Afriat's Theorem and Some Extensions to Choice under Uncertainty

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Abstract

The first part of the paper reviews the methodology developed by Sydney Afriat for determining whether a finite set of price and quantity data are consistent with utility maximizing behavior by a consumer. Some extensions of his basic model to models of consumer behavior where the structure of preferences is restricted in some way are also explained. Examples of special structures are homotheticity, separability and quasilinearity of the utility function. The second half of the paper is devoted to developing Afriat type consistency tests for expected and nonexpected utility maximizing behavior.

Keywords

Revealed preference theory, Afriat inequalities, nonparametric approach to demand theory, homotheticity, separability, quasilinearity, testing for maximizing behavior, choice under uncertainty, nonexpected utility, investing, insurance.

JEL Classification Numbers

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1. Introduction

Sydney Afriat has been working on index number theory and consumer demand theory for over 55 years and so it is appropriate that the Royal Economic Society recognize his contributions by having a plenary session on the foundations of revealed preference theory at its annual meeting in 2011.[2]

The seminal paper is Afriat (1967).[3] In this paper, Afriat showed how a finite set of price and quantity data pertaining to a consumer who might or might not be maximizing utility subject to a budget constraint for say T periods could be *tested* for its consistency with maximizing behavior. Afriat's test can be formulated as a problem where a certain set of inequalities needs to be satisfied in order for the data to be consistent. Moreover, if his test passed, the results of the testing procedure could be used in order to construct a utility function that would be consistent with the given data. Thus Afriat developed a *nonparametric approach* to the estimation of consumer preferences. Finally, Afriat showed that in the case where his test passed, the constructed utility function, which was consistent with the data, turned out to be a concave, increasing function even though these hypotheses were not required for consistent behavior.[4]

In the first half of this paper, we will review the essentials of Afriat's methodology and in later sections, we will develop some new applications of his basic approach to choice under uncertainty. Thus in section 2, we will present the basic Afriat methodology and in section 3, we will establish the necessity of the Afriat inequalities in the case where it is assumed that the consumer's utility function is concave.[5]

In the remainder of the paper, various additional restrictions on the decision maker's preferences are imposed and various tests for consistency with maximizing behavior are developed. Thus in section 4, the extra assumption that the consumer have homothetic preferences is added. In section 5, a test for additive separability is developed while in section 6, a test for a form of quasilinearity is presented.

It turns out that the assumption of quasilinearity is very useful in the context of estimating preferences when the decision maker faces uncertainty and so the remainder of the paper is devoted to decision making problems under uncertainty. Section 7 deals with estimating stochastic preferences when the *expected utility model* holds when there are only a finite number of states of nature and section 8 applies the general framework developed in section 7 to some simple applications to insurance and investing. Section 9 develops a consistency test for a *nonexpected utility model* where the stochastic preferences are assumed to be homothetic.

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[2] For an extensive bibliography of Afriat's contributions to economic theory, see Afriat (2011).

[3] For a preview of this paper, see Afriat (1961).

[4] Another very important contribution that Afriat made was to show that the existence of a solution to the Afriat inequalities is equivalent to the consistency of the revealed preference inequalities (which Afriat called cyclical consistency and Varian (1982) called the Generalized Axiom of Revealed Preference or GARP).

[5] This material is not new; it can be found in Afriat (1967; 75), Diewert and Parkan (1978) and Varian (1983a; 101).

Section 10 concludes with a few comments on violation measures if the basic consistency test is not satisfied and it also lists a few of the applications of the Afriat methodology to other areas of economics.

2. Afriat's Consistency Conditions for the Basic Utility Maximization Problem

How can one determine whether a finite body of price and quantity data is consistent with utility maximizing behavior? Afriat (1967) provided an answer and his basic methodology will now be explained.

Suppose we are given T strictly positive (N dimensional) price vectors, $p^t >> 0_N$ and T nonnegative, nonzero quantity vectors $x^t > 0_N$[6] for t = 1,...,T. Suppose x^t solves the following period t utility maximization problem for some utility function f:

(1) x^t solves $\max_x \{f(x) : p^t \cdot x \leq p^t \cdot x^t \,;\, x \geq 0_N\}$ for t = 1,...,T.

Suppose further that:

(2) f is continuous from above and is subject to local nonsatiation.[7]

Then there exist T nonnegative numbers $u^t \geq 0$ and T positive numbers $\lambda^t > 0$, t = 1,...,T such that the following inequalities are satisfied:[8]

(3) $u^s \leq u^t + \lambda^t p^t \cdot (x^s - x^t)$; s, t = 1,...,T.

Conversely, suppose $u^t \geq 0$ and $\lambda^t > 0$ exist such that the inequalities (3) are satisfied. Then the data p^t, x^t, t = 1,...,T, are consistent with the utility maximization hypothesis (1) using the following utility function f which can be constructed for $x \geq 0_N$ using the u^t and λ^t which satisfy (3):

(4) $f(x) \equiv \min_t \{f^t(x) : t = 1,...,T\}$

where the linear functions f^t are defined as follows:

(5) $f^t(x) \equiv u^t + \lambda^t p^t \cdot (x - x^t)$; t = 1,...,T.

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[6] Notation: $x \geq 0_N$ means the components of the N dimensional vector x are nonnegative, $x >> 0_N$ means that the components are positive and $x > 0_N$ means $x \geq 0_N$ and $x \neq 0_N$. If p and x are two N dimensional vectors, then $p \cdot x$ denotes the inner product $\sum_{n=1}^{N} p_n x_n$.

[7] These assumptions follow those made by Diewert (1973) in order to insure that solutions to the utility maximization problems existed and that the maximum utility occurred on the boundary of the budget set (rather than in the interior). Afriat (1967) made no assumptions whatsoever on the utility function.

[8] When s = t, the resulting inequality (3) is automatically satisfied so that these inequalities can be dropped from (3). Note also that if $x^t = x^s$ for any two observations t and s, then the inequalities in (3) will imply that u^t must equal u^s.

Proofs of this result may be found in Afriat (1967), Diewert (1973) and Varian (1982). The first part of Afriat's Theorem is the hard part. The converse part can easily be shown as follows.[9] Since f(x) defined by (4) is the minimum of the $f^s(x)$, we must have $f(x) \le f^s(x)$ for each s and thus for s = 1,...,T:

$$
\begin{aligned}
(6)\quad & \max_x \{f(x) : p^s\cdot x \le p^s\cdot x^s \,;\, x \ge 0_N\} \\
& \le \max_x \{f^s(x) : p^s\cdot x \le p^s\cdot x^s \,;\, x \ge 0_N\} \\
& \equiv \max_x \{ u^s + \lambda^s p^s\cdot(x - x^s) : p^s\cdot x \le p^s\cdot x^s \,;\, x \ge 0_N\} \\
& \qquad \text{using the definition of } f^s \\
& = u^s \quad \text{using } \lambda^s > 0.
\end{aligned}
$$

Also for s = 1,...,T, we have

$$
\begin{aligned}
(7)\ f(x^s) & \equiv \min_t \{f^t(x^s) : t = 1,...,T\} \quad \text{using the definition of f} \\
& = \min_t \{ u^t + \lambda^t p^t\cdot(x^s - x^t) : t = 1,...,T\} \\
& = u^s
\end{aligned}
$$

where the last equality follows using the Afriat inequalities (3). Thus for each s, x^s solves the period s utility maximization problem.

Note that the rationalizing f(x) defined by (4) is the minimum of a finite number of increasing linear functions of x and hence is a continuous, concave and increasing function of x. But we only assumed that f was continuous from above and was subject to local nonsatiation. Hence if the data set can be at all rationalized by a utility function satisfying minimal regularity conditions, it can be rationalized by a concave utility function.[10]

How can we check whether a solution to the Afriat inequalities (3) exists? Linear programming techniques cannot be immediately applied because we need the constraints to be weak inequalities or equalities (and not the strict inequalities $\lambda^t > 0$). However, looking at the inequalities (3), we see that they are homogeneous of degree one in the u^t and λ^t. Thus without loss of generality, we can impose the following constraint on the λ^t:

(8) $\lambda^i \ge 1$; i = 1,...,T.

Now solve the following linear program[11] with respect to the nonnegative variables s, s_{ij} for i,j = 1,...,T, λ^i and u^i for i = 1,...,T:

min s ≥ 0 subject to the T constraints (8) and the T^2 constraints (9):

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[9] This proof is due to Afriat (1967; 73-75).

[10] This amazing result was noted by Afriat (1967; 75) and emphasized by Diewert (1973; 423) and Varian (1982; 946).

[11] This linear program is due to Fleissig and Whitney (2005; 356) and it is a simplified version of Diewert's (1973; 421) linear program.

(9) $u^i = u^j + \lambda^j p^j \cdot (x^i - x^j) + s_{ij} - s$ $\qquad$ i, j = 1,...,T.

It can be seen that if the slack variable s can be driven down to 0 while satisfying the constraints (8) and (9), then the Afriat inequalities (3) will be satisfied and the data can be rationalized by a concave utility function. Conversely, if the optimized objective function for the above LP problem is positive, then the Afriat inequalities cannot be satisfied and the data cannot be rationalized by any utility function that exhibits local nonsatiation.

3. The Necessity of the Afriat Inequalities in the Concave Case

Since Afriat's Theorem tells us if the data can be rationalized, then they can be rationalized by an increasing concave utility function, why not assume the underlying utility function is increasing and concave right at the start? And while we are adding unnecessary assumptions about f, why not assume that it is differentiable as well? If we are willing to make these extra assumptions on f, then it is very easy to derive the necessity of the Afriat inequalities (3).

Thus assume f is concave, increasing[12] and differentiable over the positive orthant and consider the utility maximization problems in (1) above. Assume $x^t >> 0_N$ solves $\max_x \{f(x) : p^t \cdot x \le p^t \cdot x^t ; x \ge 0_N\}$ for t = 1,...,T. Then the Kuhn-Tucker conditions for these utility maximization problems (and our positivity assumptions) will imply that there will exist Lagrange multipliers λ^t such that the following conditions hold:[13]

(10) $\lambda^t > 0$; $\qquad$ t = 1,...,T;

(11) $\nabla f(x^t) = \lambda^t p^t$; $\qquad$ t = 1,...,T.

