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Granger's Representation Theorem and Multicointegration

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Abstract

We consider multicointegration in the sense of Granger and Lee (1990), that is, the cumulated equilibrium error cointegrates with the process itself. It is shown, that if the process is given by the cointegrated VAR model for $I(1)$ variables, then multicointegration cannot occur. If, however, the cumulated process satisfies an $I(2)$ model then multicointegration may occur. Finally conditions are given on the moving average representation for the process to exhibit multicointegration. This result generalizes the analysis of Granger and Lee.

1 Introduction

Since Engle and Granger's (1987) seminal paper the concept of cointegration has developed progressively in several ways, and many extensions of the basic concept have been made. One such extension is the definition of multicointegration which refers to the case where the cumulation of equilibrium errors cointegrates with the original $I(1)$ variables of the system. Such situations arise naturally in economic models involving stock-flow relationships. One example, analysed in detail by Granger and Lee (1989), is the case where the two $I(1)$ flow series production, X_t, and sales, Y_t, cointegrate such that inventory investment $Z_t = X_t - Y_t$ is $I(0)$. It follows that $\sum_{i=1}^{t} Z_i$ is the level of inventories (stock) which might cointegrate with X_t and Y_t such that $U_t = \sum_{i=1}^{t} Z_i - aY_t - bX_t$ is $I(0)$. Thus, there are essentially two levels of cointegration between just two $I(1)$ time series. Other examples involve consumption, income, savings, and wealth, or new housing units started, new housing units completed, uncompleted starts, and housing units under construction, see Lee (1992).

Granger (1986) anticipates the notion of multicointegration, and the concept is formally developed in Granger and Lee (1989, 1990). In particular, they prove a representation theorem stating that multicointegrated time series are generated by an error-correction model which contains both Z_{t-1} and U_{t-1} as error-correction terms. In addition, they show that multicointegration can be derived from a standard linear quadratic adjustment cost framework often used in economics.

The papers by Granger and Lee constitute an important starting point for the analysis of multicointegrated time series, but they are somewhat limited in scope since they analyse only bivariate systems. Furthermore, in the estimation procedure they assume that the cointegration vector at the first level is known and, hence, does not have to be estimated. In estimating the cointegrating vector at the second level they propose to use the simple OLS estimator which in general is not optimal and does not allow hypothesis testing using standard asymptotic theory. An asymptotically efficient two step estimation procedure for this situation can be found in Johansen (1995), whereas maximum likelihood inference in the unrestricted multicointegrated $I(2)$ model is given in Johansen (1997).

The purpose of the present paper is to provide a more detailed analysis of multicointegration in multivariate systems of $I(1)$ time series. It is shown that although the interest lies in the analysis of the $I(1)$ series one cannot achieve multicointegration in the cointegrated error correction model for $I(1)$ variables. If, however, we assume that the cumulated variables satisfy an error correction model for $I(2)$ variables then we have the possibility of modelling multicointegration for $I(1)$ variables.

Engle and Yoo (1991) also suggest to relate multicointegration to $I(2)$ cointegration, and apply the Smith-McMillan decomposition to derive the VAR representation from the MA representation. Their results are discussed by Haldrup and Salmon (1996) for multivariate processes. Equivalently one can say that the process satisfies an integral control mechanism, see Hendry and Von Ungern Sternberg (1981).

The contents of the present paper is the following. First we give a theorem about the inversion of matrix valued functions which is the essence of the Granger representation theorem. We then discuss multicointegration in the usual error correction model for $I(1)$ variables, and show that this phenomenon cannot occur. If, however, we assume that the cumulated process satisfies an $I(2)$ model, then the results about this model can be phrased in terms of multicointegration. Finally we show how the general formulation of Granger's theorem solves the problem of deriving the $I(2)$ model from the moving average representation. This generalizes the results of Granger and Lee (1990). Throughout we assume that the equations generating the process have no deterministic terms. The representation results given are easily generalized, but the statistical analysis becomes more complicated, see Johansen (1995, 1997) and Paruolo (1996).

