Lecture Notes [55, 57, 80], the review papers [85, 86, 116], and the books [88, 70, 66, 69, 77, 94, 108, 118, 119] with references therein.

2. Relaxation equations of fractional order

The different roles played by the R-L and C derivatives and by the intermediate H derivative are clear when one wants to consider the corresponding fractional generalization of the first-order differential equation governing the phenomenon of (exponential) relaxation. Recalling (in non-dimensional units) the initial value problem

$$\frac{du}{dt} = -u(t)\,, \quad t \geq 0\,, \quad \text{with} \quad u(0^+) = 1 \tag{2.1}$$

whose solution is

$$u(t) = \exp(-t)\,, \tag{2.2}$$

the following three alternatives with respect to the R-L and C fractional derivatives with $\mu \in (0,1)$ are offered in the literature:

$$ {}_*D_t^\mu\, u(t) = -u(t)\,, \quad t \geq 0\,, \quad \text{with} \quad u(0^+) = 1\,. \tag{2.3a}$$

$$D_t^\mu\, u(t) = -u(t)\,, \quad t \geq 0\,, \quad \text{with} \quad \lim_{t\to 0^+} J_t^{1-\mu}\, u(t) = 1\,, \tag{2.3b}$$

$$\frac{du}{dt} = -D_t^{1-\mu}\, u(t)\,, \quad t \geq 0\,, \quad \text{with} \quad u(0^+) = 1\,, \tag{2.3c}$$

In analogy to the standard problem (2.1) we solve these three problems with the Laplace transform technique, using respectively the rules (1.13), (1.14) and (1.15). The problems (a) and (c) are equivalent since the Laplace transform of the solution in both cases comes out as

$$\widetilde{u}(s) = \frac{s^{\mu-1}}{s^\mu + 1}\,, \tag{2.4}$$

whereas in the case (b) we get

$$\widetilde{u}(s) = \frac{1}{s^\mu + 1} = 1 - s\frac{s^{\mu-1}}{s^\mu + 1}\,. \tag{2.5}$$

The Laplace transforms in (2.4)-(2.5) can be expressed in terms of functions of Mittag-Leffler type, for which we provide essential information in Appendix. In fact, in virtue of the Laplace transform pairs, that here we report from Eqs. (A.4) and (A.8),

$$\mathcal{L}\{E_\mu(-\lambda t^\mu); s\} = \frac{s^{\mu-1}}{s^\mu + \lambda}\,, \tag{2.6}$$

$$\mathcal{L}\{t^{\nu-1}\, E_{\mu,\nu}(-\lambda t^\mu); s\} = \frac{s^{\mu-\nu}}{s^\mu + \lambda}\,, \tag{2.7}$$

with $\mu, \nu \in \mathbf{R}^+$ and $\lambda \in \mathbf{R}$, we have: in the cases (a) and (c),

$$u(t) = \Psi(t) := E_\mu(-t^\mu)\,, \quad t \geq 0\,, \quad 0 < \mu < 1\,, \tag{2.8}$$

and in the case (b), using the identity (A.10),

$$u(t) = \Phi(t) := t^{-(1-\mu)} E_{\mu,\mu}\left(-t^\mu\right) = -\frac{d}{dt} E_\mu\left(-t^\mu\right),\ t \geq 0,\ 0 < \mu \leq 1. \tag{2.9}$$

It is evident that for $\mu \to 1^-$ the solutions of the three initial value problems (2.3) reduce to the standard exponential function (2.8).

We note that the case (b) is of little interest from a physical view point since the corresponding solution (2.9) is infinite in the time-origin.

We recall that former plots of the Mittag-Leffler function $\Psi(t)$ are found (presumably for the first time in the literature) in the 1971 papers by Caputo and Mainardi [28][2] and [29] in the framework of fractional relaxation for viscoelastic media, in times when such function was almost unknown. Recent numerical treatments of the Mittag-Leffler functions have been provided by Gorenflo, Loutschko and Luchko [56] with *MATHEMATICA*, and by Podlubny [97] with *MATLAB*.

The plots of the functions $\Psi(t)$ and $\Phi(t)$ are shown below in **Figure 1** and **Figure 2**, respectively, for some rational values of the parameter μ, by adopting linear and logarithmic scales.

[2]Ed. Note: This paper has been now reprinted in this same FCAA issue, under the kind permission of Birkhäuser Verlag AG.

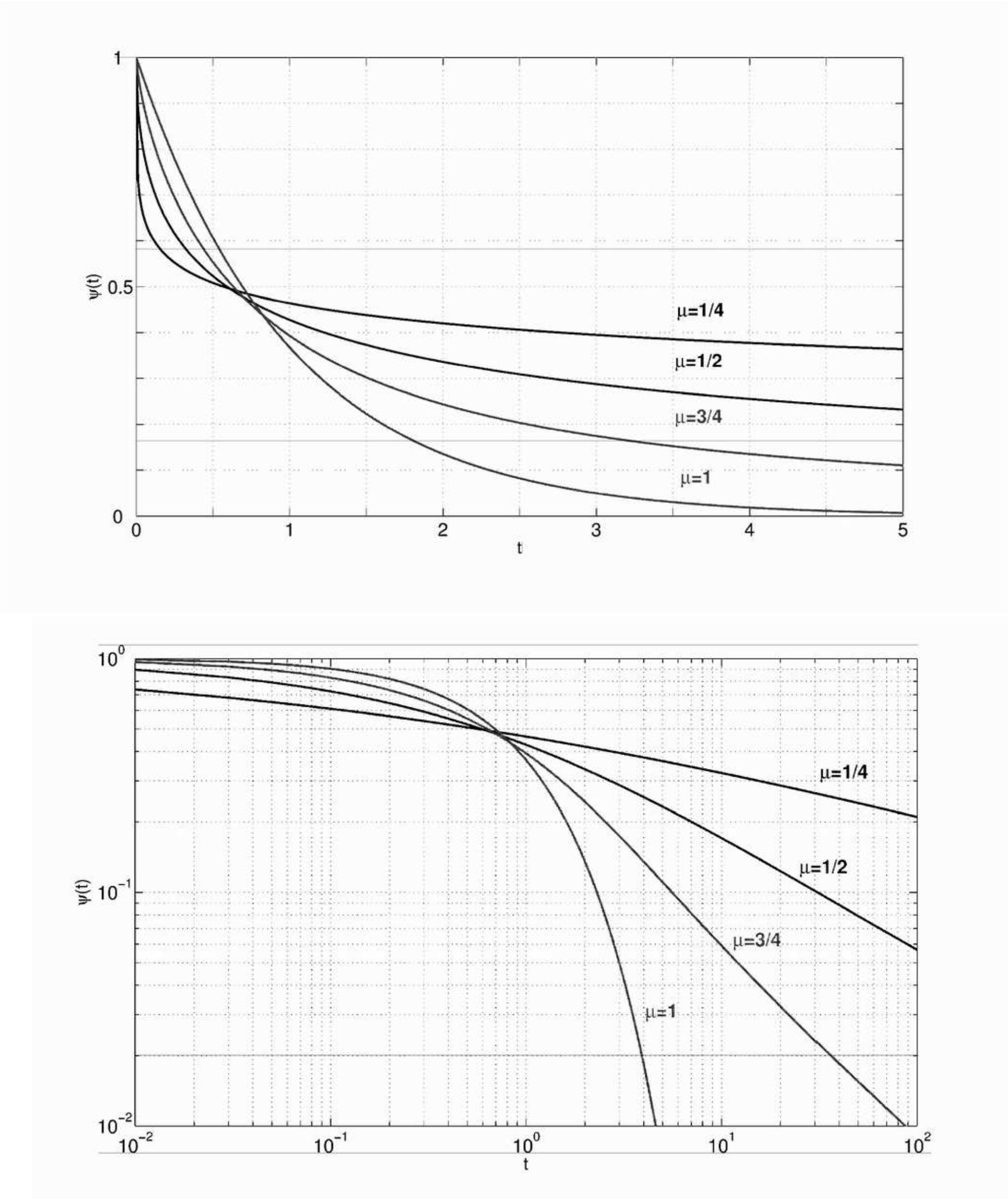

Figure 1: Plots of the function $\Psi(t)$ with $\mu = 1/4, 1/2, 3/4, 1$ versus t; top: linear scales ($0 \leq t \leq 5$); bottom: logarithmic scales ($10^{-2} \leq t \leq 10^{2}$).

