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Lower Bounds on Externalities in Sunspot Models

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Abstract

A method for describing preferences independently of the functional form of utility is developed for log-linearized models and applied to finding general conditions for indeterminacy in the presence of externalities. It is shown that if the elasticity of output with respect to labor in the social production function is less than unity and externalities arise from output then there are no concave utility functions consistent with indeterminacy. Considering the same assumption in the more general setting where externalities to labor and capital can be different, we prove that there are no utility functions separable in consumption and labor and no utility functions in the KPR class consistent with indeterminacy.

These results show that there is an error in Bennett and Farmer (JET, 2000), who claimed that indeterminacy could occur with low externalities for non-separable preferences of the KPR form. Finally, we show that there can be sunspots below the former lower bounds, if we continue to allow for factor-specific externalities in the inputs and consider the admissible parameterspace for preferences, not restricted by functional forms.

1 Overview

It[1] turns out that the existence of sunspots in a one-sector economy depends on the specification of preferences and the way imperfections enter the model. We start from a setting where there is no restriction on functional forms of preferences whatsoever and where imperfections can be due to output externalities or factor-specific input externalities. The toolkit for dealing with preferences not restricted by any functional form is developed in a note on the δ_{xy} notation in the appendix.

The strategy to find the results presented here was to solve the problem in its most general form and then use numerical methods, most importantly random searches over the parameter space, that led to guesses about theorems that might hold, that were then subject to analytic proofs. This turned out to be a very useful way to go and the theorems and their proofs are presented here.

Section 2 defines the model.

In section 3 it is shown that in a steady state, around which the stability properties are evaluated, the admissible preference parameters are subject to a specific restriction.

In section 4 we prove that under the assumption $\alpha = am < 1$, $\beta = bm < 1$ (where α and β are the elasticities of output with respect to capital and labor in the social production function, a and b are the shares of capital and labor in the private production function) there are no concave utility functions that would lead to indeterminacy. This means that under empirically plausible assumptions there are no rational expectations stationary sunspot equilibria when imperfections are modelled as externalities in output.

Next, we relax the assumption that social and private partial elasticities of output are related by the same factor of proportionality for both capital and labor. That is, we consider $\alpha = an < 1$, $\beta = bm < 1$.

[1]I thank Roger Farmer, Thomas Steinberger and the audience in a presentation at UCLA for comments. Of course, all remaining errors are my own.

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This can be interpreted as imperfections arising from externalities specific to the use of inputs in production.

In section 5 we prove that in this more general environment indeterminacy is precluded by two standard classes of preferences. One theorem relates to separable preferences to other to preferences of the KPR form.

In section 6 we continue to allow for this more general formulation of imperfections. It turns out that there exist parametrizations of utility which are consistent with indeterminacy when $\alpha = an < 1$, $\beta = bm < 1$. We find that, even when there are no externalities in labor, there are preferences consistent with indeterminacy if capital externalities are high enough, while capital externalities are still too low to allow for endogenous growth.

Section 7 concludes the paper.

2 A nesting model

The following model nests the models of Benhabib and Farmer (1994) and Bennett and Farmer (2000) and will be used to search for a credible pure sunspot economy in a more general class of preferences.

The technology is the same as in the two above-mentioned nested models. They assume a large number of competitive firms, each of which produces a homogenous commodity using a constant-returns-to-scale technology.

$$Y = K^a l^b E, \tag{1}$$

where $a+b=1$ and $E>0$. Each firm takes E as given, however, in practice E is determined by the activity of other firms. This imperfection is modelled by the equation

$$E = \overline{K}^{\alpha-a} \overline{l}^{\beta-b}, \tag{2}$$

where $\overline{K}$ and $\bar{l}$ denote economy wide averages. It is further assumed that $1 > \alpha \geq a$, $\beta \geq b$, and $\alpha + \beta > 1$. Combining (1) and (2) we get the social production function

$$Y = K^{\alpha} l^{\beta}. \tag{3}$$

Factor markets are competitive and factors of production receive fixed shares of national income,

$$b = \frac{wl}{Y}, \tag{4}$$

$$a = \frac{rK}{Y}, \tag{5}$$

where w is the wage rate and r is the rental rate, both measured in terms of the consumption good.

Without loss of generality, imperfections are implicitly parametrized by the parameters n and m, where $n \geq 1$, $m \geq 1$, $n + m > 2$, such that[2]

$$\alpha = an,$$

$$\beta = bm.$$

The representative consumer maximizes the present value of utility

$$\int_0^{\infty} u(c, l) e^{-\rho t} dt,$$

where $u(.,.)$ is assumed to be concave,

subject to the budget constraint

[2]When we let $n = m$, m can be interpreted as the returns-to-scale parameter; more specifically, as the elasticity of scale at the social level, while at the firm level returns-to-scale are constant.

$$\dot{K} = (r - \delta)K + wl - c,$$

an initial condition for capital and the usual no Ponzi scheme constraint.

We start by setting up the Hamiltonian of the problem:

$$H = u(c, l) + \Lambda \left[(r - \delta) K + wl - c \right].$$

The necessary and, by the assumption of concavity of $u(c, l)$, sufficient optimality conditions are:

$$\frac{\partial H}{\partial c} = \frac{\partial u(c, l)}{\partial c} - \Lambda = 0 \Rightarrow \Lambda = \frac{\partial u(c, l)}{\partial c} \tag{6}$$

$$\frac{\partial H}{\partial l} = \frac{\partial u(c, l)}{\partial l} + w\Lambda = 0 \Rightarrow w\Lambda = -\frac{\partial u(c, l)}{\partial l} \tag{7}$$

Note that we can combine (4), (6) and (7) to describe the labor market in the model as

$$b\frac{Y}{l} = w = \frac{-\frac{\partial u(c,l)}{\partial l}}{\frac{\partial u(c,l)}{\partial c}},$$

and using (3) as

$$bK^{\alpha} l^{\beta - 1} = w = \frac{-\frac{\partial u(c,l)}{\partial l}}{\frac{\partial u(c,l)}{\partial c}}. \tag{8}$$

In the optimum the co-state variable moves according to:

$$\dot{\Lambda} = \rho\Lambda - \frac{\partial H}{\partial K} = \rho\Lambda + \Lambda\delta - \Lambda r,$$

$$\dot{\Lambda} = \Lambda \left(\rho + \delta - \frac{aY}{K} \right). \tag{9}$$

We can substitute from (4) and (5) into the law of motion for capital and get:

$$\dot{K} = \left(\frac{aY}{K} - \delta \right) K + \frac{bY}{l} l - c,$$

$$\dot{K} = Y - \delta K - c. \tag{10}$$

The transversality condition of the problem is

$$\lim_{T \to \infty} e^{-\rho T} \Lambda = 0.$$

(9) and (10) can be restated as

$$\frac{\dot{\Lambda}}{\Lambda} = \rho + \delta - \frac{aY}{K},$$

$$\frac{\dot{K}}{K} = \frac{Y}{K} - \delta - \frac{c}{K}.$$

We then define

$$\lambda = \log \Lambda, \;\; k = \log K,$$

$$\widetilde{l} = \log l, \;\; \widetilde{c} = \log c, \;\; y = \log Y,$$

to reformulate the previous system as

$$\dot{\lambda} = \rho + \delta - a e^{y-k}, \tag{11}$$

$$\dot{k} = e^{y-k} - \delta - e^{\widetilde{c}-k}, \tag{12}$$

which together with $k(0) = k_0$ and $\lim_{T \to \infty} e^{\lambda - \rho T} = 0$, completely describes the dynamics of the system. Next, we let hats over variables define their deviations from the steady state, whose existence we postulate, and find the log-linearizations of the contemporaneous side conditions:

$$\widehat{y} = \alpha\widehat{k} + \beta\widehat{l}, \tag{13}$$

$$(1 + \delta_{ll} - \delta_{cl})\,\widehat{l} + (\delta_{lc} - \delta_{cc})\,\widehat{c} = \widehat{y}, \tag{14}$$

$$\widehat{\lambda} = \delta_{cc}\widehat{c} + \delta_{cl}\widehat{l}. \tag{15}$$

This is where our δ_{xy} parameters, as introduced in the appendix of this paper, come in and allow us to work with utility at its most general level. (13) comes from the social production function, (14) and (15) come from the first-order conditions for labor and consumption.