Now concave differentiable functions have the property that their first order Taylor series approximations are tangent to or lie above the graph of the function. Thus for each t, the following inequality is valid for each $x \ge 0_N$:

(12) $f(x) \le f(x^t) + \nabla f(x^t) \cdot (x - x^t)$; $\qquad$ t = 1,...,T.

Now define $u^t \equiv f(x^t)$ for t = 1,...,T, set $x = x^s$ in (12), substitute (11) into (12) and we obtain the Afriat inequalities:

(13) $u^s \le u^t + \lambda^t p^t \cdot (x^s - x^t)$; $\qquad$ s, t = 1,...,T.

This type of argument can be found in Afriat (1967; 75) and Diewert and Parkan (1978) (1985; 128-129) and Varian (1983a; 101). Varian and Diewert and Parkan used this "easy" method for deriving the necessity of the Afriat inequalities in more complicated settings involving separability or special structures for the utility function.

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12 The function f is *increasing* over the nonnegative orthant if $0_N \le x^1 < x^2$ implies $f(x^2) > f(x^1)$.

13 In order to obtain the positivity of the Kuhn-Tucker multipliers (1951) λ^t, it is necessary to use the fact that f(x) is an increasing and concave function of x. The assumption that $x^t >> 0_N$ can be replaced by $x^t > 0_N$ but the arguments become more complex and so in the interests of brevity, we assume strict positivity of the x^t.

The assumption that f be differentiable is not required in order to obtain the Afriat inequalities (13) if we draw on the concept of a supergradient. Thus the vector b is a *supergradient* to the function of N variables f defined over S at the point $x^0 \in S$ if and only if the following inequalities hold:

$$(14)\ f(x) \le f(x^0) + b\cdot(x - x^0) \qquad \text{for all } x\in S.$$

Note that the function on the right hand side of (13) is a linear function of x which takes on the value $f(x^0)$ when $x = x^0$. This linear function acts as an upper bounding function to f.

Rockafellar (1970; 217) showed that if f is a concave function defined over a convex subset S of R^N, then for every x^0 in the interior of S, f has at least one supergradient vector b^0 to f at the point x^0. Denote the set of all such supergradient vectors as $\partial f(x^0)$. Then Rockafellar also showed that $\partial f(x^0)$ is a nonempty, closed, bounded convex set.

Now assume that f is concave and increasing over the nonnegative orthant and consider the utility maximization problems in (1) above. Assume that $x^t >> 0_N$ solves $\max_x \{f(x) : p^t\cdot x \le p^t\cdot x^t ; x \ge 0_N\}$ for $t = 1,...,T$. Using the Uzawa (1958) Karlin (1959; 201-203) Saddle Point Theorem, we can deduce the existence of supergradient vectors ρ^t and positive[14] Lagrange multipliers $\lambda^t > 0$ such that

$$(15)\ \rho^t\in\partial f(x^t) ; \rho^t = \lambda^t p^t ; \qquad t = 1,...,T.$$

Since $\rho^t\in\partial f(x^t)$, using the definition of a supergradient (14), the following inequalities hold for all $x \ge 0_N$:

$$(16)\ f(x) \le f(x^t) + \rho^t\cdot(x - x^t) \qquad t = 1,...,T$$
$$= u^t + \lambda^t p^t\cdot(x - x^t)$$

where the last equality follows using the equations in conditions (15) and the definitions $u^t \equiv f(x^t)$ for $t = 1,...,T$. For $s = 1,...,T$, set $x = x^s$ in (16), and we have derived the necessity of the Afriat inequalities (3). The sufficiency of the Afriat inequalities (3) and (10) follows in the usual way.

In the following sections, we will generally assume that the preference function is concave and differentiable but it should be understood that the differentiability assumption is not required: first order necessary conditions like (11) can be replaced by the more general supergradient conditions (15).

4. The Afriat Inequalities when Preferences are Homothetic

[14] It is necessary to use the fact that f(x) is an increasing function of x in order to establish the positivity of the λ^t. The assumption that $p^t\cdot x^t > 0$ implies that the Slater (1950) constraint qualification condition is satisfied.

Homothetic preferences can be represented by a linearly homogenous utility function so we will assume that the f in (1) above is linearly homogeneous.[15] In addition to the linear homogeneity assumption, we will assume that f is a continuous, increasing, concave function defined over the nonnegative orthant.[16] If we also assume that f is differentiable,[17] then Euler's Theorem on homogenous functions implies that the following equalities hold:

(17) $f(x^t) = \nabla f(x^t)\cdot x^t$; $t = 1,...,T$.

The above equalities mean that the first order Taylor approximations to f around x^t pass through the origin. Now substitute the first order conditions (11) into (17) and we obtain the following equalities:

(18) $u^t = f(x^t) = \nabla f(x^t)\cdot x^t = \lambda^t p^t\cdot x^t$; $\quad t = 1,...,T$.

Thus the u^t and the λ^t are linearly related (and if the price data are normalized so that $p^t\cdot x^t = 1$ for $t = 1,...,T$, then $u^t = \lambda^t > 0$ for $t = 1,...,T$) and so either the u^t or the λ^t can be replaced in the Afriat inequalities (3) or (13) and we now have to satisfy these inequalities with only T free variables rather than the previous 2T variables. Hence the test for homothetic utility maximization is more stringent (and can be rejected more easily). This test for homothetic utility maximization was noted by Diewert (1973; 424) and Varian (1983; 102-103). Afriat (1972a; 35-42), (1977) (1981) and Varian (1983a; 103-104) provide other equivalent tests of homothetic utility maximization.

Using equations (18) to solve for the λ^t in terms of the u^t and then eliminating the λ^t from equations (3) leads to the following equations:[18]

(19) $u^s \le u^t p^t\cdot x^s/p^t\cdot x^t$; $\quad s, t = 1,...,T$;

[15] The function f defined over the nonnegative orthant is (positively) linearly homogeneous if $x \ge 0_N$, $\lambda \ge 0$ implies $f(\lambda x) = \lambda f(x)$. Note that this definition implies that $f(0_N) = 0$. A homothetic function has the form $g[f(x)]$ where $g(z)$ is an increasing, continuous function of one variable and f is positively linearly homogeneous. The concept of homotheticity is due to Shephard (1953; 4).

[16] The concavity assumption on f can be replaced by the assumption of quasiconcavity since using a result due to Berge (1963; 208), if f is a positive, linearly homogeneous and quasiconcave function defined over the positive orthant, then f is also concave over this domain of definition. Fenchel (1953; 74) showed that a concave function is continuous over the interior of its domain of definition and the Fenchel (1953; 78) closure operation can be used to extend the definition of the concave function to the closure of its domain of definition. The results of Gale, Klee and Rockafellar (1968) and Rockafellar (1970; 85) can be used to show that this extension is continuous. Thus the assumption that f be continuous simply rules out discontinuities on the boundary of the nonnegative orthant.

[17] If f is not differentiable, then $\nabla f(x^t)$ in (17) and (18) must be replaced by a supergradient vector $\rho^t \in \partial f(x^t)$ for $t = 1,...,T$. Using the definition of a supergradient vector (14), it can be seen that $f(x) \le f(x^t) + \rho^t\cdot(x - x^t)$ for all $x \ge 0_N$. Since f is linearly homogeneous, $f(\lambda x^t) = \lambda f(x^t)$ for all $\lambda > 0$. Now replace x by λx^t in the above supergradient inequality and we obtain the inequality $\lambda f(x^t) \le f(x^t) + \rho^t\cdot(\lambda x^t - x^t)$ or $(\lambda-1)f(x^t) \le (\lambda-1)\rho^t\cdot x^t$ for all $\lambda \ge 0$. Choosing λ below 1 and above 1 leads to the equalities $f(x^t) = \rho^t\cdot x^t$ for $t = 1,...,T$. Thus even in the nondifferentiable case, we obtain equations (18) with $\nabla f(x^t)$ replaced with ρ^t.

[18] Recall that our assumptions that $p^t >> 0_N$ and $x^s >> 0_N$ for all s and t imply that $p^t\cdot x^s > 0$

(20) $u^t > 0$; $t = 1,...,T.$

The inequalities (20) follow from equations (18), $p^t \cdot x^t > 0$ and $\lambda^t > 0$. The conditions (19) and (20) are necessary and sufficient for homothetic utility maximization under our regularity conditions on the utility function f.

Note that the inequalities (19) and (20) are homogeneous in the u^t; i.e., if u^{t*} for t = 1,...,T satisfies (19) and (20), then so do λu^{t*} for any $\lambda > 0$. Thus we can impose an arbitrary normalization on solutions to (19) and (20) such as:

(21) $u^1 = u^{1*} > 0$

where u^{1*} is an arbitrary positive number. If this normalization is imposed, then the strict inequalities in (20) can be replaced by the following weak inequalities:[19]

(22) $u^t \geq 0$; $t = 1,...T.$

Thus conditions (19), (21) and (22) are necessary and sufficient to imply homothetic utility maximizing behavior under our regularity conditions on the utility function f.

A linear program can be set up to test whether the inequalities (19), (21) and (22) hold and if they do hold, the set of u^t that satisfy the inequalities can be used in order to construct a linearly homogeneous utility function f that rationalizes the data using definitions (4) and (5) above.[20] A linear program which will do the task of testing the data is the following one:

(23) $\min_{z, z_{st}, u^t} \{z: z \geq 0; z_{st} \geq 0 \text{ for } s, t = 1,...,T; u^t \geq 0 \text{ for } t = 1,...,T; u^1 = u^{1*} > 0 \text{ and } (24)\}$

where the linear constraints (24) are defined as follows:

(24) $u^s = u^t p^t \cdot x^s / p^t \cdot x^t + z_{st} - z$; $s, t = 1,...,T; s \neq t.$

If the objective function in (23) attains its lower bound of 0, then the data are consistent with homothetic utility maximization and the solution values for the u^t can be used to construct an explicit utility function f that rationalizes the data.[21] If the optimal objective function for the linear program (23) is positive, then the data cannot be rationalized by a homothetic, quasiconcave utility function.

5. Testing for Additive Separability

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[19] Suppose (19) and (21) are satisfied. Then let s = 1 in (19) and using (21), we have $0 < u^1 \leq u^t p^t \cdot x^1 / p^t \cdot x^t$ for t = 2,3,...,T, which in turn implies that all u^t are positive.