2 A general formulation of Granger's representation theorem

In this section we consider $n \times n$ matrix valued functions $A(z)$ with entries that are power series in a complex argument z. Let $|A(z)|$ denote the determinant and $adj A(z)$ the adjoint matrix.

Assumption 1 *The power series*

$$A(z) = \sum_{i=0}^{\infty} A_i z^i$$

is convergent for $|z| < 1 + \delta$, *and satisfies the condition that if* $|A(z)| = 0$, *then either* $|z| > 1 + \gamma$ *or* $z = 1$. *Here* $0 < \gamma < \delta$. *We assume further that* $A(z) = I$ *for* $z = 0$.

We are concerned with the power series for the function $A^{-1}(z)$. This function will have a power series expansion in a neighborhood of the origin, since $A(0) = I$ implies that $|A(z)| \neq 0$ for z sufficiently small, and hence $A^{-1}(z)$ exists.

We give a theorem that summarizes the Granger representation theorems for $I(0)$, $I(1)$ and $I(2)$ variables given in Johansen (1992). We give the results a purely analytic formulation without involving any probability theory, since the basic structure is then more transparent. The result allows a direct identification of the relevant coefficients of the inverse function in terms of the coefficients of the matrix function, and gives conditions for the presence of poles of the order $0, 1$, and 2 respectively. The result will be applied below to derive the autoregressive representation from the moving average representation and vice versa, and the explicit formulae allow one to discuss the coefficients in the moving average representation in terms of the estimated coefficients from the autoregressive model.

We expand the function $A(z)$ around $z = 1$ and define the coefficients $\dot{A}(1)$ and $\ddot{A}(1)$ by the expansion

$$A(z) = A(1) + (z-1)\dot{A}(1) + \frac{1}{2}(z-1)^2\ddot{A}(1) + \cdots,$$

which is convergent for $|z - 1| < \delta$. Thus

$$\dot{A}(1) = \left.\frac{dA(z)}{dz}\right|_{z=1},\ \ddot{A}(1) = \left.\frac{d^2A(z)}{d^2z}\right|_{z=1}.$$

For any $n \times m$ matrix a of full rank $m < n$, we denote by $a_\perp$ an $n \times (n-m)$ matrix of rank $n - m$ such that $a'a_\perp = 0$. We define $\bar{a} = a(a'a)^{-1}$, such that $a'\bar{a} = I$, and $\bar{a}a'$ is the projection of R^n onto the space spanned by the columns of a. For notational convenience we let $a_\perp = I$ if $a = 0$, and $m = 0$. Note that if we can write $a' = (a_1, a_2)$, where a_1 $(m \times m)$ has full rank, then we can choose

$$a_\perp = \begin{pmatrix} -a_1^{-1}a_2 \\ I_{n-m} \end{pmatrix}.$$

Note also that the choice of orthogonal complement is not unique. If $a_\perp^0$ and $a_\perp^1$ are any two choices, then $a_\perp^0 = a_\perp^1\xi$ for some ξ $(n-m) \times (n-m)$ of full rank.

Let $A(z)$ be a matrix power series which satisfies Assumption 1. Then the following results hold for the function $A^{-1}(z)$.

1. If $z = 1$ is not a root, then $A^{-1}(z)$ is a power series with exponentially decreasing coefficients.

2. If $z = 1$ is a root then $A(1)$ is of reduced rank $m < n$, and $A(1) = \xi\eta'$, where ξ and η are of dimension $n \times m$ and rank m. If further

$$|\xi'_\perp \dot{A}(1)\eta_\perp| \neq 0, \tag{1}$$

then

$$A^{-1}(z) = C\frac{1}{1-z} + C^*(z),$$

where $C^*(z)$ is a power series with exponentially decreasing coefficients, and where

$$C = -\eta_\perp \left(\xi'_\perp \dot{A}(1)\eta_\perp\right)^{-1} \xi'_\perp.$$

