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Comonotonic Approximations for a Generalized Provisioning Problem with Application to Optimal Portfolio Selection

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Abstract

In this paper we discuss multiperiod portfolio selection problems related to a specific provisioning problem. Our results are an extension of Dhaene et al. (2005), where optimal constant mix investment strategies are obtained in a provisioning and savings context, using an analytical approach based on the concept of comonotonicity. We derive convex bounds that can be used to estimate the provision to be set up at a specified time in the future, to ensure that, after having paid all liabilities up to that moment, all liabilities from that moment on can be fulfilled, with a high probability. We give some interpretations of this additional reserve, and apply our results to optimal portfolio selection.

1 Introduction

In this paper we discuss multiperiod portfolio selection problems related to a specific provisioning problem. Our results are an extension of Dhaene et al. (2005), where optimal constant mix investment strategies are obtained in a provisioning and savings context, using an analytical approach based on the concept of comonotonicity. In this analytical framework, we derive convex bounds that can be used to estimate the provision to be set up at a specified time t in the future, to ensure that, after having paid all liabilities up to time t, all liabilities from t on can be fulfilled, with a high probability.

We explain how this additional provision can be used to estimate the influence of a temporary change in market parameters. We see how an insurer can get an idea how much a temporary 'crisis' will cost him, and how this will influence his optimal investment portfolio. Also, if an insurer's investment portfolio is not optimal, the results of this paper can be used to check whether postponing rebalancing is acceptable, and if so, for how many years.

We apply our results to optimal portfolio selection problems, and illustrate this with numerical examples.

In the following sections a brief introduction is given to respectively risk measures, the theory of comonotonicity, convex bounds of random variables and the framework of optimal portfolio selection in a lognormal setting. Next the general provisioning problem is discussed, and applied to optimal portfolio selection.

1.1 Risk measures and Comonotonicity

A *risk measure* is defined as a mapping from a set of random variables, representing the risks at hand, to the real numbers. In other words, a risk measure summarizes the distribution function of a random variable in one single real number. The common notation for a risk measure associated with a random variable X is $\rho[X]$. A risk measure ρ quantifies the riskiness of X: the larger $\rho[X]$, the more 'dangerous' the risk X.

Throughout this paper we assume to be working with (conditioning) random variables such that all (conditional) expectations that are used are well-defined and finite.

In this paper the main focus will be on the quantile risk measure, or Value-at-Risk (VaR). The VaR at level p will be denoted by $Q_p(X)$ or $VaR_p(X)$, and is defined as:

$$Q_p(X) = VaR_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in (0,1), \tag{1}$$

with $F_X(x) = \Pr(X \leq x)$. By convention, we take $\inf \varnothing = +\infty$.

Value-at-Risk measures the worst expected loss under normal market conditions over a specific time interval. It can be used to determine how much can be lost with a given probability over a predetermined time horizon.

Other well-known risk measures are for example Tail Value-at-Risk (TVaR), Conditional Tail Expectation (CTE) and Expected Shortfall (ESF). More information on risk measures can be found e.g. in Kaas et al. (2008) or Denuit et al. (2005).

A random vector $\underline{X} = (X_1, X_2, ..., X_n)$ is said to be *comonotonic* if the individual variables X_i are non-decreasing functions (or all are non-increasing functions) of the same random variable:

$$\underline{X} \stackrel{d}{=} (g_1(Z), g_2(Z), ..., g_n(Z)) \tag{2}$$

for some common random variable Z and non-decreasing (or non-increasing) functions g_i. Intuitively, comonotonicity corresponds to an extreme form of positive dependency between the individual variables: increasing the outcome of Z will lead to a simultaneous increase in the different outcomes of $g_i(Z)$.

Comonotonicity of $\underline{X}$ can also be characterized by

$$\underline{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U)\right), \tag{3}$$

with U uniformly distributed on the unit interval.

For more characterizations and an overview of the theory of comonotonicity and its many applications in actuarial science and finance we refer to Dhaene et al. (2002a,b) and Dhaene et al. (2008).

The following result of the comonotonic dependency structure will be crucial in our setting:

Theorem 1 (Additivity of quantile risk measure for sums of comonotonic risks) *If the random vector* $(X_0, X_1, ..., X_n)$ *is comonotonic, we have that*

$$Q_p\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Q_p\left(X_i\right), \tag{4}$$

for all $p \in (0,1)$.

This additivity property holds in general for all distortion risk measures, such as Tail Value-at-Risk and Expected Shortfall. In case the variables X_i are continuous, the same property holds for the Conditional Tail Expectation. A proof of this theorem and more information about the relationship between risk measures and comonotonicity can be found in Dhaene et al. (2006).

1.2 Convex Order Bounds for Sums of Random Variables

An extensive introduction to ordering of (distributions of) random variables, including actuarial applications, can be found in Denuit et al. (2005). We recall the definition of *stop-loss order* and *convex order*:

Definition 1 (Stop-Loss Order) *A random variable* X *is said to precede a random variable* Y *in* stop-loss order *if* X *has lower stop-loss premiums than* Y*:*

$$E\left[(X-d)_+\right] \leq E\left[(Y-d)_+\right], \tag{5}$$

for all $d \in (-\infty, +\infty)$*. We denote this as* $X \leq_{sl} Y$.

Definition 2 (Convex Order) *A random variable* X *is said to precede a random variable* Y *in* convex order *if* $X \leq_{sl} Y$ *and* $E[X] = E[Y]$*. We denote this as* $X \leq_{cx} Y$.