[20] The functions $f^t(x)$ defined by (5) now simplify to $f^t(x) \equiv u^t p^t \cdot x / p^t \cdot x^t$ for t = 1,...,T.

[21] If we denote the u^t solution values to (23) by u^{t*}, then the rationalizing f is defined by $f(x) \equiv \min_t \{u^{t*} p^t \cdot x / p^t \cdot x^t : t = 1,...,T\}$.

We follow Varian's (1983a; 107-108) exposition here (with an extension to the nondifferentiable case). Let x and y be two consumption vectors and suppose the decision maker is maximizing the *additive utility function* u(x) + v(y). The *utility maximization hypothesis* is now the following one: the observed period t quantity vectors x^t and y^t solve the following period t utility maximization problem, where the observed price vectors are p^t and q^t:

(25) x^t, y^t solves $\max_{x,y} \{u(x) + v(y) : p^t\cdot x + q^t\cdot y \leq p^t\cdot x^t + q^t\cdot y^t\}$ for t = 1,...,T.

Assume that u(x) and v(y) are continuous, concave, increasing and differentiable functions, defined for $x \geq 0_N$ and $y \geq 0_M$ respectively. Assume as well that the price vectors p^t and q^t and the quantity vectors x^t and y^t are all strictly positive vectors.

Using the differentiability of u and x, if x^t, y^t solves the period t utility maximization problem in (25), then the following first order conditions must be satisfied:

(26) $\lambda^t > 0$; t = 1,...,T;
(27) $\nabla u(x^t) = \lambda^t p^t$; $\nabla v(y^t) = \lambda^t q^t$; t = 1,...,T.

The concavity property of u(x) implies that for any $x \geq 0_N$, the following inequality will hold for t = 1,...,T:

(28) $$u(x) \leq u(x^t) + \nabla u(x^t)\cdot(x - x^t)$$
$$= u(x^t) + \lambda^t p^t\cdot(x - x^t)$$

where the equality follows using (27). Define $u^t \equiv u(x^t)$ for t = 1,...,T. Let $x = x^s$ and substitute this value of x into the t^{th} inequality in (28) and we obtain the following system of inequalities:

(29) $u^s \leq u^t + \lambda^t p^t\cdot(x^s - x^t)$; s, t = 1,...,T.

Define $v^t \equiv v(y^t)$ for t = 1,...,T. Using the concavity of v(y) and using the same method of proof that established (29), we can deduce that the following inequalities must hold:

(30) $v^s \leq v^t + \lambda^t q^t\cdot(y^s - y^t)$; s, t = 1,...,T.

It can be seen that if (25) is satisfied, then it must be the case, under our regularity conditions on the functions u(x) and v(y), that there must exist numbers $u^1,...,u^T$, $v^1,...,v^t$ and $\lambda^1,...,\lambda^T$ such that the inequalities (26), (29) and (30) hold. Thus these conditions are necessary for the additive utility maximization hypothesis under our regularity conditions.[22]

[22] As usual, the differentiability assumptions on u(x) and v(y) are not required: conditions (27) can be replaced by the following supergradient conditions: $\rho^t \in \partial u(x^t)$, $\rho^t = \lambda^t p^t$, $\sigma^t \in \partial v(y^t)$, $\sigma^t = \lambda^t q^t$, t = 1,...,T.

It is straightforward to show that the existence of a solution to the inequalities (26), (29) and (30) is also sufficient to imply the existence of functions u(x) and v(y) such that the observed data are consistent with the additive utility maximization hypothesis (25) as we shall now show.

Suppose a solution $u^1,...,u^T$, $v^1,...,v^t$ and $\lambda^1,...,\lambda^T$ to the inequalities (26), (29) and (30) exists. Define the functions $u^t(x)$, $u(x)$, $v^t(y)$ and $v(y)$ for $x \geq 0_N$ and $y \geq 0_M$ as follows:

$$(31)\ u^t(x) \equiv u^t + \lambda^t p^t \cdot (x - x^t)\ ; \qquad t = 1,...,T;$$
$$(32)\ u(x) \equiv \min_t \{u^t(x) : t = 1,...,T\}\ ;$$
$$(33)\ v^t(y) \equiv v^t + \lambda^t q^t \cdot (y - y^t)\ ; \qquad t = 1,...,T;$$
$$(34)\ v(y) \equiv \min_t \{v^t(y) : t = 1,...,T\}.$$

Since the scalars u^t satisfy the inequalities (29) and the scalars v^t satisfy the inequalities (30), using definitions (31)-(34), it can be seen that the following equalities hold:

$$(35)\ u(x^s) = \min_t \{u^t + \lambda^t p^t \cdot (x^s - x^t) : t = 1,...,T\} = u^s\ ; \qquad s = 1,...,T;$$
$$(36)\ v(y^s) = \min_t \{v^t + \lambda^t q^t \cdot (y^s - y^t) : t = 1,...,T\} = v^s\ ; \qquad s = 1,...,T.$$

We now show that x^t, y^t solves the period t utility maximization problem in (25) where the functions u(x) and v(y) are defined by (32) and (34) for t = 1,...,T:

$$\begin{aligned}(37)\ &\max_{x,y} \{u(x) + v(y) : p^t \cdot x + q^t \cdot y \leq p^t \cdot x^t + q^t \cdot y^t\} \\ &\leq \max_{x,y} \{ u^t + \lambda^t p^t \cdot (x - x^t) + v^t + \lambda^t q^t \cdot (y - y^t) : p^t \cdot x + q^t \cdot y \leq p^t \cdot x^t + q^t \cdot y^t\} \\ &\qquad \text{since } u(x) \leq u^t + \lambda^t p^t \cdot (x - x^t) \text{ and } v(y) \leq v^t + \lambda^t q^t \cdot (y - y^t) \\ &= \max_{x,y} \{ u^t + v^t + \lambda^t [p^t \cdot (x - x^t) + q^t \cdot (y - y^t)] : p^t \cdot x + q^t \cdot y \leq p^t \cdot x^t + q^t \cdot y^t\} \\ &\leq u^t + v^t\end{aligned}$$

where the last inequality follows from $\lambda^t > 0$ and $p^t \cdot (x - x^t) + q^t \cdot (y - y^t) \leq 0$. Thus $u^t + v^t$ is an upper bound to utility for the period t utility maximization problem using the u and v defined by (32) and (34). But (35) and (36) show that this upper bound is attained for $x = x^t$ and $y = y^t$ and so x^t, y^t solves the period t utility maximization problem for t = 1,...,T using the u(x) and v(y) defined by (32) and (34).

Note that the inequalities (26), (29) and (30) are homogeneous in the u^t, v^t and λ^t. Thus a normalization like (8) can be imposed on the λ^t without loss of generality. It can also be seen that the u^t and v^t can be restricted to be nonnegative without loss of generality; i.e., if $u^1,...,u^T$, $v^1,...,v^t$ and $\lambda^1,...,\lambda^T$ satisfy (8), (29) and (30), then $u^1+\lambda,...,u^T+\lambda$, $v^1+\mu,...,v^t+\mu$ and $\lambda^1,...,\lambda^T$ will also satisfy (8), (29) and (30) where λ and μ are arbitrary numbers.

As in the previous two sections, a linear program can be set up to test whether the inequalities (8), (29) and (30) hold and if they do hold, the set of u^t, v^t and λ^t that satisfy the inequalities can be used in order to construct utility functions u(x) and v(y) that rationalize the data using definitions (31)-(34) above.

The extension to more than two additive groups is straightforward.

The material in this section can be viewed as a method for testing a finite data set for consistency with maximizing an additively separable utility function.[23] For additional tests for other types of separable structures, see Varian (1983a; 104-106), Diewert and Parkan (1978) (1985) and Fleissig and Whitney (2007) (2008).

6. Testing for Generalized Quasilinearity

In this section, we consider the maximization of utility functions of the following form:[24]

$$(38)\quad F(x_1,x_2,...,x_K) = \sum_{k=1}^{K} \alpha_k f(x_k)$$

where $f(x)$ is a function of N variables, $x \equiv [x_1,...,x_N]$, defined for $x \geq 0_N$ and the scalars α_k satisfy the following positivity restrictions:

$$(39)\quad \alpha_k > 0\ ; \qquad k = 1,...,K.$$

We assume that f is a differentiable, increasing, continuous and concave function of N variables defined over the nonnegative orthant.[25]

The *utility maximization hypothesis* is now the following one:[26] the observed period t quantity vector x^t solves the following period t utility maximization problem, where the observed price and quantity vectors are $p^t \equiv [p_1^t,...,p_K^t]$ and $x^t \equiv [x_1^t,...,x_K^t]$:[27]

$$(40)\quad x^t \text{ solves } \max_x \{\sum_{k=1}^{K} \alpha_k f(x_k) : \sum_{k=1}^{K} p_k^t \cdot x_k \leq \sum_{k=1}^{K} p_k^t \cdot x_k^t\} \text{ for } t = 1,...,T.$$

The utility function f which appears in (40) is assumed to be defined over the nonnegative orthant, to be continuous, increasing and concave and for the moment, differentiable. Using the differentiability of f, if x^t solves the period t utility maximization problem in (40), then the following first order conditions must be satisfied:

$$(41)\quad \lambda^t > 0\ ; \qquad t = 1,...,T;$$
$$(42)\quad \alpha_k \nabla f(x_k^t) = \lambda^t p_k^t\ ; \qquad k = 1,...,K;\ t = 1,...,T.$$

23 See Blackorby, Primont and Russell (1978) for a comprehensive treatment of separability.

24 Browning (1989) considered a special case of this model (where all of the α_k were equal) and also used revealed preference techniques to test the consistency of a data set.

25 If N = 1 and f(x) is a strictly monotonic function of one variable, Eichhorn (1978; 32) defined $f^{-1}(\sum_{k=1}^{K} \alpha_k f(x_k))$ to be a *quasilinear function*. If N = 1, the α_k are positive and sum to one and f is continuous strictly monotonic function of one variable, Hardy, Littlewood and Polya (1934; 65) called $f^{-1}(\sum_{k=1}^{K} \alpha_k f(x_k))$ a *general mean formed with the function f*. If N = 1 and f is monotonically increasing and continuous, then maximizing $f^{-1}(\sum_{k=1}^{K} \alpha_k f(x_k))$ is equivalent to maximizing $\sum_{k=1}^{K} \alpha_k f(x_k)$.