3. If $z = 1$ is a root such that $A(1) = \xi\eta'$ and if

$$\xi'_\perp \dot{A}(1)\eta_\perp = \phi\zeta',$$

is of reduced rank, where ϕ and ζ are $(n-m) \times k$ matrices of rank $k < n-m$, and if

$$\left|\phi'_\perp \xi'_\perp (\frac{1}{2}\ddot{A}(1) - \dot{A}(1)\bar{\eta}\bar{\xi}'\dot{A}(1))\eta_\perp\zeta_\perp\right| \neq 0, \tag{2}$$

then

$$A^{-1}(z) = C_2\frac{1}{(1-z)^2} + C_1\frac{1}{(1-z)} + C^{**}(z),$$

where $C^{**}(z)$ is a power series with exponentially decreasing coefficients. Expressions for the coefficients C_1 and C_2 can be found in Johansen (1992, Theorem 3). Here we give the expression

$$C_2 = \eta_\perp\zeta_\perp \left(\phi'_\perp \xi'_\perp (\frac{1}{2}\ddot{A}(1) - \dot{A}(1)\bar{\eta}\bar{\xi}'\dot{A}(1))\eta_\perp\zeta_\perp\right)^{-1} \phi'_\perp \xi'_\perp.$$

The proof can be found in the above mentioned reference for the case when $A(z)$ is a polynomial. The proof for the general case where infinitely many terms are allowed is the same. Note that the conditions (1), (2), and the expressions for the matrices C, C_1, and C_2 are invariant to the choice of orthogonal complement, such that it does not matter which orthogonal complement is chosen. Obviously the parameters ϕ and ζ will depend on the choice of orthogonal complement chosen for ξ and η.

Note that it is not enough to assume that the roots are outside the unit circle or equal to 1, since we could have infinitely many roots converging to the unit circle, which would ruin the proof. Hence we assume that the roots are bounded away from the unit disk or equal to 1. If $z = 1$ is a root then $A^{-1}(z)$ has a pole at the point $z = 1$, since

$$A^{-1}(z) = \frac{adjA(z)}{|A(z)|},$$

and $adjA(z)$ is a matrix valued power series with exponentially decreasing coefficients. The $I(1)$ condition (1) is necessary and sufficient for the pole to be of order 1. The function $C\frac{1}{1-z}$ has a pole of order 1 at $z = 1$ and the theorem says that the difference is a convergent power series. Thus the pole can be removed by subtracting the function $C\frac{1}{1-z}$. The $I(2)$ condition (2) is necessary and sufficient for the pole to be of order 2, in which case it can be removed by subtracting the function $C_2\frac{1}{(1-z)^2} + C_1\frac{1}{1-z}$, which also has a pole of order 2.

In order to apply this result in the autoregressive model

$$X_t = \sum_{i=1}^{k} \Pi_i X_{t-i} + \varepsilon_t,$$

define $A(z)$ to be the matrix polynomial

$$A(z) = I - \sum_{i=1}^{k} \Pi_i z^i.$$

Then $A^{-1}(z)$ gives the solution to the equations, that is, the coefficients in the expansion for $A^{-1}(z)$ determine X_t as a function of the errors ε_t. The translation of the result is via the lag operator, such that for a function $C(z) = \sum_{i=0}^{\infty} C_i z^i$ with exponentially decreasing coefficients and a sequence of i.i.d. variables ε_t, we define the stationary process

$$C(L)\varepsilon_t = \sum_{i=0}^{\infty} C_i \varepsilon_{t-i}.$$

For the expression $\frac{1}{1-z}$, we use the interpretation

$$(1 - L)^{-1}\varepsilon_t = \Delta^{-1}\varepsilon_t = \sum_{i=1}^{t} \varepsilon_i,$$

and $\frac{1}{(1-z)^2}$ is translated into

$$(1-L)^{-2}\varepsilon_t = \Delta^{-2}\varepsilon_t = \sum_{j=1}^{t}\sum_{i=1}^{j}\varepsilon_i.$$

The result of Theorem 1 can be used to check whether a given example of an autoregressive process is $I(0), I(1)$ or $I(2)$. It is the fundamental tool in building $I(1)$ and $I(2)$ models for autoregressive processes as we shall show below.

3 Multicointegration in the I(1) model

In the following we apply these results to discuss the problem of multicointegration as defined by Granger and Lee (1989, 1990).

