ON LIMITING CASE OF THE STEIN-WEISS TYPE INEQUALITY FOR THE $B$-RIESZ POTENTIALS *)

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Abstract

In this paper we study the Riesz potentials ($B$-Riesz potentials) generated by the Laplace-Bessel differential operator $\Delta_B = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{\gamma}{x_n} \frac{\partial}{\partial x_n}$, $\gamma > 0$, in the weighted Lebesgue spaces $L_{p,|x|^{\beta},\gamma}$. We establish an inequality of Stein-Weiss type for the $B$-Riesz potentials in the limiting case, and obtain the boundedness of the $B$-Riesz potential operator from the space $L_{p,|x|^{\beta},\gamma}$ to $BMO_{|x|^{-\lambda},\gamma}$.

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Introduction

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{\gamma}{x_n} \frac{\partial}{\partial x_n}, \quad \gamma > 0$$

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have been the research areas of many mathematicians such as K. Stempak [11], I. Kipriyanov [8], A.D. Gadjej and I.A. Aliev [1], A.D. Gadjej and V.S. Guliyev [2], E.V. Guliyev [3], V.S. Guliyev [4]-[6] and others.

In this paper we study Riesz potentials (B-Riesz potentials) generated by the Laplace-Bessel differential operator $\Delta_B$ in weighted Lebesgue spaces. We establish the inequality of Stein-Weiss type (see [10]) for B-Riesz potentials in the limiting case. We obtain the boundedness of the B-Riesz potential operator from the spaces $L^p, |x|^{-\lambda, \gamma}$ to $BMO, |x|^{-\lambda, \gamma}$ in the limiting case.

1. Definitions, notation and preliminaries

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), x_n > 0\}$ and $B(x, r) = \{y \in \mathbb{R}^n_+ : |x - y| < r, r > 0\}, B_r \equiv B(0, r)$, and let $B(x, r)$.

For a measurable set $A \subset \mathbb{R}^n_+$, let $|A|_\gamma = \int_A x_\gamma dx$, then $|B_r|_\gamma = \omega(n, \gamma) r^n + 1$, where

$$\omega(n, \gamma) = \int_{B_1} x_\gamma dx = \frac{\pi^{(n-1)/2} \Gamma((\gamma + 1)/2)}{2 \Gamma((n + \gamma - 2)/2)}.$$

Denote by $T^y$ the generalized shift operator (B-shift operator) acting according to the law

$$T^y f(x) = C_\gamma \int_0^\pi f(x' - y', (x_n, y_n)\beta) \sin^{\gamma - 1} \beta d\beta,$$

where

$$(x_n, y_n)\beta = \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos \beta} \quad \text{and} \quad C_\gamma = \frac{\Gamma((\gamma + 1)/2)}{\sqrt{\pi} \Gamma(\gamma/2)} = \frac{2}{\pi} \omega(2, \gamma).$$

We remark that the generalized shift operator $T^y$ is closely connected with the Laplace-Bessel differential operator $\Delta_B$ (for example, $n = 1$ – see [9], and $n > 1$ – [8] for details).

Let $L_{p,\gamma}(\mathbb{R}^n_+)$ be the space of measurable functions on $\mathbb{R}^n_+$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_+)} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_\gamma dx\right)^{1/p}, \quad 1 \leq p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_+)$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_{\infty}} = \text{ess sup}_{x \in \mathbb{R}^n_+} |f(x)|.$$

**Lemma 1.** ([2]) Let $0 < \alpha < n + \gamma$. Then

$$|T^y|x|^{\alpha-n-\gamma} - |y|^{\alpha-n-\gamma} \leq 2^{\alpha+\gamma+1-\alpha}|y|^{\alpha-n-\gamma-1}|x| $$

for $2|x| \leq |y|$.
ON LIMITING CASE OF THE STEIN-WEISS TYPE . . . 41

Definition 1. Let \( 1 \leq p < \infty \). We denote by \( W_{L_{p,\gamma}}(\mathbb{R}^n) \) the weak \( L_{p,\gamma} \) space defined as the set of locally integrable functions \( f \) with the finite norms

\[
\|f\|_{W_{L_{p,\gamma}}} = \sup_{r>0} \frac{f^{1/p}(r)}{r^{\gamma}},
\]

where \( f^{1/p}(r) = \left| \{ x \in \mathbb{R}^n_+ : |f(x)| > r \} \right|^{\gamma} \).