26 Problems of this type occur in the context of intertemporal utility maximization.

27 Each p_k^t and x_k^t is assumed to be an N dimensional strictly positive row vector for t = 1,...,T and k = 1,...,K.

The concavity property of f(x) implies that for any $x \geq 0_N$, the following inequality will hold for k = 1,...,K and t = 1,...,T:

$$(43)\ f(x) \leq f(x_k^t) + \nabla f(x_k^t)\cdot(x - x_k^t)$$
$$= f(x_k^t) + \alpha_k^{-1}\lambda^t p_k^t\cdot(x - x_k^t)$$

where the equality follows using (42). Define $u_k^t \equiv f(x_k^t)$ for k = 1,...,K and t = 1,...,T. Let $x = x_j^s$ and substitute this value of x into the inequality (43) for t and k and we obtain the following system of K^2T^2 inequalities:

$$(44)\ u_j^s \leq u_k^t + \alpha_k^{-1}\lambda^t p_k^t\cdot(x_j^s - x_k^t)\ ; \qquad j,k = 1,...,K;\ s, t = 1,...,T.$$

Thus under our regularity conditions on f, there must exist KT numbers u_k^t for k = 1,...,K and t = 1,...,T, $\lambda^1,...,\lambda^T$ and $\alpha_1,...,\alpha_K$ such that the inequalities (39), (41) and (44) hold. Thus these conditions are necessary for the quasilinear utility maximization hypothesis under our regularity conditions.[28]

It is straightforward to show that the existence of a solution to the inequalities (39), (41) and (44) is also sufficient to imply the existence of a function f(x) such that the observed data are consistent with the quasilinear utility maximization hypothesis (40) as we shall now show.

Suppose a solution u_k^t for k = 1,...k and t = 1,...,T, $\alpha_1,...,\alpha_K$ and $\lambda^1,...,\lambda^T$ to the inequalities (39), (41) and (44) exists. Define f(x) and the KT functions $f_k^t(x)$ for $x \geq 0_N$ as follows:

$$(45)\ f_k^t(x) \equiv u_k^t + \alpha_k^{-1}\lambda^t p_k^t\cdot(x - x_k^t)\ ; \qquad k = 1,...,K\ ;\ t = 1,...,T;$$
$$(46)\ f(x) \equiv \min_{t,k}\ \{f_k^t(x) : k = 1,...,K;\ t = 1,...,T\}.$$

Note that f(x) is a continuous, concave and increasing function of N variables x. Since the scalars u_k^t satisfy the inequalities (44), using definitions (45) and (46), it can be seen that the following equalities hold:

$$(47)\ f(x_j^s) = \min_{t,k}\ \{u_k^t + \alpha_k^{-1}\lambda^t p_k^t\cdot(x_j^s - x_k^t) : k = 1,...,K;\ t = 1,...,T\}$$
$$= u_j^s\ ; \qquad j = 1,...,K\ ;\ s = 1,...,T.$$

We now show that for t = 1,...,T, $x^t \equiv [x_1^t,x_2^t,...,x_K^t]$ solves the period t utility maximization problem in (40) where the function f(x) is defined by (45) and (46):

$$(48)\ \max_x\{\textstyle\sum_{k=1}^K \alpha_k f(x_k) : \sum_{k=1}^K p_k^t\cdot x_k \leq \sum_{k=1}^K p_k^t\cdot x_k^t\}$$
$$\leq \max_x\ \{\textstyle\sum_{k=1}^K \alpha_k[u_k^t + \alpha_k^{-1}\lambda^t p_k^t\cdot(x - x_k^t)] : \sum_{k=1}^K p_k^t\cdot x_k \leq \sum_{k=1}^K p_k^t\cdot x_k^t\}$$
$$\text{since } f(x_k) \leq u_k^t + \alpha_k^{-1}\lambda^t p_k^t\cdot(x - x_k^t) \text{ using (45) and (46)}$$
$$= \max_x\ \{\textstyle\sum_{k=1}^K \alpha_k u_k^t + \sum_{k=1}^K \lambda^t p_k^t\cdot(x - x_k^t) : \sum_{k=1}^K p_k^t\cdot x_k \leq \sum_{k=1}^K p_k^t\cdot x_k^t\}$$
$$\leq \textstyle\sum_{k=1}^K \alpha_k u_k^t$$

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[28] As usual, the differentiability assumption on f(x) is not required: conditions (42) can be replaced by the following supergradient conditions: $\rho_k^t \in \partial f(x_k^t)$, $\rho_k^t = \alpha_k^{-1}\lambda^t p_k^t$, k = 1,...,K; t = 1,...,T.

where the last inequality follows from $\lambda^t > 0$ and $\sum_{k=1}^{K} p_k^t\cdot(x_k - x_k^t) \leq 0$. Thus $\sum_{k=1}^{K} \alpha_k u_k^t$ is an upper bound to utility for the period t utility maximization problem using the f defined by (45) and (46). But the equalities (47) show that this upper bound is attained for $x_k = x_k^t$ for $k = 1,...,K$ and so $x^t \equiv [x_1^t,x_2^t,...,x_K^t]$ solves the period t utility maximization problem for $t = 1,...,T$ using the $f(x)$ defined by (45) and (46).

Suppose that the quasilinear utility maximization hypothesis (40) is satisfied for some function f. Then it can be seen that if we replace $f(z)$ by $a + bf(z) \equiv g(z)$ where $b > 0$, then the data will also satisfy (40) where f is replaced by g.

If the α_k are known, then the inequalities (44) are standard Afriat type linear inequalities and we can set up a linear program to check whether these conditions (39), (41) and (44) are satisfied or not for a given set of data. As usual, we can replace the positivity restrictions (41) with the inequalities $\lambda^t \geq 1$ for $t = 1,...,T$ and the u_k^t can be restricted to be nonnegative; see the last paragraph above.

In the case where the α_k are not known, then the weak inequality restrictions (44) are nonlinear and so nonlinear programming methods will have to be used in order to determine whether a solution to (39), (41) and (44) exists. If a solution α_k^*, λ^{t*}, u_k^{t*} does exist, then it can be seen that by scaling the α_k^* and λ^{t*} by an arbitrary positive number μ, we can obtain u_k^{t*}, $\mu\alpha_k^*$ and $\mu\lambda^{t*}$ as another solution to (39), (41) and (44). In fact, it can be seen that not all of the α_k can be identified and so we need to set at least one of them equal to a positive constant, say $\alpha_1 = 1$. Thus at least one additional normalization on the α_k and λ^t can be added to (39), (41) and (44) without loss of generality and we can replace the strict inequalities (41) by the following weak inequalities:

(49) $\lambda^t \geq 1$; $\quad t = 1,...,T.$

It can be seen that if a solution α_k^*, λ^{t*}, u_k^{t*} to (39), (44) and (49) exists, then α_k^*, λ^{t*}, $u_k^{t*} + \mu$ will also be a solution to (39), (44) and (49) where μ is an arbitrary scalar. Hence in addition to the inequalities (39), (44) and (49), we can also add the following nonnegativity conditions on the u_k^t without loss of generality:

(50) $u_k^t \geq 0$; $\quad k = 1,...,K ; t = 1,...,T.$

Let the slack variables z and z_k^t for $k = 1,...,K$ and $t = 1,...,T$ be nonnegative and rewrite the K^2T^2 inequality restrictions (44) in the following equality format:

(51) $u_j^s = u_k^t + \alpha_k^{-1}\lambda^t p_k^t\cdot(x_j^s - x_k^t) + z_k^t - z$; $\quad j,k = 1,...,K;\ s, t = 1,...,T.$

Thus it appears that a nonlinear program that could be used to test the consistency of the data with quasilinear utility maximization is the problem of minimizing $z \geq 0$ with respect to the nonnegative variables z, $\lambda^1,...,\lambda^T$, $\alpha_1,...,\alpha_K$ and u_k^t and z_k^t for $k = 1,...,K$ and $t = 1,...,T$ subject to the restrictions (49) and (51). If the optimal $z^* = 0$, then the data are consistent but if $z^* > 0$, then the hypothesis of quasilinear utility maximization (40) is rejected where f(x) is assumed to be a continuous, concave and increasing function of N variables. However, this nonlinear program will not do the job of testing for consistency. The problem is this: there is nothing in the constraints to rule out the α_k from approaching plus infinity and thus $z = 0$, $\alpha_k = +\infty$ for $k = 1,...,K$, $\lambda^t = 1$ for $t = 1,...,T$ and $u_k^t = z_k^t = 0$ for $k = 1,...,K$ and $t = 1,...,T$ will be a solution to the programming problem for arbitrary price and quantity data, p_k^t and x_k^t. This is not a finite solution to (44) and (49). However, this infinite solution will be ruled out if we impose a normalization on the α_k such as:

(52) $\alpha_1 = 1$.

It appears that it will be necessary to bound the positive α_k from above and below in order to obtain a useful nonlinear program to test the quasilinear utility maximization hypothesis (40) when the α_k are not (completely) known. Thus assume that there are 2K – 2 numbers a_k and b_k are such that $0 < a_k \leq b_k$ for $k = 2,3,...,K$ and the α_k satisfy the following bounds:

(53) $a_k \leq \alpha_k \leq b_k$; $\qquad$ $k = 2,3,...,K$.

A nonlinear program that can be used to test the consistency of the data with quasilinear utility maximization, where the α_k satisfy (52) and (53) is the problem of minimizing $z \geq 0$ with respect to the nonnegative variables z, $\lambda^1,...,\lambda^T$, $\alpha_1,...,\alpha_K$ and u_k^t and z_k^t for $k = 1,...,K$ and $t = 1,...,T$ subject to the restrictions (49)-(53). If the optimal $z^* = 0$, then the data are consistent but if $z^* > 0$, then the hypothesis of (strongly increasing) quasilinear utility maximization (40) is rejected where f(x) is assumed to be a continuous, concave and increasing function of N variables and in addition, the α_k satisfy (52) and (53).

It can be seen that testing a data set for consistency with quasilinear utility maximization when the α_k are not completely known is much more difficult than testing for consistency when the α_k are known.

In the following section, we will modify the model used in this section by assuming that the α_k are known but they can vary as the period t changes. It turns out that this modified framework is very useful when studying choice under uncertainty.