Our next goal is to obtain a relationship of the form

$$\begin{pmatrix} \widehat{y} - \widehat{k} \\ \widehat{c} - \widehat{k} \end{pmatrix} = \Phi \begin{pmatrix} \widehat{\lambda} \\ \widehat{k} \end{pmatrix}, \tag{16}$$

with

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix}. \tag{17}$$

Then we can approximate

$$\begin{pmatrix} \dot{\lambda} \\ \dot{k} \end{pmatrix} = J \begin{pmatrix} \widehat{\lambda} \\ \widehat{k} \end{pmatrix}, \tag{18}$$

where

$$J = \begin{bmatrix} -a\phi_1\frac{\rho+\delta}{a} & -a\phi_2\frac{\rho+\delta}{a} \\ \phi_1\frac{\rho+\delta}{a} - \phi_3\left(\frac{\rho+\delta}{a} - \delta\right) & \phi_2\frac{\rho+\delta}{a} - \phi_4\left(\frac{\rho+\delta}{a} - \delta\right) \end{bmatrix}. \tag{19}$$

We write (13) and (15) in the following form:

$$\begin{bmatrix} 1 & 0 \\ 0 & -\delta_{cc} \end{bmatrix} \begin{pmatrix} \widehat{y} - \widehat{k} \\ \widehat{c} - \widehat{k} \end{pmatrix} + \begin{pmatrix} -\beta \\ -\delta_{cl} \end{pmatrix} \widehat{l} + \begin{bmatrix} 0 & 1-\alpha \\ 1 & -\delta_{cc} \end{bmatrix} \begin{pmatrix} \widehat{\lambda} \\ \widehat{k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and substitute for $\widehat{l}$, using (14) and (15), by $\widehat{l} = \frac{\widehat{\lambda} + \widehat{y} - \delta_{lc}\widehat{c}}{1+\delta_{ll}}$, to give us a system of the form

$$A \begin{pmatrix} \widehat{y} - \widehat{k} \\ \widehat{c} - \widehat{k} \end{pmatrix} + B \begin{pmatrix} \widehat{\lambda} \\ \widehat{k} \end{pmatrix} = 0,$$

where

$$A = \begin{bmatrix} 1 + \delta_{ll} - \beta & \beta\delta_{lc} \\ -\delta_{cl} & -\delta_{cc} - \delta_{cc}\delta_{ll} + \delta_{cl}\delta_{lc} \end{bmatrix}, \tag{20}$$

$$B = \begin{bmatrix} -\beta & 1 + \delta_{ll} - \beta + \beta\delta_{lc} - \alpha - \alpha\delta_{ll} \\ 1 + \delta_{ll} - \delta_{cl} & -\delta_{cl} - \delta_{cc} - \delta_{cc}\delta_{ll} + \delta_{cl}\delta_{lc} \end{bmatrix}. \tag{21}$$

Solve for $\left(\widehat{y} - \widehat{k}\right)$ and $\left(\widehat{c} - \widehat{k}\right)$ in terms of $\widehat{\lambda}$ and $\widehat{k}$ to get the expression in (16) with

$$\Phi = -A^{-1}B. \tag{22}$$

We have now arrived at a point where the approximated dynamics of the system are fully specified in terms of the parameters of the model, by combining (16), (17), (18), (19), (20), (21), (22).

This puts us into a position to check for determinacy of the dynamic equilibrium of the economy. The economy is a sunspot economy iff both eigenvalues of J have negative real parts, or equivalently, if the determinant of J is positive and the trace of J is negative.

The analytic expressions for the trace and determinant of J are:

$DetJ = -(\rho + \delta(1-a))\frac{\rho+\delta}{a}\frac{1-\alpha-\delta_{cl}+\delta_{ll}-\beta+\delta_{cl}\alpha-\delta_{cc}\beta+\delta_{lc}\beta-\alpha\delta_{ll}}{(-\delta_{cc}\delta_{ll}-\delta_{cc}+\beta\delta_{cc}+\delta_{cl}\delta_{lc})}$

$TraceJ = \frac{1}{\tau}\left(\delta_{cc} - \delta_{lc}\right)\beta\left(\rho+\delta\right) - \frac{\alpha}{a\tau}\left\{\left[\delta_{cc}\delta_{ll} + \delta_{cc} - \delta_{cl}(\delta_{lc} - 1)\right]\left(\rho+\delta\right)\right\} + \frac{\alpha\delta_{cl}\delta}{\tau} - \delta,$

with $\tau = (-\delta_{cc}\delta_{ll} - \delta_{cc} + \beta\delta_{cc} + \delta_{cl}\delta_{lc})$

3 A restriction on δ_{xy} in the steady state from the symmetry of the Hessian

Consider the definitions of δ_{cl} and δ_{lc}, as given in the appendix on the δ_{xy} notation:

$$\delta_{cl} = \frac{l}{u_c}\frac{\partial u_c}{\partial l} = \frac{l}{u_c}u_{cl} \tag{23}$$

$$\delta_{lc} = \frac{c}{u_l}\frac{\partial u_l}{\partial c} = \frac{c}{u_l}u_{lc} \tag{24}$$

By the symmetry of the Hessian, $u_{cl} = u_{lc}$, we get

$$-\frac{\delta_{cl}}{\delta_{lc}} = -\frac{u_l l}{u_c c}. \tag{25}$$

Note that this is the only restriction we can derive from the definitions of δ_{xy} that involves not other expressions than those present in the equilibrium conditions.

Solve the two dynamic equilibrium conditions for the steady state:

$$\frac{\dot{\Lambda}}{\Lambda} = \rho + \delta - \frac{aY}{K} = 0$$

$$\frac{\dot{K}}{K} = \frac{Y}{K} - \delta - \frac{c}{K} = 0$$

This yields the steady state solutions:

$$\frac{Y}{K} = \frac{\rho + \delta}{a} \tag{26}$$

$$\frac{c}{K} = \frac{\rho + \delta(1 - a)}{a} \tag{27}$$

And hence also:

$$\frac{Y}{c} = \frac{\rho + \delta}{\rho + \delta(1-a)} \tag{28}$$

Consider the labor market equation:

$$b\frac{Y}{l} = w = \frac{-\frac{\partial u(c,l)}{\partial l}}{\frac{\partial u(c,l)}{\partial c}},$$

multiply by l and divide by c to obtain

$$b\frac{Y}{c} = \frac{wl}{c} = \frac{-u_l l}{u_c c}. \tag{29}$$

Substituting from (28) on the LHS and using the relationship in (25) on the RHS, we have

$$\frac{b(\rho + \delta)}{\rho + \delta(1-a)} = -\frac{\delta_{cl}}{\delta_{lc}}, \tag{30}$$

and finally

$$\delta_{cl} = -\frac{b\rho + b\delta}{\rho + b\delta}\delta_{lc} \equiv k\delta_{lc}. \tag{31}$$

4 A general impossibility theorem and the way out

We first consider the case where imperfections in the economy are modelled es externalities in output by[3]

$$E = \bar{Y}^{\frac{m-1}{m}}, \tag{32}$$

[3]Bars over variables denote economy-wide averages, equal to the representative firm's values in a symmetric equilibrium.

or, equivalently, to fit into the model as defined above, as

$$E = \left(\bar{K}^a \bar{l}^b\right)^{m-1}, \tag{33}$$

where $m > 1$, such that

$$Y = K^a l^b E \tag{34}$$

becomes

$$Y = K^{am} l^{bm} = K^\alpha l^\beta. \tag{35}$$

Definition 1 *A utility function $u(c, l)$ is said to satisfy the concavity assumption C if and only if the corresponding δ_{xy} parameters satisfy: $\delta_{cc} < 0, \delta_{ll} \geq 0, \delta_{cc}\delta_{ll} \leq \delta_{cl}\delta_{lc}$.*[4] *(See the appendix on the δ_{xy} notation for a derivation).*

Theorem 1 *Under the assumption $\alpha = am < 1$, $\beta = bm < 1$ there exist no utility functions $u(c, l)$ satisfying the concavity assumption C that are compatible with the existence of stationary sunspot equilibria.*

Proof. For simplicity we introduce the shorthand notation: $\delta_{cc} = u, \delta_{cl} = v, \delta_{lc} = w, \delta_{ll} = x$.