Let \( v \) be a non-negative and measurable function on \( \mathbb{R}^n_+ \), and \( L_{p,v,\gamma}(\mathbb{R}^n_+) \) be the weighted \( L_{p,\gamma} \)-space of all measurable functions \( f \) on \( \mathbb{R}^n_+ \) for which

\[
\|f\|_{L_{p,v,\gamma}} \equiv \|vf\|_{L_{p,\gamma}}(\mathbb{R}^n_+) < \infty.
\]

We denote by \( W_{L_{p,v,\gamma}}(\mathbb{R}^n_+) \) the weighted weak Lebesgue space which is the class of all measurable functions \( f : \mathbb{R}^n_+ \to \mathbb{R} \), for which

\[
\|f\|_{W_{L_{p,v,\gamma}}} \equiv \|vf\|_{W_{L_{p,\gamma}}}(\mathbb{R}^n_+) < \infty.
\]

The \( B - BMO \) space (see [5]) \( BMO_\gamma(\mathbb{R}^n_+) \), and weighted \( B - BMO \) space, \( BMO_{w,\gamma}(\mathbb{R}^n_+) \), are defined as the set of locally integrable functions \( f \) with finite norms

\[
\|f\|_{*,\gamma} = \sup_{r>0, \ x \in \mathbb{R}^n_+} |B_r|^{-1} \int_{B_r} |T_y f(x) - f_{B_r}(x)| y_\gamma^n dy < \infty,
\]

and

\[
\|f\|_{*,w,\gamma} = \sup_{r>0, \ x \in \mathbb{R}^n_+} w(B_r)^{-1} \int_{B_r} |T_y f(x) - f_{B_r}(x)| y_\gamma^n dy < \infty,
\]

respectively, where

\[
f_{B_r}(x) = |B_r|^{-1} \int_{B_r} T_y f(x) y_\gamma^n dy \quad \text{and} \quad w(B_r) = \int_{B_r} w(x) x_\gamma^n dx.
\]

Consider the \( B \)-Riesz potential

\[
I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_+} T_y |x|^{n-\alpha-\gamma} f(y) y_\gamma^n dy, \quad 0 < \alpha < n + \gamma
\]

and the modified \( B \)-Riesz potential

\[
\tilde{I}_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_+} \left( T_y |x|^{n-\alpha-\gamma} - |y|^{n-\alpha-\gamma} \chi_{B_1}(y) \right) f(y) y_\gamma^n dy.
\]

For the \( B \)-Riesz potential the following Stein-Weiss type theorem was proved by A.D. Gadjiev and V.S. Guliyev in [2].
Theorem A. Let $0 < \alpha < n + \gamma$, $1 \leq p \leq q < \infty$, $\beta < \frac{n+\gamma}{p}$ ($\beta \leq 0$, if $p = 1$), $\lambda < \frac{n+\gamma}{q}$ ($\lambda \leq 0$, if $q = \infty$), $\alpha \geq \beta + \lambda \geq 0$ ($\beta + \lambda > 0$, if $p = q$).

1) If $1 < p < \frac{n+\gamma}{\alpha - \beta - \lambda}$, then the condition $\frac{1}{p} = \frac{1}{q} = \frac{\alpha - \beta - \lambda}{n+\gamma}$ is necessary and sufficient for the boundedness of $I_{\alpha, \gamma}$ from $L_{p, |x|^\alpha, \gamma}(\mathbb{R}^n_+)$ to $L_{q, |x|^{-\lambda}, \gamma}(\mathbb{R}^n_+)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha - \beta - \lambda}{n+\gamma}$ is necessary and sufficient for the boundedness of $I_{\alpha, \gamma}$ from $L_{1, |x|^\alpha, \gamma}(\mathbb{R}^n_+)$ to $WL_{q, |x|^{-\lambda}, \gamma}(\mathbb{R}^n_+)$.