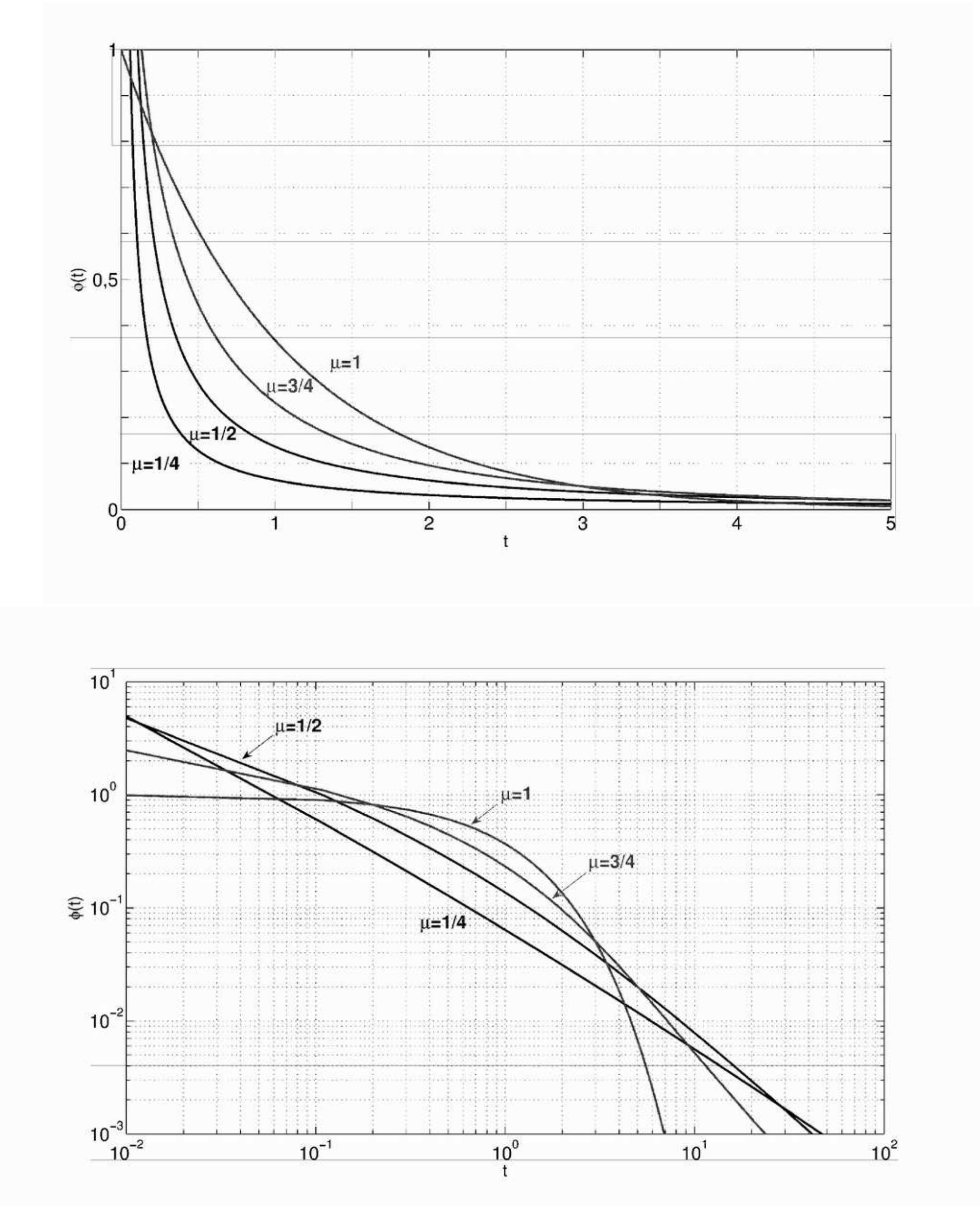

Figure 2: Plots of the function $\Phi(t)$ with $\mu = 1/4, 1/2, 3/4, 1$ versus t; top: linear scales ($0 \leq t \leq 5$); bottom: logarithmic scales ($10^{-2} \leq t \leq 10^2$).

For the use of the Hilfer intermediate derivative in fractional relaxation we refer to Hilfer himself, see [67], p.115, according to whom the Mittag-Leffler type function

$$u(t) = t^{(1-\nu)(\mu-1)} \, E_{\mu,\mu+\nu(1-\mu)}(-t^{\mu}) \, , \quad t \geq 0 \, , \tag{2.10}$$

is the solution of

$$D_t^{\mu,\nu} \, u(t) = -u(t) \, , \quad t \geq 0 \, , \quad \text{with} \quad \lim_{t \to 0^+} \, J_t^{(1-\mu)(1-\nu)} \, u(t) = 1 \, . \tag{2.11}$$

In fact, Hilfer has shown that the Laplace transform of the solution of (2.11) is

$$\widetilde{u}(s) = \frac{s^{\nu(\mu-1)}}{s^{\mu}+1} \, , \tag{2.12}$$

so, as a consequence of (2.7), we find (2.10). For plots of the function in (2.10) we refer to [115].

3. Constitutive equations of fractional order in viscoelasticity

In this section we present the fundamentals of linear Viscoelasticity restricting our attention to the one-axial case and assuming that the viscoelastic body is quiescent for all times prior to some starting instant that we assume as $t = 0$. For the sake of convenience both stress $\sigma(t)$ and strain $\epsilon(t)$ are intended to be normalized, *i.e.* scaled with respect to a suitable reference state $\{\sigma_0 \, , \, \epsilon_0\}$. After this necessary introduction we shall consider the main topic concerning viscoelastic models based on differential constitutive equations of fractional order.

3.1. Generalities

According to the linear theory, the viscoelastic body can be considered as a linear system with the stress (or strain) as the excitation function (input) and the strain (or stress) as the response function (output). In this respect, the response functions to an excitation expressed by the Heaviside step function $\Theta(t)$ are known to play a fundamental role both from a mathematical and physical point of view. We denote by $J(t)$ the strain response to the unit step of stress, according to the *creep test* and by $G(t)$ the stress response to a unit step of strain, according to the *relaxation test.* The functions $J(t) \, , \, G(t)$ are usually referred to as the *creep compliance* and *relaxation modulus* respectively, or, simply, the *material functions* of the viscoelastic body. In view of the causality requirement, both functions are causal, *i.e.* vanishing for $t < 0$.

