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The Mathematization of Macroeconomics*
A Recursive Revolution

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*I think it may be useful to clear up a few historical misconceptions. The word *Macroeconomics* was coined, and first used, to the best of my knowledge, by Jan Tinbergen, in 1936 ([63]). Tinbergen used it again, in the Preface to his famous *League of Nations* monograph, in 1939 ([64]. John Fleming in late 1938 and Erik Lindahl in 1939, were two of the other pioneers in the coining of the word ([16], p.333 and [31], p.52). All three introduced the word with a hyphen, as Macro-economics (Lindhal's *Preface* was dated June and Tinbergen's, January, 1939). It was, however, Lindahl who introduced it, contrasting the word, and the subject it was to circumscribe, with *micro-economics*. Lindahl had coined the word several years before the final publication of the English translation of his celebrated work in 1939. He had, previously and subsequently, debated with Bertil Ohlin on the desirability of the words *macro-* and *micro-economics* as against Ohlin's suggestion of *partial-* and *total*-economics, respectively (cf. [32], p.243). Contrary to unscholarly remarks, even by a serious scholar like the late Leif Johansen, in his 'Nobel Article' on Frisch ([25], p.306), the word does not appear in Frisch's celebrated **Cassel Festschrift** article. All that Johansen had to do was to read *Propagation Problems and Impulse Problems in Dynamic Economics* – without simply 'quoting' it. Even worse, in a recent textbook by two Danish authors, ([57], p.558-9), **Introducing Advanced Macroeconomics**, Slutzky is, incredibly, asserted to be an '*Italian statistician*' and Wicksell's '*rocking horse*' metaphor is given a wholly incorrect first reference and date. Frisch, himself, gave an incorrect reference when he first referred to it in his **Cassel Festschrift** paper. I make these points only because these careless mistakes are made by Norwegian and Danish economists, who - more than anyone else - should be able to verify serious references in the original languages without any difficulty. Just for completion, and to save future students the toil I had to undergo to straighten simple facts, the first mention of the rocking horse metaphor by Wicksell was in 1918, in a review article of a little known book by an even lesser known economist by the name of Karl Petander ([69], p.71, footnote 1).

†I am deeply indebted to my friends and colleagues, Professors Gianni Pegoretti and Stefano Zambelli for straightening and strengthening my scholarship on many issues dealt with in this paper. Alas, no one but – I mean, *not even* – Professor Zambelli can be blamed for the remaining infelicities in this paper.

Abstract

Frank Ramsey's classic framing of the dynamics of *optimal savings*, [51] as one to be solved as a problem in the *calculus of variations* and Ragnar Frisch's imaginative invoking of *a* felicitous *Wicksellian metaphor* to provide the *impulse-propagation dichotomy*, in a *stochastic dynamic framework*, for the tackling the problem of *business cycles*[17], have come to be considered the twin fountainheads of the mathematization of macroeconomics in its dynamic modes – at least in one dominant tradition. The *intertemporal optimization* framework of a *rational agent*, viewed as a signal processor, facing the *impulses* that are *propagated* through the mechanisms of a *real* economy, provide the underpinnings of the *stochastic dynamic general equilibrium* (*SDGE*) model that has become the benchmark and frontier of current macroeconomics.

In this paper, on the 80th anniversary of *Ramsey's* classic and the 75th anniversary of *Frisch's Cassel Festschrift* contribution, an attempt is made to characterize the mathematization of macroeconomics in terms of the frontier dominance of *recursive methods.* There are, of course, other - probably more enlightened – ways to tell this fascinating story. However, although my preferred method would have been to tell it as an evolutionary development, since I am not sure that where we are represents progress, from where we were, say 60 years ago, I have chosen refuge in some *Whig fantasies.*

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1 Preamble

"....[A]s economic analysts we are directed by, if not prisoners of, the mathematical tools we possess."

Thomas J. Sargent: **Macroeconomic Theory** (2nd Ed.), p. xix.

Macroeconomics, almost by definition, is about the dynamics of aggregate variables. The aggregates, in modern macroeconomics, are required to be underpinned by microeconomic foundations; the dynamics, considered from an abstract mathematical point of view, could be either deterministic or stochastic. The frontiers of mathematical macroeconomics are dominated by the mathematical methodology of *recursive macroeconomics*.

The qualification 'recursive' here has nothing to do with formal 'recursion theory'. Instead, this is a reference to the mathematical formalizations of the rational economic agent's intertemporal optimization problems, in terms of *Markov Decision Processes* (*MDP*), (Kalman) *Filtering* (*KF*) and *Dynamic Programming* (*DP*) where a kind of 'recursion' is invoked in the solution methods. The pioneers[1] associated with the development of these three 'theoretical technologies', the current modelling workhorses of mathematical macroeconomics are, Abraham Wald, Rudolf Kalman and Richard Bellman, respectively. The metaphor of the rational economic agent as a 'signal processor', implementing MDP, using the KF within the DP framework underpins the recursive macroeconomic paradigm.

The *Stochastic Dynamic General Equilibrium model* (henceforth, *SDGE*), although developed within the recursive macroeconomic paradigm, has come to be acknowledged as the benchmark for *all* mainstream macroeconomic theories. Significantly, for a subject that, at its renascence at the hands of Wicksell, Lindahl, Myrdal, Hayek and Keynes was intrinsically monetary in nature, the *SDGE* model is devoid of monetary content. Therefore, in telling a story of the mathematization of macroeconomics, I shall have to ignore all theoretical technologies that were part of the development of monetary macrodynamics. This will explain, although not excuse, two relative absences: the neglect of any of the mathematical developments – particularly (*non-linear*) *dynamical systems theory* – associated with *overlapping generations models* (*OLG*)[2]; and, even more unfortunately, all issues of *combinatorial optimization*[3]. But these two mathematical technologies – nonlinear dynamics and combinatorial mathe-

[1] However, as in the case of many pioneers, the very act of defining and delineating a subject conceptually and technically leads to discoveries of prior giants on whose shoulders we may be standing, without always realizing it explicitly. In this case, Richard Day pioneered the *recursive programming* approach to modelling *adaptively economizing* agents in the face of *disequilibria* in price and quantity variables many decades before the formal appearance of recursive macroeconomics (cf., for example, [8]).

[2] Except in one simple instance, in section 4, where a stylised OLG model is used as a 'vehicle' to suggest a particular recursive mode of learning a rational expectations equilibrium (*REE*).

[3] Here, my main concern is the neglect of the innovative combinatorial – number theoretic – mathematical lessons to be learned from the monetary model developed by Clower and Howitt ([6]).

matics (whether in an optimization context or not) – are impossible to avoid in any discipline in which the digital computer is ubiquitous.

And economic theory, at every level and at almost all frontiers - be it microeconomics or macroeconomics, game theory or IO - *is* now almost irreversibly dominated by *digitally* determined – i.e., based on the processing of economic data via the digital computer – *computational*, *numerical*[4] and *experimental* considerations. This means, willy-nilly, intrinsic nonlinear dynamical and combinatorial mathematics have to be considered, simply because the mathematics of the computer invokes it. Curiously, though, none of the macroeconomic frontier emphasis from any one of these three points of view - *computational*, *numerical* or *experimental* - is underpinned by the natural algorithmic mathematics of either *computability* theory or *constructive* analysis[5]. In particular, the much vaunted field of *Computable General Equilibrium* theory, with explicit claims that it is based on constructive and computable foundations is neither the one, nor the other[6]. The dominance of computational and numerical analysis, powerfully underpinned by serious *approximation theory* – for example in the definition and derivation of a *recursive competitive equilibrium*, the norm of the *SDGE* model – is devoid of formal *algorithmic* foundations. This curious case of a *Hamlet without* the *Prince* can, in my opinion, only be explained by the historical accident of economic theory having been formalized and mathematized by *classical real analysis*; a typical example of 'lock-in' and the pernicious influences of the *QWERTY* principle.

The story of the *mathematization of economic theory*[7] has been told by the *doyen* of 20th century mathematical economics, Gerard Debreu, in a series of exceptionally clear articles[8], if also written with a particularly narrow vision and understanding of the nature and scope of mathematics (cf., [10], [11], [12])

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[4] By this I aim to refer to *classical numerical analysis*, which has only in recent years shown tendencies of merging with computability theory - for example through the work of Steve Smale and his many collaborators (cf. for example [2]).

[5] For excellent expositions of numerical and computational methods in economics, *particularly macroeconomics*, see [3], [26] and [38].

[6] A complete and detailed analysis of the false claims – from the point of view of computability and constructivity – of the proponents and practitioners of CGE modelling is given in my recent paper devoted explicitly to the topic (cf. [67]).

[7] Without distinguishing between microeconomics and macroeconomcis or between alternative economic theories, the discussion is as if there is an unambiguous and universally accepted core of economic theory.

[8] Debreu, however, does not recognise – perhaps is not aware of – the power, suitability and relevance of either *constructive* or *computable* analysis, for the mathematization of economic theory. However, his above articles do invoke and rely on both *real* and *non-standard* analysis to provide examples of the successes of necessary mathematization of fundamental aspects of economic theory. Could this be because the mathematization of economic theory in a constructive or computable mode might imply serious reformulations and reconsiderations of the basic constructs of economic theory? For example, the *second fundamental theorem of welfare economics* is proved by invoking the *Hahn-Banach theorem*, which, in the form used in mathematical economics, is invalid in constructive analysis! Essentially, from the point of view of the foundations of mathematics and mathematical logic, the mathematics of economic theory that Debreu discusses is founded upon (Zermelo-Fraenkel) *Set Theory* (plus the *axiom of choice*) and a very narrow part of *Model Theory* to the total neglect of *Proof Theory* and *Recursion Theory*.

– albeit in a *Whig mode*. One way, therefore, for me to make my own story for the mathematization of macroeconomics comparable would be to mimic the strategy adopted by Debreu. His strategy was, in a nutshell, as follows. Debreu identified three functions of prices in a decentralized economy – the function of prices in the efficient allocation of resources; prices equalizing supply and demand; and prices acting to prevent the formation of destabilizing coalitions. These three roles of prices, in turn, were mathematized by the use of *convex analysis* (and classic non-constructive *functional analysis*[9]), *fixed point theory* and *non-standard analysis* (and a version of formal integration theory)[10], respectively. In spite of widespread claims to the contrary, as briefly mentioned in the previous paragraph, too, the mathematics invoked to formalize these three roles of prices cannot be algorithmized – i.e., cannot be made the basis of numerical, computational or experimental – for empirical analysis, particularly for underpinning the efficiency of policy analysis using *CGE* models, for computing equilibria or even for analysing the theory of competition, i.e., market structure. In other words, if the paradigm for the mathematization of economic theory is the one suggested by Debreu – and accepted by an overwhelming majority of the profession – then, if followed in a story of the mathematization of macroeconomics, the same dissonance will be inevitable; the dissonance between a mathematical theory that is non-numerical, non-computational and impossible to implement experimentally with the use of a digital computer and the claims and efforts of numerically, computationally and experimentally oriented mathematical macroeconomics that is, moreover, serious about approximation theory, both formally and in an applied sense.