Definition 2. The weight function $w$ belongs to the class $A_{p, \gamma}(\mathbb{R}^n_+)$ for $1 < p < \infty$, if

$$
\sup_{x \in \mathbb{R}^n_+, r > 0} \left( \frac{|B(x, r)|^{-1}}{B(x, r)} \int_{B(x, r)} w(y) y_\gamma^p dy \right) \times \left( \frac{|B(x, r)|^{-1}}{B(x, r)} \int_{B(x, r)} w^{-\frac{1}{p-1}}(y) y_\gamma^\gamma dy \right)^{p-1} < \infty,
$$

and $w$ belongs to $A_{1, \gamma}(\mathbb{R}^n_+)$, if there exists a positive constant $C$ such that for any $x \in \mathbb{R}^n_+$ and $r > 0$

$$
|B(x, r)|^{-1} \int_{B(x, r)} w(y) y_\gamma^\gamma dy \leq C \operatorname{ess sup}_{y \in B(x, r)} w(y).
$$

The properties of the class $A_{p, \gamma}(\mathbb{R}^n_+)$ are analogous to those of the Muckenhoupt classes. In particular, if $w \in A_{p, \gamma}(\mathbb{R}^n_+)$, then $w \in A_{p, \gamma}(\mathbb{R}^n_+)$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p, \gamma}(\mathbb{R}^n_+)$ for any $p > p_1$.

Note that, $|x|^{\alpha} \in A_{p, \gamma}(\mathbb{R}^n_+)$, $1 < p < \infty$, if and only if $-\frac{n+\gamma}{p} < \alpha < \frac{n+\gamma}{p}$ and $|x|^{\alpha} \in A_{1, \gamma}(\mathbb{R}^n_+)$, if and only if $-n - \gamma < \alpha \leq 0$.

For the $B$-maximal function (see [4, 5])

$$
M_{\gamma}f(x) = \sup_{r > 0} |B_r|^{-1} \int_{B_r} T^\gamma f(x) y_\gamma^\gamma dy
$$

the following analogue of Muckenhoupt theorem (see [7]) was proved by E.V. Guliyev in [3].

Theorem B. 1. If $f \in L_{1, w, \gamma}(\mathbb{R}^n_+)$, $w \in A_{1, \gamma}$, then $M_{\gamma}f \in WL_{1, w, \gamma}(\mathbb{R}^n_+)$ and

$$
\|M_{\gamma}f\|_{WL_{1, w, \gamma}} \leq C_{1, w, \gamma}\|f\|_{L_{1, w, \gamma}},
$$

where $C_{1, w, \gamma}$ depends only on $w$, $\gamma$ and $n$. 


2. If \( f \in L_{p,w,\gamma}(\mathbb{R}^n_+) \), \( w \in A_{p,\gamma} \), \( 1 < p < \infty \), then \( M_\gamma f \in L_{p,w,\gamma}(\mathbb{R}^n_+) \) and
\[
\|M_\gamma f\|_{L_{p,w,\gamma}} \leq C_{p,w,\gamma}\|f\|_{L_{p,w,\gamma}},
\]
where \( C_{p,w,\gamma} \) depends only on \( w \), \( p \), \( \gamma \) and \( n \).

2. Main result

Our main result is the following Stein-Weiss type theorem for the \( B \)-Riesz potential in the limiting case \( p = (n + \gamma)/(\alpha - \beta - \lambda) \). We prove that the modified \( B \)-Riesz potential operator \( \tilde{I}_\alpha \) is bounded from the space \( L_{p,|x|^{\beta},\gamma}(\mathbb{R}^n_+) \) to the weighted \( B \)-BMO space \( BMO_{|x|^{-\lambda},\gamma}(\mathbb{R}^n_+) \).

**Theorem 1.** Let \( 0 < \alpha < n + \gamma \), \( 1 < p = (n + \gamma)/(\alpha - \beta - \lambda) \), \( \beta < (n + \gamma)/p' \), \( \alpha \geq \beta + \lambda \geq 0 \). Then the operator \( \tilde{I}_\alpha \) is bounded from \( L_{p,|x|^{\beta},\gamma}(\mathbb{R}^n_+) \) to \( BMO_{|x|^{-\lambda},\gamma}(\mathbb{R}^n_+) \).