The limiting values of the material functions for $t \to 0^+$ and $t \to +\infty$ are related to the instantaneous (or glass) and equilibrium behaviours of the viscoelastic body, respectively. As a consequence, it is usual to denote $J_g := J(0^+)$ the *glass compliance*, $J_e := J(+\infty)$ the *equilibrium compliance*, and $G_g := G(0^+)$ the *glass modulus* $G_e := G(+\infty)$ the *equilibrium modulus.* As a matter of fact, both the material functions are non-negative. Furthermore, for $0 < t < +\infty$, $J(t)$ is a *non decreasing function* and $G(t)$ is a *non increasing function.* The monotonicity properties of $J(t)$ and $G(t)$ are related respectively to the physical phenomena of strain *creep* and stress *relaxation*[3]. Under the hypotheses of causal histories, we get the stress-strain relationships

$$\begin{cases} \epsilon(t) = \displaystyle\int_{0^-}^t J(t-\tau)\, d\sigma(\tau) = \sigma(0^+)\, J(t) + \int_0^t J(t-\tau)\, \frac{d}{d\tau}\sigma(\tau)\, d\tau \,, \\ \sigma(t) = \displaystyle\int_{0^-}^t G(t-\tau)\, d\epsilon(\tau) = \epsilon(0^+)\, G(t) + \int_0^t G(t-\tau)\, \frac{d}{d\tau}\epsilon(\tau)\, d\tau \,, \end{cases} \tag{3.1}$$

where the passage to the RHS is justified if differentiability is assumed for the stress-strain histories, see also the excellent book by Pipkin [95]. Being of convolution type, equations (3.1) can be conveniently treated by the technique of Laplace transforms so they read in the Laplace domain

$$\widetilde{\epsilon}(s) = s\, \widetilde{J}(s)\, \widetilde{\sigma}(s)\,, \quad \widetilde{\sigma}(s) = s\, \widetilde{G}(s)\, \widetilde{\epsilon}(s)\,, \tag{3.2}$$

from which we derive the *reciprocity relation*

$$s\, \widetilde{J}(s) = \frac{1}{s\, \widetilde{G}(s)}\,. \tag{3.3}$$

Because of the limiting theorems for the Laplace transform, we deduce that $J_g = 1/G_g,\ J_e = 1/G_e$, with the convention that 0 and $+\infty$ are reciprocal to each other. The above remarkable relations allow us to classify the viscoelastic bodies according to their instantaneous and equilibrium responses in four types as stated by Caputo & Mainardi in their 1971 review paper [29] in Table 3.1.

[3] For the mathematical conditions that the material functions must satisfy to agree with the most common experimental observations (physical realizability) we refer to the recent paper by Hanyga [63] and references therein. We also note that in some cases the material functions can contain terms represented by *generalized functions* (distributions) in the sense of Gel'fand-Shilov [49] or *pseudo-functions* in the sense of Doetsch [36]

Type	J_g	J_e	G_g	G_e
I	> 0	$< \infty$	$< \infty$	> 0
II	> 0	$= \infty$	$< \infty$	$= 0$
III	$= 0$	$< \infty$	$= \infty$	> 0
IV	$= 0$	$= \infty$	$= \infty$	$= 0$

Table 3.1 The four types of viscoelasticity.

We note that the viscoelastic bodies of type I exhibit both instantaneous and equilibrium elasticity, so their behaviour appears close to the purely elastic one for sufficiently short and long times. The bodies of type II and IV exhibit a complete stress relaxation (at constant strain) since $G_e = 0$ and an infinite strain creep (at constant stress) since $J_e = \infty$, so they do not present equilibrium elasticity. Finally, the bodies of type III and IV do not present instantaneous elasticity since $J_g = 0$ $(G_g = \infty)$. Other properties will be pointed out later on.

3.2. The mechanical models

To get some feeling for linear viscoelastic behaviour, it is useful to consider the simpler behaviour of analog *mechanical models*. They are constructed from linear springs and dashpots, disposed singly and in branches of two (in series or in parallel). As analog of stress and strain, we use the total extending force and the total extension. We note that when two elements are combined in series [in parallel], their compliances [moduli] are additive. This can be stated as a combination rule: *creep compliances add in series, while relaxation moduli add in parallel.*

The mechanical models play an important role in the literature which is justified by the historical development. In fact, the early theories were established with the aid of these models, which are still helpful to visualize properties and laws of the general theory, using the combination rule. Now, it is worthwhile to consider the simple models of Fig. 3 providing their governing stress-strain relations along with the related material functions.

The spring (Fig. 3a) is the elastic (or storage) element, as for it the force is proportional to the extension; it represents a perfect elastic body obeying the Hooke law (ideal solid). This model is thus referred to as the *Hooke* model. If we denote by m the pertinent elastic modulus we have

$$\textit{Hooke model} \ : \quad \sigma(t) = m\,\epsilon(t)\,, \quad \text{and} \quad \begin{cases} J(t) = 1/m\,, \\ G(t) = m\,. \end{cases} \tag{3.4}$$

In this case we have no creep and no relaxation so the creep compliance and the relaxation modulus are constant functions: $J(t) \equiv J_g \equiv J_e = 1/m$; $G(t) \equiv G_g \equiv G_e = 1/m$.

The dashpot (Fig. 3b) is the viscous (or dissipative) element, the force being proportional to rate of extension; it represents a perfectly viscous body obeying the Newton law (perfect liquid). This model is thus referred to as the *Newton* model. If we denote by b the pertinent viscosity coefficient, we have

$$Newton\ model\ : \quad \sigma(t) = b_1 \frac{d\epsilon}{dt}, \quad \text{and} \quad \begin{cases} J(t) = \dfrac{t}{b_1}, \\ G(t) = b_1\, \delta(t). \end{cases} \tag{3.5}$$

In this case we have a linear creep $J(t) = J_+ t$ and instantaneous relaxation $G(t) = G_-\, \delta(t)$ with $G_- = 1/J_+ = b_1$.

We note that the *Hooke* and *Newton* models represent the limiting cases of viscoelastic bodies of type I and IV, respectively.

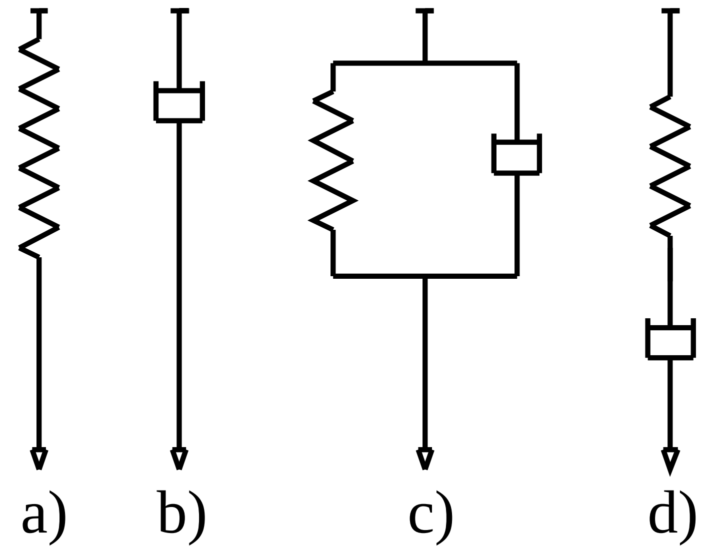

Figure 3: The representations of the basic mechanical models: a) spring for Hooke, b) dashpot for Newton, c) spring and dashpot in parallel for Voigt, d) spring and dashpot in series for Maxwell.

