

\textbf{p–SEQUENTIAL SPACES AND CLEAVABILITY}

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\textsc{Abstract.} We consider some relations between \( p \)-sequential-like properties and cleavability of topological spaces. Under a special assumption we give an very easy proof of the following result of A.V. Arhangel’skii (the main result in \cite{1}): if a (countably) compact space \( X \) is cleavable over the class of sequential spaces, then \( X \) is also sequential.

All spaces in this paper are assumed to be Hausdorff. Recall some definitions that we shall use.

Let \( \mathcal{F} \) be a filter on \( \omega \). A sequence \((x_n : n \in \omega)\) in a space \( X \), \( \mathcal{F}\text{-converges} \) to a point \( x \) in \( X \) if for every neighbourhood \( U \) of \( x \), the set \( \{ n \in \omega : x_n \in U \} \) belongs to \( \mathcal{F} \) [2]. We shall consider \( p \)-sequential and \( p \)-Fréchet-Urysohn spaces for \( p \in \omega^* = \beta\omega \setminus \omega \). A space \( X \) is said to be \( p\text{-sequential} \) if for every non-closed subset \( A \) of \( X \) there exist a point \( x \in X \setminus A \) and a sequence \((x_n)\) in \( A \) which \( p\text{-converges} \) to \( x \). \( X \) is an \( FU(p)\text{-space} \) if for every \( A \subset X \) and every \( x \in \overline{A} \) there

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is a sequence \((x_n)\) in \(A\) which \(p\)-converges to \(x\) (see [10]; different generalizations of these notions were considered in [7, 8, 3, 4, 11]).

If \(\mathcal{P}\) is a class of topological spaces and \(\mathcal{M}\) is a class of (continuous) mappings, then a space \(X\) is said to be \(\mathcal{M}\)-cleavable (resp. \(\mathcal{M}\)-pointwise cleavable) over \(\mathcal{P}\) if for every \(A \subset X\) (resp. every \(x \in X\)) there exist \(Y \in \mathcal{P}\) and \(f \in \mathcal{M}\), \(f : X \to Y\), such that \(f(X) = Y\) and \(f^{-1}f(A) = A\) (resp. \(f^{-1}f(x) = \{x\}\)) (see [1, 9]).

**Definition 1.** Let \(p \in \omega^*\).

(a) ([2]) A space \(X\) is said to be \(p\)-compact provided every sequence in \(X\) has a \(p\)-limit point. If \(X\) is \(p\)-compact for every \(p \in \omega^*\) one says that \(X\) is ultracompact.

(b) ([5]) A space \(X\) is called \(p\)-closed if every \(p\)-compact subspace of \(X\) is closed.

It was remarked in [5] that if a space \(X\) admits a continuous bijection onto a \(p\)-closed space, then \(X\) is \(p\)-closed. We give the following (simple, but useful in what follows) generalization of this fact.

**Proposition 2.** If a space \(X\) is cleavable over the class \(\mathcal{K}\) of all \(p\)-closed spaces, then \(X\) is a \(p\)-closed space.

**Proof.** Let \(A\) be a \(p\)-compact subspace of \(X\). Choose a \(p\)-closed space \(Y\) and a continuous mapping \(f : X \to Y\) such that \(f^{-1}f(A) = A\). The set \(f(A)\) is \(p\)-compact in \(Y\) and thus it is closed. Then the set \(f^{-1}f(A)\) is closed in \(X\), i.e. \(X\) is a \(p\)-closed space. \(\square\)

**Proposition 3.** If a \(p\)-compact space \(X\) is cleavable over the class of \(p\)-closed spaces, then \(X\) is \(p\)-sequential.

**Proof.** By the previous proposition \(X\) is \(p\)-closed. But \(p\)-compact \(p\)-closed spaces are precisely \(p\)-sequential spaces [5]. \(\square\)

Every \(p\)-sequential space is \(p\)-closed. Therefore, we have this

**Corollary 4.** If a \(p\)-compact space \(X\) is cleavable over the class of \(p\)-sequential spaces, then \(X\) is \(p\)-sequential.
Theorem 5. If a compact space $X$ is cleavable over the class $\mathcal{K}$ of $\text{ccc}$ $p$-sequential spaces, then $X$ is weakly $FU(\omega^*)$-space (i.e. $X$ is a $FU(q)$-space for some $q \in \omega^*$).

Proof. By Corollary 4 $X$ is $p$-sequential, so its tightness is countable. On the other hand, every $Y \in \mathcal{K}$ has cardinality $\leq 2^\omega$ because $Y$ is a compact $p$-sequential space and for such spaces $Y$ we have $|Y| \leq 2^{c(Y)}$ [6, Th. 3]. Hence, $X$ is cleavable over a class of spaces having cardinality $\leq 2^\omega$. According to a known result [1, 9] the cardinality of $X$ is $\leq 2^\omega$. Theorem 3.12 in [3] guarantees now that there is a $q \in \omega^*$ for which $X$ is a $FU(q)$-space. $\square$

A similar result is the following one.

Theorem 6. If a separable $p$-compact space $X$ is cleavable over the class $\mathcal{K}$ of $p$-closed spaces, then $X$ is a weakly $FU(\omega^*)$-space (and $p$-sequential).