In this paper we will encounter random variables of the form

$$S = \sum_{i=0}^{n} \alpha_i \, e^{Z_i} \tag{6}$$

where α_i are deterministic constants, and Z_i are linear combinations of the components of a multivariate normal random vector $(Y_1, Y_2, ..., Y_n)$: suppose $Z_i = \sum_{j=1}^{n} \lambda_{ij} Y_j$ for $i = 0, ..., n$.

The random variable (6) is a sum of dependent lognormal random variables. As it is impossible to determine the distribution function of such a sum analytically, we use approximations. Several approximation techniques have been proposed throughout the literature, see e.g. Asmussen & Rojas (2005), Dufresne (2004), Milevsky & Posner (1998) and Milevsky & Robinson (2000). In this paper we will use convex upper and lower bounds based on comonotonicity, see e.g. Kaas et al. (2000) and Dhaene et al. (2002a,b). See also Huang et al. (2004) or Vanduffel et al. (2005) for a comparison of some of the approximation techniques.

The approximations of Kaas et al. (2000) are based on the following result:

Theorem 2 (Convex bounds for sums of random variables) *For any random vector $(X_0, X_1, ..., X_n)$ and any random variable Λ, we have that*

$$S^l = \sum_{i=0}^{n} E\left[X_i|\Lambda\right] \leq_{cx} S = \sum_{i=0}^{n} X_i \leq_{cx} \sum_{i=0}^{n} F_{X_i}^{-1}(U) = S^c, \tag{7}$$

with U a uniformly distributed random variable on the unit interval.

As can be seen from (3), the sum S^c is comonotonic. The special case (6) where S is a sum of dependent lognormal random variables is discussed in detail in Dhaene et al. (2002a,b). Expressions for S^c and S^l are derived in case the cash-flows α_i are positive. The *comonotonic upper bound* S^c is given by

$$S^c = \sum_{i=0}^{n} \alpha_i e^{E[Z_i] + \sigma_{Z_i} \Phi^{-1}(U)}. \tag{8}$$

For the lower bound approximation, the conditioning variable Λ is typically chosen as a linear combination of the variables Y_i. Assume that $\Lambda = \sum_{j=1}^{n} \beta_j Y_j$. In this case the lower bound S^l can be written as:

$$S^l = \sum_{i=0}^{n} \alpha_i e^{E[Z_i] + \frac{1}{2}\left(1 - r_i^2\right)\sigma_{Z_i}^2 + r_i \sigma_{Z_i} \Phi^{-1}(U)}, \tag{9}$$

where r_i is the correlation between Z_i and Λ. If all coefficients r_i are positive, S^l is a comonotonic sum, in which case we call S^l the *comonotonic lower bound.*

If all Y_i are i.i.d., the correlation coefficients are given by

$$r_i = \frac{\sum_{j=1}^{n} \lambda_{ij} \beta_j}{\sqrt{\sum_{j=1}^{n} \lambda_{ij}^2} \sqrt{\sum_{j=1}^{n} \beta_j^2}}, \qquad i = 1, ..., n. \tag{10}$$

Maximizing the variance of S^l leads, as explained in Dhaene et al. (2005), to the optimal Λ, which is given by

$$\Lambda = \sum_{i=0}^{n} \alpha_i e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i. \tag{11}$$

In Van Weert et al. (2009) conditions are derived for the lower bound to be comonotonic in case the cash-flows α_i have changing signs.

As can be seen from Theorem 1, a crucial advantage of the comonotonic bounds is the additivity property, which makes it straightforward to apply risk measures such as quantiles (VaR), TVaR and CTE to S^c and S^l, and hence to determine their distribution function. In Dhaene et al. (2005) expressions are given for the most commonly used risk measures associated with (8) and (9).

1.3 Optimal Portfolio Selection in a Lognormal Framework

Throughout this paper we assume the classical continuous-time framework of Merton (1971), also known as the Black & Scholes (1973) setting. See e.g. Björk (1998) for more details on this Black & Scholes setting. We use the same notations and terminology as in Dhaene et al. (2005).

Assume there are m risky assets or asset classes available in which we can invest. In our examples we assume there is no risk-free asset class available. The return of the risky assets is modelled by a multivariate geometric Brownian motion: investing an amount of 1 at time $k-1$ in risky asset i grows to $e^{Y_k^i}$ at time k. For a fixed asset i, the random variables Y_k^i are assumed i.i.d., normally distributed with mean $\mu_i - \frac{1}{2}\sigma_i^2$ and variance σ_i^2. This means that the return of an asset is not influenced by its return in the past. However, within any year, the returns of the different assets are correlated. We have that:

$$Cov\left[Y_k^i, Y_l^j\right] = \begin{cases} 0 & k \neq l \\ \sigma_{ij} & k = l \end{cases} \tag{12}$$

The drift vector and the variance-covariance matrix of the risky assets are denoted as $\underline{\mu}^T = (\mu_1, \ldots, \mu_m)$ and $\underline{\Sigma}$ respectively.

We restrict to *constant mix strategies*: the fractions invested in the different assets remain constant over time, due to continuous rebalancing. A vector describing the portfolio process is denoted as $\underline{\pi}^T = (\pi_1, \ldots, \pi_m)$, where π_i is the proportion invested in risky asset i, with $\sum_{i=1}^m \pi_i = 1$. Although our results also hold in the general case, we assume short-selling is not allowed, which means $0 \leq \pi_i \leq 1$ for all i. The drift and volatility corresponding to an investment portfolio $\underline{\pi}$ are written as $\mu(\underline{\pi})$ and $\sigma^2(\underline{\pi})$, and are given by:

$$\mu(\underline{\pi}) = \underline{\pi}^T \underline{\mu} \text{ and } \sigma^2(\underline{\pi}) = \underline{\pi}^T \cdot \underline{\Sigma} \cdot \underline{\pi}. \tag{13}$$

As both the time period and the investment horizon that we consider are typically long, the use of a Gaussian model for the stochastic returns can be justified by the Central Limit Theorem, see e.g. Cesari & Cremonini (2003) and Levy (2004) for some empirical evidence.