A branch constituted by a spring in parallel with a dashpot is known as the *Voigt* model (Fig. 3c). We have

$$Voigt\ model\ :\ \sigma(t) = m\,\epsilon(t) + b_1\,\frac{d\epsilon}{dt}\,, \tag{3.6}$$

and

$$\begin{cases} J(t) = J_1\left[1 - \mathrm{e}^{-t/\tau_\epsilon}\right], & J_1 = \dfrac{1}{m}\,,\ \tau_\epsilon = \dfrac{b_1}{m}\,, \\ G(t) = G_e + G_-\,\delta(t)\,, & G_e = m\,,\ G_- = b_1\,, \end{cases}$$

where τ_ϵ is referred to as the *retardation time.*

A branch constituted by a spring in series with a dashpot is known as the *Maxwell* model (Fig. 3d). We have

$$Maxwell\ model\ :\ \sigma(t) + a_1\,\frac{d\sigma}{dt} = b_1\,\frac{d\epsilon}{dt}\,, \tag{3.7}$$

and

$$\begin{cases} J(t) = J_g + J_+\,t\,, & J_g = \dfrac{a_1}{b_1}\,,\ J_+ = \dfrac{1}{b_1}\,, \\ G(t) = G_1\,\mathrm{e}^{-t/\tau_\sigma}\,, & G_1 = \dfrac{b_1}{a_1}\,,\ \tau_\sigma = a_1\,, \end{cases}$$

where τ_σ is is referred to as the *the relaxation time.*

The *Voigt* and the *Maxwell* models are thus the simplest viscoelastic bodies of type *III* and *II*, respectively. The *Voigt* model exhibits an exponential (reversible) strain creep but no stress relaxation; it is also referred to as the retardation element. The *Maxwell* model exhibits an exponential (reversible) stress relaxation and a linear (non reversible) strain creep; it is also referred to as the relaxation element.

Based on the combination rule, we can continue the previous procedure in order to construct the simplest models of type *I* and *IV* that require three parameters.

The simplest viscoelastic body of type *I* is obtained by adding a spring either in series to a Voigt model or in parallel to a Maxwell model (Fig. 4a and Fig 4b, respectively). So doing, according to the combination rule, we add a positive constant both to the Voigt-like creep compliance and to the Maxwell-like relaxation modulus so that we obtain $J_g > 0$ and $G_e > 0$. Such a model was introduced by Zener [120] with the denomination of *Standard Linear Solid* (*S.L.S.*). We have

$$\textit{Zener model} \; : \quad \left[1 + a_1 \, \frac{d}{dt}\right] \sigma(t) = \left[m + b_1 \, \frac{d}{dt}\right] \epsilon(t) \, , \tag{3.8}$$

and

$$\begin{cases} J(t) = J_g + J_1 \left[1 - \mathrm{e}^{-t/\tau_\epsilon}\right], & J_g = \dfrac{a_1}{b_1}, \; J_1 = \dfrac{1}{m} - \dfrac{a_1}{b_1}, \; \tau_\epsilon = \dfrac{b_1}{m}, \\ G(t) = G_e + G_1 \, \mathrm{e}^{-t/\tau_\sigma}, & G_e = m, \; G_1 = \dfrac{b_1}{a_1} - m, \; \tau_\sigma = a_1 \, . \end{cases}$$

We point out the condition $0 < m < b_1/a_1$ in order J_1, G_1 be positive and hence $0 < J_g < J_e < \infty$ and $0 < G_e < G_g < \infty$. As a consequence, we note that, for the *S.L.S.* model, the retardation time must be greater than the relaxation time, *i.e.* $0 < \tau_\sigma < \tau_\epsilon < \infty$.

Also the simplest viscoelastic body of type IV requires three parameters, *i.e.* a_1 , b_1 , b_2 ; it is obtained adding a dashpot either in series to a Voigt model or in parallel to a Maxwell model (Fig. 4c and Fig 4d, respectively). According to the combination rule, we add a linear term to the Voigt-like creep compliance and a delta impulsive term to the Maxwell-like relaxation modulus so that we obtain $J_e = \infty$ and $G_g = \infty$. We may refer to this model to as the *anti-Zener* model. We have

$$\textit{anti} - \textit{Zener model} \; : \quad \left[1 + a_1 \, \frac{d}{dt}\right] \sigma(t) = \left[b_1 \, \frac{d}{dt} + b_2 \, \frac{d^2}{dt^2}\right] \epsilon(t) \, , \tag{3.9}$$

and

$$\begin{cases} J(t) = J_+ t + J_1 \left[1 - \mathrm{e}^{-t/\tau_\epsilon}\right], & J_+ = \dfrac{1}{b_1}, \; J_1 = \dfrac{a_1}{b_1} - \dfrac{b_2}{b_1^2}, \; \tau_\epsilon = \dfrac{b_2}{b_1}, \\ G(t) = G_- \, \delta(t) + G_1 \, \mathrm{e}^{-t/\tau_\sigma}, & G_- = \dfrac{b_2}{a_1}, \; G_1 = \dfrac{b_1}{a_1} - \dfrac{b_2}{a_1^2}, \; \tau_\sigma = a_1. \end{cases}$$

We point out the condition $0 < b_2/b_1 < a_1$ in order J_1, G_1 be positive. As a consequence, we note that, for the *anti-Zener* model, the relaxation time must be greater than the retardation time, *i.e.* $0 < \tau_\epsilon < \tau_\sigma < \infty$, on the contrary of the Zener (*S.L.S.*) model.

In Fig. 4 we exhibit the mechanical representations of the Zener model (3.8), see a), b), and of the anti-Zener model (3.9)), see c), d).

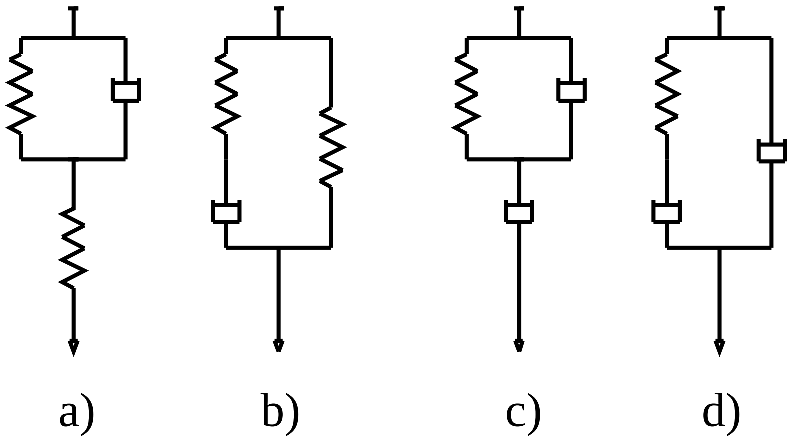

Figure 4: The mechanical representations of the Zener [a), b)] and anti-Zener[c), d)] models: a) spring in series with Voigt, b) spring in parallel with Maxwell, c) dashpot in series with Voigt, d) dashpot in parallel with Maxwell.

Based on the combination rule, we can construct models whose material functions are of the following type

$$\begin{cases} J(t) = J_g + \sum\limits_n J_n \left[1 - \mathrm{e}^{-t/\tau_{\epsilon,n}}\right] + J_+ \, t \,, \\ G(t) = G_e + \sum\limits_n G_n \, \mathrm{e}^{-t/\tau_{\sigma,n}} + G_- \, \delta(t) \,, \end{cases} \tag{3.10}$$

where all the coefficient are non-negative, and interrelated because of the *reciprocity relation* (3.3) in the Laplace domain. We note that the four types of viscoelasticity of Table 3.1 are obtained from Eqs. (3.10) by taking into account that

$$\begin{cases} J_e < \infty \quad \Longleftrightarrow \quad J_+ = 0 \,, \quad J_e = \infty \iff J_+ \neq 0 \,, \\ \\ G_g < \infty \quad \Longleftrightarrow G_- = 0 \,, \quad G_g = \infty \iff G_- \neq 0 \,. \end{cases} \tag{3.11}$$

Appealing to the theory of Laplace transforms, we write

$$\begin{cases} s\widetilde{J}(s) = J_g + \sum_n \dfrac{J_n}{1 + s\,\tau_{\epsilon,n}} + \dfrac{J_+}{s}\,, \\ s\widetilde{G}(s) = (G_e + \beta) - \sum_n \dfrac{G_n}{1 + s\,\tau_{\sigma,n}} + G_-\,s\,, \end{cases} \tag{3.12}$$

where we have put $\beta = \sum_n G_n$.