7. Application to Expected Utility Maximization

Assume that a decision maker has a continuous and increasing utility function, f(x), that is applicable when there is no uncertainty. Now assume that there are K states of nature. Denote the consumption vector of the decision maker if state k occurs by x_k. Then following Arrow (1951) (1964) and Debreu (1959; 101), it is natural to assume that the decision maker has preferences over the state contingent commodities that can be represented by a continuous and increasing utility function $F(x_1,...,x_K)$. Following Samuelson (1952; 674), it is also natural to assume that the state contingent utility function F has the following structure:

(54) $F(x_1,...,x_K) = M[f(x_1),...,f(x_K)]$

where f is the *certainty utility function* and M is the *stochastic utility function* that describes the decision maker's attitude towards risk. If the state contingent consumption vectors are all equal, so that $x_1 = ... = x_K = x$, then it is natural to require that M have the following property:

(55) $M[f(x),...,f(x)] = f(x)$.

But property (55) implies that the stochastic preference function M is a mean function.[29]

Blackorby, Davidson and Donaldson (1977; 352-354) and Diewert (1993; 402-404) assumed that M satisfied various separability properties and under their assumptions, they were able to establish the existence of an increasing, continuous function of one variable, ϕ, such that M has the following *quasilinear representation*:

(56) $M[f(x_1),...,f(x_K)] = \phi^{-1}\{\sum_{k=1}^{K} \alpha_k\phi[f(x_k)]\}$

where $\alpha_k > 0$ is the probability that state k will occur for $k = 1,...,K$. It is assumed that $\sum_{k=1}^{K} \alpha_k = 1$. Since ϕ^{-1} is also a continuous, monotonically increasing function of one variable, it can be seen that maximizing $M[f(x_1),...,f(x_K)]$ is equivalent to maximizing $\sum_{k=1}^{K} \alpha_k\phi[f(x_k)]$ and the latter expression can be interpreted as *expected utility*.[30]

For many applications of the expected utility maximization model (see the following section), the certainty utility function f can be taken to be a function of one variable and it can be normalized so that $f(x) \equiv x$. We will make this simplification in the remainder of this section.

We assume that we can observe the prices and quantities that pertain to a decision maker over T periods as usual. Our *expected utility maximization hypothesis* is the following one: the observed period t quantity vector x^t solves the following period t expected utility maximization problem, where the observed price and quantity vectors are the vectors $p^t \equiv [p_1^t,...,p_K^t] >> 0_K$ and $x^t \equiv [x_1^t,...,x_K^t] >> 0_K$:

(57) x^t solves $\max_x \{\sum_{k=1}^{K} \alpha_k^t\phi(x_k) : \sum_{k=1}^{K} p_k^t x_k \leq \sum_{k=1}^{K} p_k^t x_k^t\}$ for $t = 1,...,T$

where the period t probabilities for state k, α_k^t, are also known and satisfy the following restrictions:

(58) $\alpha_k^t > 0$; $\quad k = 1,...,K;\ t = 1,...,T;$
(59) $\sum_{k=1}^{K} \alpha_k^t = 1$; $\quad t = 1,...,T.$

[29] See Diewert (1993) for properties of mean functions.
[30] There are many alternative ways of deriving the expected utility maximization model.

Varian (1983b) considered a variant of this model and used the Afriat methodology in much the same way as will be done below.[31] Special cases of this framework have been considered in the experimental economics literature and Afriat type tests for consistency have been derived; see Choi, Fisman, Gale and Kariv (2007a) (2007b) and the references in these papers.

We assume that the function $\phi(z)$ is a continuous, concave,[32] and increasing function of one variable. We temporarily assume that ϕ is differentiable. Using the differentiability of ϕ, if $x^t >> 0_K$ solves the period t utility maximization problem in (57), then the following first order conditions must be satisfied:

(60) $\lambda^t > 0$; $\qquad t = 1,...,T;$
(61) $\alpha_k^t \phi'(x_k^t) = \lambda^t p_k^t$; $\qquad k = 1,...,K; t = 1,...,T$

where $\phi'(x_k^t) \equiv d\phi(x_k^t)/dx$ is the derivative of $\phi(z)$ evaluated at $z = x_k^t > 0$. The concavity property of $\phi(z)$ implies that for any $z \geq 0$, the following inequality will hold for k = 1,...,K and t = 1,...,T:

$$(62)\ \phi(z) \leq \phi(x_k^t) + \phi'(x_k^t)(z - x_k^t)$$
$$= \phi(x_k^t) + [\alpha_k^t]^{-1}\lambda^t p_k^t(z - x_k^t)$$

where the last equality follows using (61).[33] Define $u_k^t \equiv \phi(x_k^t)$ for k = 1,...,K and t = 1,...,T. Let $z = x_j^s$ and substitute this value of z into the inequality (62) for t and k and we obtain the following system of K^2T^2 inequalities:

$$(63)\ u_j^s \leq u_k^t + [\alpha_k^t]^{-1} \lambda^t p_k^t(x_j^s - x_k^t) ; \qquad j,k = 1,...,K; s, t = 1,...,T.$$

Thus under our regularity conditions on ϕ, there must exist u_k^t for k = 1,...,K, t = 1,...,T and $\lambda^1,...,\lambda^T$ such that the inequalities (60) and (63) hold. Thus these conditions are necessary for the expected utility maximization hypothesis under our regularity conditions.

As usual, it is straightforward to show that the existence of a solution to the inequalities (60) and (63) is also sufficient to imply the existence of an increasing, continuous and concave function ϕ such that the observed data are consistent with the expected utility maximization hypothesis (57). Suppose a solution u_k^t for k = 1,...k and t = 1,...,T and $\lambda^1,...,\lambda^T$ to the inequalities (60) and (63) exists. Define $\phi(z)$ and the KT functions $\phi_k^t(z)$ for $z \geq 0$ as follows:

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31 Varian considered a portfolio choice model and assumed that the probabilities remained constant over time. He also did not deal with the nondifferentiable case.

32 In order to have a quasiconcave preference function, it is necessary for ϕ to be concave; see Yaari (1977; 1184). Arrow (1964) and Debreu (1959; 60-61) noted that the quasiconcavity assumption implied risk aversion on the part of the decision maker.

33 As usual, the differentiability assumption on ϕ is not required: conditions (61) can be replaced by the following supergradient conditions: $\rho_k^t \in \partial\phi(x_k^t)$, $\rho_k^t = [\alpha_k^t]^{-1}\lambda^t p_k^t$, k = 1,...,K; t = 1,...,T.

$$(64)\ \phi_k^t(z) \equiv u_k^t + [\alpha_k^t]^{-1}\lambda^t p_k^t(z - x_k^t)\ ; \qquad k = 1,...,K\ ;\ t = 1,...,T;$$
$$(65)\ \phi(z) \equiv \min_{t,k}\ \{\phi_k^t(z) : k = 1,...,K;\ t = 1,...,T\}.$$

Note that $\phi(z)$ is a continuous, concave and increasing function of the scalar variable z. Since the scalars u_k^t satisfy the inequalities (63), using definitions (64) and (65), it can be seen that the following equalities hold:

$$(66)\ \phi(x_j^s) = \min_{t,k}\ \{u_k^t + [\alpha_k^t]^{-1}\lambda^t p_k^t(x_j^s - x_k^t) : k = 1,...,K;\ t = 1,...,T\}$$
$$= u_j^s\ ; \qquad j = 1,...,K\ ;\ s = 1,...,T.$$

We now show that for $t = 1,...,T$, $x^t \equiv [x_1^t,x_2^t,...,x_K^t]$ solves the period t utility maximization problem in (57) where the function $\phi(x)$ is defined by (64) and (65):

$$(67)\ \max_x\{\sum_{k=1}^K \alpha_k^t\phi(x_k) : \sum_{k=1}^K p_k^t x_k \leq \sum_{k=1}^K p_k^t x_k^t\}$$
$$\leq \max_x\ \{\sum_{k=1}^K \alpha_k^t[u_k^t + [\alpha_k^t]^{-1}\lambda^t p_k^t(x_k - x_k^t)] : \sum_{k=1}^K p_k^t x_k \leq \sum_{k=1}^K p_k^t x_k^t\ \}$$
$$\text{since } \phi(x_k) \leq u_k^t + \alpha_k^{-1}\lambda^t p_k^t(x_k - x_k^t) \text{ using (64) and (65)}$$
$$= \max_x\ \{\sum_{k=1}^K \alpha_k u_k^t + \sum_{k=1}^K \lambda^t p_k^t(x_k - x_k^t) : \sum_{k=1}^K p_k^t x_k \leq \sum_{k=1}^K p_k^t x_k^t\}$$
$$\leq \sum_{k=1}^K \alpha_k u_k^t$$

where the last inequality follows from $\lambda^t > 0$ and $\sum_{k=1}^K p_k^t(x_k - x_k^t) \leq 0$. Thus $\sum_{k=1}^K \alpha_k u_k^t$ is an upper bound to utility for the period t utility maximization problem using the ϕ defined by (64) and (65). But the equalities (66) show that this upper bound is attained for $x_k = x_k^t$ for $k = 1,...,K$ and so $x^t \equiv [x_1^t,x_2^t,...,x_K^t]$ solves the period t utility maximization problem for $t = 1,...,T$ using the $\phi(x)$ defined by (64) and (65).

As usual, a linear program can be set up to determine whether the system of inequalities (60) and (63) has a solution. Since the inequalities (63) are homogeneous in the u_k^t and the λ^t, drop the positivity restrictions (60) and replace them with the inequality restrictions, $\lambda^t \geq 1$ for $t = 1,...,T$. It is also the case that we can impose nonnegativity restrictions on the u_k^t without loss of generality.

In the following section, we will illustrate how the decision to purchase insurance or to invest can be modeled using the expected utility model.

8. Applications to Insurance and Investing

We first consider a simple model that looks at the decision to insure some property against the possibility of loss during a period.