For example, two current macroeconomic examples – analogous to the role attributed to prices in Debreu's story – would be *REE* and *dynamic programming* (in particular the '*value function*' and the '*Bellman equation*', in which it appears). Corresponding to the above three roles of prices, there would be the following issues to be considered in the former: the *existence* of REE, the *learning* of REE, the *dynamics* of REE and the *computation* of REE (with the possible *approximations* of REE for exact computation or, vice versa – i.e., approximation of the computation process for an exact REE). Each of these will invoke a different kind of mathematization, exactly as in the case of the three roles of prices invoking convex analysis, functional analysis and integration theory (non-standard analysis). In the case of REE it would be fixed point theory (yet again!), (stochastic) approximation theory, (deterministic and stochastic) dynamical systems theory and numerical analysis. In the case of dynamic programming, the value function and the 'Bellman equation', it would be almost

[9] In particular, duality theory in the form of the *Hahn-Banach theorem*, to mathematically demonstrate the validity of the second fundamental theorem of welfare economics (but, see also the previous footnote).

[10] Debreu's 'historical' remarks accompanying these illustrations are inaccurate. For example it is quite preposterous to state that 'nonstandard analysis [was] founded at the beginning of the 1960's by Abraham Robinson' ([12], p.3). There are still other surprisingly inaccurate remarks on mathematics in the Debreu papers referenced above.

the same: fix point theory, for example in the form of a contraction mapping theorem, computation of equilibiria and approximate computation (of an exact equilibrium) or an exact computation (of an approximate equilibrium).

Therefore, simply to maintain a comparison and for the sake of uniformity of method, I shall follow the strategy that I have identified with Debreu,above and shall refer to it, for want of a better name, as the '*orthodox methodology*'.

The rest of this paper is structured as follows. Section 2 is devoted to a 'panoramic' and, hence, loose narrative of some issues in the development of macroeconomics since Wicksell, mainly to make the case that any *Whig history* of the subject is to be avoided. A very general methodological discussion of the philosophy and epistemology of mathematizing macroeconomics – from Wicksell to Prescott, via Samuelson, Patinkin, Clower, Lucas, Romer and Woodford. In section 3, taking the example of the crucial role played by dynamic programming formulations of macroeconomic decision problems, I try to describe a pattern that conforms to the 'orthodox methodology'. In doing so, I try to keep in mind the explicit aims of the current mathematizing enterprise in macroeconomics: a quantitative underpinning for policy and dynamics via computational, numerical and experimental analysis. In section 4, an attempt is made to make my own contribution towards recursive macroeconomics, using the example of rational expectations equilibrium (*REE*). Nothing can be more recursive, nor more computational, than a recursion theoretic approach to proving the algorithimc existence of, and learning, REE. The concluding section is a reflection and a retrospective of the 'orthodox methodology' from the point of view of its own avowed goals of making macroeconomics quantitative. However, the section is also a manifesto for the *correct* recursive mathematization of macroeconomics, if quantitative, numerical, computational concerns are imperative.

In general, however, there is, in any case, a close parallel between the way *economic theory* was mathematized in the sense of Debreu and the *mathematization of macroeconomics*. This is because the general theoretical and modelling strategy is, on the whole, very similar: proving the existence of an equilibrium; if the model displays multiple equilibria - as in the case, say, of orthodox *OLG* models – then a possible *ad hoc* learning algorithm to select (a subset) of equilibria; at some point an approximation procedure to compute either the equilibrium that has been proved to exist or to compute the (selected) equilibrium; the approximation procedure is also introduced in arbitrary ways – i.e., there is no clear *systematic procedure* guiding the perplexed graduate student to a disciplined modelling strategy. All kinds of 'hand-waving' appeals to simplicity – *pace Ockham* – are invoked to claim the proverbial mantra: **w.o.l** (*without loss of generality*), while the approximation linearizes something or the other. The difference with the world of Debreu, to which allegiance is always claimed, is that he, at least, does not profess to be driven by quantitative criteria – numerical, computational, experimental – to underpin computable (sic!) policy prescriptions to be derived via the welfare theorems[11].

[11] Soiling the mathematical economist's hands in that direction is left to the CGE theorist.

2 From *Geldzins und Güterpreise* to *Interest and Prices* – An Ultra-Brief Macroeconomic Retrospect[12]

> "To go beyond ..... to questions involving the efficiency of alternative kinds of stabilization policy involving moving to – or, I would say, beyond – the current frontiers of macroeconomics."
>
> Robert E. Lucas, Jr.: **Models of Business Cycles**, p. 106 [34].

In March 1952, during a lecture in Stockholm, Eli Heckscher recalled, that, on 14 April 1898, Wicksell 'somewhat unexpectedly revealed before the [Stockholm Economic] Society what was perhaps his greatest theoretical achievement, his theory of the connection between interest rate and money value'. Thus was born modern macroeconomics, a thesis to be substantiated in this research program

It is, proverbially, a new name for an old subject. However, it was Wicksell – and, to a lesser extent, Fisher - not Keynes nor Hayek, who first stamped it with modernism in an unmistakable way – the modernism we associate with providing microfoundations for aggregate variables and behaviour. This he provided for the twin horns of macroeconomics – the real and the monetary sides; for the former on the basis of Austrian capital theory, which he almost single-handedly and rigorously re-wrote and re-did for Menger, Böhm-Bawerk and von Wieser; for the latter, on the basis of a wholly new approach to monetary theory by devising an innovative thought-experiment - *gedankenexperiment* - which obviated the need for a reliance on the quantity theory of money to explain inflation. This thought-experiment constructed a pure credit economy in which monetary transactions were conducted in an imaginary giro system.

The crucial event that spurred him to these conceptual innovations was the 20-year deflation – not recession – experienced, without exception, by all the advanced industrial nations, from the mid-1870s to the mid-1890s. He was – as Fisher was - deeply concerned that this deflation meant an unwarranted redistribution of wealth and income between lenders and borrowers. The only conceptual tool that was available for policy purposes was the quantity theory of money. A reliance on this would have meant a further deepening of the deflationary process and an exacerbation of the unjust income and wealth distributions. He had to devise an alternative vision of the monetary mechanism in such a way that it would yield policy perspectives and tools that would *stabilize the price level*, whilst preserving consistency with the *microeconomics of relative prices* in a situation of *deflationary dynamics*. Thus was born the Wicksellian (analogue of the Malthusian mechanism): the discrepancy between the *money rate of interest*, determined by Banking Policy, and the *natural rate of profit* resulting from the capital structure of the production system. In-

[12] I am, of course, referring to the original German title of Wicksell's classic, [68], and its purported 'updating' by Woodford, [70]. That the former wrote in German and the latter in English is itself a testimony to the altered dominances in the profession!

dependently, and motivated by the same events and concerns, Irving Fisher had suggested an alternative mechanism for the interpretation and resolution of the same problem. In a sense, modern macroeconomics is an uncoordinated amalgam of Fisher's expectational mechanism and Wicksell's capital theoretic underpinnings for monetary macroeconomic thought-experiments.

But here is a puzzle: Wicksell observes a 20-year deflation and constructs an *unstable* model of inflation for stabilization purposes! Why has modern macroeconomics, built on Wicksellian conceptual foundations, abandoned notions of unstable equilibria?

I believe Macroeconomists – notoriously fickle in their allegiances and admirably unruly in their beliefs – are united in recognising, in Wicksell, a common progenitor of their subject, in its modern form[13]. I also believe, with caveats along lines suggested above, and a respectful nod to Irving Fisher's *Appreciation and Interest*, [13], on one side, and to the paucity of the empirical underpinnings of its famous theoretical proposition(s)[23], *Geldzins und Güterpreise* is the acknowledged fountainhead of the conceptual foundations for modern macroeconomics[14].

The emergence of *SDGE* model as the dominant paradigm is as much due to innovative contributions on a new synthesis of microeconomics and macroeconomics as to the uncompromising mathematization of macroeconomics. No one questions the fact that the architects of general equilibrium theory, with the notable exception of Menger, envisaged the subject in its mathematical mode *ab initio*. This was definitely not the case in macroeconomics. Macroeconomics only gradually became a mathematical subject. Wicksell's thoughts on this particular issue, expressed clearly in *Geldzins und Güterpreise*, are worth recalling:

> I have on this occasion made next to no use of the mathematical method. This does not mean that I have changed my mind in regard to its validity and applicability, but simply that my subject does not appear to me to be ripe for methods of precision. In most other fields of political economy there is unanimity concerning at least the

[13] Naturally, the origional *Classical Economists*, Smith, Ricardo and Malthus, were also macroeocnomists, in almost every sense in which the subject is practised today, except, perhaps, in its fundamental commitment to microfoundations.

[14] Prescott in an interview, soon after receiving his Nobel Memorial Prize in Economic Science, in December, 2004, had this interesting remark about Wicksell as a progenitor ([49], p.5):

> "By the way, it turns out our real-shock story is an old one – Knut Wicksell and Arthur Cecil Pigou were famous economists who adhered to that view. Ijust got a little note from Paul Samuelson telling me to look at Wicksell, so I got Wicksell's book out of the library."

I do wonder which of Wicksell's works was singled out by Paul Samuelson. Surely, Samuelson, scholar *par excellence* that he is, would have been more specific about which of the great Wicksell's works would be relevant for an RBC interpretation? Remarkably, it does seem that Prescott had never read Wicksell before the note arrived from Samuelson! I suspect those of us trying to ride two horses – a respect for our intellectual heritage and a mastery of theoretical technology – will forever remain Neanderthals, at least because time will not be on our side to do both adequately fast enough for Stockholm to .....!!

direction in which one cause or another reacts on economic processes; the next step must then lie in an attempt to introduce more precise quantitative relations. But in the subject to which this book is devoted the dispute still rages about *plus* as opposed to *minus*."

[68], p. xxx; italics in original.

At the frontiers of macroeconomics the conscious *Wicksellian vision* of the foundations of monetary policy is encapsulated, within a squarely *SDGE* modelling framework, by Woodford's massive work titled, as Richard Kahn's (incomplete) translation of the title of the Wicksellian *opus*: *Interest & Prices* (perhaps in homage to Wicksell!). The difference in economic content between the two books, especially if Wicksell's classic is supplemented by the later works of his Swedish followers, Lindahl and Myrdal in particular, is significant, but not too dramatic that a latter-day incarnation of Wicksell will not be able to read, appreciate and even agree with (at least parts of it). I think Wicksell would wholly endorse the relentless mathematization of the subject he almost single-handedly founded.

However, the contrast in the analytical methods is dramatic; the Wicksellian scepticism, captured in the quote above, has been banished beyond doubt. How did this happen? When did it happen? Why did it happen? Was it inevitable that it would happen?

Without answering these question, telling the story of the mathematization of macroeconomics as if a *Whig historian* would say it, will not be able to make sense of 'roads not taken' of the following type. Myrdal's *Monetary Equilibrium* ([40]), together with Lindhal's classics ([30], [31]), the classic on the theory of economic policy by Myrdal ([39])[15], the Keynesian *magnum opus* ([27]) and Hayek's three pioneering contributions ([20], [21] and [22]) were the fountainheads for the concepts that came to underpin the macroeconomic basis of newclassical economics, especially in its codification via *SDGE* modelling: *intertemporal equilibrium*, *rational expectations*, *monetary equilibrium*, *credibility of policy*, *time inconsistency*, *policy invariance*, etc. Only the concept of the 'natural rate', applied to the labour market, was obviously absent. But the *Monetary Equilibrium* defined by Myrdal was for a *cumulative process* that was an *unstable* and (*stochastically*) *non-stationary dynamic process*[16]. Moreover, Myrdal (and Lindahl) were explicit in *differentiating*, conceptually and analytically, a *monetary equilibrium* from the *general equilibrium* (*of a Walrasian*

[15] Still available only in the Swedish original – indeed, only in its version as a Government memorandum – although made a part of the folklore of the classical framework of the theory of economic policy by Ragnar Frisch, Bent Hansen and Jan Tinbergen, in the eraly post-war years.