Furthermore, as a consequence of (3.12), $s\widetilde{J}(s)$ and $s\widetilde{G}(s)$ turn out to be *rational* functions in $\mathbf{C}$ with simple poles and zeros on the negative real axis and, possibly, with a simple pole or with a simple zero at $s = 0$, respectively.

In these cases the integral constitutive equations (3.1) can be written in differential form. Following Bland [9] with our notations, we obtain for these models

$$\left[1 + \sum_{k=1}^{p} a_k \frac{d^k}{dt^k}\right] \sigma(t) = \left[m + \sum_{k=1}^{q} b_k \frac{d^k}{dt^k}\right] \epsilon(t)\,, \tag{3.13}$$

where q and p are integers with $q = p$ or $q = p + 1$ and m, a_k, b_k are non-negative constants, subjected to proper restrictions in order to meet the physical requirements of realizability. The general Eq. (3.13) is referred to as the *operator equation* of the mechanical models.

In the Laplace domain, we thus get

$$s\widetilde{J}(s) = \frac{1}{s\widetilde{G}(s)} = \frac{P(s)}{Q(s)}\,, \quad \text{where} \quad \begin{cases} P(s) = 1 + \sum\limits_{k=1}^{p} a_k\, s^k\,, \\ Q(s) = m + \sum\limits_{k=1}^{q} b_k\, s^k\,. \end{cases} \tag{3.14}$$

with $m \ge 0$ and $q = p$ or $q = p + 1$. The polynomials at the numerator and denominator turn out to be *Hurwitz polynomials* (since they have no zeros for $Re\,\{s\} > 0$) whose zeros are alternating on the negative real axis ($s \le 0$). The least zero in absolute magnitude is a zero of $Q(s)$. The four types of viscoelasticity then correspond to whether the least zero is ($J_+ \ne 0$) or is not ($J_+ = 0$) equal to zero and to whether the greatest zero in absolute magnitude is a zero of $P(s)$ ($J_g \ne 0$) or a zero of $Q(s)$ ($J_g = 0$).

In Table 3.2 we summarize the four cases, which are expected to occur in the *operator equation* (3.13), corresponding to the four types of viscoelasticity.

Type	*Order*	m	J_g	G_e	J_+	G_-
I	$q = p$	> 0	a_p/b_p	m	0	0
II	$q = p$	$= 0$	a_p/b_p	0	$1/b_1$	0
III	$q = p + 1$	> 0	0	m	0	b_q/a_p
IV	$q = p + 1$	$= 0$	0	0	$1/b_1$	b_q/a_p

Table 3.2: The four cases of the operator equation.

We recognize that for $p = 1$, Eq. (3.13) includes the operator equations for the classical models with two parameters: Voigt and Maxwell, illustrated in Fig. 3, and with three parameters: Zener and anti-Zener, illustrated in Fig. 4. In fact we recover the Voigt model (type III)for $m > 0$ and $p = 0, q = 1$, the Maxwell model (type II) for $m = 0$ and $p = q = 1$, the Zener model (type I) for $m > 0$ and $p = q = 1$, and the anti-Zener model (type IV) for $m = 0$ and $p = 1, q = 2$.

Remark. We note that the initial conditions at $t = 0^+$, $\sigma^{(h)}(0^+)$ with $h = 0, 1, \dots p - 1$ and $\epsilon^{(k)}(0^+)$ with $k = 0, 1, \dots q - 1$, do not appear in the operator equation but they are required to be compatible with the integral equations (3.1). In fact, since Eqs (3.1) do not contain the initial conditions, some compatibility conditions at $t = 0^+$ must be *implicitly* required both for stress and strain. In other words, the equivalence between the integral Eqs. (3.1) and the differential operator Eq. (3.13) implies that when we apply the Laplace transform to both sides of Eq. (3.13) the contributions from the initial conditions are vanishing or cancel in pair-balance. This can be easily checked for the simplest classical models described by Eqs (3.6)-(3.9). It turns out that the Laplace transform of the corresponding constitutive equations does not contain any initial conditions: they are all hidden being zero or balanced between the RHS and LHS of the transformed equation. As simple examples let us consider the Voigt model for which $p = 0$, $q = 1$ and $m > 0$, see Eq. (3.6), and the Maxwell model for which $p = q = 1$ and $m = 0$, see Eq. (3.7).

For the Voigt model we get $s\widetilde{\sigma}(s) = m\widetilde{\epsilon}(s) + b\,]s\widetilde{\epsilon}(s) - \epsilon(0^+)]$, so, for any causal stress and strain histories, it would be

$$s\widetilde{J}(s) = \frac{1}{m + bs} \iff \epsilon(0^+) = 0\,. \tag{3.15}$$

We note that the condition $\epsilon(0^+) = 0$ is surely satisfied for any reasonable stress history since $J_g = 0$, but is not valid for any reasonable strain history; in fact, if we consider the relaxation test for which $\epsilon(t) = \Theta(t)$ we have

$\epsilon(0^+) = 1$. This fact may be understood recalling that for the Voigt model we have $J_g = 0$ and $G_g = \infty$ (due to the delta contribution in the relaxation modulus).

For the Maxwell model we get $\widetilde{\sigma}(s) + a\,[s\widetilde{\sigma}(s) - \sigma(0^+)] = b\,[s\widetilde{\epsilon}(s) - \epsilon(0^+)]$, so, for any causal stress and strain histories it would be

$$s\widetilde{J}(s) = \frac{a}{b} + \frac{1}{bs} \iff a\sigma(0^+) = b\epsilon(0^+)\,. \tag{3.16}$$

We now note that the condition $a\sigma(0^+) = b\epsilon(0^+)$ is surely satisfied for any causal history both in stress and in strain. This fact may be understood recalling that for the Maxwell model we have $J_g > 0$ and $G_g = 1/J_g > 0$.

Then we can generalize the above considerations stating that the compatibility relations of the initial conditions are valid for all the four types of viscoelasticity, as far as the creep representation is considered. When the relaxation representation is considered, caution is required for the types III and IV, for which, for correctness, we would use the generalized theory of integral transforms suitable just for dealing with generalized functions.

3.3. The time spectral functions

From the previous analysis of the classical mechanical models in terms of a finite number of basic elements, one is led to consider two *discrete* distributions of characteristic times (the *retardation* and the *relaxation* times), as stated in (3.10). However, in more general cases, it is natural to presume the presence of *continuous* distributions, so that, for a viscoelastic body, the material functions turn out to be of the following form:

$$\begin{cases} J(t) = J_g + \alpha \int_0^\infty R_\epsilon(\tau)\,\left(1 - \mathrm{e}^{-t/\tau}\right)\,d\tau + J_+\,t\,, \\[2ex] G(t) = G_e + \beta \int_0^\infty R_\sigma(\tau)\,\mathrm{e}^{-t/\tau}\,d\tau + G_-\,\delta(t)\,. \end{cases} \tag{3.17}$$

where all the coefficients and functions are non-negative. The function $R_\epsilon(\tau)$ is referred to as the *retardation spectrum* while $R_\sigma(\tau)$ as the *relaxation spectrum*. For the sake of convenience we shall replace the suffix ϵ or τ with $*$ to denote anyone of the two spectra that we refer simply to as the *time-spectral function*. We require $R_*(\tau)$ to be locally integrable in $\mathbf{R}^+$, with the supplementary normalization condition $\int_0^\infty R(\tau)\,d\tau = 1$ if the integral in $\mathbf{R}^+$ turns out to be convergent.

