Some results on stability concepts for matching models *

Ester Camiña†

December 3, 2010

Abstract

We consider a general class of two-sided matching markets, called many-to-one matching markets with money. For a special case of these markets, where each seller owns different objects, we prove that stable outcomes cannot be characterized by the non-existence of unsatisfied pairs. Moreover, we restore the dual lattice structure in markets with more than one seller using a connection with an assignment game.

Key words: matching, assignment, stability.

JEL Classification Numbers: C71, C78.

*I am very grateful to Carmelo Rodríguez-Álvarez for his useful comments and suggestions. This research was undertaken with support from the Spanish Ministry of Education (SEJ2007-65722) and Universidad Complutense de Madrid (GR35/940435). The responsibility for errors is solely mine.

†Departamento de Fundamentos del Análisis Económico II. Fac. Ciencias Económicas. Universidad Complutense (Campus de Somosaguas). Pozuelo de Alarcón 28223 (Madrid). Spain. Phone: (34) 91 394 25 11. Fax: (34) 91 394 25 91. E-mail: ecaminac@ccee.ucm.es

1 Introduction

We consider a general class of two-sided matching markets, called *many-to-one matching markets with money.*[1] We consider a set of sellers and buyers (that can also be seen as a set of firms and workers) which has the characteristic that each seller owns a set of possibly different objects, and each buyer wants to buy at most one object. The main concern of these models is to establish what coalitions we can expect to observe in the market and how they will divide their gains. In this regard, stability is the central solution requirement. Loosely speaking, an outcome, defined by a matching between buyers and objects and the price that each buyer pays to the seller, is *stable* if it is individually rational (no buyer or seller would prefer to cancel some of her transactions) and satisfies a no blocking requirement. There are two main ways for a set of agents to block a given outcome: considering deviations only of pairs of agents, that is, a buyer and a seller would like to create a new transaction or replace a previous joint one while possibly canceling other transactions and possibly keeping other ones to obtain a strictly higher payoff (pairwise stability), or deviations of groups of agents, that is, a set of buyers and sellers, by making new trades only among themselves, possibly dissolving some transactions and possibly keeping some, can all obtain a strictly higher payoff (setwise stability).[2]

The main contribution of this study within the matching literature is the following result: The pairwise stable set and the setwise stable set do not coincide when we introduce the possibility that each seller owns a set of different objects. These two sets trivially coincide in one-to-one models, such as the Shapley and Shubik Assignment Game, (Shapley and Shubik, 1972) where these two concepts coincide since the only sensible coalition that can block a given outcome is formed by, at most, two agents. For the many-to-one models without money studied in Blair (1988) and Martínez et al. (2001), among others, pairwise stable matchings are immune to group deviations. Also, in Sotomayor (1992) she studies two different many-to-many models where she assumes that all the objects a seller owns are equal, showing that (setwise) stable outcomes can be identified by the non-existence of unsatisfied pairs. But, when we study a model where each seller can have different objects, introduced in Camiña (2006), the coincidence of the pairwise stable set and the setwise stable set is no longer true. Consequently, the non-existence of bloking pairs does not characterize stability for our model and we cannot concentrate for this kind of models on "small" coalitions formed by pairs of agents. In particular, we prove

[1] For the purpose of our study, the one-to-one analysis is trivial and the many-to-many extension escapes from the scope of this paper.

[2] A version of this definition, which was called group stability, was defined in Roth (1985) in a different context.

that setwise stability is a sufficient, but not a necessary condition, for pairwise stability. This relationship has to do with the fact that the total gain that one seller and one buyer can share is not always the same, but depends on the object bought. In fact, since the pairwise stable set and the setwise stable set do not coincide for this kind of models, the choice of an appropriate solution concept for this class of games becomes crucial. This choice is reinforced by the coincidence of the setwise stable set with the core of the game.

In Camiña (2006), the existence of setwise stable outcomes was proved and this set was endowed with a lattice structure under the partial ordering of the buyers. Moreover, she proves that it was not possible to do the same for a dual partial ordering of the sellers. In this paper, we recover this discussion and show that, in a model where there are more that one seller, we are not able to endow the set of (setwise) stable payoffs with a lattice structure if we let *only* the sellers choose. Further, we consider a related one-to-one market assignment game to restore the dual lattice structure for a proper subset of the set of (setwise) stable payoffs.

This paper is organized as follows: Section 2 presents the formal model. In section 3 we propose a definition of pairwise stability adapted to our framework and reproduce the definition of setwise stability in Camiña (2006), and establish their relationship by means of a counter example. Section 4 presents the results for the analysis of the lattice structure.

2 The model

We consider a general version of the Generalized Assignment Game studied in Camiña (2006), where each seller may own different objects, and compare it with the special case where all the objects of the same seller are equal.