(1) Concavity

Concavity requires

$ux \leq kw^2$

or $kw^2 - ux \geq 0$

since k is negative, and $u < 0, x \geq 0$, from the concavity assumption, for given u and x the concavity region in the space of w is the closed interval between the roots of the polynomial whose graph is a parabola open from below, since k is negative; and existence of a solution is guaranteed by $u < 0, x \geq 0$.

The bounds are $\underline{w_c} = -\sqrt{\frac{ux}{k}}, \overline{w_c} = +\sqrt{\frac{ux}{k}}$.

[4]The strictness of the first inequality ensures non-trivial dynamics.

(2) Trace

We show that the trace cannot be negative on the domain of concavity.

From the solution of the model above we have

$TraceJ = \frac{1}{\tau}(u-w)\beta(\rho+\delta) - \frac{\alpha}{a\tau}\{[ux+u-v(w-1)](\rho+\delta)\} + \frac{\alpha v\delta}{\tau} - \delta,$

with $\tau = (-ux - u + \beta u + vw)$.

Concavity and the assumption that $\beta = bm < 1$ can be shown to ensure that τ is positive:

Rearranging and using the restriction $v = kw$ yields

$\tau = -ux + kw^2 - u(1-mb)$.

Since the concavity condition says that $kw^2 - ux \geq 0$, and $u < 0$, and we have assumed $mb < 1$, positivity of τ follows.

Hence, the negativity of the trace can be expressed as

$(u-w)mb(\rho+\delta) - n[ux+u-v(w-1)](\rho+\delta) + nav\delta - \tau\delta < 0.$

We substitute for $v = kw$ and rearrange to get

$-u\{x[n\rho + (n-1)\delta] + \delta(n-1) + \rho(n-mb)\} - wb(\rho+\delta)(m-n) + w^2k[n\rho + (n-1)\delta] < 0.$

For any given values of all the parameters except w, the LHS is a quadratic function in w. Given that $u < 0, x \geq 0$ (by concavity), $\rho > 0, \delta > 0, n \geq 1, mb < 1$ we see that the value at $w = 0$ (the intercept) is positive and that the coefficient of w^2 is negative, such that the graph of the LHS is a parabola open from below.

Consider now the case $m = n$:

The trace condition becomes

$-u\{x[m\rho + (m-1)\delta] + \delta(m-1) + m\rho(1-b)\} + w^2k[m\rho + (m-1)\delta] < 0.$

Note that the term linear in w has cancelled, such that the graph of the parabola is now symmetric in w. It is this symmetry that is crucial in completing the proof.

We will show that at no point of the concavity domain the above condition can hold. Because of the graph of the LHS is a symmetric parabola open from below and the concavity domain is also symmetric in w, we can establish positivity of the LHS on the entire concavity domain, by considering its values at $w = 0$ and at the border of the concavity domain.

The value of the LHS at $w = 0$ is $-u\left\{x\left[m\rho + (m-1)\,\delta\right] + \delta(m-1) + m\rho(1-b)\right\}$, which is positive, since $u < 0, x \geq 0$ (by concavity), $\rho > 0, \delta > 0, mb < 1$.

The value of the LHS at the borders of the domain of concavity, that is at $w = \pm\sqrt{\frac{ux}{k}}$, is $-u[\delta(n-1)+m\rho(1-b)]$, which is unambiguously positive too. So, the trace condition cannot be satisfied for concave utility functions if $m = n$. ■

The proof of this theorem also gives the intuition of how to find parametrizations of the economy that lead to indeterminacy. Relaxing the restriction $m = n$, and letting m and n differ by a high enough value, the parabola will lose its symmetry in w and shift. If the shift in some direction is big enough the trace condition can be satisfied on the domain of concavity. This is the idea behind the results below in section 6.

This result implies that Benhabib and Farmer (1994) hit the lower bound of externalities in production required for indeterminacy, by requiring $bm > 1$ for a standard utility function that is logarithmic in consumption and additively separable in labor, where disutility of labor is linear. There are no other concave utility functions that would be compatible with indeterminacy for lower levels of the externality parameter m, when imperfections are modelled as externalities in production.

The plausibility of assuming $bm > 1$ was put into question by empirical work. Basu and Fernald (1997) estimated $m = 1.09$, meaning that the inequality would not hold for reasonable values of the labor-share parameter b.

Bennett and Farmer (2000) claimed to have shown that indeterminacy can occur for lower levels of the output externality by assuming preferences in a more general class utility functions of the so-called KPR form, named after King, Plosser and Rebelo (1988). They develop a condition which generalizes the condition of Benhabib and Farmer (1994). The latter required the slope of the labor demand curve to be greater than the slope of the constant consumption labor supply curve. The more recent condition of Bennett and Farmer (2000) required the slope of the labor demand curve to be greater than the slope of the Frisch labor supply curve.[5]This condition is nested in the solution above. It guarantees that the denominator of the determinant is negative, i.e. $\tau = (-\delta_{cc}\delta_{ll} - \delta_{cc} + \beta\delta_{cc} + \delta_{cl}\delta_{lc}) < 0$, or equivalently $\beta - 1 > \delta_{ll} - \frac{\delta_{cl}\delta_{lc}}{\delta_{cc}}$. Note that the LHS is the slope of the log-approximated labor demand curve and the RHS is the slope of the log-approximated Frisch labor supply curve as derived in the appendix on the δ_{xy} notation. Hence, the condition is correct but cannot be applied to cases where $\beta = mb < 1$, since then the negativity of the LHS would require negativity of the RHS. However, as shown in the appendix on the δ_{xy} notation, the slope of the Frisch labor supply curve is negative if and only if the utility function $u(c, l)$ is not concave.[6]

5 Factor-specific externalities and standard utility functions

We now allow for the possibility that externalities are specific to the factors of production.[7] This corresponds to modelling

[5]The Frisch labor supply curve coincides with the constant consumption labor supply curve in the case of a utility function that is logarithmic in consumption and separable in labor, which explains the generalization achieved by Bennett and Farmer (2000). This can readily be seen from the propositions on Frisch labor supply and constant consumption labor supply in the appendix on the δ_{xy} notation.

[6]Bennett and Farmer (2000) calibrate $k = -1$. However, it was shown above that k is determined by parameters of the model by the assumption of a steady state such that there is no room for further calibration. For $0 < b < 1, \rho > 0, \delta > 0$, k is always greater than -1.

[7]Factor-specific externalities are proposed and motivated by Harrison and Weder (forthcoming), who also check for the relative importance of scale economies from labor and capital. However, they require utility to be logarithmic in consumption and additively separable and linear in labor. They conclude that it is primarily the exter-

$$E = \bar{K}^{a(n-1)}\bar{l}^{b(m-1)}, \tag{36}$$

where $n \geq 1, m \geq 1, n+m > 2$, such that

$$Y = K^a l^b E \tag{37}$$

becomes

$$Y = K^{an} l^{bm} = K^{\alpha} l^{\beta}. \tag{38}$$

5.1 Separable preferences preclude indeterminacy

Theorem 2 *There are no preferences described by the class of utility functions*

$$\begin{aligned} u(c,l) &= U(c) - V(l), \\ U'(c) &> 0, U''(c) < 0, V'(l) > 0, V''(l) \geq 0, \end{aligned} \tag{39}$$

which are compatible with indeterminacy if $\beta = bm < 1, \alpha = an <$ 1.