The yearly returns $Y_i(\underline{\pi})$ of an investment portfolio $\underline{\pi}$ are independent and normally distributed random variables, with expected value $\mathrm{E}[Y_i(\underline{\pi})] = \mu(\underline{\pi}) - \frac{1}{2}\sigma^2(\underline{\pi})$ and variance $\mathrm{Var}[Y_i(\underline{\pi})] = \sigma^2(\underline{\pi})$.

When no confusion is possible, we omit the dependence on the investment portfolio $\underline{\pi}$ in the notations. Hence the yearly returns are modelled by the i.i.d., normally distributed random variables Y_i, with mean $\mu - \frac{1}{2}\sigma^2$ and standard deviation σ.

2 Generalized Provisioning Problem

In this section we discuss the main topic of this paper. We want to determine (an estimate of) the provision to be set up at certain time in the future, to ensure that, after having paid the first liabilities, all liabilities from then on can be fulfilled with a high probability. First a general description of the problem is given, followed by the derivation of a solution based on convex order comonotonic bounds. Next the problem is applied to optimal portfolio selection, and illustrated with numerical examples. In the final part of this section some practical interpretations of this provision are described and illustrated.

2.1 Problem Description

Consider a series of deterministic liabilities α_i due at time i, for $i = 1, \ldots, n$, with $\alpha_i \geq 0$ for all i. Suppose we have an initial capital $K_0 > 0$ available at time 0. Assume that during the first m years, with $0 < m < n$, an investment strategy $\underline{\pi}_1$ is followed where the return in year i is described by the random variable $Y_i(\underline{\pi}_1)$, with $E[Y_i(\underline{\pi}_1)] = \mu(\underline{\pi}_1) - \frac{1}{2}\sigma^2(\underline{\pi}_1)$ and $Var[Y_i(\underline{\pi}_1)] = \sigma^2(\underline{\pi}_1)$. The random variables $Y_i(\underline{\pi}_1)$ are iid and normally distributed, for $i = 1, ..., m$. After m years, a different investment strategy is followed, with return in year j equal to $Y_j(\underline{\pi}_2)$. The random variables $Y_j(\underline{\pi}_2)$ are iid and normally distributed, with $\mathrm{E}[Y_j(\underline{\pi}_2)] = \mu(\underline{\pi}_2) - \frac{1}{2}\sigma^2(\underline{\pi}_2)$ and $\mathrm{Var}[Y_j(\underline{\pi}_2)] = \sigma^2(\underline{\pi}_2)$. We assume that the random variables $Y_i(\underline{\pi}_1)$ and $Y_j(\underline{\pi}_2)$ are independent for all i and j.

We want to determine (an estimate of) the provision to be set up at time m, with $0 < m < n$, to ensure that, after having paid the first m liabilities, all future liabilities can be fulfilled, incorporating a certain ruin probability ϵ. We denote this additional reserve at time m by K_m. Formally, we want to determine K_m such that:

$$\Pr\left[K_0 e^{\sum_{i=1}^m Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^m Y_j(\underline{\pi}_1)} + K_m \geq \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^i Y_j(\underline{\pi}_2)}\right] \geq (1-\epsilon), \quad (14)$$

for some small ϵ. In other words, the reserve K_m is equal to the following quantile:

$$K_m = Q_{1-\epsilon}\left[K_0 e^{\sum_{i=1}^m Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^m Y_j(\underline{\pi}_1)} - \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^i Y_j(\underline{\pi}_2)}\right]. \quad (15)$$

Note that K_m is not necessarily positive. A negative K_m means that the investor can withdraw an amount $-K_m$ from the account at time m, and still fulfill future liabilities, incorporating a ruin probability ϵ. If $K_m = 0$, no additional reserve is needed at time m, but at the same time nothing can be withdrawn from the account.

Note that (14) is a long-term survival probability, over the whole investment period of n years. For example a survival probability of 85% over a period of 30 years corresponds to a yearly survival probability of approximately 99.46%, since $0.85 \approx (0.9946)^{30}$.

In the following Section expressions are derived for respectively the convex upper bound and lower bound approximation.

2.2 Derivation of Convex Bounds

Within the quantile (15) we have sums of dependent lognormal random variables. As explained in Section 1.2, it is impossible to determine the distribution function of these sums exactly. Therefore we derive analytical approximations, based on the concept of comonotonicity, which are easy to compute. The results in this Section are a generalization of Dhaene et al. (2005).

The bounds derived in Dhaene et al. (2005) can not be applied directly to compute (15), as the terms within the quantile have different signs. Also, Theorem 1 from Van Weert et al. (2009) can not be applied here, since the conditions of the theorem are not necessarily satisfied. Therefore we have to use a different approach to determine a value for the reserve K_m.

Denote $Z = \sum_{i=1}^{m} Y_i(\underline{\pi}_1)$. Applying the law of total probability, conditioning on Z, the left hand side of inequality (14) becomes:

$$\int_{-\infty}^{\infty} \Pr\left[\sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z} + \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)} \leq K_0 e^z + K_m \right] \frac{1}{\sigma_Z} \phi\left(\frac{z-\mu_Z}{\sigma_Z}\right) dz, \tag{16}$$

with $\mu_Z = \mathrm{E}[Z] = m\mu(\underline{\pi}_1)$ and $\sigma_Z = \sqrt{m}\sigma(\underline{\pi}_1)$.