The buyer-seller market consists of m buyers and t sellers. Each seller owns a number of possibly different objects, and each buyer wants to buy at most one object. Formally, there are two finite disjoint sets of agents, P and S, containing m and t agents, respectively, and a set Q of n objects. Let $P = \{p_1, p_2, ..., p_m\}$ be the set of buyers. Generic buyers will be denoted by p_i and p_k. The payoff of buyer $p_i \in P$ will be denoted by u_i. Let $S = \{s_1, s_2, ..., s_t\}$ be the set of sellers. Generic sellers are denoted by s_r and s_d, and the payoff of seller $s_r \in S$ is denoted by w_r. Let $Q = \{q_1, q_2, ..., q_n\}$ be the set of indivisible objects. Generic objects are denoted by q_j and q_h, and the price of object $q_j \in Q$ is v_j. We also define a function $f : Q \to S$ that assigns each object to the seller who owns it, i.e., $f(q_j) = s_r$ if and only if seller s_r owns object q_j. We denote by $Q_r \equiv \{q_j \in Q : f(q_j) = s_r\}$

the set of objects that seller s_r owns, and by $|Q_r|$ the quota of seller s_r, that is, the number of objects he owns.

Associated to each possible pair $(p_i, q_j) \in P \times Q$ there is a nonnegative real number, α_{ij}, which denotes the maximum price that buyer p_i is willing to pay for object q_j, which is her reservation value. We may interpret it as if she had in hand an offer of α_{ij} from a client who will purchase the object from the buyer at that price. For simplicity, we assume, without loss of generality, that the reservation price of seller s_r for every object $q_j \in Q_r$ is zero (that is, if the seller offers any of his objects to an outside party, he will obtain zero). Therefore, α_{ij}, denotes the potential gains from trade between the buyer p_i and the seller $f(q_j)$ if the object sold is q_j. We denote by α the $m \times n$ matrix $(\alpha_{ij})_{i=1,\ldots m;j=1,\ldots n}$. We also assume that there are no monetary transfers among agents of the same side, which is a natural and usual assumption in these kind of models and agents' preferences are concerned only with their monetary payoffs. Thus, if buyer p_i buys the object q_j at a price v_j then the resulting payoffs are $u_i = \alpha_{ij} - v_j$ for the buyer and v_j for seller $s_r = f(q_j)$. The total payoff of seller s_r, denoted by w_r, is the sum of all the prices of the objects he sells. For technical convenience, we introduce one artificial null object, q_0, and one dummy player, seller s_0. Several buyers may buy this null object. This convention allows us to treat a buyer p_i that does not buy any object as if she has bought q_0. We assume that $f(q_0) = s_0$, (and $Q_0 = \{q_0\}$), so p_i will be matched to the dummy player s_0 if she buys no object. We also assume that the value α_{i0} is zero to all buyers, and the price of the object q_0 is always zero, $v_0 = 0$. Hence, if buyer p_i buys q_0 she obtains a utility $u_i = \alpha_{i0} - v_0 = 0$.

Therefore, a market M is determined by (P, S, Q, f, α).

3 Pairwise and setwise stable sets

An outcome of the market described specifies a matching between buyers and objects (and, hence, with sellers) and the price that each buyer pays to the owner of the object she is buying. Note that in this particular many-to-one matching model with money, the gain of a given partnership of a buyer and a seller depends on the object bough. This fact is crucial to understand the posterior results. First, we define a feasible matching as a function that assigns buyers to objects, and its associated matching as a correspondence that assigns buyers and sellers to an agent from the opposite side of the market, and then we define what a feasible outcome is.

Definition 1 *A* ***feasible matching*** μ *for a market* $M \equiv (P, S, Q, f, \alpha)$ *is a function* *from the set $P \cup Q$ into the set $P \cup Q \cup \{q_0\}$ such that:*

(i) For any $p_i \in P$, $\mu(p_i) \in Q \cup \{q_0\}$,

(ii) For any $q_j \in Q$, either $\mu(q_j) \in P$ or $\mu(q_j) = q_j$,

(iii) For any $(p_i, q_j) \in P \times Q$, $\mu(p_i) = q_j$ if and only if $\mu(q_j) = p_i$.

We say that buyer p_i is unmatched if $\mu(p_i) = q_0$. Similarly, we say that object q_j is unsold if $\mu(q_j) = q_j$.

Definition 2 *For any given feasible matching μ, we define its* ***associated matching*** *μ_s as a correspondence from the set $P \cup S$ into the set of non-empty subsets of $P \cup S \cup \{s_0\}$, such that:*

(I) $\mu_s(p_i) = f(q_j)$ if and only if $\mu(p_i) = q_j$,

(II) $\mu_s(s_r) = \{p_i \in P : \mu(p_i) \in Q_r\}$.

Given a feasible matching μ, a vector $(u, w, v) \in \Re_+^m \times \Re_+^t \times \Re_+^n$ of utilities for the agents and prices is *compatible with* μ if:[3]

(i) $u_i = \alpha_{i\mu(p_i)} - v_{\mu(p_i)}$, for every $p_i \in P$, and

(ii) $w_r = \sum_{q_j \in Q_r} v_j = \sum_{q_j \in Q_r} \left(\alpha_{\mu(q_j)j} - u_{\mu(q_j)}\right)$, for every $s_r \in S$.