The discrete distributions of the classical mechanical models, see (3.10), can be easily recovered from (3.17); in fact, assuming $\alpha \neq 0\,, \beta \neq 0\,,$ we have to put

$$\begin{cases} R_\epsilon(\tau) = \dfrac{1}{\alpha} \displaystyle\sum_n J_n \, \delta(\tau - \tau_{\epsilon,n}) \,, \quad \alpha = \sum_n J_n \,, \\ R_\sigma(\tau) = \dfrac{1}{\beta} \displaystyle\sum_n G_n \, \delta(\tau - \tau_{\sigma,n}) \,, \quad \beta = \sum_n G_n \,. \end{cases} \tag{3.18}$$

We devote particular attention to the time-dependent contributions to the material functions (3.17) which are provided by the continuous spectra, *i.e.*

$$\begin{cases} \Psi(t) := \alpha \displaystyle\int_0^\infty R_\epsilon(\tau) \left(1 - \mathrm{e}^{-t/\tau}\right) d\tau \,, \\ \Phi(t) := \beta \displaystyle\int_0^\infty R_\sigma(\tau) \, \mathrm{e}^{-t/\tau} \, d\tau \,. \end{cases} \tag{3.19}$$

We recognize that $\Psi(t)$ (that we refer as the *creep function with spectrum*) is a non-decreasing, non-negative function in $\mathbf{R}^+$ with limiting values $\Psi(0^+) = 0$, $\Psi(+\infty) = \alpha$ or ∞, whereas $\Phi(t)$ (that we refer as the *relaxation function with spectrum*) is a non-increasing, non-negative function in $\mathbf{R}^+$ with limiting values $\Phi(0^+) = \beta$ or ∞, $\Phi(+\infty) = 0$. More precisely, in view of their spectral representations (3.19), we have

$$\begin{cases} \Psi(t) \ge 0 \,, \quad (-1)^n \, \dfrac{d^n \Psi}{dt^n} \le 0 \,, \\ \Phi(t) \ge 0 \,, \quad (-1)^n \, \dfrac{d^n \Phi}{dt^n} \ge 0 \,, \end{cases} \quad t \ge 0 \,, \quad n = 1, 2, \ldots \,. \tag{3.20}$$

In other words, $\Phi(t)$ is a *completely monotonic* function and $\Psi(t)$ is a *Bernstein function* (namely a non-negative function with a completely monotonic derivative). These properties have been investigated by several authors, including Molinari [87] and more recently by Hanyga [63]. The determination of the *time-spectral functions* starting from the knowledge of the creep and relaxation functions is a problem which can be formally solved through the Titchmarsh inversion formula of the Laplace transform theory, see *e.g.* [62],[29].

3.4. Fractional viscoelastic models

The straightforward way to introduce fractional derivatives in linear viscoelasticity is to replace the first derivative in the constitutive equation (3.5) of the Newton model with a fractional derivative of order $\nu \in (0, 1)$, that, being $\epsilon(0^+) = 0$, may be intended both in the Riemann-Liouville or Caputo sense. Some people call the fractional model of the Newtonian dashpot with the suggestive name *pot*: we prefer to refer such model to as *Scott-Blair* model, to give honour to the scientist who already in the middle of the past century proposed such a constitutive equation to explain a material property that is intermediate between the elastic modulus (Hooke solid) and the coefficient of viscosity (Newton fluid), see *e.g.* [111, 113, 114]. We note that Scott-Blair was surely a pioneer of the fractional calculus even if he did not provide a mathematical theory accepted by mathematicians of his time!

The use of *fractional calculus* in linear viscoelasticity leads to generalizations of the classical mechanical models: the basic Newton element is substituted by the more general Scott-Blair element (of order ν). In fact, we can construct the class of these generalized models from Hooke and Scott-Blair elements, disposed singly and in branches of two (in series or in parallel). Then, extending the procedures of the classical mechanical models (based on springs and dashpots), we will get the *fractional operator equation* (that is an operator equation with fractional derivatives) in the form which properly generalizes (3.13), *i.e.*

$$\left[1+\sum_{k=1}^{p} a_k \frac{d^{\nu_k}}{dt^{\nu_k}}\right] \sigma(t) = \left[m+\sum_{k=1}^{q} b_k \frac{d^{\nu_k}}{dt^{\nu_k}}\right] \epsilon(t)\,, \quad \nu_k = k+\nu-1\,. \tag{3.21}$$

so, as a generalization of (3.10),

$$\begin{cases} J(t) \;=\; J_g + \displaystyle\sum_n J_n \left\{1 - \mathrm{E}_\nu\left[-(t/\tau_{\epsilon,n})^\nu\right]\right\} + J_+ \dfrac{t^\nu}{\Gamma(1+\nu)}\,, \\ G(t) \;=\; G_e + \displaystyle\sum_n G_n\, \mathrm{E}_\nu\left[-(t/\tau_{\sigma,n})^\nu\right] + G_- \dfrac{t^{-\nu}}{\Gamma(1-\nu)}\,, \end{cases} \tag{3.22}$$

where all the coefficient are non-negative. Of course, for the fractional operator equation (3.21) the four cases summarized in Table 3.2 are expected to occur in analogy with the operator equation (3.13).

We conclude with some considerations on the presence of the Mittag-Leffler function in the material functions of the fractional models. For this purpose let us consider the following *creep* and *relaxation* functions with the

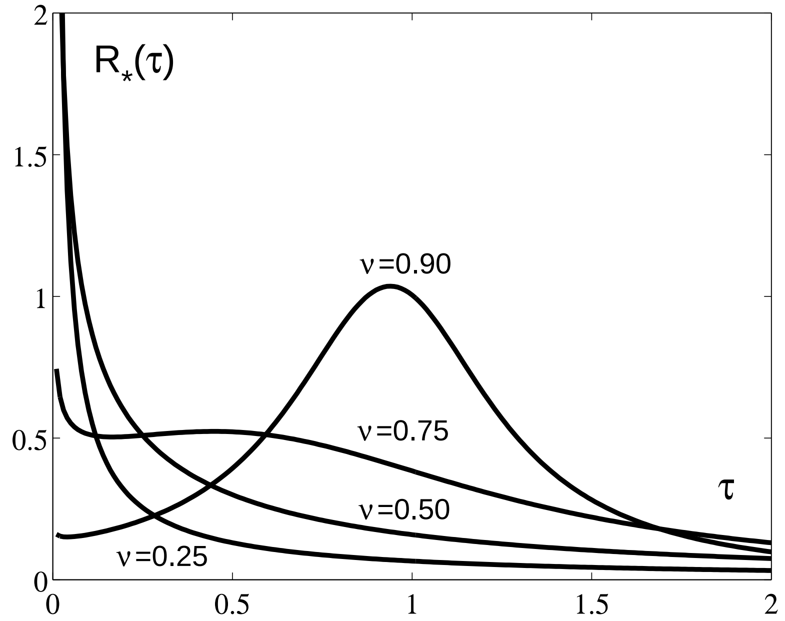

Figure 5: Spectral function $R_*(\tau)$ for $\nu = 0.25,\, 0.50,\, 0.75,\, 0.90$ and $\tau_* = 1$.

corresponding *time-spectral* functions

$$\begin{cases} \Psi(t) = & \alpha \left\{1 - \mathrm{E}_\nu \left[-(t/\tau_\epsilon)^\nu\right]\right\} = \alpha \displaystyle\int_0^\infty R_\epsilon(\tau) \left(1 - \mathrm{e}^{-t/\tau}\right) d\tau\,, \\ \Phi(t) = & \beta\, \mathrm{E}_\nu \left[-(t/\tau_\sigma)^\nu\right] = \beta \displaystyle\int_0^\infty R_\sigma(\tau)\, \mathrm{e}^{-t/\tau}\, d\tau\,. \end{cases} \tag{3.23}$$