Moreover, for \( f \in L_{p,|x|^{\beta},\gamma}(\mathbb{R}^n_+) \) the integral \( I_{\alpha,\gamma}f \) exists almost everywhere, then \( I_{\alpha,\gamma} \in BMO_{|x|^{-\lambda},\gamma}(\mathbb{R}^n_+) \) and the following inequality is valid
\[
\|I_{\alpha,\gamma}f\|_{BMO_{|x|^{-\lambda},\gamma}} \leq C\|f\|_{L_{p,|x|^{\beta},\gamma}},
\]
where \( C > 0 \) is independent of \( f \).

**Proof.** Let \( f \in L_{p,|x|^{\beta},\gamma}(\mathbb{R}^n_+) \), \( 1 < p = (n + \gamma)/(\alpha - \beta - \lambda) \). For given \( t > 0 \) we denote
\[
f_1(x) = f(x)\chi_{B_{2t}}(x), \quad f_2(x) = f(x) - f_1(x),
\]
where \( \chi_{B_{2t}} \) is the characteristic function of the set \( B_{2t} \). Then
\[
\tilde{I}_{\alpha,\gamma}f(x) = \tilde{I}_{\alpha,\gamma}f_1(x) + \tilde{I}_{\alpha,\gamma}f_2(x) = F_1(x) + F_2(x),
\]
where
\[
F_1(x) = \int_{B_{2t}} \left( T_{y,|x|^{\alpha-n-\gamma}} - |y|^{\alpha-n-\gamma}\chi_{B_1}(y) \right) f(y)y_\alpha dy,
\]
\[
F_2(x) = \int_{\mathbb{R}^n} (y,|x|^{\alpha-n-\gamma}) f(y)y_\alpha dy.
\]
Note that the function \( f_1 \) has compact support and thus
\[
a_1 = -\int_{B_{2t}\setminus B_{\min(1,2t)}} |y|^{\alpha-n-\gamma} f(y)y_\alpha dy
\]
is finite.
Note also that
\[ F_1(x) - a_1 = \int_{B_2t} T^y|\alpha - n - \gamma f(y)y_n^\gamma dy - \int_{B_2t \setminus B_{\min(1,2t)}} |y|^{\alpha - n - \gamma} f(y)y_n^\gamma dy \]
\[ + \int_{B_2t \setminus B_{\min(1,2t)}} |y|^{\alpha - n - \gamma} f(y)y_n^\gamma dy = \int_{\mathbb{R}_t^n} T^y|\alpha - n - \gamma f_1(y)y_n^\gamma dy = I_{n,\gamma}f_1(x). \]

Therefore
\[ |F_1(x) - a_1| \leq \int_{\mathbb{R}_t^n} |y|^{\alpha - n - \gamma} |T^y f_1(x)| y_n^\gamma dy = \int_{B(x,2t)} |y|^{\alpha - n - \gamma} |T^y f(x)| y_n^\gamma dy. \]

Further, for \( x \in B_t, \ y \in B(x,2t) \) we have
\[ |y| \leq |x| + |x - y| < 3t. \]

Consequently, we have
\[ |F_1(x) - a_1| \leq \int_{B_{3t}} |y|^{\alpha - n - \gamma} |T^y f(x)| y_n^\gamma dy, \tag{5} \]
if \( x \in B_t. \)

By Theorem B and inequality (5), for \((\alpha - \beta - \lambda)p = n + \gamma\) we have
\[ t^{\alpha - n - \gamma - \lambda} \int_{B_t} |T^z F_1(x) - a_1| z_n^\gamma dz \]
\[ \leq Ct^{\alpha - n - \gamma - \lambda} \int_{B_{3t}} T^z \left( \int_{B_{3t}} |y|^{\alpha - n - \gamma} T^y |f(x)| y_n^\gamma dy \right) z_n^\gamma dz \]
\[ \leq Ct^{\alpha - n - \gamma - \lambda} \cdot t^{(n+\gamma)/p'} \left( \int_{B_t} T^z (M_\gamma(f(x)))^p z_n^\gamma dz \right)^{1/p} \]
\[ \leq Ct^\beta \left( \int_{B_t} T^z (M_\gamma(f(x)))^p z_n^\gamma dz \right)^{1/p} \leq C \left( \int_{B_t} |z|^{\beta p} T^z (M_\gamma(f(x)))^p z_n^\gamma dz \right)^{1/p} \]
\[ = C \left( \int_{\mathbb{R}_t^n} T^z \chi_{B_t} |x|^{\beta p} (M_\gamma(f(x)))^p z_n^\gamma dz \right)^{1/p} \]
\[ = C \left( \int_{\mathbb{R}_t^n} |z|^{\beta p} (M_\gamma(f(x)))^p z_n^\gamma dz \right)^{1/p} \leq C \|f\|_{L_p,|x|^{\beta,\gamma}}. \tag{6} \]