[16]

"Our central statement of the problem .. is .. the following: From the standpoint of the fundamental idea of Wicksell's monetary theory, what do the properties of a price situation in a *non-stationary* course of events have to be in order that this situation can be characterized as a position of monetary equilibrium." Myrdal, op.cit, p.42; italics added.

system)[17]:

> "This monetary equilibrium, which is stated precisely with respect to a certain accrual or hypothetical price situation, has by no means the same character as the conditions of perfect general equilibrium of prices in the static analysis of price formation. Wicksell emphasised this."
>
> op.cit., p. 35.

How did these considerations get subverted and replaced by the *SDGE* modelling framework of *stationary*, *stable*, *stochastic dynamic processes* with the *Walrasian equilibrium* as the benchmark? I do not believe there is any other way to account for this *impoverishment of economic insight* than by understanding the mathematization of macroeconomics. This is not an isolated episode of impoverishment of economic insight in the process of the mathematization of macroeconomics. Similar episodes happened at crucial forks in the development of mathematical macroeconomics in business cycle theory, monetary theory (and growth theory).

On the other hand, there are also edifying episodes in the emergence of mathematical macroeconomics where the lack of, or ignorance of, theoretical technologies stunted the development of articulated theories. The particular case of Robertson's theory of the cycle, first formulated in 1915 and then finessed in the context of an innovative monetary policy framework consistent with that being developed by Wicksell's followers in Sweden, is paradigmatic. Robertson's theory of the business cycle is *RBC* theory in embryo. However, Robertson's theory was no match for the emerging, endogenous, Keynesian theories of the cycle, where powerful *non-linear mathematical theories* were harnessed to encapsulate the multiplier-accelerator model. The full force of the mathematically formal RBC theory was required to challenge the dominance of the endogenous theory and, in the process, install the *SDGE* modelling strategy as the canonical model of mathematical macroeconomics. Such a re-formalization of Robertson's equilibrium, real, theory of the business cycle can easily enrich *RBC* theory by allowing the monetary – in fact, banking – principles to which the former was attached, to be incorporated in the latter with the powerful theoretical technologies now at hand. This is in line with a neo-Wicksellian synthesis that can enrich, also, the foundations of monetary policy in technically more interesting and realistic ways – those that were intrinsic to the original cumulative process:

[17] Leijonhufvud ([29], p.155; italics in the original), on the other hand, popularised the diametrically opposite view (perhaps thereby, inadvertently, influencing Woodford to propagate the same vision):

> "The central concepts of Wicksell's analytical apparatus are, of course, the *market rate* and the *natural rate* of interest. The terms are names for two values of the same variable."

Nowhere in the Wicksellian classics, nor in any of the monetary macroeconomic writings of Lindahl or Myrdal have I found any evidence to substantiate this assertion.

instability, stochastic non-stationarity circumscribed by an equilibrium dynamic process.

The irony in this particular case is that monetary theory and the foundations of monetary policy were the impetus for the creation of the subject of macroeconomics in the imaginative hands of Wicksell. However, with the emergence of *SDGE* modelling as the canonical method, it is business cycle theory, buttressed by growth theory, that act as the foundations of macroeconomics; monetary theory and the foundations of monetary policy are handmaidens to the benchmark that is provided by the *SDGE*. Paradoxically, however, this reversal of roles can be justified on Wicksellian grounds, as well . In other words, the mathematization of macroeconomics had a natural development towards placing business cycle theory at its core simply because the focus on dynamics was most natural in that setting.

One final methodological issue must be mentioned. The developments in the mathematization of macroeconomics, particularly at the hands of Lucas, Sargent and Kydland & Prescott, called forth also a fresh look at what had become standard econometric practice: the *Cowles Foundation Methodology*, and revived the old debate between Rutledge Vining and Tjalling Koopmans. If the preoccupation was with Keynesian ghosts and their slaying during the consolidation of the canonical newclassical macroeconomic model, then this '*After-Keynesian Macroeconomics*' period can almost, be said to characterize the slaying of the scepter of the Cowles Foundation Methodologies. Once again, new metaphors were forged and new concepts invented, foremost among them being the computational, numerical, approximation theoretic and recursive metaphors, in almost every macroeconomic corner. This, in turn, has led to new perspectives on accounting categories, as well.

In telling this kind of story of the mathematization of the subject, it also provides coherence to a narrative of a 'century of macroeconomic theory', which is no longer a story of competing schools of thought; nor one of *Whig history*.

3 Recursive Macroeconomics and Dynamic Programming: Possibilities and Impossibilities

> "The recursive competitive equilibrium is particularly convenient .... because it fits naturally into the dynamic programming approach to solving optimization problems."
>
> Thomas Cooley & Edward Prescott: '*Economic Growth and Business Cycles*', ch. 1, p.9, in: [7].

This is a typical Debreu-type approach to the mathematization of macroeconomics. An equilibrium concept seems to have been defined, the *Recursive Competitive Equilibrium* (*RCE*), with a particular mathematical construct in mind – in this case, dynamic programming; just as the role of prices equilibrating supply and demand was transformed into a fix point problem to facilitate a specific kind of mathematization of economic theory[18]. Thereby, the original and historical approach to the problem of supply-demand equilibrium was lost and fix point theory became the bread & butter mathematical technique and framework for almost every equilibrium problem in economics[19].

The *RCE* concept emerged via the particular macroeconomic exercise of modelling economies experiencing balanced growth as dynamic general equilibrium descriptions. Unlike the original motivation for the Arrow-Debreu approach to the mathematization of a Walrasian Equilibrium[20], the RCE was, *ab initio*, justified on grounds of computation, simulation, policy analysis and other quantitative issues.

Moreover, the *RCE* construct for *SDGE* modelling provides a direct link with the two fundamental theorems of welfare economics and grounds a particularly powerful mathematization of macroeconomics in core mathematical economic theory. Theory, computation, simulation, approximation and policy seem all to be seamlessly knit together in a rich tapestry of mathematical macroeconomics in this approach. Let us see.

We imagine a representative agent economy, with preferences that are additively separable and defined over consumption at every date in a discrete

[18] A comparison with Paul Romer's vision on this methodological issue seems to show that he was slightly less than prescient! In Romer's view it was *merely* a dynamic general equilibrium model (*DGE*) which was to rely entirely on *static* optimization techniques ([52], p. 70-1; italics added):

> ""Growth is a general equilibrium process. ... A growth theorist *must* therefore construct a dynamic general equilibrium model, starting with a specification of preferences and the technology and specifying an equilibrium concept. ......Either explicitly or implicitly, the central tool used in the characterization of dynamic competitive equilibrium models is the Kuhn-Tucker theorem. It offers a general *procedure* for reducing the problem of calculating a competitive equilibrium to the problem of solving maximization problems."

It does nothing of the sort! But explicating that will require a whole monograph on computability theory.

[19] It took an outsider, a distinguished mathematician and a 'part-time' mathematical economist, Steve Smale, to point out that the proverbial 'emperor' was less than well clad:

> "We return to the subject of equilibrium theory. The existence theory of the static approach is deeply rooted to the use of the mathematics of fixed point theory. Thus one step in the liberation from the static point of view would be to use a mathematics of a different kind. Furthermore, proofs of fixed point theorems traditionally use difficult ideas of algebraic topology, and this has obscured the economic phenomena underlying the existence of equilibria. Also the economic equilibrium problem presents itself most directly and with the most tradition not as a fixed point problem, but as an equation, supply equals demand. Mathematical economists have translated the problem of solving this equation into a fixed point problem.
>
> I think it is fair to say that for the main existence problems in the theory of economic equilibrium, one can now bypass the fixed point approach and attack the equations directly to give existence of solutions, with a simpler kind of mathematics and even mathematics with dynamic and algorithmic overtones."
>
> [58], p.290; bold emphasis added.

[20] Where the issue of the computability of a Walrasian equilibirum was an ex post construct, but also – ostensibly – motivated by policy considerations.

economy[21]:

$$u(c_0, c_1......) = \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1 \tag{1}$$

where:
$U \in C^1 : \Re^+ \to \Re$, strictly concave, increasing and $lim_{c \to 0} = \infty$.
The aggregate production function is[22]:

$$Y_t = F(K_t, H_t) \tag{2}$$

where:
K_0 : initial endowment of capital
K_t : household supply of period t capital to firms;
H_t : household supply of period t labour to firms;
F satisfies the Inada conditions w.r.t H and K; and $F \in C^1$, increasing and concave w.r.t K and H and is homogeneous of degree one.

The aggregate resource constraint is:

$$C_t + K_{t-1} - (1-\delta) K_t \leq F(K_t, H_t), \qquad \forall t \tag{3}$$

Now, assuming that this is a problem to be solved by one of those mythical creatures – with which economics is richly endowed, the Walrasian Auctioneer being the prime example – the so-called *benevolent social planner*, and that no output is 'wasted', thereby converting the aggregate resource constraint, (3), to an equality, the problem becomes:

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[f(K_t) - K_{t+1}]$$
$$s.t \quad 0 \leq K_{t+1} \leq f(K_t), \quad t = 0, ... \textit{ and given } K_0 > 0 \tag{4}$$

The mathematization of macroeconomics that I am focusing on – paralleling the Debreu approach, to keep emphasising – is how this 'traditional' and very standard optimization problem was transformed, in the new matehamtical macroeconomics into a *recursive form* so that it can be solved by *dynamic programming*. The benevolent social planners task is then represented as a classic

[21] I follow, in the sequel, the standard expositions in [7] and [36] and concentrate on the deterministic optimal growth case. The stochastic case is, of course, a straightforward analog and only requires minor notational and conceptual finessing. However, the full generality of the stochastic case is, in my opinion, most illuminatingly treated in [37], chapter 7.