Following Caputo and Mainardi [28, 29] and denoting with star the suffixes ϵ , σ , we obtain

$$R_*(\tau) = \frac{1}{\pi\, \tau}\, \frac{\sin \nu\pi}{(\tau/\tau_*)^\nu + (\tau/\tau_*)^{-\nu} + 2\, \cos\, \nu\pi}\,. \tag{3.24}$$

From Fig 5 reporting the plots of $R_*(\tau)$ for some values of ν we can easily recognize the effect of the variation of ν on the character of the spectrum. For $0 < \nu < \nu_0$, where $\nu_0 \approx 0.736$ is the solution of the equation $\nu = \sin\, \nu\pi$, the spectrum $R_*(\tau)$ is a decreasing function of τ; subsequently, with increasing ν, it first exhibits a minimum and then a maximum; for $\nu \to 1$ it becomes steeper and steeper near its maximum approaching a delta function. In fact we have $\lim_{\nu\to 1} R_*(\tau) = \delta(\tau - \tau_*)$, where we have set $\tau_* = 1$, being τ_* the single retardation/relaxation time exhibited by the corresponding creep/relaxation function. Formula (3.24) was formerly obtained by Gross [61] in 1947, when, in the attempt to eliminate the faults which a power law shows for the creep function, he proposed the Mittag-Leffler function as an empirical law for both the creep and relaxation functions. In their 1971 papers, Caputo and Mainardi derived the same result by introducing into the stress-strain relations a Caputo derivative that implies a *memory mechanism* by means of a convolution between the first-order derivative and a power of time, see $(1.6')$.

In fractional viscoelasticity governed by the operator equation (3.21) the corresponding material functions are obtained by using the combination rule valid for the classical mechanical models. Their determination is made easy if we take into account the following *correspondence principle* between the classical and fractional mechanical models, as introduced by Caputo & Mainardi in 1971. Taking $0 < \nu \le 1$ such correspondence principle can be formally stated by the following three equations where transitions from Laplace transform pairs are outlined:

$$\begin{cases} \delta(t) \div 1 \;\Rightarrow\; \dfrac{t^{-\nu}}{\Gamma(1-\nu)} \div s^{1-\nu}\,, \\ t \div \dfrac{1}{s^2} \;\Rightarrow\; \dfrac{t^{\nu}}{\Gamma(1+\nu)} \div \dfrac{1}{s^{\nu+1}} \\ \mathrm{e}^{-t/\tau} \div \dfrac{1}{s+1/\tau} \;\Rightarrow\; \mathrm{E}_\nu[-(t/\tau)^\nu] \div \dfrac{s^{\nu-1}}{s^\nu + 1/\tau}\,, \end{cases} \tag{3.25}$$

where $\tau > 0$ and E_ν denotes the Mittag-Leffler function of order ν.

Remark. We note that the initial conditions at $t = 0^+$ for the stress and strain do not explicitly enter into the fractional operator equation (3.21) if they are taken in the same way as for the classical mechanical models reviewed in Subsection 3.2. This means that the approach with the Caputo derivative, which requires in the Laplace domain the same initial conditions as the classical models is quite correct. However, if we assume the same initial conditions, the approach with the Riemann-Liouville derivative provides the same results since, in view of the corresponding Laplace transform rule (1.15), the initial conditions do not appear in the Laplace domain. The equivalence of the two approaches has also been noted for the fractional Zener model in a recent note by Bagley [5]. We refer the reader to the paper by Heymans and Podubny [65] for the physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville derivatives, especially in viscoelasticity. In such field, however, we prefer to adopt the Caputo derivative since it requires the same initial conditions as in the classical cases, as pointed out in [81]: by the way, for a physical view point, these initial conditions are more accessible than those required in the more general Riemann-Liouville approach, see (1.14).

4. Some historical notes

In this section we provide some historical notes on how the Caputo derivative came out as a contrast to the Riemann-Liouville derivative and a sketch about the generic use of fractional calculus in viscoelasticity.

4.1. The origins of the Caputo derivative

Here we find it worthwhile and interesting to say something about the commonly used attributes to Riemann and Liouville and to Caputo for the two types of fractional derivatives, that have been discussed.

Usually names are given to pay honour to some scientists who have provided main contributions, but not necessarily to those who have as the first introduced the corresponding notions. Surely Liouville, *e.g.* [75, 76] (starting from 1832), and then Riemann [105] (as a student in 1847) have given important contributions towards fractional integration and differentiation, but these notions had previously a story. As a matter of fact it was Abel who solved his celebrated integral equations by a fractional integration of order $1/2$ and $\alpha \in (0, 1)$, respectively in 1823 and 1826, see [1, 2]. So, Abel, using the operators that nowadays are ascribed to Riemann and Liouville, preceded these eminent mathematicians by at least 10 years!

We remind that the Laplace transform rule (1.13) was practically the starting point of Caputo [12, 13] in defining his generalized derivative in the late Sixties of the past century, and ignoring the existence of the classical Riemann-Liouville derivative. Please keep in mind that the first treatise devoted to the so-called fractional calculus appeared only in 1974, published by Oldham & Spanier [94] who were unaware of the alternative form (1.6) of the fractional derivative and of its property (1.13) with respect to the Laplace transform. However the form used by Caputo is found in a paper by Liouville himself as recently noted by Butzer and Westphal [10] but Liouville disregarded this notion because he did not recognize its role.

As far as we know, up to the middle of the past century the authors did not take care of the difference between the two forms (1.5)-(1.6) and of the possible use of the alternative form (1.6). Indeed, in the classical book on Differential and Integral Calculus by the eminent mathematician R. Courant the two forms of the fractional derivative were considered as equivalent, see [34], pp. 339-341. Only in the late Sixties it seems that the relevance of the alternative form was recognized. In fact, in 1968 Dzherbashyan and Nersesian [38] used the alternative form for dealing with Cauchy problems of differential equations of fractional order. In 1967, just one year earlier, Caputo [12], see also [13], used this form to generalize the usual rule for the Laplace transform of a derivative of integer order and to solve some problems in Seismology. Soon later, this derivative was adopted by Caputo and Mainardi in the framework of the theory of *Linear Viscoelasticity*, see [28, 29].

Starting with the Seventies many authors, very often ignoring the works by Dzherbashian-Nersesian and Caputo-Mainardi, have re-discovered and used the alternative form, recognizing its major utility for solving physical problems with standard (namely integer order) initial conditions. Although there appeared several papers by different authors, including Caputo [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], Gorenflo et al. [55, 56, 57, 58], Kochubei [71, 72], Mainardi [78, 79, 80, 83], where the alternative derivative was adopted, it was mainly with the 1999 book by Podlubny [96] that it became popular: indeed, in that book it was named as Caputo derivative.

The notation adopted here was introduced in a systematic way by us in our 1996 CISM lectures [57], partly based on the 1991 book on Abel Integral Equations by Gorenflo & Vessella [59].

4.2. Fractional calculus in viscoelasticity in the XX-th century

Starting from the past century a number of authors have (implicitly or explicitly) used the *fractional calculus* as an empirical method of describing the properties of viscoelastic materials: Gemant [50, 52], Scott-Blair [114, 113, 112], Gerasimov [53] were early contributors. Particular mention is due to the theory of hereditary solid mechanics developed by Rabotnov [102, 103, 104], see also [107], that (*implicitly*) requires fractional derivatives.

In the late Sixties, Caputo [11, 12, 13] and then Caputo and Mainardi [28, 29] suggested that derivatives of fractional order (of Caputo type) could be successfully used to model the dissipation in seismology and in metallurgy.