Proof. Let the preference parameters be defined as

$$\begin{aligned} \delta_{cc} &= -r < 0, \\ \delta_{ll} &= \chi \geq 0. \end{aligned}$$

The assumption of additive separability of utility in consumption and labor is reflected in $\delta_{cl} = 0$, $\delta_{lc} = 0$.

The dynamic system from the model as solved above is indeterminate iff its trace is negative and the determinant is positive.

Consider the trace:

$$TraceJ = \frac{1}{\tau}(\delta_{cc} - \delta_{lc})\beta(\rho+\delta) - \frac{\alpha}{a\tau}\{[\delta_{cc}\delta_{ll} + \delta_{cc} - \delta_{cl}(\delta_{lc} - 1)](\rho+\delta)\} + \frac{\alpha\delta_{cl}\delta}{\tau} - \delta,$$

nalities associated with labor that generate indeterminacy in the one-sector model. Below we find that in a general setting, where we do not restrict preferences to be representable by standard classes of utility functions, we can well have indeterminacy with externalities in capital only.

where $\tau = \left(-\delta_{cc}\delta_{ll} - \delta_{cc} + \beta\delta_{cc} + \delta_{cl}\delta_{lc}\right)$.

In the case of separability, with $\delta_{cl} = 0$ and $\delta_{lc} = 0$, this becomes:

$$\begin{aligned} TraceJ &= \frac{1}{\tau}\delta_{cc}\beta\left(\rho+\delta\right) - \frac{\alpha}{a\tau}\delta_{cc}\left(1+\delta_{ll}\right)\left(\rho+\delta\right) - \delta, \\ \tau &= \delta_{cc}\left(\beta - 1 - \delta_{ll}\right), \end{aligned}$$

which we can combine to

$$TraceJ = -\frac{\beta}{1+\delta_{ll}-\beta}\left(\rho+\delta\right) + \frac{\alpha}{a\left(1+\delta_{ll}-\beta\right)}\left(1+\delta_{ll}\right)\left(\rho+\delta\right) - \delta.$$

For indeterminacy this expression needs to be negative:

$$TraceJ = \left(\rho+\delta\right)\frac{1}{1+\delta_{ll}-\beta}\left[\frac{\alpha}{a}\left(1+\delta_{ll}\right) - \beta\right] - \delta < 0$$

Above we assumed that $\beta - 1 < 0$, or $1 - \beta > 0$. $\delta_{ll} > 0$, by concavity. $\alpha = an$, so $\frac{\alpha}{a} = n \geq 1$. Hence, the inequality above cannot hold, meaning that indeterminacy requires non-separable utility if $\beta < 1$. ■

5.2 KPR preferences preclude indeterminacy

Theorem 3 *There are no preferences described by the class of utility functions, as specified in King, Plosser and Rebelo (1988),*

$$\begin{aligned} u(c,l) &= \frac{c^{1-\sigma}}{1-\sigma}v(l), \ \textit{for } \sigma > 0, \sigma \neq 1 \\ u(c,l) &= \log c - v(l), \ \textit{for } \sigma = 1 \end{aligned}$$

which are compatible with indeterminacy if $\beta = bm < 1, \alpha = an < 1$.

Proof. Substituting the utility function into the definition of the δ_{xy} parameters, we find the restriction imposed on preference parameters by this functional form of utility. KPR preferences imply $\delta_{lc} = 1 + \delta_{cc}$. We reintroduce the shorthand notation $\delta_{cc} = u, \delta_{cl} = v, \delta_{lc} = w, \delta_{ll} = x$. In functional form of KPR preferences implies $w = 1 + u$.

Consider the domain of concavity: $ux \leq kw^2$, which then becomes $x \geq k\frac{(1+u)^2}{u}$.

We show that $DetJ = -(\rho+\delta(1-a))\frac{\rho+\delta}{a}\frac{1-\alpha-\delta_{cl}+\delta_{ll}-\beta+\delta_{cl}\alpha-\delta_{cc}\beta+\delta_{lc}\beta-\alpha\delta_{ll}}{(-\delta_{cc}\delta_{ll}-\delta_{cc}+\beta\delta_{cc}+\delta_{cl}\delta_{lc})}$ cannot be positive.

Given concavity and the assumption $mb < 1$, $\tau = (-\delta_{cc}\delta_{ll} - \delta_{cc} + \beta\delta_{cc} + \delta_{cl}\delta_{lc})$ is unambiguously positive. Hence, $1 - \alpha - \delta_{cl} + \delta_{ll} - \beta + \delta_{cl}\alpha - \delta_{cc}\beta + \delta_{lc}\beta - \alpha\delta_{ll}$ would have to be negative. We show that this cannot be the case. Rearrange the previous expression to get

$$-\beta(1 + \delta_{cc} - \delta_{lc}) + (1 - \alpha)(1 - \delta_{cl} + \delta_{ll}) < 0$$

By the KPR restriction, the first term drops out:

$$-\beta[1 + u - (1 + u)] + (1 - \alpha)[1 - k(1 + u) + x] < 0$$

Since $\alpha < 1$, by assumption, we are left with

$1 - k - ku + x < 0$. We substitute for x by its lower limit on the concavity domain:

$1 - k(1 + u) + \frac{k(1+u)^2}{u} < 0$, simplify to obtain

$u + k(1 + u) > 0$, substituting for k we get

$u\rho(1-b)-b(\rho+\delta) > 0$, which cannot hold, given $u < 0$ (concavity), $0 < b < 1, \rho > 0, \delta > 0$. ■

The previous two theorems show that even if we relax the assumption that $m = n$, we cannot find indeterminacy in economies where $mb < 1$ in two standard classes of utility functions, namely the separable class and the KPR class.[8]

[8]KPR preferences were derived by King, Plosser and Rebelo (1988) to make a neoclassical accumulation economy with labor-leisure choice compatible with steady state growth. Rebelo (1991) points out another way to make an economy with exogenous growth compatible with labor-leisure choice. This is by the assumption of utility depending on the state of technology S by $u(c, lS)$. It is this way that can be followed in order to make non-KPR preferences, like the parametrizations that lead to indeterminacy in the next section, fit into a model that includes exogenous growth.

6 Factor-specific externalities and general parametrized preferences

Given the solution of the model and its stability properties as summarized in $DetJ$ and $TraceJ$, it can be checked whether there are parametrizations of the economy that lead to indeterminacy when allowing for weaker assumptions than in the three non-existence theorems above. For given values of a, b, ρ, δ, we search over the space of admissible values of the externality parameters m and n, and the space of the preference parameters δ_{xy} for combinations that lead to indeterminacy and satisfy the other assumptions of the model (e.g. concavity of $u(c, l)$).

The results presented here are for values of $a = 0.3, b = 0.7, \rho = 0.065, \delta = 0.1$.[9] In the parameter space of preferences we consider $(\delta_{cc,}\delta_{lc,}\delta_{ll}) \in D = \{[-10, 0] \times [-15, 15] \times [0, 10]\}$.[10]

It turns out that by allowing for both factor-specific externalities and more general preferences than in the standard classes considered above there are economies with $\alpha = an < 1$, $\beta = bm < 1$ that have stationary sunspot equilibria.

Thus, in contrast and complementary to, the previous three non-existence theorems we can state the following positive existence theorem:

Theorem 4 *There are preferences*[11]*, as parametrized by δ_{xy}, and values for the externality parameters m and n, satisfying $\alpha = an < 1$, $\beta =$*

[9] This benchmark calibration is also used by Benhabib and Farmer (1994) and Bennett and Farmer (2000).

[10] Note that in the KPR case $-\delta_{cc}$ corresponds to the coefficient of relative risk aversion, σ, and that in the separable case δ_{ll} defines the curvature of disutility of labor. The box above thus is wide enough to contain parametrizations of utility that are usually considered reasonable. The range along the δ_{lc} dimension then guarantees that we take into account the entire concave domain for given δ_{cc} and δ_{ll}.