Denoting $S(z) = S_1(z) + S_2$, with $S_1(z) = \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z}$, and $S_2 = \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)}$, we can rewrite (16) as

$$\int_{-\infty}^{\infty} \Pr\left[S(z) \leq K_0 e^z\right] \frac{1}{\sigma_Z} \phi\left(\frac{z-\mu_Z}{\sigma_Z}\right) dz \tag{17}$$

To approximate the distribution function of $S(z)$, we can use its comonotonic upper bound $S^c(z)$ or lower bound $S^l(z)$, as defined by (8) and (9). Important is that here it is possible to apply the results from Dhaene et al. (2005), because all terms in $S(z)$ are of the same sign.

2.2.1 Upper Bound Approximation

To compute the probability within integral (16), we can approximate $S(z)$ by its comonotonic upper bound $S^c(z)$ as follows:

$$S(z) \leq_{cx} S^c(z) = S_1^c(z) + S_2^c = \sum_{i=1}^{m} \alpha_i F^{-1}_{e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z}}(U) + \sum_{i=1}^{n-m} \alpha_{m+i} F^{-1}_{e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)}}(U), \tag{18}$$

with U uniformly distributed on the unit interval.

As shown in Dhaene et al. (2002b), the random variables $\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z = z$ are normally distributed for any z. It can easily be seen that its expected value and variance are given by:

$$\mathrm{E}\left[\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z = z\right] = \frac{m-i}{m} z \ \text{ and } \mathrm{Var}\left[\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z = z\right] = \frac{i(m-i)}{m}\sigma^2(\underline{\pi}_1) \tag{19}$$

Using (8) and (19), $S_1^c(z)$ can be rewritten as:

$$S_1^c(z) = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m} z + \sqrt{\frac{i(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(U)\right). \tag{20}$$

We also have an expression for S_2^c:

$$S_2^c = \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-i\left(\mu(\underline{\pi}_2) - \frac{1}{2}\sigma^2(\underline{\pi}_2)\right) + \sqrt{i}\sigma(\underline{\pi}_2)\Phi^{-1}(U)\right). \tag{21}$$

Hence, using the additivity property (see Theorem 1), we can compute the quantiles of $S^c(z)$ as:

$$\begin{aligned} Q_{1-p}[S^c(z)] = &\sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m} z - \sqrt{\frac{i(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(p)\right) \\ &+ \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-i\left(\mu(\underline{\pi}_2) - \frac{1}{2}\sigma^2(\underline{\pi}_2)\right) - \sqrt{i}\sigma(\underline{\pi}_2)\Phi^{-1}(p)\right). \end{aligned} \tag{22}$$

This result can be used to determine the distribution function of $S^c(z)$, which can then be used to approximate integral (16).

2.2.2 Lower bound approximation

We can also approximate $S(z)$ using convex lower bounds. We have that

$$S_1(z) \geq_{cx} S_1^l(z) = E[S_1(z)|\Lambda_1(z)]. \tag{23}$$

Using (11) and (19) we get:

$$\Lambda_1(z) = \sum_{i=1}^{m} \alpha_i e^{\frac{m-i}{m}z + \frac{1}{2}\frac{i(m-i)}{m}\sigma^2(\underline{\pi}_1)} \left(\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1) | Z = z \right). \tag{24}$$

Using (9) and (19), we can write the lower bound S_1^l as:

$$S_1^l(z) = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m}z + \frac{1}{2}(1-r_i^2)\frac{i(m-i)}{m}\sigma^2(\underline{\pi}_1) + r_i\sqrt{\frac{i(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(U_1) \right), \tag{25}$$

with U_1 uniformly distributed on the unit interval. The correlation coefficients r_i can be determined using (10). Using the additivity property explained in Theorem 1, the quantiles of $S_1^l(z)$ can be determined as:

$$Q_{1-p}[S_1^l(z)] = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m}z + \frac{1}{2}(1-r_i^2)\frac{i(m-i)}{m}\sigma^2(\underline{\pi}_1) - r_i\sqrt{\frac{i(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(p) \right). \tag{26}$$

S_2 can be approximated by a convex lower bound S_2^l in a similar way:

$$S_2 \geq_{cx} S_2^l = E[S_2|\Lambda_2]. \tag{27}$$

The conditioning variable Λ_2 is given by

$$\Lambda_2 = \sum_{i=1}^{n-m} \alpha_{m+i} e^{-i\mu(\underline{\pi}_2) + \frac{1}{2}i\sigma^2(\underline{\pi}_2)} \left(-\sum_{j=1}^{i} Y_j(\underline{\pi}_2) \right). \tag{28}$$

Using (9) we get the following expression for S_2^l:

$$S_2^l = \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-i\mu(\underline{\pi}_2) + (1 - \frac{1}{2}r_i'^2)i\sigma^2(\underline{\pi}_2) + r_i'\sqrt{i}\sigma(\underline{\pi}_2)\Phi^{-1}(U_2) \right), \tag{29}$$

with U_2 uniformly distributed on the unit interval. The correlation coefficients r_i' can be determined using (10). Using the additivity property, the quantiles of S_2^l can be determined using:

$$Q_{1-p}[S_2^l] = \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-i\mu(\underline{\pi}_2) + (1 - \frac{1}{2}(r_i')^2)i\sigma^2(\underline{\pi}_2) - r_i'\sqrt{i}\sigma(\underline{\pi}_2)\Phi^{-1}(p) \right). \tag{30}$$

We approximate $S(z) = S_1(z) + S_2$ by the sum $S^l(z) = S_1^l(z) + S_2^l$. The approximation $S^l(z)$ is a convex lower bound for $S(z)$, since convex order is closed under convolution for independent risks (see e.g. Denuit et al. (2005)). The quantiles of $S^l(z)$ can be computed by adding (26) and (30). This allows us to compute the distribution function of $S^l(z)$, and hence to approximate integral (16).