[22] Of course, the implications of the 'Cambridge Controversies in Capital Theory' have passed over the current generation of mathematical macroeconomists like water off of a duck's back – or even worse, [50], p. 523; italics added:

> "In the 1960s there was the famous Cambridge capital controversy. This controversy bears on the issue 'What is money?' The Cambridge capital controversy was a silly one, as pointed out so clearly by Arrow.. . .'

dynamic programming problem in terms of the *Bellman equation* (or the *functional equation*) and the *value function*, V, as (writing the production function in its intensive form):

$$V(K_0) = \max_{0 \leq K_1 \leq f(K_0)} \{U[f(K_0) - K_1] + \beta V(K_1)\} \tag{5}$$

This can be rewritten, purely formally, in per capita and intensive forms, as:

$$V(k) = \max_{0 \leq y \leq f(k)} \{U[f(k) - y] + \beta v(y)\} \tag{6}$$

Remark 1 *From the functional equation we can derive an analogue of the Euler equations and, thus, 'return' to the Ramsey marginal conditions*[23]*. Thus, if v is assumed to be differentiable and if the maximizing value of y, say $v^*(k)$ was an interior value, then:*

$$\begin{aligned} U'[f(k) - v^*(k)] &= \beta v'[v^*(k)] \\ v'(k) &= f'(k) U'[f(k) - v^*(k)] \end{aligned} \tag{7}$$

The above formulation is the optimum allocation problem of our benevolent social planner. To be able to interpret the solution to the 'centralized planned economy' dynamic programming problem (5) (or (6)) as the solution brought about by a competitive, decentralized, market economy, one invokes the fundamental theorems of welfare economics, thereby exploiting the much 'maligned' relationship between *Pareto Optima* and *Competitive Equilibria* (and, thus, underpinning this alleged macroeconomic framework in solid microeconomic foundations). The first fundamental theorem of welfare economics enables the conclusion that any (decentralized) competitive equilibrium allocation is Pareto

[23] The knowledgeable, discerning reader will recall Ramsey's explanatory note immediately after deriving his famous result ([51], p.547):

> "Mr Keynes, to whom I am indebted for several other suggestions, has shown me that this result can also be obtained by the following simple reasoning."

Those, like me, who are deeply disturbed by the ahistorical scholarship of the stalwarts of Newclassical economics and wince at their ignorant pronouncements, will like to recall this handsome acknowledgement to Keynes by one of the greatest logicians and mathematicians of the 20th Century. In saying this, and remembering that just Cambridge at the time of Keynes was richly endowed with greatness in mathematics, mathematical physics, mathematical logic and applied mathematics – G.H. Hardy, Bertrand Russell, A.N. Whitehead, Ludwig Wittgenstein, Harold Jeffreys, Paul Dirac, Arthur Eddington, Lord Rutherford, J.E. Litlewood, A.S. Besicovich and legions of others. How, then, can Lucas make the following preposterous suggestion ([35], p.149; italics added):

> When Marshall was educated, and even when Keynes was educated, England was a mathematical backwater. If they had been educated in France, Germany or Russia, working with people like Kolmogorov, Borel or Cantor, they would have thought differently. Walras, Pareto and Slutzky thought differently. The people who were giving birth to mathematical economics were mainly on the continent at that time."

Were they, indeed? And were they taught in the traditions that were fostered by Cantor and Borel; Kolmogorov and Andronov; Volterra and Cantelli; etc?

Optimal; conversely, the second fundamental theorem of welfare economics supports the mythical benevolent social planner's Pareto Optimal allocation as a competitive equilibrium[24]. In other words, the price system associated with the latter can support the optimal allocation derived by the former. To obtain, therefore, the necessary (decentralized) competitive price system, we revert to the framework of the individual agent's decision problem.

The household's decision problem is:

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\begin{aligned} s.t \quad \sum_{t=0}^{\infty} p_t \left[c_t + K_{t+1}\right] &\leq \sum_{t=0}^{\infty} p_t \left[w_t + (r_t + 1 - \delta) K_t\right] \\ c_t &\leq 0, \quad K_{t+1} \geq 0 \end{aligned} \tag{8}$$

From the FOCs for the household's decision problem we can derive:

$$\frac{p_t}{p_{t+1}} = \frac{U'(c_t)}{\left[\beta \cdot U'(c_{t+1})\right]} \tag{9}$$

Similarly, from the firm's analogous decision problem:

$$\max_{K_t, H_t} p_t \cdot \left[F(K_t, H_t) - r_t K_t - w_t H_t\right], \qquad \forall t \tag{10}$$

we get, again from the FOCs, $\forall t$, the optimum values for the real wage rate, w_t and the real rate of return on capital, r_t, respectively:

$$w_t = F_2(K_t, H_t) \tag{11}$$

$$r_t = F_1(K_t, H_t) \tag{12}$$

Now, to transform the above standard formulation of the household and firm decision problems to a dynamic programming formulation, denote by lower case letters those variables over which an individual household has immediate control; upper case letters for variables that are their aggregate counterparts. Thus, for example: k is an individual households capital stock; and K is the economy-wide per capita capital stock. In equilibrium, of course, $K = k$. Thus the state variables for the individual households are the pair (k, K) .Then, denote by $v(k, K)$, the individual households optimum value function; and, if primes denote values one period later, the households dynamic programming decision problem will be to choose a path for investment, say x, and consumption c, that

[24] Surely, Hayek, Robbins, von Mises, Lange and other participants of the much-debated 'socialist calculation debate' of the 1930s, must be turning and twiting in their noble graves!

solves the problem:

$$\begin{aligned} v(k,K) &= \max_{c,x\geq 0} \{u(c)+\beta v[k',K']\} \\ s.t \quad c+x &\leq r(K)k+w(K) \\ k' &= (1-\delta)k+x \\ K' &= (1-\delta)K+X(K) \end{aligned} \tag{13}$$

Suppose $d(k,K)$ is the optimal policy function; since this is a representative agent economy, this implies: $d(k,K)=D(K)$, in equilibrium. Then:

Definition 2 *A Recursive Competitive Equilibrium (*RCE*), (v,d,D,r,w), is characterized by the following set of conditions:*

1. a value function $v(k,K):\Re_2^+\rightarrow\Re$;
2. a policy function $d(k,K):\Re_2^+\rightarrow\Re^+$ giving decisions on $c(k,K)$ and $x(k,K)$;
3. analogous to $d(k,K)$ above, an aggregate policy function $D(K):\Re^+\rightarrow\Re^+$, giving the aggregate decisions, $C(K)$ and $D(K)$,respectively;
4. factor price functions, $r(K)$ and $w(K)$,both $\Re^+\rightarrow\Re^+$, satisfying, (13), (11)–(12, the aggregate resource constraint and the consistency between the individual and aggregate decisions[25].

Remark 3 *If (v,d,D,r,w) is a **RCE**, then an implication of the second fundamental theorem of welfare economics is that $v(k,K)=V(K)$.*

The power this particular dynamic extension of the traditional equilibrium concept plays a significant role in the mathematized macroeconomy is best described in the words of two of the frontier practitioners of the subject:

> "Another great advantage of the RCE approach is that for an increasingly rich class of model economies, *the equilibrium process can be computed* and *can be simulated to generate equilibrium paths for the economy.* These paths can be studied to see whether model economies mimic the behavior of actual economies and can be used to provide quantitative answers to questions of economic welfare."
> [7], p.9; italics added.

A brief reflection on the foundations of computable general equilibrium, at this point, may make this apparently laudable aim more clear. The real power of a formal CGE model, from a computational point of view, relies entirely on *Uzawa's Equivalence Theorem* (cf.[61], chapter 11). It is this theorem that

[25] There is no question of the fallacy of composition driving a wedge between individual and aggregate decisions in this kind of macroeconomics!

proves the formal equivalence between a Walrasian Equilibrium Existence Theorem and the Brouwer (or Kakutani) Fix Point Theorem. This equivalence enables any computational process – i.e., an algorithm – constructed for computing a Brouwer fix point gives also the Walrasian (Exchange) Equilibrium. Roles analogous to the Uzawa Equivalence Theorem are played, in the above claims by Cooley and Prescott, particularly when, at the stage of policy analysis, efficiency propositions are also imputed to the processes.

Now, there are three problems with these claims and aims. First of all, and trivially, no where in the literature on mathematical economics, mathematical macroeconomics or even in formal computability theory is there any proposition on the efficiency of processes; in fact, it is quite easy to show that the dynamic programming formulation above, for the RCE, is, in fact computationally intractable in a precise sense. Secondly, neither the first nor the second welfare theorems are computationally feasible in the precise senses of computability theory and constructive analysis. Thirdly, the approximation procedures used, in computing the relevant RCE are provable intractable, simply because the equilibrium is uncomputable!

I shall only deal with the second of these infelicities in this paper. Companion pieces to this work tackle the whole set of issues more systematically.

The *First Fundamental Theorem of Welfare Economics* asserts the that *a competitive equilibrium is Pareto optimal.* A textbook formulation of the theorem is as follows ([61], p. 145):

Theorem 4 *Assume Weak monotonicity and continuity of preferences; Let $p^* \in \Re_+^N$ be a competitive equilibrium price vector of the economy. Let ω^{0i}, $i \in H$, be he associated individual consumption bundles, and let y^{0j}, $j \in F$,be the associated firm supply vectors. Then ω^{0i} is Pareto efficient.*

where:

F : set of firms.

Proof. See [61], p. 145-6. ■

Remark 5 *The theorem is proved non-constructively, using an uncomputable equilibrium price vector to compute an equilibrium allocation. Therefore, the contradiction step in the proof requires a comparison between an uncomputable allocation and an arbitrary allocation, for which no computable allocation can be devised. Moreover, the theorem assumes the intermediate value theorem in its non-constructive form. Finally, even if the equilibrium price vector is computable, the contradiction step in the proof invokes the law of the excluded middle and is, therefore, unacceptable constructively (because it requires algorithmically undecidable disjunctions to be employed in the decision procedure).*

The *Second Fundamental Welfare Theorem* establishes the proposition that *any Pareto optimum can, for suitably chosen prices, be supported as a competitive equilibrium.* The role of the Hahn-Banach theorem in this proposition is in establishing *the suitable price system.*

Lucas and Stokey state 'their' version of the Hahn-Banach Theorem in the following way[26]:

Theorem 6 *Geometric form of the Hahn-Banch Theorem.*

Let S be a normed vector space; let $A, B \subset S$ be convex sets. Assume:

(a). Either B has an interior point and $A \cap \mathring{B} = \emptyset$, $\left(\mathring{B} : \text{ closure of } B\right)$;

(b). Or, S is finite dimensional and $A \cap B = \emptyset$;

Then: $\exists$ a continuous linear functional ϕ, not identically zero on S, and a constant c s.t:

$\phi(y) \leq c \leq \phi(x)$, $\forall x \in A$ *and* $\forall y \in B$.

Next, I state the economic part of the problem in merciless telegraphic form as follows:

There are$I$ consumers, indexed $i = 1, ..., I$;

S is a vector space with the usual norm;

Consumer i chooses from commodity set $X_i \subseteq S$, evaluated according to the utility function $u_i : X_i \to \Re$;

There are j firms, indexed $j = 1, ..., J$;

Choice by firm j is from the technology possibility set, $Y_j \subseteq S$; (evaluated along profit maximizing lines);

The mathematical structure is represented by the following absolutely standard assumptions:

1. $\forall i, X_i$ *is convex*;
2. $\forall i$, *if* $x, x' \in C_i, u_i(x) > u_i(x')$, *and if* $\theta \in (0, 1)$, *then* $u_i[\theta x + (1-\theta)x'] > u_i(x')$;
3. $\forall i, u_i : X_i \to \Re$ *is continuous*;
4. *The set* $Y = \sum_j Y_j$ *is convex*;
5. *Either the set* $Y = \sum_j Y_j$ *has an interior point*, or S *is finite dimensional*;

Then, the *Second Fundamental Theorem of Welfare Economics* is:

Theorem 7 *Let assumptions $1 - 5$ be satisfied; let $\left[\left(x_i^0\right), \left(y_j^0\right)\right]$ be a Pareto Optimal allocation; assume, for some $h \in \left\{\bar{1}, ...\bar{I}\right\}, \exists \hat{x}_h \in X_h$ with $u_h(\hat{x}_h) > u_h(x_h^0)$. Then $\exists$ a continuous linear functional $\phi : S \to \Re$, not identically zero on S, s.t:*

(a). $\forall i, x \in X_i$ *and* $u_i(x) \geq u_i(x^0) \Longrightarrow \phi(x) \geq \phi(x_i^0)$;

(b). $\forall j, y \in Y_j \Longrightarrow \phi(j) \leq \phi(y_i^0)$;

[26] Essentially, the 'classical' mathematician's Hahn-Banach theorem guarantees the extension of a bounded linear functional, say ρ, from a linear subset Y of a separable normed linear space, X, to a functional, η, on the whole space X, with exact preservation of norm; i.e., $|\rho| = |\eta|$. The constructive Hahn-Banach theorem, on the other hand, cannot deliver this pseudo-exactness and preserves the extension as: $|\rho| \leq |\eta| + \varepsilon$, $\forall \varepsilon > 0$. The role of the positive ε in the constructive version of the Hahn-Banach theorem is elegantly discussed by Nerode, Metakides and Constable in their beautiful piece in the Bishop Memorial Volume ([42], pp. 85-91). Again, compare the difference between the 'classical' IVT and the constructive IVT to get a feel for the role of ε.