From then, applications of fractional calculus in viscoelsticity were considered by an increasing number of authors. Restricting our attention to a few significant papers published in the past century, let us quote the contributions by Bagley & Torvik [4, 6, 117], Caputo [14, 16, 18, 20, 25, 27], Friedrich & associates [40, 41, 42, 43, 44], Graffi [60], Gaul, Kempfle & associates [8, 45, 46, 47], Heymans [64], Koeller [73, 74], Mainardi [78, 80], Meshkov and Rossikhin [84], Nonnenmacher & associates [91, 92, 54], Pritz [99, 100], Rossikhin and Shitikova [106].

Additional references up to nowadays can be found in the huge (even not exhaustive) bibliography of the forthcoming book by Mainardi [82].

We like to recall the formal analogy between the relaxation phenomena in viscoelastic and dielectric bodies; in this respect the pioneering works by Cole and Cole [32, 33] in 1940's on dielectrics can be considered as precursors of the implicit use of fractional calculus in that area, see *e.g.* [86].

Appendix: The functions of Mittag-Leffler type

A.1. The classical Mittag-Leffler function

The Mittag-Leffler function $E_\mu(z)$ (with $\mu > 0$) is an entire transcendental function of order $1/\mu$, defined in the complex plane by the power series

$$E_\mu(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu\, k+1)}\,, \quad z \in \mathbf{C}\,. \qquad (A.1)$$

It was introduced and studied by the Swedish mathematician Mittag-Leffler at the beginning of the XX-th century to provide a noteworthy example of entire function that generalizes the exponential (to which it reduces for $\mu = 1$).

Details on this function can be found *e.g.* in the treatises by Davis [35], Dzherbashyan [37], Erdelyi et al. [39], Kilbas et al. [69], Kiryakova [70], Podlubny [96], Samko et al. [108], Sansone and Gerretsen [109]. Concerning earlier applications of the Mittag-Leffler function in physics let us quote the contributions by K. S. Cole, see [31], mentioned in the 1936 book by Davis, see [35], p. 287, in connection with nerve conduction, and by Gross, see [61], in connection with creep and relaxation in viscoelastic media.

We note that the function $E_\mu(-x)$ ($x \ge 0$) is a completely monotonic function of x if $0 < \mu \le 1$, as was formerly conjectured by Feller on probabilistic arguments and later in 1948 proved by Pollard [98]. This property still holds with respect to the variable t if we replace x by $\lambda\, t^\mu$ $(t \ge 0)$ where λ is a positive constant. Thus in its dependence on t the function $E_\mu(-\lambda t^\mu)$ preserves the *complete monotonicity* of the exponential $\exp(-\lambda t)$: indeed, for $0 < \mu < 1$ it is represented in terms of a real Laplace transform (of a real parameter r) of a non-negative function (that we refer to as the spectral function)

$$E_\mu(-\lambda t^\mu) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{\,-rt}\, \frac{\lambda r^{\mu-1}\,\sin(\mu\pi)}{\lambda^2 + 2\lambda\, r^\mu\,\cos(\mu\pi) + r^{2\mu}}\, dr\,. \qquad (A.2)$$

We note that as $\mu \to 1^-$ the spectral function tends to the generalized Dirac function $\delta(r-\lambda)$.

We point out that the Mittag-Leffler function (A.2) starts at $t = 0$ as a stretched exponential and decreases for $t \to \infty$ like a power with exponent $-\mu$:

$$E_\mu(-\lambda t^\mu) \sim \begin{cases} 1 - \lambda\,\dfrac{t^\mu}{\Gamma(1+\mu)} \sim \exp\left\{-\dfrac{\lambda t^\mu}{\Gamma(1+\mu)}\right\}, & t \to 0^+\,, \\ \dfrac{t^{-\mu}}{\lambda\,\Gamma(1-\mu)}\,, & t \to \infty\,. \end{cases} \qquad (A.3)$$

The integral representation (A.2) and the asymptotic (A.3) can also be derived from the Laplace transform pair

$$\mathcal{L}\{E_\mu(-\lambda t^\mu); s\} = \frac{s^{\mu-1}}{s^\mu + \lambda}\,. \qquad (A.4)$$

In fact it it sufficient to apply the Titchmarsh theorem for contour integration in the complex plane ($s = re^{i\pi}$) for deriving (A.2) and the Tauberian theory ($s \to \infty$ and $s \to 0$) for deriving (A.3).

If $\mu = 1/2$ we have for $t \ge 0$:

$$E_{1/2}(-\lambda\sqrt{t}) = \mathrm{e}^{\,\lambda^2 t}\,\mathrm{erfc}(\lambda\sqrt{t}) \sim 1/(\lambda\sqrt{\pi\, t})\,,\; t \to \infty\,, \qquad (A.5)$$

where erfc denotes the *complementary error* function, see *e.g.* [3].

A.2. The generalized Mittag-Leffler function

The Mittag-Leffler function in two parameters $E_{\mu,\nu}(z)$ ($\Re\{\mu\} > 0,\ \nu \in \mathbf{C}$) is defined by the power series

$$E_{\mu,\nu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu\, k + \nu)}\,, \quad z \in \mathbf{C}\,. \tag{A.6}$$

It generalizes the classical Mittag-Leffler function to which it reduces for $\nu = 1$. It is an entire transcendental function of order $1/\Re\{\mu\}$ on which the reader can inform himself by again consulting the treatises cited for the classical Mittag-Leffler function.

The function $E_{\mu,\nu}(-x)$ $(x \geq 0)$ is completely monotonic in x if $0 < \mu \leq 1$ and $\nu \geq \mu$, see *e.g.* [89, 90, 110]. Again this property still holds with respect to t if we replace the variable x by $\lambda\, t^\mu$ where λ is a positive constant. In this case the asymptotic representations as $t \to 0^+$ and $t \to +\infty$ read

$$E_{\mu,\nu}(-\lambda t^\mu) \sim \begin{cases} \dfrac{1}{\Gamma(\nu)} - \lambda\, \dfrac{t^\mu}{\Gamma(\nu+\mu)}, & t \to 0^+, \\ \dfrac{1}{\lambda}\, \dfrac{t^{-\mu+\nu-1}}{\Gamma(\nu-\mu)}\,, & t \to \infty. \end{cases} \tag{A.7}$$

We point out the Laplace transform pair, see [96],

$$\mathcal{L}\{t^{\nu-1}\, E_{\mu,\nu}(-\lambda t^\mu); s\} = \frac{s^{\mu-\nu}}{s^\mu + \lambda}\,, \tag{A.8}$$

with $\mu, \nu \in \mathbf{R}^+$. By aid of this Laplace transform, with $0 < \mu = \nu \leq 1$, we can obtain the useful identity

$$t^{-(1-\mu)} E_{\mu,\mu}\left(-\lambda\, t^\mu\right) = -\frac{1}{\lambda}\, \frac{d}{dt} E_\mu\left(-\lambda\, t^\mu\right)\,, \quad 0 < \mu \leq 1\,. \tag{A.9}$$

To see this it is sufficient to write

$$\mathcal{L}\left\{t^{-(1-\mu)} E_{\mu,\mu}\left(-\lambda\, t^\mu\right)\right\} = \frac{1}{s^\mu + \lambda} = -\frac{1}{\lambda}\left[s\, \frac{s^{\mu-1}}{s^\mu + \lambda} - 1\right]\,, \tag{A.10}$$

and invert the Laplace transforms. Of course the identity (A.9) can be proved directly by differentiating term by term the power series of the classical Mittag-Leffler function, but, as often in matters of fractional calculus, it is simpler to work with the Laplace transform technique.

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