[11] The three theorems above are about non-existence. Preferences are represented by the utility function $u(c, l)$ from which we derive (unique) parameters δ_{xy}, the only properties of utility that matter for our problem. Hence, having shown that there exist no δ_{xy} consistent with indeterminacy, it was clear that there are no utility functions and preferences consistent with indeterminacy.

However, this theorem is about existence in a positive sense. We show that there

$bm < 1$ such that the model is indeterminate and, hence, stationary sunspot equilibria exist. In particular, indeterminacy can occur without any externality in labor, i.e. $m = 1$, when the externality in capital is high enough.

Figure 1 shows combinations of the labor externality parameter, m, and the capital externality parameter, n, for which we could find admissible preference parameters in the box D consistent with indeterminacy.[12] We see that there are no hits on the dashed line, corresponding to the case of an output externality, where $m = n$. This corresponds to Theorem 1, proven analytically above. Moving away from the dashed line, we can find combinations of the labor and capital externality, satisfying the assumption $\alpha = an < 1$, $\beta = bm < 1$, that are consistent with indeterminacy for some triple $(\delta_{cc,}\delta_{lc,}\delta_{ll})$ belonging to D. Note that Theorem 2 and Theorem 3 tell us that these δ_{xy} cannot result from preferences described by utility functions of the separable or the KPR form.

It shows that indeterminacy can hold when there are externalities to just one of the factors. The case without externalities in the labor

are δ_{xy} parameters, derived from utility, consistent with indeterminacy. We must then make sure that we can map back the δ_{xy} we found into an appropriate utility function describing the preferences of the representative agent. Since the δ_{xy} are defined (see the appendix) in terms of steady state levels of c and l and first and second partial derivatives of $u(c, l)$ evaluated at this point, this is, in principle, a problem of solving a system of second-order partial differential equations. We can take the steady state levels of c and l as given and also take the first partial derivatives as given, in order to match any given real wage. By the steady state restriction from the symmetry of the Hessian, as derived above, there are three free δ_{xy} parameters. We can obtain the value of each triple by adjusting the second partial derivatives (the elements of the Hessian) accordingly. This means that the existence of δ_{xy} parameters implies the existence of an appropriate utility function representing preferences.

[12]The search over D was implemented in a way such that the points at which a check for determinacy was performed are distributed uniformly over D. For each point (m, n)on the grid of labor and capital externalities up to a millon trials over D were done. At each (m, n) the random search was stopped as soon as the first hit had occurred. This combination of random search and stopping rule proved quite useful since it helped speeding up the computations to get results all the points on the grid that were computationally affordable.

input seems particularly interesting. A capital externality that is not strong enough to allow for endogenous growth can be sufficiently high to lead to endogenous cycles, while not requiring any further imperfection in the economy. We found hits both above and below the dashed line, where $m = n$. Using the propositions in the appendix about the properties of δ_{xy} preference parameters, it turned out that all the hits with $m > n$ implied that both consumption and leisure were normal goods; whereas for all hits with $m < n$ consumption was a normal good and leisure an inferior good.

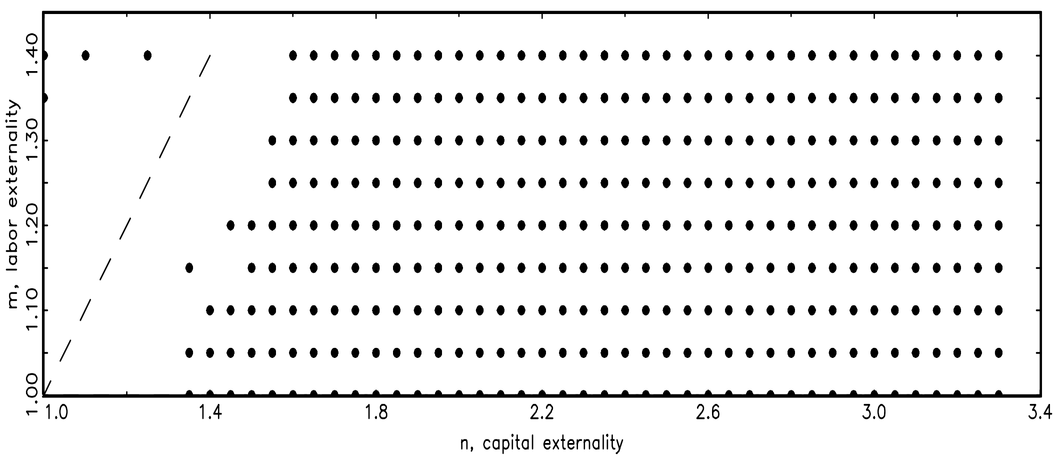

Figure 1: Combinations (n, m) leading to indeterminacy for some δ_{xy}

In the next two figures we restrict attention to specific points in the externality space, but instead have a closer look at the preference space. Related work points out the importance of two labor market equations for indeterminacy in models that are special cases of the model considered here. Benhabib and Farmer (1994) show that the slope of the constant consumption labor supply curve is crucial for the indeterminacy. Bennett and Farmer (2000) emphasize the role of the Frisch labor supply curve. In the appendix it is shown how to express these curves in terms of the δ_{xy} notation. We can thus map the hits in the space of preference parameters into a well-known framework.

For the combinations $m = 1$and $n = 1.5$, $m = 1$and $n = 2$, $m = 1$and $n = 3$, we checked each point on a grid of step-size 0.1 in the box D of preference parameters, as defined above, for indeterminacy. The hits obtained in the preference space were then mapped into the coefficients of the following two equations, derived in the appendix:

The approximated constant consumption labor supply curve:

$$\widetilde{w} = \text{ constant } + (\delta_{ll} - \delta_{cl})\,\widetilde{l} + (\delta_{lc} - \delta_{cc})\,\widetilde{c}. \tag{40}$$

The approximated Frisch labor supply curve:

$$\widetilde{w} = \text{ constant } + \left(\delta_{ll} - \frac{\delta_{cl}\delta_{lc}}{\delta_{cc}}\right)\widetilde{l} + \left(\frac{\delta_{lc}}{\delta_{cc}} - 1\right)\widetilde{\lambda}. \tag{41}$$

Figure 2 expresses the preference parameters that led to indeterminacy in terms of the coefficients of the constant consumption labor supply curve.

We see that the region of coefficients in the constant consumption labor supply curve that corresponds to preference parameters leading to indeterminacy increases considerably in size as we increase capital externalities from $n = 1.5$ (for which we use solid squares) to $n = 3$ (for which we use pluses), while having no labor externality in the model. The constant consumption labor supply curve slopes up in all the cases. As shown in the appendix, a downward sloping curve would imply that consumption is an inferior good.

In Figure 3 the preference parameters that led to indeterminacy are represented in terms of the coefficients of the Frisch labor supply curve.

Again, we see how the indeterminacy region gets larger as we let the capital externality increase in a model without labor externality. It is important to point out that the coefficient on labor is positive in all the cases. As shown in the appendix, a downward sloping would mean non-concave utility.