Suppose that an individual has insurable property which has value $W^t > 0$ and there are two states of nature in period t for $t = 1,...,T$. In State 1, there is no damage to the property and in State 2, the property is totally destroyed. In period t, the probability that State 1 occurs is $\alpha_1^t > 0$ and the probability that State 2 occurs is $\alpha_2^t = 1 - \alpha_1^t > 0$ for $t = 1,...,T$. The decision maker can purchase property insurance $i^t \geq 0$ in period t at the premium rate of δ^t for each dollar of insurance purchased where $0 < \delta^t < 1$. There is a limit on the amount of insurance that can be purchased in period t; i.e., $i^t \leq W^t$ so that the insurable wealth in period t is an upper bound to the amount of insurance coverage that can be purchased. Let x_1^t be the end of period t wealth if State 1 occurs and let x_2^t be the end of period t wealth if State 2 occurs for t = 1,..,T. It can be seen that x_1^t and x_2^t depend on the amount of insurance purchased in period t, i^t, in the following manner:

$$(68)\quad x_1^t \equiv W^t - \delta^t i^t \; ; \; x_2^t \equiv (1-\delta^t)i^t \; ; \; 0 \leq i^t \leq W^t \; ; \qquad t = 1,\ldots,T.$$

Note that if the decision maker fully insures in period t so that $i^t = W^t$, then end of period wealth is equalized no matter which state of nature occurs; i.e., we have $x_1^t = x_2^t = (1 - \delta^t)W^t$.

Now suppose that the decision maker's preferences over contingent commodities can be represented by the expected utility maximization model explained in the previous section where in the present context, K = 2 so we have only two states of nature. Assume that the decision maker has a continuous, concave and increasing function of one variable ϕ such that (x_1^t,x_2^t) solves (57) for t = 1,...,T, where K = 2 and the (x_1^t,x_2^t) are defined by (68) above. We cannot immediately apply the model in the previous section because we have not defined the prices p_1^t and p_2^t that correspond to the state contingent quantities x_1^t and x_2^t defined by (68). This problem is easily remedied: we use the two equations in (68) in order to eliminate the insurance decision variable i^t. This will leave us with a single budget constraint equation for each period t involving x_1^t and x_2^t. These equations turn out to be the following ones:

$$(69)\quad (1-\delta^t)x_1^t + \delta^t x_2^t = (1-\delta^t)W^t \; ; \qquad t = 1,\ldots,T.$$

Using (69), we can now define the positive period t prices, p_1^t and p_2^t, for the two contingent commodities:

$$(70)\quad p_1^t \equiv 1 - \delta^t \; ; \; p_2^t \equiv \delta^t \; ; \qquad t = 1,\ldots,T.$$

With the x_k^t and p_k^t defined by (68) and (70), the consistency hypothesis (57) for our insurance model can be tested using the analysis developed in the previous section.

As a second example of how the methodology developed in the previous section can be used, we conclude this section by considering a simple model of investing under risk.

Suppose a decision maker has wealth $W^t > 0$ that he or she wishes to invest in two assets. The first asset is a riskless asset which returns $r^t > 0$ per dollar invested in period t. The second asset is a risky asset which pays the interest rate r_1^t in period t if State 1 occurs and the rate r_2^t if State 2 occurs in period t. We assume that the three interest rates satisfy the following inequalities:

$$(71)\quad 0 < r_2^t < r^t < r_1^t \; ; \qquad t = 1,\ldots,T.$$

Thus for each period t, State 1 is the favourable state for the investor when the risky asset delivers the rate of return r_1^t which is above the safe asset rate of r^t whereas in State 2, the risky asset delivers the rate of return r_2^t which is below the safe asset rate of return. In period t, the probability that State 1 occurs is $\alpha_1^t > 0$ and the probability that State 2 occurs is $\alpha_2^t = 1 - \alpha_1^t > 0$ for t = 1,...,T. The decision variable of the investor in period t is s^t, the share of beginning of period t wealth that is invested in the safe asset. We assume that $0 \le s^t \le 1$ for each period t.

Let x_1^t be the end of period t wealth if State 1 occurs and let x_2^t be the end of period t wealth if State 2 occurs for t = 1,..,T. It can be seen that x_1^t and x_2^t depend on the share of initial wealth s^t that the investor allocates to the safe asset in the following manner:

(72) $x_1^t \equiv s^t(1+r^t)W^t + (1-s^t)(1+r_1^t)W^t$; $x_2^t \equiv s^t(1+r^t)W^t + (1-s^t)(1+r_2^t)W^t$; $\quad$ t = 1,...,T.

Now suppose that the decision maker's preferences over contingent commodities can be represented by the expected utility maximization model explained in the previous section where K = 2 so we have only two states of nature. Assume that the decision maker has a continuous, concave and increasing function of one variable ϕ such that (x_1^t,x_2^t) solves (57) for t = 1,...,T, where K = 2 and the (x_1^t,x_2^t) are defined by (72) above. Now use the two equations in (72) in order to eliminate the portfolio allocation variable s^t. This will leave us with a single budget constraint equation for each period t involving x_1^t and x_2^t. These equations turn out to be the following ones:

(73) $(r^t - r_2^t)x_1^t + (r_1^t - r^t)x_2^t = (1 + r^t)(r_1^t - r_2^t)W^t$; $\quad$ t = 1,...,T.

Using (73), we can now define the positive period t prices, p_1^t and p_2^t, for the two contingent commodities:

(74) $p_1^t \equiv r^t - r_2^t$; $p_2^t \equiv r_1^t - r^t$; $\quad$ t = 1,...,T.

With the x_k^t and p_k^t defined by (72) and (74), the consistency hypothesis (57) for the above investment model can be tested using the algebra developed in the previous section.

The expected utility model for making choices under uncertainty has some limitations and so many nonexpected utility models have been proposed to remedy these shortcomings. In the following section, we will show how the Afriat inequalities can be adapted to a useful nonexpected utility model.

9. Implicitly Defined Stochastic Preference Functions: The Homothetic Case

Recall that in section 7, the stochastic preference function $M(x_1,...,x_K)$ was explicitly defined as $\phi^{-1}[\sum_{k=1}^K \alpha_k\phi(x_k)]$ where $\alpha_k > 0$ was the probability that state k will occur for k = 1,...,K and ϕ was a continuous, concave, increasing function of one variable. In the present section, the stochastic preference function will be defined *implicitly* as the solution $u = M(x_1,...,x_K)$ to the following equation:

(75) $\sum_{k=1}^{K} \alpha_k\gamma(x_k/u) = \gamma(1)$

where $\gamma(z)$ is a continuous, concave and increasing function of one variable, defined for $z \geq 0$. Axiomatic justifications for this implicitly defined stochastic preference function may be found in Chew (1989; 287) and Diewert (1993; 397-415) (1995).[34] Note that if we let $\gamma(z) \equiv z^r$ for $r > 0$ or $\gamma(z) \equiv -z^r$ for $r < 0$, the stochastic preference function becomes the *mean of order r* defined as $M(x) \equiv [\sum_{k=1}^{K} \alpha_k x_k^{r}]^{1/r}$ and if we let $\gamma(z) \equiv \ln z$, then M becomes the *weighted geometric mean*, $M(x) \equiv \prod_{k=1}^{K} x^{\alpha_k}$.[35]

We first show that a unique solution u to (75) exists if $x > 0_K$. Define the function of K + 1 variables, F(x,u), for $x \equiv [x_1,...,x_K] \geq 0_K$ and $u > 0$ as follows:

(76) $F(x,u) \equiv \sum_{k=1}^{K} \alpha_k\gamma(x_k/u)$.

Using the continuity properties of γ, it can be seen that F(x,u) is jointly continuous in x, u for $x \geq 0_K$ and $u > 0$. Moreover, since each $\alpha_k > 0$ and $\gamma(z)$ is an increasing function, F(x,u) is strictly increasing in each x_k and strictly decreasing in u. Let $x \equiv [x_1,...,x_K] > 0_K$ and define $u^* \equiv \min_k\{x_k : k = 1,...,K\} > 0$ and $u^{**} \equiv \max_k\{x_k : k = 1,...,K\} \geq u^*$. Then $x_k/u^* \geq 1$ and $x_k/u^{**} \leq 1$ for each k. Using the monotonicity of γ and the fact that $\sum_{k=1}^{K} \alpha_k = 1$, we deduce that

(77) $\gamma(1) \leq F(x,u^*)$; $F(x,u^{**}) \leq \gamma(1)$.

Since F(x,u) is monotonically decreasing and continuous in u for $u^* \leq u \leq u^{**}$, the inequalities (77) imply the existence of a unique solution $u = M(x)$ to (75). The above inequalities also show that M satisfies the following inequalities: for $x > 0_K$,

(78) $0 < \min_k\{x_k : k = 1,...,K\} \leq M(x) \leq \max_k\{x_k : k = 1,...,K\}$.

Thus if $x > 0_N$, then (78) implies that M(x) is positive. The inequalities (78) also imply that if all components of x are equal to the same positive number $\lambda > 0$ say, then M reproduces this positive number (and thus M is a mean function):[36]

(79) $M(\lambda 1_K) = \lambda$.

---

34 See also the closely related models of Dekel (1986), Chew and Epstein (1989).

35 See Hardy, Littlewood and Polya (1934) for the properties of means of order r. Diewert (1993; 399) noted the advantage of the present specification of stochastic preferences over the expected utility specification if it is desirable to have homothetic preferences. Using the expected utility framework, the only linearly homogeneous stochastic preference functions are the means of order r whereas in the present implicit expected utility framework, a much larger class of linearly homogeneous stochastic preference functions can be accommodated.

36 The inequalities (78) can also be used to show that M(x) tends to 0 as $x > 0_K$ tends to 0_K. Thus we define $M(0_K) \equiv 0$. It can be shown that M(x) is continuous for $x \geq 0_K$.

The monotonicity and continuity properties of γ along with the assumption that the α_k are all positive imply that M(x) is a continuous, strictly increasing function[37] over the nonnegative orthant.

The function M has another important property; namely, it is (positively) linearly homogeneous so that for $x > 0_K$ and $\lambda > 0$, we have:

(80) $M(\lambda x) = \lambda M(x)$.

To establish (80), let $x > 0_K$ and let $u \equiv M(x)$. Thus u and x satisfy equation (75). Now replace x by λx in equation (75). It can be seen that if we replace u by λu, then $u^* \equiv \lambda u$ and $x^* \equiv \lambda x$ will also satisfy (75) and thus $\lambda M(x) = \lambda u = u^* = M(x^*) = M(\lambda x)$, which establishes (80). Thus the stochastic preferences are *homothetic* under our assumptions on γ.