2.3 Numerical Illustration

Assume $n = 30$, $\alpha_i = 10$ for $i = 1, ..., 30$, $K_0 = 200$ and $m = 5$. Furthermore, assume as a first example $\mu_X = \mu_Y = 0.05$ and $\sigma_X = \sigma_Y = 0.1$. Using this setting we can compute the probability of shortfall

$$\Pr\left[K_0 e^{\sum_{i=1}^{m} Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)} + K_m \leq \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)}\right] \tag{31}$$

for a range of reserves K_m. In Figure 1 our lower and upper bound approximations are compared to simulated results. We observe that both approximations perform very well, especially the lower bound. The figure also illustrates the intuitive fact that increasing the additional reserve K_m decreases the probability of shortfall. As a second example, suppose a more conservative strategy is followed after 5 years. More precisely, assume $\mu(\underline{\pi}_1) = 0.05$, $\sigma(\underline{\pi}_1) = 0.1$, $\mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$. Computing the probability of shortfall (31) for different reserves K_m leads to Figure 2. In this second example we see that our approximations are even closer to the simulated results, as it is almost not possible to distinguish the lines. Detailed numerical results of these examples can be found in Table 5 and Table 6 in Appendix A.

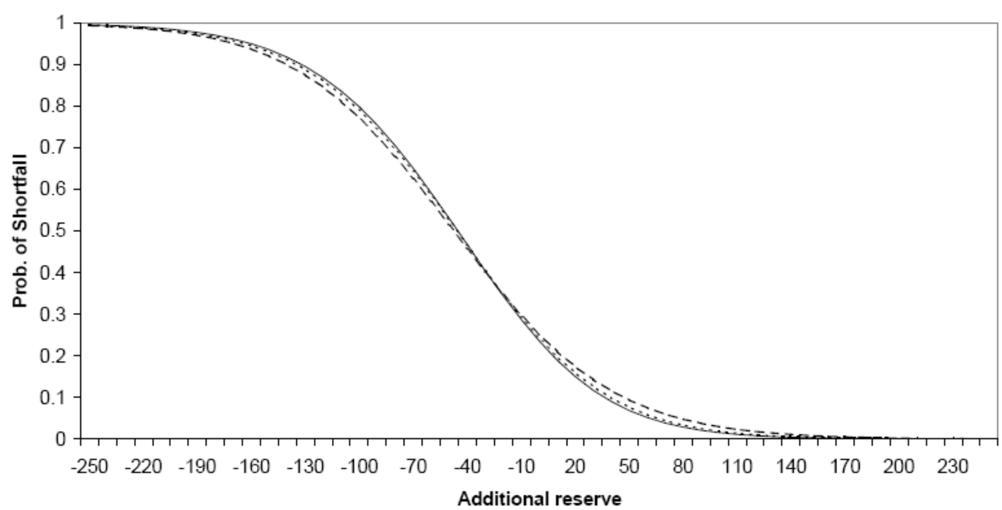

Figure 1: Comparison of upper bound (dashed line) and lower bound (dotted line) to simulated results (solid line), $\mu(\underline{\pi}_1) = \mu(\underline{\pi}_2) = 0.05$ and $\sigma(\underline{\pi}_1) = \sigma(\underline{\pi}_2) = 0.1$.

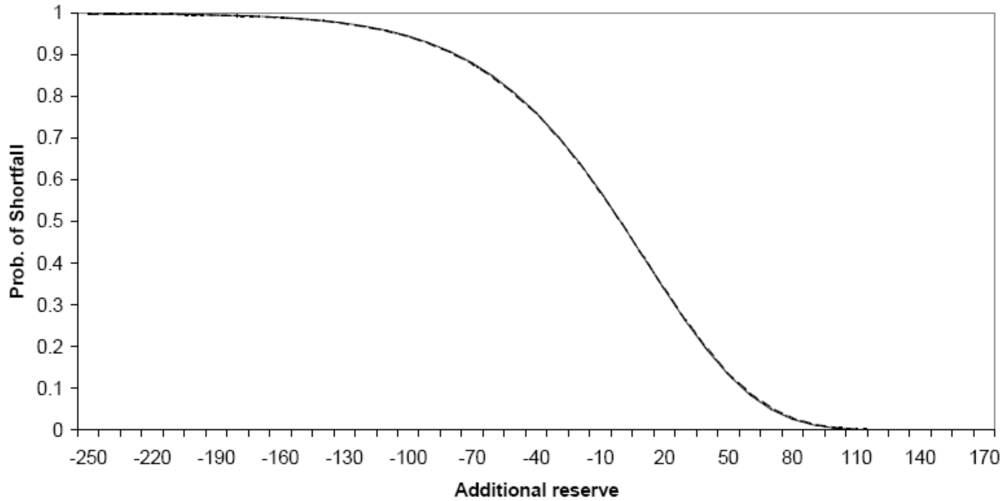

Figure 2: Comparison of upper bound (dashed line) and lower bound (dotted line) to simulated results (solid line), $\mu(\underline{\pi}_1) = 0.05$, $\sigma(\underline{\pi}_1) = 0.1$, $\mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$.