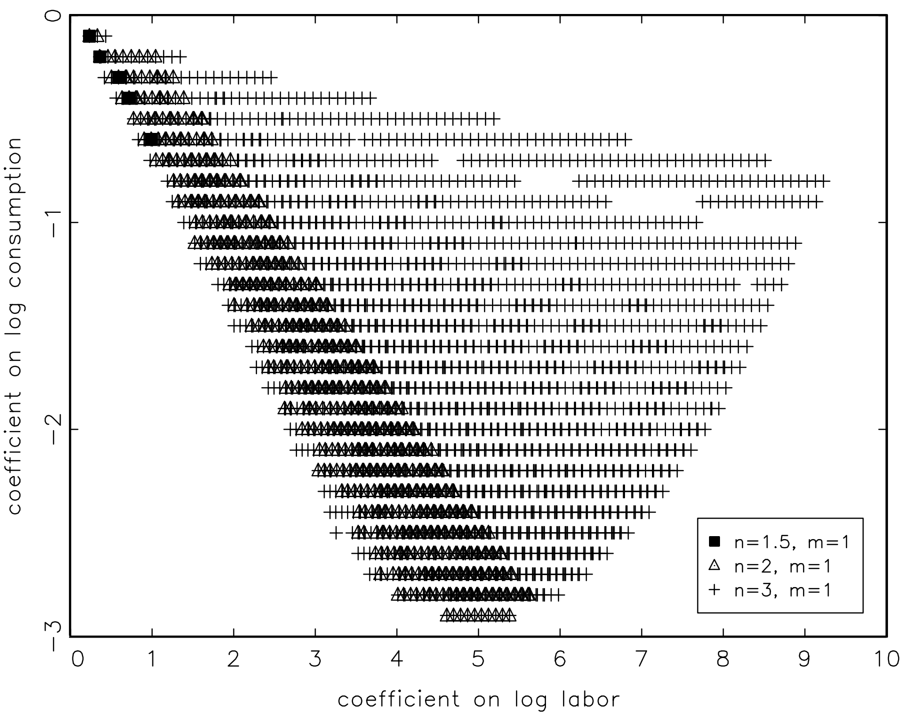

Figure 2: Indeterminacy and constant consumption labor supply

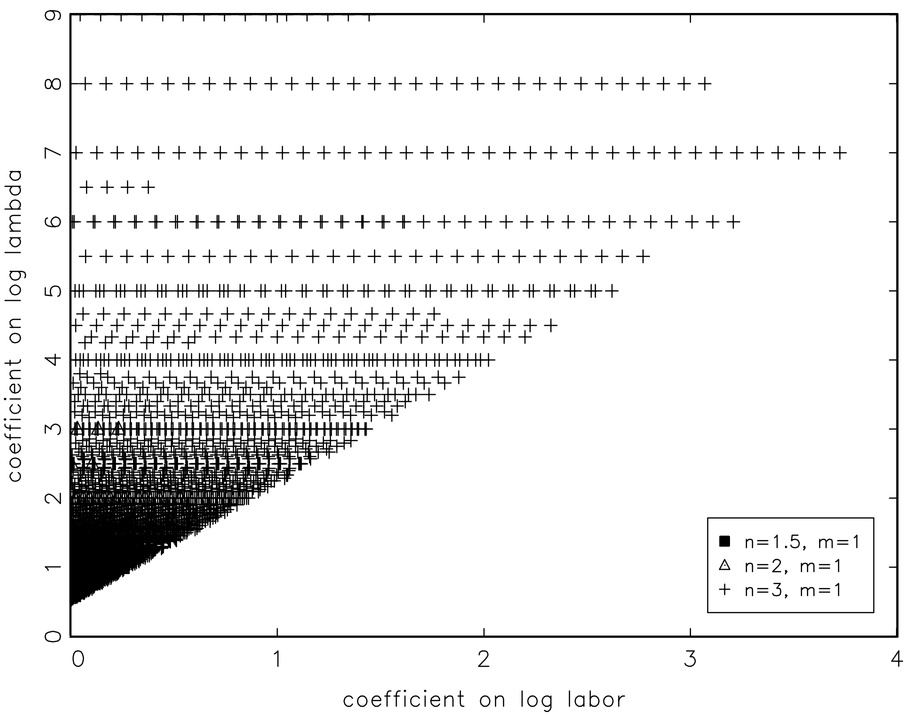

Figure 3: Indeterminacy and Frisch labor supply

7 Conclusions

We conclude that, by considering preferences which are logarithmic in consumption and additively separable in labor, and hence also belong to the KPR class, Benhabib and Farmer (1994) hit three lower bounds on the degree of imperfections required for a sunspot economy. This follows from the theorems, which state non-existence of indeterminacy for levels of the externality below the level they found. The lowest level of the output externality $m = n$ such that indeterminacy can exist for any concave utility function is just high enough to make $bm > 1$ hold true. The lowest level of the labor externality m consistent with indeterminacy for any separable utility function is just high enough to make $bm > 1$ hold true. Similarly, the lowest level of the labor externality m consistent with indeterminacy for any KPR utility function is just high enough to make $bm > 1$ hold true. The first and the third statement are in contrast to Bennett and Farmer (2000).

The models considered in the first three theorems differ in terms of flexibility in two respects: In Theorem 1 there is no restriction to any certain functional form of utility but at require externalities to be due to output, or $m = n$. In Theorem 2 and in Theorem 3 we relax the assumption on externalities by admitting factor-specific externalities, where m can be different from n, considering two specific classes of utility functions. We proved that none of these settings is compatible with indeterminacy under our assumptions.

We have shown that by allowing for more flexibility in both modelling imperfections and preferences we can find sunspot economies below the bounds found from the first three theorems in this paper.

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A Appendix : Describing preferences using the δ_{xy} notation

It has turned out that in log-linearized versions of models only the following δ_{xy} parameters of utility matter:

Let the utility function be given by $U(c, L)$, where the arguments are consumption and leisure, justifying the assumption that both $\frac{\partial U(c,L)}{\partial c}$ and $\frac{\partial U(c,L)}{\partial L}$ are positive. In the appendix the same framework is applied to the case when the second argument of utility is labor instead of leisure.

Definition 2

$$U_c = \frac{\partial U(c, L)}{\partial c} \tag{42}$$

$$U_L = \frac{\partial U(c, L)}{\partial L} \tag{43}$$

$$U_{cc} = \frac{\partial^2 U(c, L)}{\partial c^2} \tag{44}$$

$$U_{cL} = \frac{\partial^2 U(c, L)}{\partial c \partial L} \tag{45}$$

$$U_{Lc} = \frac{\partial^2 U(c, L)}{\partial L \partial c} \tag{46}$$

$$U_{LL} = \frac{\partial^2 U(c, L)}{\partial L^2} \tag{47}$$

$$\delta_{cc} = \frac{c}{U_c}\frac{\partial U_c}{\partial c} = \frac{c}{U_c}U_{cc} \tag{48}$$

$$\delta_{cL} = \frac{L}{U_c}\frac{\partial U_c}{\partial L} = \frac{L}{U_c}U_{cL} \tag{49}$$

$$\delta_{Lc} = \frac{c}{U_L}\frac{\partial U_L}{\partial c} = \frac{c}{U_L}U_{Lc} \tag{50}$$

$$\delta_{LL} = \frac{L}{U_L}\frac{\partial U_L}{\partial L} = \frac{L}{U_L}U_{LL} \tag{51}$$

The definition tells us that the δ_{xy} can be interpreted as the elasticities of marginal utility. It seems worth pointing out that the δ_{xy} are not invariant with respect to monotonic transformations of the utility function.

Properties of the utility function and of what it implies can be formulated using the δ_{xy} notation:

A.1 Concavity

The following proposition gives a condition in terms of δ_{xy} that translates into concavity as a local property of the utility function.