The $n-$dimensional $I(1)$ process X_t is said to be multicointegrated with coefficient τ if $\tau' X_t$ is stationary and if the process $\sum_{i=1}^{t}\tau' X_i$ cointegrates with X_t, such that there exist coefficients ρ and ψ, that is, $\rho'\sum_{i=1}^{t}\tau' X_i + \psi' X_t$ is stationary.

We want to prove that multicointegration cannot take place in the error correction model for $I(1)$ variables

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1}\Gamma_i \Delta X_{t-i} + \varepsilon_t,\ t = 1, \dots, T, \tag{3}$$

where α and β are $n \times r$, where $r < n$.

Multicointegration cannot appear in the $I(1)$ model (3) if the process X_t is $I(1)$, that is, if

$$|\alpha'_{\perp}(I - \sum_{i=1}^{k-1}\Gamma_i)\beta_{\perp}| \neq 0. \tag{4}$$

We apply Theorem 1 to the polynomial

$$A(z) = (1-z)I - \alpha\beta' z - \sum_{i=1}^{k-1}\Gamma_i(1-z)z^i.$$

Here $A(1) = -\alpha\beta'$ and $\dot{A}(1) = -\alpha\beta' - I + \sum_{i=1}^{k-1}\Gamma_i$, such that the $I(1)$ condition (1) becomes the condition (4). The inverse polynomial has the expression as

$$A^{-1}(z) = C\frac{1}{1-z} + C^*(z)$$
$$= C\frac{1}{1-z} + C^* + (1-z)C_1^*(z),$$

such that the process has the representation

$$X_t = C\sum_{i=1}^{t}\varepsilon_i + C^*\varepsilon_t + \Delta Y_t + A, \tag{5}$$

where A depends on initial conditions, $\beta' A = 0$, and $Y_t = C_1^*(L)\varepsilon_t$ is a stationary process, see Johansen (1995). The matrix C is given by

$$C = \beta_\perp \left(\alpha'_\perp (I - \sum_{i=1}^{k-1}\Gamma_i)\beta_\perp\right)^{-1}\alpha'_\perp.$$

The cumulated equilibrium error has the form

$$\sum_{i=1}^{t}\beta' X_i = \beta' C^* \sum_{i=1}^{t}\varepsilon_t + \beta' Y_t - \beta' Y_0.$$

The common trends in the expression for X_t are of the form $\alpha'_\perp \sum_{i=1}^{t}\varepsilon_i$, and the common trends in the cumulated equilibrium error are of the form $\beta' C^* \sum_{i=1}^{t}\varepsilon_t$. In order to see if these cointegrate we have to find the matrix $\beta' C^*$. From the relation $A(z)A^{-1}(z) = I$ we find

$$\left((1-z)I - \alpha\beta' z - \sum_{i=1}^{k-1}\Gamma_i(1-z)z^i\right)\left(C\frac{1}{1-z} + C^* + (1-z)C_1^*(z)\right) = I.$$

or

$$\left(I - \sum_{i=1}^{k-1}\Gamma_i z^i\right)C + A(z)(C_1 + (1-z)C_1^*(z)) = I.$$

For $z = 1$ we find

$$\left(I - \sum_{i=1}^{k-1}\Gamma_i z^i\right)C - \alpha\beta' C^* = I,$$

which when multiplied by $\bar{\alpha}'$ gives

$$\beta' C^* = \bar{\alpha}'\Gamma\beta_\perp \left(\alpha'_\perp \Gamma\beta_\perp\right)^{-1}\alpha'_\perp - \bar{\alpha}',$$

where $\Gamma = I - \sum_{i=1}^{k-1} \Gamma_i z^i$. This shows that the non-stationarity of the cumulated equilibrium errors is given in part by the $n-r$ common trends $\alpha'_\perp \sum_{i=1}^t \varepsilon_i$ of the process X_t, and in part by r random walks $\alpha' \sum_{i=1}^t \varepsilon_i$ which do not appear in X_t. Thus any linear combination of X_t and $\sum_{i=1}^t \beta' X_i$ will necessarily contain the trends $\alpha' \sum_{i=1}^t \varepsilon_i$ and hence be non-stationary.