2.4 Application to Optimal Portfolio Selection

We can easily use our results in an optimal portfolio selection setting. For example, suppose we have an initial capital K_0 available at time 0, and suppose we know that we will add an extra capital K_m at time m. Suppose also that the investment strategy followed during the first m years is fixed, and given by $\underline{\pi}_1$. In this case we can optimize the investment strategy to be followed from year m on. The optimal portfolio is the one leading to a maximal survival probability p^*:

$$p^* = \max_{\underline{\pi}} \Pr\left[K_0 e^{\sum_{i=1}^{m} Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)} + K_m \geq \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi})}\right] \quad (32)$$

As a second and perhaps more relevant optimization, suppose we have given an initial capital K_0 and a ruin probability ϵ. Suppose again that the investment strategy followed during the first m years is fixed, and given by $\underline{\pi}_1$. We can then optimize the investment strategy to be followed from year m on, by looking for the portfolio leading to a minimal additional reserve K_m^*at time m:

$$K_m^* = \min_{\underline{\pi}} Q_{1-\epsilon}\left[K_0 e^{\sum_{i=1}^{m} Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)} - \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi})}\right] \quad (33)$$

We illustrate this second optimization numerically in the following paragraph.

2.4.1 Numerical Illustration

Assume $n = 20$, and $\alpha_i = 10$ for $i = 1, ..., 20$. Suppose we have two risky asset classes available in which we can invest, with drift vector $\underline{\mu}^T = (0.06, 0.10)$, standard deviations

$\underline{\sigma}^T = (0.10, 0.20)$ and correlations $\rho_{1,2} = 0.50$. Furthermore, take $K_0 = 150$ and suppose during the first 5 years a strategy $\underline{\pi}_1$ is followed where the return is modelled by the i.i.d. normal random variables $Y_i(\underline{\pi}_1)$, with parameters $\mu(\underline{\pi}_1) = 0.07$ and $\sigma(\underline{\pi}_1) = 0.15$. In this paragraph, we use the lower bound approximation as defined in Section 2.2.2 to determine the optimal investment strategy $\underline{\pi}^*$, leading to a minimal reserve K_m^*, as described by (33). As illustrated in Section 2.3, this lower bound approximation is in general significantly more accurate than the upper bound.

Assuming $m = 5$, the results of our optimization for different values of ϵ are given in Table 1. These results show that increasing the certainty level leads to a more conservative optimal strategy, and a higher required additional reserve K_5^* at time 5. For example, if from year 5 on the strategy $(0.6030, 0.3970)$ is followed, and if an amount of 22.63 is put on the account at time 5, there is 90% certainty that all liabilities can be paid. Following any other investment strategy, or adding less than 22.63 at time 5, would lead to a survival probability of less than 90%.

	ϵ			
	0.15	0.10	0.05	0.01
π_1^*	52.76%	60.30%	68.34%	78.39%
π_2^*	47.24%	39.70%	31.66%	21.61%
$\mu(\underline{\pi}^*)$	7.89%	7.59%	7.27%	6.86%
$\sigma(\underline{\pi}^*)$	11.92%	12.14%	11.40%	10.68%
K_5^*	9.30	22.63	41.47	76.70

Table 1: Minimal reserves K_5^* and optimal strategies for given certainty levels ϵ.

2.5 Interpretations of Additional Reserve

In this section we give interpretations for the reserve K_m, illustrated with numerical examples. Throughout this section, we use the comonotonic lower bound approximations derived in the previous sections to solve the optimization problems.

2.5.1 Effect of a temporary change in market parameters

Estimating the additional reserve K_m can be useful to quantify the effect of temporary changes in the market parameters. Suppose an insurance company has determined its investment portfolio using long-term estimates for the parameters describing the financial market. To estimate the influence of a temporary change in market parameters, assume that during the first m years the market behaves differently, with different parameters μ and σ.

If we assume a temporary improvement of market conditions (asset classes with higher drifts and/or lower variances), the reserve K_m as defined by (15) can be interpreted as the amount of money that will be available on the account at time m due to these favourable short-term market conditions (assuming we use the same ruin probability ε).

Similarly, if we would assume temporary adverse market conditions, the reserve K_m is an estimate of the amount of money the insurer will have to put on the account at time m in order to recover for this short-term "crisis".

Also, the insurer can see how these temporary (un)favourable parameters change its optimal investment strategy: assuming the market behaves unusually well (bad) during the first years, how will the optimal investment strategy look like afterwards. For a given reserve K_m (e.g. $K_m = 0$), the influence of (un)favourable temporary market conditions on the ruin probability ε can also be investigated.

Example Take $K_0 = 175$, $n = 30$, and $\alpha_i = 10$ for $i = 1, ..., 30$, and assume we have the 2 asset classes as in Section 2.4.1. Maximizing the survival probability, which is the probability of being able to pay all the liabilities, leads to an optimal investment strategy $\underline{\pi} = (0.5804, 0.4196)$, with $\mu(\underline{\pi}) = 0.0768$, $\sigma(\underline{\pi}) = 0.1236$ and corresponding maximal survival probability 85%. Note that this is a survival probability over the whole investment period of 30 years, corresponding to a yearly survival probability of approximately 99.46%. Assume the insurer invests according to this optimal strategy.

Suppose the insurer wants to check the influence of unusual short-term market conditions. In Table 2, additional reserves K_m are given for different market assumptions, and different values of m. In all examples, the survivial probability is 85%. For example, if every asset class has a drift 2% higher than normal for a period of 5 years, an amount of 16.20 can be withdrawn from the account at time 5. If the standard deviations of the asset classes is double for a period of 10 years, the insurer will have to put 96.29 on the account at time 10 in order to keep the same survival probability of 85%.