Proposition 5 *The utility function $U(c, L)$ is concave if and only if δ_{cc}, $\delta_{LL} \leq 0$ and*

$$\delta_{cc}\delta_{LL} \geq \delta_{cL}\delta_{Lc}. \tag{52}$$

Proof: Define the Hessian matrix H as

$$H = \begin{bmatrix} U_{cc} & U_{cL} \\ U_{Lc} & U_{LL} \end{bmatrix}. \tag{53}$$

The function $U(c, L)$ is concave if and only if H is negative semidefinite. To check for negative semidefiniteness we have to consider the principal minors of H. The following conditions must hold:

a) $U_{cc} \leq 0$; rearranging the definition of δ_{cc} and using the assumption that $U_c > 0$ we get $\delta_{cc} \leq 0$.

b) $U_{LL} \leq 0$; rearranging the definition of δ_{LL} and using the assumption that $U_L > 0$ we get $\delta_{LL} \leq 0$.

c) $U_{cc}U_{LL} - U_{Lc}U_{cL} \geq 0$; substituting for U_{xy} from the definitions of δ_{xy} we get

$$\frac{U_c}{c}\delta_{cc}\frac{U_L}{L}\delta_{LL} \geq \frac{U_c}{L}\delta_{cL}\frac{U_L}{c}\delta_{Lc.} \tag{54}$$

The condition in the proposition follows immediately, using the assumption that U_c, $U_L > 0$.■

A.2 Quasiconcavity

Proposition 6 *a) If $\delta_{cL} > 0$ (and hence also $\delta_{Lc} > 0$) and*

$$1 > \frac{1}{2}\frac{\delta_{cc}}{\delta_{Lc}} + \frac{1}{2}\frac{\delta_{LL}}{\delta_{cL}}, \tag{55}$$

then $U(c, L)$ is quasiconcave; conversely if $U(c, L)$ is quasiconcave then the above inequality holds weakly.

b) If $\delta_{cL} < 0$ (and hence also $\delta_{Lc} < 0$) and

$$1 < \frac{1}{2}\frac{\delta_{cc}}{\delta_{Lc}} + \frac{1}{2}\frac{\delta_{LL}}{\delta_{cL}}, \tag{56}$$

then $U(c, L)$ is quasiconcave; conversely if $U(c, L)$ is quasiconcave then the above inequality holds weakly.

c) If $\delta_{cL} = 0$ (and hence also $\delta_{Lc} = 0$) and

$$\delta_{cc} \leq 0, \delta_{LL} \leq 0, \tag{57}$$

with at least one of the inequalities holding strictly, then $U(c, L)$ is quasiconcave.

Proof: We consider the bordered Hessian, $\overline{H}$, defined as

$$\overline{H} = \begin{bmatrix} 0 & U_c & U_L \\ U_c & U_{cc} & U_{cL} \\ U_L & U_{Lc} & U_{LL} \end{bmatrix}. \tag{58}$$

If the determinant of $\overline{H}$ is positive, then $U(c, L)$ is quasiconcave. Conversely, if $U(c, L)$ is quasiconcave then the determinant is ≥ 0.

Positivity of the determinant means

$$U_c U_{cL} U_L + U_L U_c U_{Lc} - U_L U_{cc} U_L - U_{LL} U_c U_c > 0. \tag{59}$$

Rearranging and using the definitions of δ_{xy} we get

$$U_{Lc} + U_{cL} > U_{cc} \frac{\delta_{cL}}{\delta_{Lc}} \frac{c}{L} + U_{LL} \frac{\delta_{Lc}}{\delta_{cL}} \frac{L}{c}. \tag{60}$$

The result in c) follows from this stage of the proof.

Assuming $U_{Lc} > 0$, and using the symmetry of the Hessian and the definitions δ_{xy} again, yields the result in a).

The result in b) is obtained analogously by assuming $U_{Lc} = U_{cL} < 0$, which implies $\delta_{Lc} < 0$, $\delta_{cL} < 0$.■

A.3 Inferiority

Proposition 7 *Leisure is an inferior good if and only if*

$$\frac{\delta_{cL} - \delta_{LL}}{\delta_{Lc} - \delta_{cc}} < -\frac{\delta_{cL}}{\delta_{Lc}}. \tag{61}$$

Proof: We maximize $U(c, L)$, such that $c + wL = B$, which leads to the following system of equations:

$$B - c - wL = 0 \tag{62}$$

$$U_c - \lambda = 0 \tag{63}$$

$$U_L - w\lambda = 0 \tag{64}$$

Totally differentiating we get:

$$-1dc - wdL = Ldw - dB \tag{65}$$

$$-1d\lambda + U_{cc}dc + U_{cL}dL = 0 \tag{66}$$

$$-wd\lambda + U_{Lc}dc + U_{LL}dL = \lambda dw \tag{67}$$

This provides the basis for the use of the implicit function theorem in several dimensions in the following:

$$\begin{bmatrix} 0 & -1 & -w \\ -1 & U_{cc} & U_{cL} \\ -w & U_{Lc} & U_{LL} \end{bmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial B} \\ \frac{\partial c}{\partial B} \\ \frac{\partial L}{\partial B} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \tag{68}$$

We then apply Cramer's rule to get

$$\frac{\partial L}{\partial B} = \frac{1}{|J|} \begin{vmatrix} 0 & -1 & -1 \\ -1 & U_{cc} & 0 \\ -w & U_{Lc} & 0 \end{vmatrix}, \tag{69}$$

where $|J|$ denotes the determinant of the matrix on the left-hand side of the previous equation.

$$\frac{\partial L}{\partial B} = \frac{U_{Lc} - wU_{cc}}{wU_{cL} + wU_{Lc} - w^2 U_{cc} - U_{LL}} \tag{70}$$

We then use the first order conditions to replace w by $\frac{U_L}{U_c}$.

$$\frac{\partial L}{\partial B} = \frac{U_{Lc} - \frac{U_L}{U_c} U_{cc}}{\frac{U_L}{U_c} U_{cL} + \frac{U_L}{U_c} U_{Lc} - \left(\frac{U_L}{U_c}\right)^2 U_{cc} - U_{LL}} \tag{71}$$

$$\frac{\partial L}{\partial B} = \frac{U_{Lc} - \frac{U_L}{U_c} U_{cc}}{\frac{U_L}{U_c} \left(U_{Lc} - \frac{U_L}{U_c} U_{cc}\right) + \frac{U_L}{U_c} U_{cL} - U_{LL}} \tag{72}$$

Simplify and impose inferiority:

$$\frac{\partial L}{\partial B} = \frac{1}{\frac{U_L}{U_c} + \frac{\frac{U_L}{U_c} U_{cL} - U_{LL}}{U_{Lc} - \frac{U_L}{U_c} U_{cc}}} < 0 \tag{73}$$

$$\frac{U_L}{U_c} + \frac{\frac{U_L}{U_c} U_{cL} - U_{LL}}{U_{Lc} - \frac{U_L}{U_c} U_{cc}} < 0 \tag{74}$$

$$\frac{U_{cL} - \frac{U_c}{U_L} U_{LL}}{U_{Lc} - \frac{U_L}{U_c} U_{cc}} < -1 \tag{75}$$

Now use the definitions of δ_{xy} and simplify to get

$$\frac{\frac{U_c}{L}}{\frac{U_L}{c}} \frac{\delta_{cL} - \delta_{LL}}{\delta_{Lc} - \delta_{cc}} < -1, \tag{76}$$

where the double fraction can again be replaced using the definitions of δ_{xy}, giving the result stated in the proposition.■

Proposition 8 *Consumption is an inferior good if and only if*

$$\frac{\delta_{Lc} - \delta_{cc}}{\delta_{cL} - \delta_{LL}} < -\frac{\delta_{Lc}}{\delta_{cL}}. \tag{77}$$

Proof: The proof is analogous to the one for inferiority of leisure. The symmetry between consumption and leisure leads to the symmetry between the two inferiority conditions.■

A.4 Constant consumption leisure demand

Let tildes over variables denote their logarithms. Start from the first order condition

$$\frac{U_L}{U_c} = w \tag{78}$$

$$\widetilde{U_L} - \widetilde{U_c} = \widetilde{w} \tag{79}$$

Totally differentiating we get:

$$\frac{1}{U_L} U_{LL} L d\widetilde{L} + \frac{1}{U_L} U_{Lc} c d\widetilde{c} - \frac{1}{U_c} U_{cc} c d\widetilde{c} - \frac{1}{U_c} U_{cL} L d\widetilde{L} = d\widetilde{w} \tag{80}$$

Using the definitions of δ_{xy} we get

$$(\delta_{LL} - \delta_{cL})\, d\widetilde{L} + (\delta_{Lc} - \delta_{cc})\, d\widetilde{c} = d\widetilde{w}, \tag{81}$$

leading to the following

Proposition 9 *The constant consumption leisure demand curve can be approximated by*

$$\widetilde{w} = \text{ constant } + (\delta_{LL} - \delta_{cL})\, \widetilde{L} + (\delta_{Lc} - \delta_{cc})\, \widetilde{c}. \tag{82}$$

A.5 Frisch leisure demand

The Frisch demand curve corresponds to the demand curve holding constant the marginal utility of consumption.