This shows that multicointegration can not appear in the $I(1)$ model. Another way of formulating this result is that no process $\mu' X_t$, where X_t is generated by the $I(1)$ model, will be $I(-1)$. In order to see this assume that there is a stationary process Z_t, say, and a coefficient vector $\mu \in R^n$ such that $\mu' X_t = \Delta Z_t$. From (5) we find that we must have $\mu' C = \mu' C^* = 0$. Hence $\mu = \beta\kappa$ for some vector κ and $\mu' C^* \alpha = \kappa' \beta' C^* \alpha = -\kappa' = 0$ shows the impossibility. We therefore next discuss the $I(2)$ model for the cumulated variables.

4 Multicointegration in the I(2) model

Next we want to prove a more constructive result where we take as a starting point that the cumulated processes are generated by an $I(2)$ model, see Engle and Yoo (1991). Thus we define

$$S_t = \sum_{i=1}^t X_i,$$

and assume that this new process is given by an autoregressive model, restricted such that it generates $I(2)$ variables. This model can be parametrized in many ways. A parametrization that allows freely varying parameter is given by

$$\Delta^2 S_t = \alpha(\rho' \tau' S_{t-1} + \psi' \Delta S_{t-1}) + \Omega \alpha_\perp (\alpha'_\perp \Omega \alpha_\perp)^{-1} \kappa' \tau' \Delta S_{t-1} + \varepsilon_t. \quad (6)$$

The parameters in this model are $(\alpha, \rho, \tau, \psi, \kappa, \Omega)$ and it is assumed that all parameters vary freely. This gives the possibility to derive the maximum likelihood estimators and find their asymptotic distributions, see Johansen (1997). We can add a lag polynomial applied to $\Delta^2 S_t$, to account for more short term dynamics.

The characteristic polynomial is given by

$$A(z) = (1-z)^2 I - \alpha(\rho' \tau' z + \psi'(1-z)z) - \Omega \alpha_\perp (\alpha'_\perp \Omega \alpha_\perp)^{-1} \kappa' \tau' (1-z) z,$$

such that

$$A(1) = -\alpha\rho'\tau', \ \dot{A}(1) = -\alpha\rho'\tau' + \alpha\psi' + \Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau'.$$

With $\beta = \tau\rho$ we find that $A(1) = -\alpha\beta'$ is of reduced rank and that

$$\alpha_\perp'\dot{A}(1)\beta_\perp = \kappa'\tau'\beta_\perp = \kappa'(\bar{\rho}_\perp\rho_\perp' + \bar{\rho}\rho')\tau'\beta_\perp = (\kappa'\bar{\rho}_\perp)(\rho_\perp'\tau'\beta_\perp),$$

since $\rho'\tau'\beta_\perp = \beta'\beta_\perp = 0$. This matrix is of reduced rank, and we can define $\phi = \kappa'\bar{\rho}_\perp$ and $\zeta = \beta_\perp'\tau\rho_\perp$. If further condition (2) is satisfied, the process S_t is $I(2)$, which implies that $X_t = \Delta S_t$ is $I(1)$. It is a consequence of the results in Johansen (1992) that $\tau'\Delta S_t = \tau' X_t$ is stationary, and furthermore that $\rho'\tau' S_t + \psi'\Delta S_t = \rho' \sum_{i=1}^t \tau' X_i + \psi' X_t$ is stationary. Thus we find that expressed in terms of the process X_t we have multicointegration and the error correction terms are exactly the integral correction term $\rho' \sum_{i=1}^t \tau' X_i + \psi' X_t$ and the usual error correction term $\tau' X_t$.