In Table 3 the influence of a change in short-term market conditions on the survival probability is illustrated. Suppose the insurer does not want to invest extra money at time m, or $K_m = 0$. We see from the table that the more (un-)favourable the market conditions are, and the longer these conditions last, the higher (lower) the survival probability becomes.

m	2	3	4	5	10	15
$2 * \underline{\sigma}$	28.68	39.53	49.17	58.02	96.29	132.04
$\underline{\mu} - 2\%$	7.07	11.00	14.89	18.73	37.75	57.27
$\underline{\mu} - 1\%$	3.66	5.99	8.31	10.61	21.96	33.66
$\underline{\mu} + 1\%$	−3.34	−4.45	−5.60	−6.83	−14.95	−27.30
$\underline{\mu} + 2\%$	−6.94	−9.89	−12.96	−16.20	−36.50	−66.41
$0.5 * \underline{\sigma}$	−5.68	−8.06	−10.38	−12.69	−24.53	−38.58

Table 2: Reserve K_m at time m in case of (un-)favourable short-term market conditions.

m	2	3	4	5	10	15
$2*\underline{\sigma}$	0.746	0.713	0.688	0.668	0.604	0.567
$\underline{\mu}-2\%$	0.821	0.805	0.789	0.774	0.715	0.676
$\underline{\mu}-1\%$	0.836	0.827	0.818	0.811	0.782	0.765
$\underline{\mu}+1\%$	0.862	0.866	0.869	0.871	0.884	0.895
$\underline{\mu}+2\%$	0.874	0.882	0.889	0.896	0.920	0.935
$0.5*\underline{\sigma}$	0.884	0.898	0.910	0.920	0.958	0.977

Table 3: Survival probability in case of (un-)favourable short-term market conditions ($K_m = 0$).

2.5.2 Postponing rebalancing of investment portfolio

Suppose an insurance company knows that its current investment portfolio is not optimal. Assume however the insurer does not want to change to a different investment strategy immediately, but prefers to wait for a period of m years. In this case, the reserve K_m as defined by (15) can serve as an estimate of the cost of postponing the rebalancing (incorporating, of course, a certain ruin probability). Also the influence of postponing rebalancing on the optimal investment strategy can be investigated.

Similarly, suppose the insurer knows how much money he will have available in m years to put on its account (e.g. $K_m = 0$ if he does not want to invest extra money). In that case, the insurer can determine the influence of postponing the rebalancing on the ruin probability and on the optimal investment strategy. This way the insurer can get an idea of the maximum number of years m for which postponing changing its investment strategy is acceptable.

Example Take $K_0 = 175$, $n = 30$, and $\alpha_i = 10$ for $i = 1, ..., 30$, and assume we have the 2 asset classes as in Section 2.4.1. Maximizing the survival probability leads to an optimal investment strategy $\underline{\pi} = (0.5804, 0.4196)$, with $\mu(\underline{\pi}) = 0.0768$, $\sigma(\underline{\pi}) = 0.1236$ and corresponding maximal survival probability 85%.

Suppose the insurer has currently an investment portfolio given by $(0.25, 0.75)$, with corresponding drift 0.09 and standard deviation 0.1639. In other words, the insurer's current portfolio is more risky than the optimal one. Suppose the insurer does not want to rebalance immediately, but would like to keep its current strategy for m years. Using (33) we can determine the optimal investment strategy $\underline{\pi}^*$, to be followed from time m on, leading to a minimal additional reserve K_m^*. For the same long-term survival probability of 85%, the results are given in Table 4. For example, for $m = 5$ we find as a result $\underline{\pi}^* = (0.4824, 0.5176)$, with $\mu(\underline{\pi}^*) = 0.0807$ and $\sigma(\underline{\pi}^*) = 0.1343$. The minimal additional reserve at time 5 amounts to $K_5^* = 8.35$. In other words, if the insurer wants to postpone rebalancing for 5 years, and if he wants to keep the same survival probability of 85%, we estimate that he has to invest an additional amount of 8.35 at time 5, and change to the strategy $\underline{\pi}^*$.

From the results in Table 4 we can see increasing m, hence delaying the moment of rebalancing, leads to an increase in the additional reserve K_m^*. Also we see that the optimal strategy to be followed from time m on becomes more risky for increasing m.

	m				
	2	3	4	5	10
π_1^*	0.5226	0.5075	0.4925	0.4824	0.4523
π_2^*	0.4774	0.4925	0.5075	0.5176	0.5477
$\mu(\underline{\pi}^*)$	0.0791	0.0797	0.0803	0.0807	0.0819
$\sigma(\underline{\pi}^*)$	0.1298	0.1314	0.1331	0.1343	0.1378
K_m^*	3.59	5.38	6.94	8.35	13.90

Table 4: Minimal reserves K_m^* and optimal strategies for different values of m.

2.6 Conclusion

In this paper we discussed a general provisioning problem. We derived approximations that can be used to determine an estimate at time 0 of the provision to be set up at a certain time in the future, to ensure, after having paid the first liabilities, that all future liabilities can be fulfilled, incorporating a specified (low) ruin probability. We derived a convex lower and upper bound based on comonotonicity to determine an accurate and easily computable approximation for this reserve. We applied our results in an optimal portfolio selection framework, and illustrated it with numerical examples.