Start from the first order condition, let again tildes denote logarithms of variables.

$$U_c = \lambda \tag{83}$$

$$\widetilde{U_c} = \widetilde{\lambda} \tag{84}$$

In the same style as in the constant consumption case above the last equation is totally differentiated as:

$$\delta_{cc} d\widetilde{c} + \delta_{cL} d\widetilde{L} = d\widetilde{\lambda}, \tag{85}$$

which can be solved for $d\widetilde{c}$ and substituted into (81) to yield the following

Proposition 10 *The Frisch leisure demand curve can be approximated by*

$$\widetilde{w} = \text{ constant } + \left(\delta_{LL} - \frac{\delta_{cL}\delta_{Lc}}{\delta_{cc}}\right)\widetilde{L} + \left(\frac{\delta_{Lc}}{\delta_{cc}} - 1\right)\widetilde{\lambda}. \tag{86}$$

A.6 Some corollaries

We are now in a position to harvest some corollaries on the propositions made.

Corollary 11 *If* $\delta_{LL} \leq 0$, *as is usually assumed and required for concavity, we must have* $\delta_{cL} < 0$ *for constant consumption leisure demand to slope up.*

Proof: The result follows immediately, when considering the slope coefficient of leisure in the (approximated) constant consumption leisure demand curve.■

Corollary 12 *If constant consumption leisure demand slopes up and utility is quasiconcave then consumption is a non-normal good.*

Proof: We want to show that the inequality in (77) holds weakly, or equivalently,

$$\frac{\delta_{Lc} - \delta_{cc}}{\delta_{cL} - \delta_{LL}} + \frac{\delta_{Lc}}{\delta_{cL}} \leq 0. \tag{87}$$

Rearranging we get

$$\frac{2\delta_{Lc}\delta_{cL} - \delta_{cc}\delta_{cL} - \delta_{Lc}\delta_{LL}}{\left(\delta_{cL} - \delta_{LL}\right)\delta_{cL}} \leq 0. \tag{88}$$

We can use the previous corollary to concentrate on the case $\delta_{cL} < 0$. Upward sloping constant consumption leisure demand implies from (82) that $\delta_{cL} - \delta_{LL} < 0$. Therefore the denominator is positive. Nonpositivity of the numerator is assured by quasiconcavity, as given in (56).■

Corollary 13 *Frisch leisure demand slopes up if and only if utility is not concave.*

Proof: Assume $\delta_{cc} < 0$ and rearrange the condition for upward sloping Frisch leisure demand, as given by

$$\delta_{LL} - \frac{\delta_{cL}\delta_{Lc}}{\delta_{cc}} > 0. \tag{89}$$

$$\delta_{cc}\delta_{LL} < \delta_{cL}\delta_{Lc} \tag{90}$$

This is the opposite of the concavity condition in (52).■

A.7 Using labor instead of leisure in the utility function

The utility function now is $u(c,l)$, where we assume that $\frac{\partial u(c,l)}{\partial c} > 0$ and $\frac{\partial u(c,l)}{\partial l} < 0$.

Definition 3

$$u_c = \frac{\partial u(c,l)}{\partial c} \tag{91}$$

$$u_l = \frac{\partial u(c,l)}{\partial l} \tag{92}$$

$$u_{cc} = \frac{\partial^2 u(c,l)}{\partial c^2} \tag{93}$$

$$u_{cl} = \frac{\partial^2 u(c,l)}{\partial c \partial l} \tag{94}$$

$$u_{lc} = \frac{\partial^2 u(c,l)}{\partial l \partial c} \tag{95}$$

$$u_{ll} = \frac{\partial^2 u(c,l)}{\partial l^2} \tag{96}$$

$$\delta_{cc} = \frac{c}{u_c}\frac{\partial u_c}{\partial c} = \frac{c}{u_c}u_{cc} \tag{97}$$

$$\delta_{cl} = \frac{l}{u_c}\frac{\partial u_c}{\partial l} = \frac{l}{u_c}u_{cl} \tag{98}$$

$$\delta_{lc} = \frac{c}{u_l}\frac{\partial u_l}{\partial c} = \frac{c}{u_l}u_{lc} \tag{99}$$

$$\delta_{ll} = \frac{l}{u_l}\frac{\partial u_l}{\partial l} = \frac{l}{u_l}u_{ll} \tag{100}$$

Proposition 14 *The utility function $u(c,l)$ is concave if and only if $\delta_{cc} \leq 0$, $\delta_{ll} \geq 0$ and*

$$\delta_{cc}\delta_{ll} \leq \delta_{cl}\delta_{lc}. \tag{101}$$

The proof is analogous to the one above in the case of leisure.

Proposition 15 *a) If $\delta_{cl} > 0$, and hence $\delta_{lc} < 0$, and*

$$1 < \frac{1}{2}\frac{\delta_{cc}}{\delta_{lc}} + \frac{1}{2}\frac{\delta_{ll}}{\delta_{cl}}, \tag{102}$$

then $u(c,l)$ is quasiconcave; conversely if $u(c,l)$ is quasiconcave then the above inequality holds weakly.

b) If $\delta_{cl} < 0$, and hence $\delta_{lc} > 0$, and

$$1 > \frac{1}{2}\frac{\delta_{cc}}{\delta_{lc}} + \frac{1}{2}\frac{\delta_{ll}}{\delta_{cl}}, \tag{103}$$

then $u(c,l)$ is quasiconcave; conversely if $u(c,l)$ is quasiconcave then the above inequality holds weakly.

c) If $\delta_{cl} = 0$ (and hence also $\delta_{lc} = 0$) and

$$\delta_{cc} \leq 0, \delta_{ll} \geq 0, \tag{104}$$

with at least one of the inequalities holding strictly, then $u(c,l)$ is quasiconcave.

The proof is analogous to the one above in the case of leisure.

Proposition 16 *Leisure is an inferior good if and only if*

$$\frac{\delta_{cl} - \delta_{ll}}{\delta_{lc} - \delta_{cc}} > -\frac{\delta_{cl}}{\delta_{lc}}. \tag{105}$$

The proof is analogous to the one above in the case of leisure.

Proposition 17 *Consumption is an inferior good if and only if*

$$\frac{\delta_{lc} - \delta_{cc}}{\delta_{cl} - \delta_{ll}} > -\frac{\delta_{lc}}{\delta_{cl}}. \tag{106}$$

The proof is analogous to the one above in the case of leisure.

Proposition 18 *The constant consumption labor supply curve can be approximated by*

$$\widetilde{w} = \text{ constant } + (\delta_{ll} - \delta_{cl})\,\widetilde{l} + (\delta_{lc} - \delta_{cc})\,\widetilde{c}. \tag{107}$$

The derivation is the same as in the case of leisure demand.

Proposition 19 *The Frisch labor supply curve can be approximated by*

$$\widetilde{w} = \text{ constant } + \left(\delta_{ll} - \frac{\delta_{cl}\delta_{lc}}{\delta_{cc}}\right)\widetilde{l} + \left(\frac{\delta_{lc}}{\delta_{cc}} - 1\right)\widetilde{\lambda}. \tag{108}$$

The derivation is the same as in the case of leisure demand.

The corollaries can be reformulated using labor instead of leisure; the proofs are analogous to those above.

Corollary 20 *If* $\delta_{ll} \geq 0$, *as is usually assumed and required for concavity, we must have* $\delta_{cl} > 0$ *for constant consumption labor supply to slope down.*

Corollary 21 *If constant consumption labor supply slopes down and utility is quasiconcave then consumption is a non-normal good.*

Corollary 22 *Frisch labor supply slopes down if and only if utility is not concave.*