Anyone can see, as anyone would have seen and has seen for the last 70 years, that an economic problem has been 'mangled' into a mathematical form to conform to the structure and form of a mathematical theorem. But this is standard practice, as we saw in Debreu's examples in section 1, for practitioners of the 'orthodox methodology'..

It is a pure mechanical procedure to verify that the assumptions of the economic problem satisfy the conditions of the Hahn-Banach Theorem and, therefore, the powerful *Second Fundamental Theorem of Welfare Economics* is 'proved'[27].

The Hahn-Banach theorem does have a constructive version, but only on subspaces of *separable* normed spaces. The standard, 'classical' version, valid on nonseparable normed spaces depends on *Zorn's Lemma* which is, of course, equivalent to the axiom of choice, and is therefore, non-constructive[28].

Schechter's perceptive comment on the constructive Hahn-Banach theorem is the precept I wish economists with a numerical, computational or experimental bent should keep in mind (ibid, p. 135; italics in original; emphasis added).:

> "[O]ne of the fundamental theorems of classical functional analysis is the *Hahn-Banach Theorem; ...* some versions assert the existence of a certain type of linear functional on a normed space X. The theorem is inherently nonconstructive, but a constructive proof can be given for a variant involving normed spaces X that are *separable* – i.e., normed spaces that have a countable dense subset. *Little is lost in restricting one's attention to separable spaces*[29]*, for in applied math most or all normed spaces of interest are separable.* The constructive version of the Hahn-Banach Theorem is more complicated, but it has the advantage that it actually *finds* the linear functional in question."

So, one may be excused for wondering, why economists rely on the 'classical' versions of these theorems? They are devoid of numerical meaning and computational content. Why go through the rigmarole of first formalizing in terms of numerically meaningless and computationally invalid concepts to then seek impossible and intractable approximations to determine uncomputable equilibria, undecidably efficient allocations, and so on?

Thus my question is: why should an economist *force* the economic domain to be a normed vector space? Why not a *separable normed vector space*? Isn't

[27]To the best of my knowledge an equivalence between the two, analogous to that between the Brouwer fix point theorem and the Walrasian equilibrium existence theorem, proved by Uzawa ([65]), has not been shown.

[28]This is not a strictly accurate statement, although this is the way many advanced books on functional analysis tend to present the Hahn-Banach theorem. For a reasonably accessible discussion of the precise dependency of the Hahn-Banach theorem on the kind of axiom of choice (i.e., whether countable axiom of choice or the axiom of dependent choice), see [41]. For an even better and fuller discussion of the Hahn-Banach theorem, both from 'classical' and a constructive points of view, Schechter's encyclopedic treatise is unbeatable ([56]).

[29]However, it must be remembered that Ishihara, [24], has shown the constructive validity of the Hahn-Banach theorem also for uniformly convex spaces.

this because of pure ignorance of constructive mathematics and a carelessness about the nature and scope of fundamental economic entities and the domain over which they should be defined?

On the other hand, the first fundamental theorem of welfare economics fails constructively and computably on three grounds: the dependence on the intermediate value theorem (non-constructive), the use of an uncomputable equilibirum price vector in the proof by contradiction (uncomputability) and the use of the law of the excluded middle in the proof by contradiction (non-constructivity).

Under these conditions, the equilibrium of the canonical *SDGE* model, *RCE*, cannot, in any formal algorithmic sense be effectively or constructively computed; therefore, no equilibrium process can effectively be determined to show convergence to a balanced growth path.

Finally, the mathematical structure of the space on which the value function and the policy function are defined is such that the existence of a fix point for the contraction operator that is invoked is non-algorithmizable. This is because *Cauchy Completeness* is assumed for the space over which the contraction is implemented. But Cauchy Completeness, stated as:

Theorem 8 *Every Cauchy sequence in* $\mathbb{R}$ *converges to an element of* $\mathbb{R}$

This theorem is, in turn, proved using the *Bolzano-Weierstrass theorem*, which contains an unconstructifiable - i.e., non-algorithmic and hence impossible to utilise in a consistent 'computational experiment' - *undecidable disjunction* in its proof!

In other words, the computational program of mathematizing macroeconomics by formulating optimal decision problems as dynamic programming problems is impossible.

4 ***Recursive***[30] Rational Expectations Equilibria

> "In dynamic contexts, we formulate a rational expectations equilibrium as a fixed point in a space of sequences of prices and quantities, or, equivalently, a fixed point in a space of *functions* that determine sequences of prices and quantities."
>
> Thomas J. Sargent: **Bounded Rationality in Macroeconomics**, p.8, [55]; italics in the original.

Thereby, hangs the same tale! Existence of a *rational expectations equilibrium* (*REE*) is established using a fix point theorem. Such a theorem is completely formal, without a shred of algorithmic content. Therefore, additional *ad hoc* algorithmic mechanisms and theories have to be constructed for learning the *REE* that has been proved to exist purely formally. One might be excused

[30] Here, for once, I am able to refer to 'recursive' in the sense of 'recursion theory'.

for wondering, at this stage, and given the aims of the mathemtizing enterprise of macroeconomics that we have identified as being quantitative in the sense of computational, numerical and experimental economics, why the theorist does not algorithmize the proof, *ab initio*! Then, a separate and *ad hoc* learning mechanism to determine the *REE* will not have to be devised.

To suggest a remedy to this infelicity, I tackle two issues pertaining to *REE* from a purely recursion theoretic point of view, where such dichotomies are never present. Anything proved to exist, in a recursion theoretic framework, comes, *pari passu*, with an algorithm to compute it. In the first part of this section, building on Spear's early work (cf. [60][31]), a recursion theoretic learning mechanism is suggested. The second part is a more direct attack on the problem of the existence of *REE*.

There are two crucial aspects to the notion of *REE,* ([55], pp.6-10): an individual optimization problem, subject to *perceived constraints,* and a system wide, autonomous, set of constraints imposing consistency across the collection of the perceived constraints of the individuals. The latter would be, in a most general sense, the accounting constraint, generated autonomously, by the logic of the macroeconomic system. In a representative agent framework the determination of *REE*s entails the solution of a general fix point problem. Suppose the representative agent's *perceived law of motion* of the macroeconomic system (as a function of state variables and exogenous 'disturbances') as a whole is given by the (topological) map H[32]. The system wide autonomous set of constraints, implied, partially at least, by the optimal decisions based on perceived constraints by the agents, on the other hand, imply an *actual law of motion* given by, say, H^0. The search for fixed-points of a mapping, T, linking the individually perceived macroeconomic law of motion, H, and the actual law of motion, H^0 is assumed to be given by a general functional relationship subject to the standard mathematical assumptions:

$$H^0 = T(H) \tag{14}$$

Thus, the fixed-points of H^* of T[33]:

$$H^* = T(H^*) \tag{15}$$

determine *REE*s.

What is the justification for T? What kind of 'object' is it? It is variously referred to as a 'reaction function', a 'best response function', a 'best response mapping', etc. But whatever it is called, eventually the necessary mathematical assumptions are imputed to it such that it is amenable to a topological interpretation whereby appeal can be made to the existence of a fix-point for it as

[31] Spear's contribution is riddled with technical infelicities that display quie a comprehensive ignorance of even classical recursion theory. How it came to be published in the Econometrica should be an interesting study in the sociology of peer-reviewed publications!

[32] Readers familiar with the literature will recognise that the notation H reflects the fact that, in the underlying optimisation problem, a Hamiltonian function has to be formed.

[33] In a space of functions.

a mapping from a structured domain into itself. So far as I know, there is no optimizing economic theoretical justification for it.

Now let me go behind the scenes, so to speak, and take one of the many possible economic worlds in which T operates, a simple Overlapping Generation Model (OLG), with standard assumptions, which generates REEs as solutions to the following type of *functional* dynamic equation (cf. [1], pp. 414-6):

$$u'(e_1 - m_t) = \mathcal{E}\left\{\frac{m_{t+1}}{m_t}\frac{L_{t+1}}{L_t}v'(e_2 + m_{t+1}\frac{L_{t+1}}{L_t}) \mid \mathbf{I}_t\right\}, \forall \mathbf{I}_t \tag{16}$$

Where:

u and v are functional notations for the additive utility functions;

The real gross yield on money, $R_t = \frac{p_t x_{t+1}}{p_{t+1}} = \frac{m_{t+1}}{m_t}\frac{L_{t+1}}{L_t}$;

The real per capita currency balances, $m_t = \frac{M_t}{p_t L_t}$;

L_t : size of generation t (a discrete random variable with standard assumptions);

M_t : aggregate stock of currency;

p_t : realized price (of the one consumption good);

p_{t+1} : future price (random variable);

e_t : endowment at time t;

$\mathbf{I}_t$: information set defined by

$$\mathbf{I}_t = I\left\{\mathbf{I}_{t-1}, L_{t-1}, x_{t-1}, p_{t-1}, \theta_t\right\} \tag{17}$$

θ_t : vector of all other residual variables that the agent believes will influence future prices;

The problem I pose is the devising of an *effective* mechanism to *learn* and *identify* the above REE solution. However, it is immediately clear that one must first ensure that the solution is itself a *recursive real*, if an *effective* mechanism is to locate it. *A priori*, and except for flukes, it is most likely that the standard solution will be a *non-recursive real*. To make it possible, therefore, to ensure a recursively real solution to the above functional dynamic equation, this OLG structure must be endowed with an appropriate recursion theoretic basis. I shall, now, indicate a possible set of minimum requirements for the required recursion theoretic basis.