We have seen that the general provisioning problem can be useful in practice. As a first plausible interpretation, the additional reserve can be used to quantify the effect of temporary changes in market conditions. We have seen for example that such changes can significantly influence the long-term survival probability. Secondly, the setting discussed in this paper can be used to see if and how long postponing rebalancing of the invesmtent portfolio can be justified.

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A Numerical Results

The tables in this Appendix contain the numerical results of the examples of Section 2.3, comparing our lower bound and upper bound approximations to results obtained using simulation. Table 5 contains the results of the first example, where $\mu(\underline{\pi}_1) = \mu(\underline{\pi}_2) = 0.05$ and $\sigma(\underline{\pi}_1) = \sigma(\underline{\pi}_2) = 0.1$. Table 6 contains the results of the second example, where $\mu(\underline{\pi}_1) = 0.05$, $\sigma(\underline{\pi}_1) = 0.1$, $\mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$.

K_m	upper bound	simulation	lower bound	K_m	upper bound	simulation	lower bound
-250	0.99495	0.99644	0.99589	10	0.21083	0.18979	0.19807
-240	0.99319	0.99512	0.99443	20	0.17456	0.15064	0.15929
-230	0.99085	0.99340	0.99248	30	0.14320	0.11729	0.12636
-220	0.98776	0.99115	0.98989	40	0.11652	0.09065	0.09900
-210	0.98372	0.98803	0.98647	50	0.09413	0.06881	0.07670
-200	0.97846	0.98383	0.98199	60	0.07557	0.05187	0.05885
-190	0.97168	0.97851	0.97615	70	0.06035	0.03886	0.04476
-180	0.96301	0.97153	0.96861	80	0.04798	0.02886	0.03380
-170	0.95205	0.96245	0.95897	90	0.03801	0.02123	0.02537
-160	0.93832	0.95100	0.94676	100	0.03002	0.01563	0.01894
-150	0.92135	0.93660	0.93147	110	0.02366	0.01136	0.01409
-140	0.90063	0.91839	0.91256	120	0.01861	0.00830	0.01044
-130	0.87569	0.89626	0.88947	130	0.01462	0.00611	0.00772
-120	0.84614	0.86913	0.86168	140	0.01148	0.00440	0.00570
-110	0.81170	0.83670	0.82879	150	0.00900	0.00322	0.00420
-100	0.77227	0.79913	0.79051	160	0.00706	0.00231	0.00309
-90	0.72801	0.75489	0.74680	170	0.00554	0.00172	0.00228
-80	0.67930	0.70569	0.69788	180	0.00435	0.00126	0.00168
-70	0.62686	0.65106	0.64431	190	0.00342	0.00091	0.00124
-60	0.57165	0.59203	0.58699	200	0.00268	0.00066	0.00091
-50	0.51485	0.53085	0.52714	210	0.00211	0.00047	0.00067
-40	0.45780	0.46770	0.46621	220	0.00166	0.00037	0.00050
-30	0.40183	0.40432	0.40581	230	0.00131	0.00027	0.00037
-20	0.34822	0.34431	0.34751	240	0.00104	0.00020	0.00027
-10	0.29803	0.28737	0.29276	250	0.00082	0.00014	0.00020
0	0.25206	0.23562	0.24268				

Table 5: Comparison of upper bound and lower bound to simulated results, with $\mu(\underline{\pi}_1) = \mu(\underline{\pi}_2) = 0.05$ and $\sigma(\underline{\pi}_1) = \sigma(\underline{\pi}_2) = 0.1$.

K_m	upper bound	simulation	lower bound	K_m	upper bound	simulation	lower bound
-250	0.99943	0.99947	0.99944	-30	0.70308	0.70566	0.70414
-240	0.99922	0.99925	0.99923	-20	0.64021	0.64259	0.64112
-230	0.99892	0.99897	0.99894	-10	0.57053	0.57252	0.57120
-220	0.99851	0.99858	0.99854	0	0.49554	0.49640	0.49586
-210	0.99795	0.99807	0.99798	10	0.41748	0.41648	0.41738
-200	0.99718	0.99733	0.99723	20	0.33928	0.33760	0.33870
-190	0.99614	0.99631	0.99620	30	0.26427	0.26126	0.26321
-180	0.99472	0.99499	0.99480	40	0.19581	0.19192	0.19437
-170	0.99280	0.99312	0.99291	50	0.13681	0.13258	0.13513
-160	0.99022	0.99059	0.99037	60	0.08922	0.08476	0.08749
-150	0.98678	0.98724	0.98696	70	0.05366	0.04952	0.05208
-140	0.98219	0.98291	0.98243	80	0.02934	0.02619	0.02806
-130	0.97612	0.97684	0.97643	90	0.01434	0.01205	0.01343
-120	0.96816	0.96919	0.96854	100	0.00614	0.00477	0.00559
-110	0.95779	0.95924	0.95827	110	0.00225	0.00155	0.00196
-100	0.94441	0.94593	0.94499	120	0.00069	0.00041	0.00056
-90	0.92733	0.92900	0.92802	130	0.00017	0.00008	0.00013
-80	0.90576	0.90784	0.90657	140	0.00003	0.00001	0.00002
-70	0.87887	0.88127	0.87979	150	0.00000	0.00000	0.00000
-60	0.84581	0.84864	0.84684	160	0.00000	0.00000	0.00000
-50	0.80584	0.80854	0.80693	170	0.00000	0.00000	0.00000
-40	0.75835	0.76116	0.75946				

Table 6: Comparison of upper bound and lower bound to simulated results, with $\mu(\underline{\pi}_1) = 0.05$, $\sigma(\underline{\pi}_1) = 0.1$, $\mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$.

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