Proof. $X$ is a $p$-compact $p$-closed space (and so $p$-sequential). By the formula $|X| \leq d(X)^\omega$ for every $p$-compact $p$-closed space $X$, we conclude $|X| \leq 2^\omega$. Since $X$ is a $p$-sequential space, its tightness is countable. Again by Theorem 3.12 in [3] we have that $X$ is a weakly $FU(\omega^*)$-space. $\square$

Theorem 7. If a space $X$ is closed pointwise cleavable over the class of $FU(p)$-spaces, then $X$ is also a $FU(p)$-space.

Proof. Let $A$ be a subset of $X$ and $x \in \overline{A}$. Choose a $FU(p)$-space $Y$ and a closed continuous mapping $f : X \rightarrow Y$ such that $f^{-1}(y) = \{x\}$. There is a sequence $(y_n) \subset f(A)$ which $p$-converges to $f(x) \in \overline{f(A)} \setminus f(A)$. For every $n \in \omega$ take a point $x_n \in f^{-1}(y_n) \cap A$. Then the sequence $(x_n) \subset A$ $p$-converges to $x$. Indeed, let $U$ be any neighbourhood of $x$. Since $f$ is closed and $\{x\} = f^{-1}(y)$ there is a neighbourhood $V$ of $f(x)$ such that $f^{-1}(V) \subset U$. Because of $\{n \in \omega : y_n \in V\} \in p$ and $\{n \in \omega : x_n \in U\} \supset \{n \in \omega : y_n \in V\}$ we have that the set $\{n \in \omega : x_n \in U\} \in p$, i.e. $(x_n)$ $p$-converges to $x$. $\square$

We need now the following lemma.

Lemma 8. Every countably compact $p$-sequential space $X$ is $p$-compact.
Proof. Let \((x_n)\) be a sequence in \(X\). Since \(X\) is countably compact, there exists an accumulation point \(x\) of this sequence. The set \(A = \{x_n : n \in \omega\} \cup \{x\}\) is not closed as \(x \in \overline{A} \setminus A\). Since \(X\) is \(p\)-sequential, there is a sequence \((a_k) \subset A\), \(a_k = x_{n_k}\), which \(p\)-converges to a point \(y \in \overline{A} \setminus A\). This means that for every neighbourhood \(U\) of \(y\) the set \(\{n_k : a_k = x_{n_k} \in U\}\) belongs to \(p\). Clearly, then because of \(\{n \in \omega : x_n \in U\} \supset \{n_k : x_{n_k} \in U\}\) we have \(\{n \in \omega : x_n \in U\} \in p\). Therefore, \((x_n)\) \(p\)-converges to \(y\) and \(X\) is \(p\)-compact. \(\square\)

Recall that a space \(X\) is called \(\omega\)-bounded if the closure of every countable subset of \(X\) is compact.

Theorem 9. If a countably compact regular space \(X\) is closed pointwise cleavable over the class \(\mathcal{C}\) of Fréchet-Urysohn spaces, then \(X\) is \(\omega\)-bounded.

Proof. Every \(Y \in \mathcal{C}\) is a \(FU(p)\)-space for every \(p \in \omega^*\) [9]. By Theorem 7, \(X\) is also a \(FU(p)\)-space for every \(p \in \omega^*\). Therefore, by Lemma 8, \(X\) is \(p\)-compact for every \(p \in \omega^*\), i.e. \(X\) is ultracom pact. According to a result of Bernstein [1, Thms 3.4 and 3.5] (see also [12]) it follows that \(X\) is \(\omega\)-bounded. \(\square\)

The following theorem is a special case of Theorem 23 in [2] but with very easy proof. The Novak number \(n(X)\) of a space \(X\) is the smallest cardinality of a family of nowhere dense subsets of \(X\) covering \(X\).

Theorem 10 \((n(\omega^*) > c)\). If a ultracom pact space \(X\) is cleavable over the class \(\mathcal{K}\) of sequential spaces, then \(X\) is also sequential.

Proof. Every \(Y \in \mathcal{K}\) is a \(p\)-sequential space for each \(p \in \omega^*\), so that every \(Y \in \mathcal{K}\) is \(p\)-closed for each \(p \in \omega^*\). By Proposition 2 \(X\) is also \(p\)-closed for every \(p \in \omega^*\). Therefore, \(X\) is a \(p\)-closed \(p\)-compact space for all \(p \in \omega^*\), hence \(X\) is \(p\)-sequential for all \(p \in \omega^*\). By a result of Malykhin (which states that under \(n(\omega^*) > c\) a space is sequential if and only if it is \(p\)-sequential for every \(p \in \omega^*\); see Theorem 1.10 in [5]), this means that \(X\) is sequential. \(\square\)

Every ultracom pact space is compact, so that we have

Corollary 11 \((n(\omega^*) > c)\). If a compact space is cleavable over the class of sequential spaces, then \(X\) is also sequential.

We end by a question regarding \(p\)-compact spaces. Every \(\omega\)-bounded space is \(p\)-compact for every \(p \in \omega^*\). A.V. Arhangel’skii has remarked that if an
$\omega$-bounded space is cleavable over the class of spaces of countable tightness, then it itself has countable tightness [1]. So, the following question is natural.

**Question 12.** Let a $p$-compact space $X$ be cleavable over the class of Hausdorff spaces of countable tightness. Is the tightness of $X$ countable?

**REFERENCES**


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