The final property that we want to establish for the stochastic preference function M is quasiconcavity, which will imply risk averting behavior on the part of the decision maker. Let $x^1 > 0_K$, $x^2 > 0_K$, $0 < \lambda < 1$, $u^1 \equiv M(x^1)$, $u^2 \equiv M(x^2)$ with $u^1 \le u^2$. For quasiconcavity of M, we need to show that

(81) $u^1 \le M(\lambda x^1 + (1-\lambda)x^2)$.

Since $x^i > 0_K$, u^i will be positive for i = 1,2. Thus u^i, x^i satisfy (75) which we rewrite as follows:

(82) $\sum_{k=1}^K \alpha_k\gamma(x_k^i/u^i) = \gamma(1)$; $\qquad$ i = 1,2.

Since $u^2 \ge u^1 > 0$, $1/u^1 \ge 1/u^2$. Using $x^2 > 0_K$ and the increasing property of γ, we have:

(83) $\gamma(x_k^2/u^1) \ge \gamma(x_k^2/u^2)$; $\qquad$ k = 1,...,K.

Using the concavity property of γ and the positivity of the α_k, we have the following inequality:

$$
\begin{aligned}
(84)\ \sum_{k=1}^K \alpha_k\gamma[(\lambda x_k^1 + (1-\lambda)x_k^2)/u^1] &\ge \sum_{k=1}^K \alpha_k\{\lambda\gamma(x_k^1/u^1) + (1-\lambda)\gamma(x_k^2/u^1)\} \\
&= \lambda\gamma(1) + (1-\lambda)\sum_{k=1}^K \alpha_k\gamma(x_k^2/u^1) && \text{using (82) for i = 1} \\
&= \lambda\gamma(1) + (1-\lambda)\sum_{k=1}^K \alpha_k\gamma(x_k^2/u^2) && \text{using (83)} \\
&= \gamma(1) && \text{using (82) for i = 2.}
\end{aligned}
$$

Let $u^* \equiv M(\lambda x^1 + (1-\lambda)x^2)$ be the solution to $\sum_{k=1}^K \alpha_k\gamma[(\lambda x_k^1 + (1-\lambda)x_k^2)/u^*] = \gamma(1)$. The inequality (84) and the increasing property of γ shows that $u^* \ge u^1$, which is (81).

---

[37] Thus if $0_K < x < y$, then $M(x) < M(y)$.

Note that (78), (80) and (81) imply that M(x) is a positive, linearly homogeneous and quasiconcave function defined over the positive orthant and so we can apply Berge's (1963; 208) result and conclude that M is also concave over this domain of definition. Extending the domain of definition to the nonnegative orthant by continuity means that M will be concave over the nonnegative orthant.

We assume that we can observe the prices and quantities that pertain to a decision maker over T periods as usual. Our *homothetic implicit expected utility maximization hypothesis* is the following one: the observed period t quantity vector x^t solves the following period t expected utility maximization problem, where the period t probability vector $\alpha^t \equiv [\alpha_1^t,...,\alpha_K^t] >> 0_K$ and the observed price and quantity vectors are the vectors $p^t \equiv [p_1^t,...,p_K^t] >> 0_K$ and $x^t \equiv [x_1^t,...,x_K^t] >> 0_K$ and the unobserved utility level is u^t:

(85) u^t,x^t solves $\max_{u,x} \{u : \sum_{k=1}^K \alpha_k^t\gamma(x_k/u) = \gamma(1) ; \sum_{k=1}^K p_k^t x_k \le \sum_{k=1}^K p_k^t x_k^t\}$; t = 1,...,T.

It is possible to show that if u^t, x^t solves (85), then x^t must be a solution to the following constrained maximization problem:

(86) x^t solves $\max_x \{\sum_{k=1}^K \alpha_k^t\gamma(x_k/u^t) : \sum_{k=1}^K p_k^t x_k \le \sum_{k=1}^K p_k^t x_k^t\}$; t = 1,...,T.

Using the inequalities (78), it can be seen that if (85) is satisfied, then the optimal period t utility level u^t must be positive and it satisfies the following observable bounds:

(87) $0 < \min_k \{x_k^t : k = 1,...,K\} \le u^t \le \max_k \{x_k^t : k = 1,...,K\}$; t = 1,...,T.

The above bounds are useful since they insure that a solution u^t to the period t utility maximization problem defined by (85) is bounded away from 0 and bounded from above.

Recall that we assumed that $\gamma(z)$ is a continuous, concave and increasing function of one variable. Assume for the moment that γ is also differentiable. Then if x^t and u^t solve (85), x^t also solves the period t problem in (86) and we can deduce that a $\lambda^t \ge 0$ must exist such that the following first order necessary conditions are satisfied:

(88) $\alpha_k^t\gamma'(x_k^t/u^t) = \lambda^t u^t p_k^t$; k = 1,...,K ; t = 1,...,T;
(89) $\sum_{k=1}^K \alpha_k^t\gamma(x_k^t/u^t) = g(1)$; t = 1,...,T

where $\gamma'(z)$ denotes the derivative of $\gamma(z)$ evaluated at z. The assumption that γ is an increasing concave function implies that the nonnegative λ^t must be positive; i.e., the following conditions must hold under our regularity assumptions:

(90) $\lambda^t > 0$; t = 1,...,T.

The concavity property of $\gamma(z)$ implies that for any $z \ge 0$, the following inequality will hold for k = 1,...,K and t = 1,...,T:

(91) $\gamma(z) \le \gamma(x_k^t/u^t) + \gamma'(x_k^t/u^t)[z - (x_k^t/u^t)]$
$= \gamma(x_k^t/u^t) + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[z - (x_k^t/u^t)]$

where the last equality follows using (88).[38] Define $v_k^t \equiv \gamma(x_k^t/u^t)$ for $k = 1,...,K$ and $t = 1,...,T$. Let $z = x_j^s/u^s$ and substitute this value of z into the inequality (91) for t and k and we obtain the following system of K^2T^2 Afriat type inequalities:

(92) $v_j^s \le v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[(x_j^s/u^s) - (x_k^t/u^t)]$; $\quad j,k = 1,...,K;\ s, t = 1,...,T.$

Since $v_k^t \equiv \gamma(x_k^t/u^t)$, the equalities (89) can be rewritten as follows:

(93) $\sum_{k=1}^K \alpha_k^t v_k^t = g(1)$; $\quad t = 1,...,T.$

Thus under our regularity conditions on γ, there must exist v_k^t for $k = 1,...,K$, $t = 1,...,T$, $u^1,...,u^T$ and $\lambda^1,...,\lambda^T$ such that equations (93) hold and the inequalities (87), (90) and (92) hold. Thus these conditions are necessary for the implicit homothetic expected utility maximization hypothesis under our regularity conditions.

There are some additional conditions that must be satisfied under the hypothesis (85) where it is assume that $\gamma(z)$ is an increasing, continuous and concave function of one variable defined for $z \ge 0$. Recall that $u = M(x_1,...,x_K)$ is the u solution to the equation (75) and the function γ generates the function M. But it can be seen if γ generates M, then the function $\gamma^*(z) \equiv \alpha + \beta\gamma(z)$ will also generate M for any scalar α and any positive scalar β and $\gamma^*(z)$ will have the same properties as $\gamma(z)$. Thus we are free to set $\gamma(1)$ equal to an arbitrary number and to set $\gamma'(1)$ equal to an arbitrary positive number. We will choose to set $\gamma(1) = 1$ and thus conditions (93) become the following conditions:

(94) $\sum_{k=1}^K \alpha_k^t v_k^t = 1$; $\quad t = 1,...,T.$

The concavity of $\gamma(z)$ means that $\gamma(z) \le \gamma(1) + \gamma'(1)(z - 1)$ for all $z \ge 0$. Thus using $\gamma(1) = 1$ and $\gamma'(1) \equiv \lambda^0 > 0$,[39] the following inequalities will be satisfied:

(95) $\lambda^0 > 0$;
(96) $v_j^s \le 1 + \lambda^0[(x_j^s/u^s) - 1]$; $\quad j = 1,...,K$; $s = 1,...,T$

[38] As usual, the differentiability assumption on γ is not required: conditions (88) can be replaced by the following supergradient conditions: $\rho_k^t \in \partial\gamma(x_k^t)$, $\rho_k^t = [\alpha_k^t]^{-1}\lambda^t u^t p_k^t$, $k = 1,...,K$; $t = 1,...,T$. A concave function of one variable has a right and left derivative at each point in the interior of its domain of definition. The set of supergradients in this case reduces to all numbers ρ that are equal to or greater than the right derivative and equal to or less than the left derivative. It is important to allow $\gamma(z)$ to be nondifferentiable at $z = 1$ to allow for stochastic preferences that exhibit first degree risk aversion; see Chew (1989; 287), Epstein and Zin (1990), Segal and Spivak (1990) and Diewert (1993; 415-423) (1995; 139).
[39] In the case where $\gamma'(1)$ does not exist, let $\lambda^0 \in \partial\gamma(1)$.

where $v_k^t \equiv \gamma(x_k^t/u^t)$ as usual. The concavity of γ and $\gamma(1) = 1$ also imply that $1 \leq \gamma(x^t/u^t) + \gamma'(x^t/u^t)[1 - (x_k^t/u^t)]$ and so the following inequalities will be satisfied:

$$(97)\ 1 \leq v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[1 - (x_k^t/u^t)] \ ; \qquad k = 1,...,K \ ; t = 1,...,T$$

where we have used equations (88).