The derivative of the second period component of the additive utility function, v, must be a *computable real function*. Roughly speaking, if the domain of v is chosen judiciously and if $v \in C^2$, and computable, then v' is computable. But, for these to be acceptable assumptions, the arguments of v',i.e., e_2, m_{t+1}, and $\frac{L_{t+1}}{L_t}$, must be computable reals. Since this is straightforward for e_2 and per capita currency balances[34],m_{t+1}, a recursion theoretic interpretation for the random variable L_t will ensure that the assumptions underlying v' are recursion theoretically sound. Now, the random variables in the OLG model above are characterized by finite means and stationary probability distributions. It

[34]Provided we assume a straightforward recursive structure for prices, which turns out, usually, to be natural.

is, therefore, easy to construct a *Probabilistic Turing Machine*, *PTM*, endowed with an extra random-bit generator which outputs, whenever necessary, the necessary element that has the pre-assigned probability distribution. Next, there is the question of the recursivity of the information set, $\mathbf{I}_t$. Given that a recursion theoretic learning model requires this information set to be *recursively* presented to the agents, it is only the element θ_t that remains to be recursively defined. However, this is a purely exogenous variable that can be endowed with the required recursive structure almost arbitrarily.

Finally, the expectations operator is interpreted as an integration process and, since integration is a computable process, this completes the necessary endowment of the elements of the above OLG model with a sufficient recursive structure to make the *REE* generated by the solution to the functional equation a *recursive real.* The minor caveat 'sufficient recursive structure' is to guard against any misconception that this is the only way to endow the elements of an OLG model as given above with the required assumptions to guarantee the generation of a recursive real as a solution. There are many ways to do so but I have chosen this particular mode because it seems straightforward and simple. Above all, these assumptions do not contradict any of the standard assumptions and can live with almost all of them, with minor and inconsequential modifications.

With this machinery at hand, I can state and prove the following theorem:

Theorem 9 *A unique, recursively real, solution to (16) can be identified as the REE and learned recursively.*

Proof. See [66], pp. 98-9. ∎

Remark 10 *The theorem is about recursive learning; nevertheless it does embody an unpleasant epistemological implication: there is no **effective** way for the learning agent to know when to stop applying the learning mechanism! Moreover, nothing in the assumptions guarantee tractable computability at any stage.*

Going back to T, clearly there is nothing sacrosanct about a topological interpretation of such an operator. It could equally well be interpreted *recursion theoretically*, which is what I shall do in the sequel. I need some unfamiliar, but elementary, formal machinery, not routinely available to the mathematical economist.

Definition 11 *An **operator** is a function:*

$$\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n \tag{18}$$

where $\mathcal{F}_k \quad (k \geqq 1)$ *is the class of all partial (recursive) functions from* $\mathbb{N}^k$ *to* $\mathbb{N}$.

Definition 12 Φ *is a **recursive operator** if there is a computable function* ϕ *such that* $\forall f \in \mathcal{F}_m$ *and* $\mathbf{x} \in \mathbb{N}^m$, $y \in \mathbb{N}$:

$$\Phi(f)(\mathbf{x}) \simeq y \ iff \ \exists \ a \ finite \ \theta \sqsubseteq f \ such \ that \ \phi\left(\widetilde{\theta}, \mathbf{x}\right) \simeq y$$

where[35] $\widetilde{\theta}$ is a standard *coding* of a *finite* function θ, which is extended by f.

Definition 13 *An operator $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ is* ***continuous*** *if, for any $f \in \mathcal{F}_m$, and $\forall \mathbf{x}, y$:*

$$\Phi(f)(\mathbf{x}) \simeq y \ iff \ \exists \ a \ finite \ \theta \sqsubseteq f \quad such \ that \ \Phi(\theta)(\mathbf{x}) \simeq y$$

Definition 14 *An operator $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_n$ is* ***monotone*** *if, whenever $f, g \in \mathcal{F}_m$ and $f \sqsubseteq g$, then $\Phi(f) \sqsubseteq \Phi(g)$.*

Theorem 15 *A recursive operator is continuous and monotone.*

Example 16 *Consider the following* ***recursive program,*** *Þ ,(also a recursive operator) over the integers:*

Þ: $F(x,y) \Longleftarrow if \ x = y \ then \ y+1, \ else \ F(x, F(x-1, y+1))$

Now replace each occurrence of F in Þ by each of the following functions:

$$f_1(x,y): \ if \ x = y \ then \ y+1, \ else \ x+1 \tag{19}$$

$$f_2(x,y): \ if \ x \geqq y \ then \ x+1, \ else \ y-1 \tag{20}$$

$$f_3(x,y): \ if \ (x \geqq y) \wedge (x-y \ even) \ then \ x+1, \ else \ undefined. \tag{21}$$

Then, on either side of $\Longleftarrow$ in Þ, we get the ***identical*** *partial functions:*

$$\forall i \, (1 \leqq i \leqq 3), \ f_i(x,y) \equiv if \ x = y \ then \ y = 1, \ else \ f_i(x-1, y+1) \tag{22}$$

Such functions f_i $(\forall i \, (1 \leqq i \leqq 3))$ are referred to as ***fixed-points*** *of the recursive program Þ (recursive operator).*

Note that these are fixed-points of functionals.

Remark 17 *Note that f_3, in contrast to f_1 and f_2, has the following special property. $\forall \langle x, y \rangle$ of pairs of integers such that $f_3(x,y)$ is defined, both f_1 and f_2 are also defined and have the same value as does f_3.*

- *f_3 is, then, said to be* ***less defined than or equal to*** *f_1 and f_2 and this property is denoted by $f_3 \sqsubseteq f_1$ and $f_3 \sqsubseteq f_2$.*
- *In fact, in this particular example, it so happens that f_3* ***is less defined than or equal to*** *all fixed points of Þ.*

[35] If $f(\mathbf{x})$ and $g(\mathbf{x})$ are expressions involving the variables $x = (x_1, x_2, ..., x_k)$, then:

$$f(\mathbf{x}) \simeq g(\mathbf{x})$$

means: for any x, $f(\mathbf{x})$ and $g(\mathbf{x})$ are either both defined or undefined, and if defined, they are equal.

- *In addition, f_3 is the **only** partial function with this property for Þ and is, therefore called the **least fixed point of Þ**.*

We now have the minimal formal machinery needed to state one of the classic theorems of recursive function theory, known variously as the *first recursion theorem*, *Kleene's theorem* or, sometimes, as the *fixed point theorem for complete partial orders.*

Theorem 18 *Suppose that $\Phi : \mathcal{F}_m \longrightarrow \mathcal{F}_m$ is a recursive operator (or a recursive program Þ). Then there is a partial function f_ϕ that is the least fixed point of Φ :*
$\Phi(f_\phi) = f_\phi$;
If $\Phi(g) = g$, then $f_\phi \sqsubseteq g$.

Remark 19 *If, in addition to being partial, f_ϕ is also total, then it is the **unique least fixed point.** Note also that a recursive operator is characterized by being continuous and monotone. There would have been some advantages in stating this famous theorem highlighting the domain of definition, i.e., complete partial orders, but the formal machinery becomes slightly unwieldy.*

Remark 20 *Although this way of stating the (first) recursion theorem almost highlights its non-constructive aspect – i.e., the theorem guarantees the **existence** of a fix-point without indicating a way of finding it – it is possible to use a slightly stronger form of the theorem to amend this 'defect'.*

Before stating formally, as a summarizing theorem, the result, it is necessary to formalize the rational agent and the setting in which rationality is exercised in the expectational domain in recursion theoretic formalisms, too. This means, at a minimum, the rational agent as a *recursion theoretic agent*[36]. The topological fix-point theorems harnessed by a rational agent are, as mentioned previously, easily done in standard economic theory where the agents themselves are *set-theoretically* formalized. There is no dissonance between the formalism in which the rational agent is defined and the economic setting in which such an agent operates. The latter setting is also set theoretically defined.

The recursion theoretic formalism introduced above presupposes that the rational agent is now recursion theoretically defined and so too the setting - i.e., the economy. Defining the rational agent recursion theoretically means defining the preferences characterizing the agent and the choice theoretic actions recursion theoretically. This means, firstly, defining the domain of choice for the agent number theoretically and, secondly, the choice of maximal (sub)sets over such a domain in a computably viable way. Such a redefinition and reformalization should mean equivalences between the rational choice of an agent over well defined preferences and the computing activities of an ideal computer, i.e., Turing

[36] This should not cause any disquiet in expectational economics, at least not to those of us who have accepted the Lucasian case for viewing agents as 'signal processors' who use optimal filters in their rational decision processing activities (cf. [33], p.9). Agents as 'signal processors' is only a special variant of being 'optimal computing units'.

Machine (or any of its own formal equivalences, by the *Church-Turing Thesis*). Since a complete formalism and the relevant equivalences are described, defined and, where necessary, rigorously proved in [66], chapter 3, I shall simply assume the interested reader can be trusted to refer to it for any detailed clarification and substantiation.

It is now easy to verify that the domain over which the recursive operator and the partial functions are defined are weaker[37] than the conventional domains over which the economist works. Similarly, the continuity and monotonicity of the recursive operator is naturally satisfied by the standard assumptions in economic theory for the reaction or response function, T. Hence, we can apply the *first recursion theorem* to equation (15), interpreting T as a recursive operator and not as a topological mapping. Then we know that there is a partial function - i.e., a computable function - f_t that is the least fixed point of T. Thus, we can summarize the desired result in the form of the following theorem:

Theorem 21 *Suppose that the reaction or response function, $T : H_m \longrightarrow H_m$ is a recursive operator (or a recursive program Γ). Then there is a computable function f_t that is a least fixed point of T :*
$T(f_t) = f_t$;
If $T(g) = g$, then $f_t \sqsubseteq g$

Remark 22 *Theorem 21 can be used directly to show that $\exists$ a (recursive) program that, under any input, outputs exactly itself. It is this program that acts as the relevant reaction or response **function** for an economy in REE. The existence of such a recursive program justifies the Newclassical methodological stand on the ubiquity of rational expectations equilibria. However, since the theorem is stated above in its non-constructive version, finding this particular recursive program requires a little effort. Hence, the need for learning processes to find this program, unless the theorem is utilized in its constructive version. Even with these caveats, the immediate advantage is that there is no need to deal with non-recursive reals or non-computable functions in the recursion theoretic formalism. In the traditional formalism the fix-point that is the REE is, except for flukes, a* non-recursive real*;* constructing learning processes to determine non-recursive reals *is either provably impossible or formally intractable (computationally complex).*

What are the further advantages of recasting the problem of solving for the *REE recursion theoretically* rather than retaining the traditional topological formalizations?

An advantage at the superficial level but nevertheless not unimportant in policy oriented economic theoretic contexts is the simple fact that, as even the name indicates, recursion encapsulates, explicitly, the idea of self-reference because functions are defined, naturally, in terms of themselves. Secondly the

[37] They are 'weaker' in a very special sense. A domain of definition that is number theoretically defined – i.e., over only the rational or the natural numbers – rather than over the whole of the real number system pose natural diophantine and combinatorial conundrums that cannot easily be resolved by the standard operators of optimization.

existence of a least fix point is a solution to the infinite-regress problem. Thus the two conceptual difficulties that bedevil the theory of rational expectations are formally encapsulated in one fell swoop, within one analytical framework - and, that too, with a computable function.

Think of the formal discourse of economic analysis as being conducted in a programming language; call it $\Im$. We know that we choose the underlying terminology for economic formalisms with particular meanings in mind for the elemental units: preferences, endowments, technology, information, expectation and so on; call the generic element of the set ς. When we form a compound economic proposition out of the ς units, the meaning is natural and clear. We can, therefore, suppose that evaluating a compound expression in $\Im$ is immediate: given an expression in $\Im$, say $\lambda(\varsigma)$, the variables in λ, when given specific values α, are to be evaluated according to the *semantics* of $\Im$. To actually *evaluate* a compound expression, $\lambda(\varsigma)$, we write a *recursive program* in the language $\Im$, the language of economic theory.