Thus this model is a general version of the error correction model derived by Granger and Lee (1990). The result shows that the general model for the $I(2)$ variable S_t can be formulated as an error correction model for $X_t = \Delta S_t$ which has both integral correction terms and equilibrium correction terms exactly as the model in Granger and Lee (1990). Model (6) can be written in this way as

$$\Delta X_t = \alpha(\rho' \sum_{i=1}^{t-1} \tau' X_i + \psi' X_{t-1}) + \Omega\alpha_\perp(\alpha_\perp'\Omega\alpha_\perp)^{-1}\kappa'\tau' X_{t-1} + \varepsilon_t.$$

Note that when the cumulated X_t satisfies an autoregressive error correction model then X_t itself does not, since the equations we find for X_t by differencing will have $\Delta\varepsilon_t$ as an error term.

5 Multicointegration and moving average models

The formulation of the result of Granger and Lee (1990) starts with the MA representation of the process and derives an (infinite) AR model for the process involving an integral correction term and an error correction term. We shall here show how Theorem 1 gives a necessary and sufficient condition for this construction to go through.

We consider the situation where we model the process in the usual form by its moving average form

$$\begin{aligned}\Delta X_t = C(L)\varepsilon_t = C_0\varepsilon_t + C_1\Delta\varepsilon_t + C^*(L)\Delta^2\varepsilon_t \\ = C_0\varepsilon_t + C_1\Delta\varepsilon_t + \Delta^2 Y_t.\end{aligned}$$

If we assume that C_0 is of reduced rank, such that $\tau' C_0 = 0$ for some $\tau \neq 0$, then we find

$$\begin{aligned}X_t &= X_0 + C_0\sum_{i=1}^t \varepsilon_i + C_1(\varepsilon_t - \varepsilon_0) + \Delta Y_t - \Delta Y_0, \\ \tau'\Delta X_t &= \tau' C_1 \Delta\varepsilon_t + \tau'\Delta^2 Y_t, \\ \tau' X_t &= \tau' X_0 + \tau' C_1(\varepsilon_t - \varepsilon_0) + \tau'(\Delta Y_t - \Delta Y_0), \\ \sum_{i=1}^t \tau' X_i &= \tau' C_1(\sum_{i=1}^t \varepsilon_i - t\varepsilon_0) + \tau'(Y_t - Y_0 - t\Delta Y_0).\end{aligned}$$

In order to find examples which exhibit multicointegration we only have to construct the matrices C_0 and C_1 such that there are coefficients ρ and ψ with the property that

$$\psi' C_0 - \rho'\tau' C_1 = 0,$$

since

$$\psi' X_t - \rho'\tau' \sum_{i=1}^t X_i,$$

does not contain any random walk. Thus there by choosing C_0 and C_1 appropriately it is easy to find examples of multicointegration. We shall now show how Theorem 1 generalizes the result of Granger and Lee.

Let the $n-$dimensional process X_t satisfy the equation

$$\Delta X_t = C(L)\varepsilon_t,$$

where $C(0) = I$, and we assume that the roots of $|C(z)| = 0$ are either bounded away from the unit disk or equal to 1.

1. If $z = 1$ is not a root, then ΔX_t satisfies an (infinite order) autoregressive equation

$$C^{-1}(L)\Delta X_t = \varepsilon_t.$$

The process X_t is $I(1)$ and does not cointegrate.

2. If $z = 1$ is a root, then $C(1) = \xi\eta'$ is of reduced rank $m < n$ and if further $\xi'_\perp \dot{C}(1)\eta_\perp$ has full rank then X_t satisfies an (infinite order) $I(1)$ model

$$\alpha\beta' X_t + A^*(L)\Delta X_t = \varepsilon_t,$$

with $\alpha\beta' = -\eta_\perp(\xi'_\perp \dot{C}(1)\eta_\perp)^{-1}\xi'_\perp$. Here $A^*(z) = C^{-1}(z) - \alpha\beta'\frac{1}{1-z}$ has exponentially decreasing coefficients.

3. If $z = 1$ is a root, such that $C(1) = \xi\eta'$ is of reduced rank $m < n$ and if further $\xi'_\perp \dot{C}(1)\eta_\perp = \phi\zeta'$ is of reduced rank $k < n - m$ and condition (2) holds then X_t satisfies an (infinite order) autoregressive model with integral and error correction terms.