As usual, it is possible to show that the existence of a solution to the equalities (94) and the inequalities (87), (90) and (92) is also sufficient to imply the existence of an increasing, continuous and concave function γ such that the observed data are consistent with the homothetic implicit expected utility maximization hypothesis (85). Suppose a solution v_k^t for $k = 1,...,K$, $t = 1,...,T$, $u^1,...,u^T$ and $\lambda^1,...,\lambda^T$ to (87), (90), (92), (94), (95) (96) and (97) exists. Define $\gamma^0(z)$, the KT functions $\gamma_k^t(z)$ and $\gamma(z)$ for $z \geq 0$ as follows:

$$(98)\ \gamma^0(z) \equiv 1 + \lambda^0(z - 1) \ ;$$
$$(99)\ \gamma_k^t(z) \equiv v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[z - (x_k^t/u^t)] \ ; \qquad k = 1,...,K \ ; t = 1,...,T;$$
$$(100)\ \gamma(z) \equiv \min_{t,k} \{\gamma^0(z), \gamma_k^t(z) : k = 1,...,K; t = 1,...,T\}.$$

Note that $\gamma(z)$ is a continuous, concave and increasing function of the scalar variable z. Since the scalars v_k^t satisfy the inequalities (92), using definitions (99), it can be seen that the following equalities hold:

$$\begin{aligned}(101)\ \min_{t,k} \{\gamma_k^t(x_j^s/u^s) &: k = 1,...,K; t = 1,...,T\} \\ &= \min_{t,k} \{v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[(x_j^s/u^s) - (x_k^t/u^t)] : k = 1,...,K; t = 1,...,T\} \\ &= v_j^s \ ; \qquad j = 1,...,K \ ; \ s = 1,...,T.\end{aligned}$$

Using definition (100), it can be seen that

$$\begin{aligned}(102)\ \gamma(x_j^s/u^s) &= \min \{\gamma^0(x_j^s/u^s), v_j^s\}; \qquad j = 1,...,K \ ; \ s = 1,...,T \\ &= \min \{1 + \lambda^0[(x_j^s/u^s) - 1], v_j^s\} \\ &= v_j^s\end{aligned}$$

where the last equality follows using (96). We also need to show that γ defined by (100) has the property that $\gamma(1) = 1$. Using definitions (99), we have

$$\begin{aligned}(103)\ \min_{t,k} \{\gamma_k^t(1) &: k = 1,...,K; t = 1,...,T\} \\ &= \min_{t,k} \{ v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[1 - (x_k^t/u^t)] \ : k = 1,...,K; t = 1,...,T\} \\ &\geq 1\end{aligned}$$

using the inequalities (97). Using definition (98), it can be seen that $\gamma^0(1) = 1$. Thus using (103) and definition (100), we have $\gamma(1) = 1$.

We now show that for $t = 1,...,T$, $x^t \equiv [x_1^t,x_2^t,...,x_K^t]$ solves the period t maximization problem in (86) (and hence u^t, x^t solves (85)) where the function $\gamma(z)$ is defined by (100).

$$
\begin{aligned}
(104)\ & \max_x \{\textstyle\sum_{k=1}^K \alpha_k^t\gamma(x_k/u^t) : \sum_{k=1}^K p_k^t x_k \le \sum_{k=1}^K p_k^t x_k^t\} \\
& \le \max_x \{\textstyle\sum_{k=1}^K \alpha_k^t[v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[(x_k/u^t) - (x_k^t/u^t)] : \sum_{k=1}^K p_k^t x_k \le \sum_{k=1}^K p_k^t x_k^t \} \\
& \qquad \text{since } \gamma(x_k/u^t) \le v_k^t + [\alpha_k^t]^{-1}\lambda^t u^t p_k^t[(x_k/u^t) - (x_k^t/u^t)] \text{ using } \alpha_k^t > 0, (99) \text{ and } (100) \\
& = \max_x \{\textstyle\sum_{k=1}^K \alpha_k v_k^t + \sum_{k=1}^K \lambda^t p_k^t(x_k - x_k^t) : \sum_{k=1}^K p_k^t x_k \le \sum_{k=1}^K p_k^t x_k^t\} \\
& \le \textstyle\sum_{k=1}^K \alpha_k v_k^t
\end{aligned}
$$

where the last inequality follows from $\lambda^t > 0$ and $\sum_{k=1}^K p_k^t(x_k - x_k^t) \le 0$. Thus $\sum_{k=1}^K \alpha_k v_k^t$ is an upper bound to utility for the period t utility maximization problem using the γ defined by (100). But the equalities (102) show that this upper bound is attained for $x_k = x_k^t$ for $k = 1,...,K$ and so $x^t \equiv [x_1^t,x_2^t,...,x_K^t]$ solves the period t utility maximization problem for $t = 1,...,T$ using the $\gamma(x)$ defined by (100).

The above analysis can be summarized as follows: necessary and sufficient conditions for the implicit expected utility hypothesis (85) where $\gamma(z)$ is an increasing, continuous and concave function defined for $z \ge 0$ with $\gamma(1) = 1$ are the existence of numbers $\lambda^0, \lambda^1,...,\lambda^T$, $u^1,...,u^T$ and v_k^t for $k = 1,...,K$ and $t = 1,....,T$ such that (87), (90), (92), (94), (95) (96) and (97) are satisfied. Note that the v_k^t are unrestricted in sign.

It is difficult to set up a nonlinear programming problem that will enable us to test for the existence of a solution to (87), (90), (92), (94), (95), (96) and (97); in particular, the strict inequality conditions (90) and (95) are difficult to deal with. However, recall our discussion on how $\gamma(z)$ can be replaced by $\alpha + \beta\gamma(z)$ with $\beta > 0$ without affecting the maximization hypothesis (85). We used this fact to scale the function γ (by choosing an α) so that $\gamma(1) = 1$. We can also choose $\beta > 0$ so that all of the slope parameters λ^0, $\lambda^1,...,\lambda^T$ are equal to or greater than an arbitrary positive number. Thus we can replace the strict inequalities (90) and (95) by the following weak inequalities:

$$(105)\ \lambda^t \ge 1 ; \qquad t = 0,1,...,T.$$

A programming problem with a linear objective function involving slack variables with some nonlinear constraints that would enable one to find a feasible solution to the constraints (87), (92), (94), (96), (97) and (105) (if one exists) can easily be constructed.

We conclude this section by considering the case where the probability vectors $\alpha^t = [\alpha_1^t,...,\alpha_K^t]$ are constant over time; i.e., assume that:

$$(106)\ \alpha^t = \alpha \equiv [\alpha_1,...,\alpha_K] >> 0_K \text{ with } \textstyle\sum_{k=1}^K \alpha_k = 1 ; \qquad t = 1,...,T.$$

Let $j = k$ and premultiply v_j^s in equations (92) by α_j and sum these equations for $j = 1,...,K$ to obtain the following equations:

$$
\begin{aligned}
(107)\ & \textstyle\sum_{j=1}^K \alpha_j v_j^s \le \sum_{j=1}^K \alpha_j v_j^t + \lambda^t[\sum_{j=1}^K p_j^t x_j^s(u^t/u^s) - \sum_{j=1}^K p_j^t x_j^t] && s, t = 1,...,T \text{ or} \\
& 1 \le 1 + \lambda^t[\textstyle\sum_{j=1}^K p_j^t x_j^s(u^t/u^s) - \sum_{j=1}^K p_j^t x_j^t] && \text{using (94) and (106).}
\end{aligned}
$$

It can be seen that the inequalities $\lambda^t > 0$ and the inequalities in (107) imply the homotheticity conditions (19) in section 4 above. Thus if the test for implicit expected utility maximization passes and the probability vectors α^t are constant over time, then the homotheticity test developed in section 4 will also pass.[40]

10. Conclusion

We conclude this paper with a few comments on extensions of the basic Afriat methodology.

Suppose that there is no solution to the Afriat inequalities (3) and (8). The question arises: how bad is the failure to find a solution?

We could try and perturb the data (as little as possible in some metric) so that the perturbed data satisfies (3). Thus Varian (1985; 449) considered the following problem:

$$(108)\quad \min_{\xi^t,u^t,\lambda^t} \{\textstyle\sum_{t=1}^T (\xi^t - x^t)\cdot(\xi^t - x^t) : \lambda^t \geq 1,\ t = 1,\ldots,T;\ u^s \leq u^t + \lambda^t p^t\cdot(\xi^s - \xi^t),\ s, t = 1,\ldots,T\}.$$

The problem with this model is that the solution to this nonlinear programming problem is not invariant to changes in the units of measurement.

A more parsimonious model that is invariant to changes in the units of measurement[41] is the following one:

$$(109)\quad \min_{e^t,u^t,\lambda^t} \{\textstyle\sum_{t=1}^T (e^t - 1)^2 : \lambda^t \geq 1,\ t = 1,\ldots,T;\ u^s \leq u^t + \lambda^t p^t\cdot(e^s x^s - e^t x^t),\ s, t = 1,\ldots,T\}.$$

Thus we proportionally perturb the quantity data so as to achieve consistency where the proportionality factor for observation t is e^t for $t = 1,\ldots,T$. In the case where the original Afriat conditions fail, the researcher can look at the distribution of the e^t and decide how bad the failure of the maximization hypothesis is.[42]

Another extension of the basic Afriat nonparametric approach to demand theory is due to Varian (1983; 108) who derived the Afriat type inequalities in the context of a rationing model, or more generally, to a maximization model where the decision maker faces two linear inequality constraints. This framework could be extended to a household

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[40] Note that the model developed in this section implies homothetic preferences *within* each period but in order to obtain homothetic preferences over all T periods, we require that the probability vectors α^t be constant over time.

[41] This model is due to Varian (1990; 131) but it builds on a related model due to Afriat (1972a) (1973). Jones and Edgerton (2009) approach the topic of violation measures in a systematic manner.

[42] We note that more research needs to be done to work out violation measures for the more complex stochastic optimization models explained in sections 7 and 9.

production context where households face a time constraint as well as a budget constraint[43] and the household production functions are concave.

Cherchye, de Rock and Vermeulen (2007) (2011) extended the Afriat nonparametric approach to demand analysis to households with multiple decision makers where only aggregate demand vectors can be observed. In a related vein, Blundell, Browning and Crawford (2003) (2008) combined the Afriat inequalities for microeconomic household data with additional information on Engel curves in order to derive nonparametric bounds to price indexes and aggregate demand responses to changing relative prices.

Finally, Afriat's influence on economic theory has not been limited to consumer theory and the related literature on consumer price indexes: following the example of Farrell (1957), he developed a similar nonparametric approach to producer theory; see Afriat (1972b).[44]

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[43] Recall the time allocation model of Becker (1965).

[44] Just as Diewert and Varian followed up Afriat's path breaking contributions to consumer theory with contributions of their own, they also made similar follow up contributions to producer theory; see Diewert and Parkan (1983), Varian (1984) and Diewert and Mendoza (2007). It is likely that Afriat's interest in developing the nonparametric approach to consumer theory was stimulated by Farrell's (1957) nonparametric approach to production theory; see Afriat (2011).

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