But that leaves a key question unanswered: what is the computable function that is implicitly defined by the recursive program? The first recursion theorem answers this question with the answer: the least fixed-point. In this case, therefore, there is a direct application of the first recursion theorem to the semantics of the language $\Im$. The artificial separation between the syntax of economic analysis, when formalized, and its natural semantics can, therefore, be bridged *effectively.*

If the language of economic theory is best regarded as a very high level programming language, $\Im$, to understand a *theorem* in economics, in recursion theoretic terms, represent the *assumptions* - i.e., *axioms* and the *variables* - as *input data* and the *conclusions* as *output data*. State the theorem as an expression in the language $\Im$.Then try to convert the proof into a program in the language $\Im$, which will take in the inputs and produce the desired output. If one is unable to do this, it is probably because the proof relies essentially on some infusion of non-constructive or uncomputable elements. This step will identify any inadvertent infusion of non-algorithmic reasoning, which will have to be resolved - sooner or later, if computations are to be performed on the variables as input data. The computations are not necessarily numerical; they can also be symbolic.

In other words, if we take algorithms and data structures to be fundamental,then it is natural to define and understand functions in these terms. If a function does not correspond to an algorithm, what can it be? The topological definition of a function is not naturally algorithmic. Therefore, the expressions formed from the language of economic theory, in a topological formalization, are not necessarily implementable by a program, except by flukes, appeal to magic or by illegitimate, intractable and vague approximations. Hence the need to dichotomize every topological existence proof. In the case of *REE,* this is the root cause of the artificial importance granted to a separate problem of learning *REE*s.

5 Reflections and Retrospectives

> "... I want to emphasize that the methodology that transformed macroeconomics is applicable to the study of virtually all fields of economics. In fact, the meaning of the word *macroeconomics* has changed *to refer to the tools being used*[38] rather than just to the study of business cycle fluctuations[39]."
>
> Edward Prescott: '*The Transformation of Macroeconomic Policy and Research*', **Nobel Prize Lecture** ([48]; second set of italics, added)

The recursivization of macroeconomics has implied its mathematization in a particular way. There is clear evidence that the frontiers of macroeconomics is recursive macroeconomics. Paradoxically, however, the main aims of the recursivization cannot be achieved by means of the particular mathematization of macroeconomics that has come to be realized. I have discussed the reasons above, in the previous sections.

Moreover, the microeconomics on which recursive macroeconomics is founded – orthodox general equilibrium theory – is intrinsically non-algorithmic and, indeed, cannot be algorithmized without drastic re-mathematization of *its* foundations.

Why have mathematically minded macroeconomists, committed to a formally quantified theory of aggregates, placing at the core of the subject computational, numerical, approximation and experimental issues, failed to realize the intrinsic non-numerical nature of the formal mathematics they use? Especially since there are at least two formal, alternative, mathematical formalisms, far superior in numerical and algorithmic content, easily available for harnessing in their noble and laudable formalization enterprise: computability theory and constructive mathematics (free of philosophical baggage of the Brouwerian stringent variety). Moreover, both of these deep and well founded and highly developed areas of mathematics have their formal metamathematical foundations as well as their analytic handmaidens: recursion theory and proof theory in the one case; computable analysis and constructive analysis in the other. In fact, there are even many varieties of each of the latter, from which a mathematically minded macroeconomist can choose, to suit his or her own purpose in any particular application. For example, there are at least three different ways to appeal to a constructive version of the Hahn-Banach theorem so as to substantiate the second fundamental theorem of welfare economics; there are at least two different ways to prove the validity of the first fundamental theorem

[38] Prescott is, of course, referring to mathematical and computational tools.

[39] I cannot resist the temptation to add, as a counter-weight to this sanguine view a trenchant observation made by a previous Nobel Laureate, who may not have been unsympathetic to the new classicals, when he reviewed the classic of an earlier generation, Paul Samuelson's Foundations of Economic Analysis ([62], p. 605):

> "... [W]ho can know what tools we need unless he knows the material on which they will be used."

of welfare economics using computable analysis. Such proofs of existence come, pari passu, with algorithmic possibilities. One illustration, for the case of *REE,* was given in section 4, where embedding the problem, *ab initio*, in a recursion theoretic setting obviated the need for the traditional two stage difficulty of first proving existence and then devising mechanisms to locate the *REE*.

I must confess I have no reasonable answer to these questions – not even conjectures.

A recent Nobel Laureate in economics, Finn Kydland, in his own 'Nobel Lecture' claims ([28], p. 341; italics added):

> "The key tool macroeconomists use is the *computational experiment*"

But he fails to have ever investigated whether any of the models he uses for computational experiments is algorithmically – i.e., computably or constructively – founded or not. How can a computational experiment be conducted, utilizing discrete data and using a non-computable, non-constructive model, for quantitative policy experiments with a digital computer?

These are the paradoxes of Recursive Macroeconomics that will have to be resolved as the mathematization of macroeconomics gathers pace and the digital revolution is approached by the recursive revolution in macroeconomics itself.

There was a time, not too long ago, when the mathematical underpinnings of macroeconomics was adequately learnable by a complete mastery of that mid-20th century classic by Paul Samuelson: *Foundations of Economic Analysis, FOA* ([46]). Any mathematically minded macroeconomist, having to read Patinkin's classic ([47], right up to its second edition, was adequately prepared with the mathematics in *FOA*. A little later, with the dominance of von Neumann growth models, turnpike theory and optimal growth theory, there were Nikaido's two admirable books ([43], [44]) that summarised the necessary mathematics for the mathematically minded macroeconomist[40]. Those who were interested in exotic macrodynamics – nonlinear trade cycle theory, for example - also had their textbooks, for example, [18], [19].

It is, however, only now that we have a sustained development of particular kind of macroeconomics, entirely driven by a commitment to a particular mathematical framework: recursive dynamics. Thus it is that the Lucas-Stokey text, [36], has replaced *FOA*, and even more comprehensively. There is one dominant macroeconomic paradigm – the *SDGE* model – and there is one integrated set of mathematical tools to be mastered to work within it, and to push its frontiers, and that set is adequately covered in one comprehensive textbook. And this story can be further substantiated by studying, carefully, as I have had to do – both as a student and as a teacher of advanced macroeconomic theory –

[40] It is interesting to recall Solow's closing lines in his enthusiastic *Foreward* to the book on growth, in 1970, by two of his own pupils ([59], p. ix, italics in original):

> "The mind boggles at the thought of the sort of books that *their* students may write."

the evolution of the economic contents and the mathematical sophistication of the series of textbooks on Macroeconomics written by a leading exponent of recursive macroeconomics: Tom Sargent ([53], [54] and [37]). In particular, the emergence of the *RCE* concept and the necessary mathematics for it.

So, it appears as if Macroeconomics, in its mathematical mode, as *Recursive Macroeconomics*, has achieved what was achieved by Debreu's classic codification of Walrasian economics, in 1959 ([9]), built on the shoulders of his pioneering work with Arrow, and that, in turn, on the mighty foundations laid by von Neumann-Morgenstern and Nash. A half century later, Macroeconomics seems to have achieved a similar codification.

Yet, there is disquiet, at least in the fringes of the frontiers, if not at the core. Noble attempts are continuing in trying to develop a macrodynamics that is consistently nonlinear in its dynamic underpinnings and evolutionary and disequilibrium in its conceptual outlook. Richard Day, Sidney Winter and Richard Nelson come to mind as the 'patrons' of such an alternative – and their followers (I have in mind the series of outstanding books in this *genre* merging from a group of researchers working with and around Peter Flaschel – cf. for example, [15], [4], [5], and the constant stream of high quality texts coming out of this 'stable', with a unified theme, both from an economic and a mathematical point of view). In contrast to the relentlessly equilibrium-dominated, stochastic dynamic, recursive mathematical macroeconomics, this alternative is disequilibrium-dominated, nonlinear dynamic endogenous macroeconomics.

The kind of mathematical macroeconomics that I myself see as emerging in the years to come is *Recursion Theoretic Macroeconomics* or *Computable Macroeconomics*[41]. The kind of mathematics that will underpin such a macroeconomics will be determined by the need to conduct the computational experiments on digital computers, using digitally available economic data. The dissonance between an economic theory developed with the mathematics of real analysis and an applied economics having to indulge in inexplicable contortions to make theory and data mesh seamlessly with the experimental tool, will have to come to an end. When it does, I hope there will be a reasonably complete Computable Macroeconomics, building on recursion theory and constructive mathematics, readily available for students to turn to, rely on and work with. When that happens, we will not have to be too seriously concerned with Maury Osborne's perceptive perplexity:

> "There are numerous other paradoxical beliefs of this society [of economists], consequent to the difference between discrete numbers .. in which data is recorded, whereas the theoreticians of this society tend to think in terms of real numbers. ...No matter how hard I looked, I never could see any actual real [economic] data that showed that [these solid, smooth, lines of economic theory] ... actually could be observed in nature. ...... At this point a beady eyed Chicken Little might ... say, 'Look here, you can't have solid lines on that

[41] I coined the phrase *Computable Macroeconomics*, in the sense discussed above, when I was working with my friend Jean-Paul Fitoussi on our piece for the **Patinkin Festschrift**, in summer, 1990 (cf. [14]).

> picture because there is always a smallest unit of money ... and in addition there is always a unit of something that you buy. ..[I]n any event we should have just whole numbers of some sort on [the supply-demand] diagram on both axes. The lines should be dotted. ... Then our mathematician Zero will have an objection on the grounds that if we are going to have dotted lines instead of solid lines on the curve then there does not exist any such thing as a slope, or a derivative, or a logarithmic derivative either. .... .
>
> If you think in terms of solid lines while the practice is in terms of dots and little steps up and down, this misbelief on your part is worth, I would say conservatively, to the governors of the exchange, at least eighty million dollars per year.
>
> [45], pp.16-34

The mind boggles at the thought of the current profits being made by the governers of the exchange, simply in view of 'our misbeliefs' about 'dots and little steps' vs.'solid lines.

References

[1] Azariadis, Costas, (1993), **Intertemporal Macroeconomics**, Blackwell Publishers, Oxford.

[2] Blum, Lenore, Felipe Cucker, Michael Shub and Steve Smale (1998), **Complexity and Real Computation**, Springer-Verlag, New York and Berlin.

[3] Canova, Fabio (2007), **Methods for Applied Macroeconomic Research**, Princeton University Press, Princeton, NJ.

[4] Chiarella, Carl and Peter Flaschel (2000), **The Dynamics of Keynesian Monetary Growth: Macrofoundations**, Cambridge University Press, Cambridge.

[5] Chiarella, Carl, Peter Flaschel, Gangolf Groh and Willi Semmler – with contributions by Carsten Köper (2000), **Disequilibrium, Growth and Labor Market Dynamics: Macro Perspectives**, Springer-Verlag, Berlin and New York

[6] Clower, Robert W and Peter W. Howitt (1978), *The Transactions Theory of the Demand for Money: A Restatement*, **Journal of Political Economy**, Vol. 86, No. 3, pp. 449-65.