1. If the roots of $|C(z)| = 0$ are all bounded away from the unit disk, then the power series of $C^{-1}(z) = \sum_{i=0}^{\infty} A_i z^i$ is convergent for $|z| < 1 + \delta$, where $\delta > 0$. This means that the coefficients in $C^{-1}(z)$ are exponentially decreasing such that the stationary process $\sum_{i=0}^{\infty} A_i \Delta X_{t-i}$ is well defined and equal to ε_t. Expanding the function $C(z)$ around $z = 1$ we find

$$C(z) = C(1) + (1 - z)C^*(z),$$

such that when summing the original equation from $s = 1$ to $s = t$ we find that

$$X_t = X_0 + C(1)\sum_{i=1}^{t} \varepsilon_i + Y_t - Y_0,$$

where $Y_t = C^*(L)\varepsilon_t$ is stationary. Thus X_t is an $I(1)$ process and since $C(1)$ has full rank it does not cointegrate.

2. Now assume that $z = 1$ is a root such that $C(1) = \xi\eta'$ is of reduced rank, but $\xi'_\perp \dot{C}(1)\eta_\perp$ has full rank. We then find from Theorem 1, that

$$(1 - z)C^{-1}(z) = A + (1 - z)A^*(z), \tag{7}$$

with $A = -\eta_\perp(\xi'_\perp \dot{C}(1)\eta_\perp)^{-1}\xi'_\perp$. Inserting this expression into (7) we find

$$AX_t + A^*(L)\Delta X_t = C^{-1}(L)\Delta X_t = \varepsilon_t.$$

This is the required result if we define $\beta = \xi_\perp$ and $\alpha = -\eta_\perp(\xi'_\perp \dot{C}(1)\eta_\perp)^{-1}$.

3. Finally assume that $C(1) = \xi\eta'$ and that $\xi'_\perp \dot{C}(1)\eta_\perp = \phi\zeta'$ is of reduced rank. In this case we have from Theorem 1

$$(1 - z)^2 C^{-1}(z) = A_2 + (1 - z)A_1 + (1 - z)^2 A^{**}(z).$$

Insert this into the equation for $S_t = \sum_{i=1}^{t} X_i$

$$\Delta^2 S_t = C(L)\varepsilon_t,$$

and we find

$$A_2 S_t + A_1 \Delta S_t + A^{**}(z)\Delta^2 S_t = C^{-1}(L)\Delta^2 S_t = \varepsilon_t.$$

Expressing this in terms of X we find

$$A_2 \sum_{i=1}^{t} X_i + A_1 X_t + A^{**}(z)\Delta X_t = \varepsilon_t.$$

This shows the occurrence of integral correction terms and error correction terms in the same model. This model can be expressed in terms of freely varying parameters as (6) by using the explicit form for the matrices A_2 and A_1 given in Johansen (1992).

6 An example

Consider the example given by Granger and Lee (1990)

$$\Delta X_t = \begin{pmatrix} a + \Delta(1-a) & -a^2(1-\Delta) \\ 1-\Delta & -a + \Delta(1+a) \end{pmatrix} \varepsilon_t$$

In this case the polynomial is

$$C(z) = \begin{pmatrix} a + (1-z)(1-a) & -a^2 z \\ z & -a + (1-z)(1+a) \end{pmatrix},$$

with

$$C(1) = \begin{pmatrix} a & -a^2 \\ 1 & -a \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -a \end{pmatrix},$$

$$\dot{C}(1) = \begin{pmatrix} -1+a & -a^2 \\ 1 & -1-a \end{pmatrix}.$$

In this case we find

$$\begin{pmatrix} 1 \\ -a \end{pmatrix}' \begin{pmatrix} -1+a & -a^2 \\ 1 & -1-a \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = 0.$$

Condition (2) is satisfied, since $\ddot{C}(1) = 0$ and we can take $\phi = \zeta = 0$. In this case the $I(1)$ condition reduces to

$$\xi_{\perp}' \dot{A}(1)\bar{\eta}\bar{\xi}' \dot{A}(1)\eta_{\perp} = 1 \neq 0.$$

Thus the cumulated X_t satisfies an $I(2)$ model which gives multicointegration.

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