Denote
\[ a_2 = \int_{B_{\max(1,2t)} \setminus B_{2t}} |y|^{\alpha - n - \gamma} f(y)y_n^\gamma dy, \]
and estimate $|F_2(x) - a_2|$ for $x \in B_t$:

$$|F_2(x) - a_2| \leq \int_{B_{2t}} |f(y)| |T^y x|^{\alpha - n - \gamma} - |y|^{\alpha - n - \gamma} y_n^\gamma dy.$$ 

Applying Lemma 1 and Hölder’s inequality we get

$$|F_2(x) - a_2| \leq 2^{n+\gamma-\alpha+1}|x| \int_{B_{2t}} |f(y)||y|^{\alpha - n - \gamma - 1} y_n^\gamma dy$$

$$\leq 2^{n+\gamma-\alpha+1}|x| \left( \int_{B_t} |y|^\beta |f(y)|^{\beta} y_n^\gamma dy \right)^{1/p} \times \left( \int_{B_t} |y|^{(-\beta+\alpha-n-\gamma-1)p'} y_n^\gamma dy \right)^{1/p'}$$

$$\leq C|x|t^{\alpha-\beta-1-n-\gamma/p} \|f\|_{L_{\gamma,|x|^\beta,\gamma}} \leq C|x|t^{\lambda-1} \|f\|_{L_{\gamma,|x|^\beta,\gamma}} \leq C|x|^{\lambda} \|f\|_{L_{\gamma,|x|^\beta,\gamma}}.$$ 

Note that if $|x| \leq t$ and $|z| \leq 2t$, then $T^z |x| \leq |x| + |z| \leq 3t$. Thus for $(\alpha - \beta - \lambda)p = Q$ we obtain

$$|T^z F_2(x) - a_2| \leq T^z |F_2(x) - a_2| \leq C|x|^{\lambda} \|f\|_{L_{\gamma,|x|^\beta,\gamma}}.$$ 

(7)

Denote

$$a_f = a_1 + a_2 = \int_{B_{\max(1,2t)}} |y|^{\alpha - n - \gamma} f(y) y_n^\gamma dy.$$ 

Finally, from (6) and (7) we have

$$\sup_{x,t} t^{n-\gamma-\lambda} \int_{B_t} T^y \tilde{I}_{\alpha,\gamma} f(x) - a_f y_n^\gamma dy \leq C \|f\|_{L_{\gamma,|x|^\beta,\gamma}}.$$ 

Thus,

$$\|\tilde{I}_{\alpha,\gamma} f\|_{BMO_{|x|^{-\lambda,\gamma}}} \leq 2C \sup_{x,t} t^{n-\gamma-\lambda} \int_{B_t} T^y \tilde{I}_{\alpha,\gamma} f(x) - a_f y_n^\gamma dy \leq C \|f\|_{L_{\gamma,|x|^\beta,\gamma}}.$$ 

Thus Theorem 1 is proved.

\[\square\]

**Corollary 1.** ([4, 5]) Let $0 < \alpha < n + \gamma$, $1 < p = (n + \gamma)/\alpha$. Then the operator $\tilde{I}_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_+)$ to $BMO_{\gamma}(\mathbb{R}^n_+)$. Moreover, for $f \in L_{p,\gamma}(\mathbb{R}^n_+)$ the integral $I_{\alpha,\gamma} f$ exists almost everywhere, then $I_{\alpha,\gamma} f \in BMO_{\gamma}(\mathbb{R}^n_+)$ and the following inequality is valid

$$\|I_{\alpha,\gamma} f\|_{BMO_{\gamma}} \leq C \|f\|_{L_{p,\gamma}},$$

where $C > 0$ is independent of $f$. 


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