[7] Cooley, Thomas F and Edward C. Prescott (1995), "*Economic Growth and Business Cycles*", in: **Frontiers of Business Cycle Research**, edited by Thomas F. Cooley, Princeton University Press, Princeton, N.J.

[8] Day, Richard H, (1963), **Recursive Programming and Production Response**, North-Holland Publishing Company, Amsterdam.

[9] Debreu, Gerard (1960), **Theory of Value - An Axiomatic Analysis of Economic Equilibrium**, John Wiley & Sons, Inc., London.

[10] Debreu, Gerard (1984), *Economic Theory in the Mathematical Mode*, **American Economic Review**, Vol. 74, No. 3, June, pp. 267-78.

[11] Debreu, Gerard (1986), *Theoretic Models: Mathematical Form and Economic Content,* **Econometrica**, Vol. 54, No.6, November, pp. 1259-70.

[12] Debreu, Gerard (1991), *The Mathematization of Economic Theory*, **American Economic Review**, Vol. 81, No. 1, March, pp. 1-7.

[13] Fisher, Irving, 1896, **Appreciation and Interest**, *Publications of the American Economic Association, Vol. XI, No. 4*, The Macmillan Co., New York.

[14] Fitoussi, Jean-Paul and K. Vela Velupillai (1993), '*Macroeconomic Perspectives*', in: **Monetary Theory and Thought: The Patinkin Festschrift**, edited by Haim Barkai, Stanley Fischer and Nissan Liviatan, Macmillan, London.

[15] Flaschel, Peter, Rainer Franke and Willi Semmler (1997), **Dynamic Macroeconomics: Instability, Fluctuation and Growth in Monetary Economies**, The MIT Press, Cambridge, Massachusetts.

[16] Fleming, J. M (1938), "*The Determination of the Rate of Interest*", **Economica**, *New Series*, Vol. 5, # 19, August, pp. 333-341.

[17] Frisch, Ragnar (1933), "*Propagation Problems and Impulse Problems in Dynamic Economics*", pp. 171-205, in: **Essays in Honour of Gustav Cassel**, George Allen & Unwin, London.

[18] Gandolfo, Giancarlo (1980), **Economic Dynamics: Methods and Models**, (Second Revised Edition), North-Holland Publishing Company, Amsterdam and New York.

[19] Gandolfo, Giancarlo (1996), **Economic Dynamics**, (Third, Completely Revised and Enlarged Edition), Springer-Verlag, Berlin and New York.

[20] Hayek, Friedrich, A (1928), '*Das Intertemporale Gleicgewichtssystem der Preise und die Bewegungen des >>Geldwertes>>*', **Weltwirtschaftliches Archiv**.

[21] Hayek, Friedrich A (1931), **Prices and Production**, George Routledge & Sons, Ltd.

[22] Hayek, Friedrich A (1932), **Monetary Theory and the Trade Cycle**, Harcourt Brace & Co., Inc., New York.

[23] Hughes, J.R.T (1968), "*Wicksell on the Facts: Prices and Interest Rates, 1844 to 1914*", chapter 8, pp. 215-255, in: **Value, Capital and Growth: Papers in Honour of Sir John Hicks**, edited by J.N.Wofe, Edinburgh University Press, Edinburgh.

[24] Ishihara, Hajime (1989), "*On the Constructive Hahn-Banach Theorem*", **Bulletin of the London Mathematical Society**, Vol. 21, pp.79-81.

[25] Johansen, Leif (1969), "*Ragnar Frisch's Contribution to Economics*", **The Swedish Journal of Economics**, Vol. 71, # 4, December, pp. 302-324.

[26] Judd, Kenneth L (1998), **Numerical Methods in Economics**, The MIT Press, Cambridge, Massachusetts.

[27] Keynes, John Maynard (1936), **The General Theory of Employment, Interest and Money**, Macmillan and Co., Limited, London.

[28] Kydland, Finn E (2004), "*Quantitative Aggregate Theory*", pp. 341-356, available in:

http://nobelprize.org/nobel_prizes/economics/laureates/2004/kydland-lecture.pdf

[29] Leijonhufvud, Axel (1981), "*The Wicksell Connection: Variation on a Theme*", chapter 7, pp. 131-202, in **Information and Coordination: Essays in Macroeconomic Theory** by Axel Leijonhufvud, Oxford University Press, Oxford.

[30] Lindahl, Erik (1930), **Penningpolitikens Medel**, C.W.K.Gleerup, Lund

[31] Lindahl, Erik (1939), **Studies in the Theory of Money and Capital**, George Allen & Unwin, Ltd., London.

[32] Lindahl, Erik (1941), '*Professor Ohlin om Dynamisk Teori: Ett Genmäle*', **Ekonomisk Tidskrift**, Vol. 43, No. 3, September, pp. 236-247.

[33] Lucas, Robert E, Jr., (1981), **Studies in Business-Cycle Theory, Basil Blackwell, Oxford.**

[34] Lucas, Robert E, Jr., (1987), **Models of Business Cycles**, Basil Blackwell, Oxford.

[35] Lucas, Robert E, Jr., (1999), "*Conversations with Brian Snowdon and Howard R. Vane*", in: **Conversations with Leading Economists: Interpreting Modern Macroeconomics**, edited by Brian Snowdon and Howard R. Vane, Edward Elgar, Cheltenham, UK.

[36] Lucas, Robert. E, Jr., and Nancy L. Stokey with Edward C. Prescott, (1989), **Recursive Methods in Economic Dynamics**, Harvard University Press, Cambridge, Massachusetts.

[37] Ljungqvist, Lars and Thomas J Sargent (2000), **Recursive Macroeconomic Theory**, The MIT Press, Cambridge, Massachusetts.

[38] Marimon, Ramon and Andrew Scott (Editors), (1999), **Computational Methods for the Study of Dynamic Economies**, Oxford University Press, Oxford.

[39] Myrdal, Gunnar (1933), **Finanspolitikens Ekonomiska Verkningar**, SOU 1934:1, P. A. Norstedt & Söner, Stockholm.

[40] Myrdal, Gunnar (1939), **Monetary Equilibrium**, William Hodge & Company Limited, London.

[41] Narici, Lawrence & Edward Beckenstein (1997), "*The Hahn-Banach Theorem*", **Topology and its Applications**, Vol. 77, pp. 193-211.

[42] Nerode, Anil, George Metakides and Robert Constable, (1985), *Recursive Limits on the Hahn-Banach Theorem*, in: **Errett Bishop - Reflections on Him and His Research**, pp.85-91, edited by Murray Rosenblatt, *Contemporary Mathematics*, Vol. 39, American Mathematical Society, Providence, Rhode Island.

[43] Nikaido, Hukukane (1969), **Convex Structures and Economic Theory**, Academic Press, New York.

[44] Nikaido, Hukukane (1970), **Introduction to Sets and Mappings in Modern Economics**, North-Holland Publishing Company, Amsterdam.

[45] Osborne, M. S. M (1977, [1995]), **The Stock Market and Finance from a Physicist's Viewpoint**, Crossgar Press, Minneapolis, MN.

[46] Samuelson, Paul Anthony (1947), **Foundations of Economic Analysis**, Harvard University Press, Cambridge, Massachusetts.

[47] Patinkin, Don (1965), **Money, Interest and Prices: An Integration of Money and Value Theory**, Second Edition, Harper & Row, Publishers, New York.

[48] Prescott, Edward C (2004b), *The Transformation of Macroeconomic Policy and Research*, **Nobel Prize Lecture,** December, 8; pp. 24.

[49] Prescott, Edward C (2004a), *Interview* (with Nina Mehta) at: *http://www.fenews.com/fen41/one_on_one/one_on_one.html*

[50] Prescott, Edward C (2005), "Comments on '*Inflation, Output and Welfare' by Ricardo Lagos and Guilleaume Rocheteau*", **International Economic Review**, Vol. 46, # 2, May, pp. 523-531.

[51] Ramsey, Frank P (1928), "*A Mathematical Theory of Saving*", **Economic Journal**, Vol. 38, # 152, December, pp.543-559.

[52] Romer, Paul E (1989), "*Capital Accumulation in the Theory of Long-Run Growth*", chapter 2, pp. 51-127, in: **Modern Business Cycle Theory** edited by Robert J. Barro, Harvard University Press, Cambridge, Massachusetts.

[53] Sargent, Thomas J (1987), **Macroeconomic Theory**, Second Edition, Academic Press, Inc., London.

[54] Sargent, Thomas J (1987), **Dynamic Macroeconomic Theory**, Harvard University Press, Cambridge, Massachusetts.

[55] Sargent, Thomas J, (1993), **Bounded Rationality in Macroeconomics**, Clarendon Press, Oxford.

[56] Schechter, Eric, (1997), **Handbook of Analysis and Its Foundations**, Academic Press, San Diego.

[57] S∅rensen, Peter Birch & Hans J∅rgen Whitta-Jacobsen (2005), **Introducing Advanced Macroeconomics: Growth & Business Cycles**, McGraw-Hill Education, Berkshire.

[58] Smale, Steve (1976), "*Dynamics in General Equilibrium Theory*", **American Economic Review**, Vol. 66, No.2, May, pp.288 – 94.

[59] Solow, Robert M (1970), "*Foreword*", to: **Mathematical Theories of Economic Growth**, The Macmillan Company/Collier-Macmillan Limited, London.

[60] Spear, S. E (1989), "*Learning Rational Expectations under Computability Constraints*", **Econometrica**, Vol. 57, # 4, July, pp. 889-910.

[61] Starr, Ross M (1977), **General Equilibrium Theory: An Introduction**, Cambridge University Press, Cambridge.

[62] Stigler, George J (1948), "*Review of Foundations of Economic Analysis*", **Journal of the American Statistical Association**, Vol. 43, # 244, December, pp. 603-5.

[63] Tinbergen, Jan (1936), "*Sur la détermination statistique de la position d'équilibre cyclique*", **Revue de l'Institut International de Statistique/Review of the INternational Statistical Institute**, Vol. 4, No. 2, July, pp. 173-186.

[64] Tinbergen, Jan (1939), **Statistical Testing of Business-Cycle Theories, Vol I: A Method and its Application to Investment Activity**, League of Nations, Economic Intelligence Service, Geneva.

[65] Uzawa, Hirofumi (1962), "*Walras' Existence Theorem and Brouwer's Fixed Point Theorem*", **The Economic Studies Quarterly**, Vol. 8, No. 1, pp. 59 – 62.

[66] Velupillai, K. Vela (2000), **Computable Economics**, Oxford University Press, Oxford.

[67] Velupillai, K. Vela (2006), "*Algorithmic Foundations of* Computable *General Equilibrium Theory*", **Applied Mathematics and Computation**, Vol. 179, # 1, August, pp. 360 – 369.

[68] Wicksell, Knut (1898, [1936]), **Interest and Prices**, translated by Richard F. Kahn, Macmillan, London.

[69] Wicksell, Knut (1918), "*Ett Bidrag till Krisernas Teori*", a Review Article of: Goda och Dåliga Tider by Karl Petander, **Ekonomisk Tidskrift**, Vol. 20, # 2, pp. 66-75.

[70] Woodford, Michael (2003), **Interest & Prices: Foundations of a Theory of Monetary Policy**, Princeton University Press, Princeton, NJ.

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