Note that compatibility requires that the total payoff of a seller is the sum of the prices of his sold objects.[4]

Definition 3 *A* ***feasible outcome****, denoted by $(u, w, v; \mu)$, is a vector of utilities (or payoffs) $(u, w) \in \Re_+^m \times \Re_+^t$, a price vector $v \in \Re_+^n$, and a feasible matching μ, such that the vector (u, w, v) of utilities and prices is compatible with μ. If $(u, w, v; \mu)$ is a feasible outcome, then (u, w) is called a* ***feasible payoff****.*

The next step is to define the solution concepts used in the matching literature adapted to our framework to comment on their relationship. Stability is the key concept in a matching model. We propose definitions of two solution concepts: pairwise stability and setwise stability. We prove that, unlike what we observe in the many-to-one matching

[3] We sometimes abuse notation by writing $\alpha_{i\mu(p_i)}$ instead of α_{ij}, where $q_j = \mu(p_i)$. Similarly for $\alpha_{\mu(q_j)j}$.

[4] We are assuming that every unsold object has zero price. This assumption simplifies notation and posterior analysis.

models where all the objects of the same seller are equal, these two sets of stable outcomes are different when we allow each seller to have different objects. Moreover, setwise stability implies efficiency while pairwise stability does not.

We start by defining the pairwise stability concept. A feasible outcome is pairwise stable if it is individually rational and there does not exist a pair of a seller and a buyer that can generate together a gain from trade that leaves both of them strictly better off. Before formally defining the concept, we analyze the individual rationality of any feasible outcome.

Definition 4 *A feasible outcome $(u, w, v; \mu)$ is* ***individually rational*** *if no buyer can obtain a higher utility by becoming unmatched, and no seller can obtain a higher utility by leaving some of his objects unsold. That is, if:*

(i) For every buyer $p_i \in P$, $u_i \geq 0$.

(ii) For every object $q_j \in Q$, $v_j \geq 0$.

Then, it is trivial that a feasible outcome is always individually rational.

Definition 5 *A feasible outcome $(u, w, v; \mu)$ is* ***pairwise stable*** *if:*

(i) For any (p_i, q_j) such that $f(q_j) \neq f(\mu(p_i))$ we have:

$$\begin{array}{ll} u_i + v_j \geq \alpha_{ij}, & \text{if } \mu(q_j) \in P \\ u_i \geq \alpha_{ij}, & \text{if } \mu(q_j) = q_j \end{array}$$

(ii) For any (p_i, q_j) such that $f(q_j) = f(\mu(p_i))$ we have:

$$\begin{array}{ll} \alpha_{i\mu(p_i)} + v_j \geq \alpha_{ij}, & \text{if } \mu(q_j) \in P \\ \alpha_{i\mu(p_i)} \geq \alpha_{ij}, & \text{if } \mu(q_j) = q_j \end{array}$$

Condition (i) is the usual requirement for pairwise stability in a two-sided matching market where each seller owns a set of equal objects and in one-to-one models. Note that in our case it is a sufficient condition for all pairs formed by a buyer and an object, but it is not necessary for those pairs where the object and the partner of the buyer belong to the same seller. For these pairs we need condition (ii). This is due to the fact that a partnership formed by a buyer and a seller can generate different gains depending on the object sold. Therefore, condition (ii) implies that each buyer is buying the object that maximizes the gain that she can share with the seller she is matched with. Also, note that we do not require that the "blocking pair" is such that they are not partners. This makes sense in models where the gain of a partnership is always the same.[5]

The definition of pairwise stability that we have proposed is, then, more general than the usual one. Note that if we restrict attention to the one-to-one Shapley and Shubik Assignment Game or to a many-to-one model where all the objects of a seller are equal, this definition coincides with the one defined for these models. This follows directly from the non-negativeness of prices, which implies that condition (ii) in Definition 5 is trivially satisfied in those cases.

For an outcome to be setwise stable, we do not only require the non-existence of blocking pairs, but also of blocking coalitions. In the many-to-one models where each seller owns a set of equal objects, these two definitions are equivalent as Sotomayor (1992) shows for a many-to-many model, and an outcome is (setwise) stable if there is no unsatisfied pair. Clearly, it is *easier* for a buyer-seller pair to join and generate a trade, than for a coalition where more than one agent has to meet. Therefore, in the case where each seller owns a set of different objects, we cannot restrict attention only to individual payments of unmatched pairs.

We denote by T a coalition of agents, and T_s and T_p will denote the sets of $S-$ and $P-$ agents in T, respectively, (i.e., the intersection of the coalition T with S and P, respectively).

Definition 6 *A feasible outcome $(u, w, v; \mu)$ is* ***setwise stable*** *if it is not blocked by any coalition. That is, if there does not exist any coalition $T = T_s \cup T_p$ of agents that, by matching among themselves, according to, say, μ', and setting a price v'_j for every $q_j \in \bigcup\limits_{s_r \in T_s} Q_r$ such that $\mu(q_j) \in T_p$, all members of T prefer this new assignment to μ.*

Definition 6 is equivalent to Definition 7 below:

Definition 7 *A feasible outcome $(u, w, v; \mu)$ is* ***setwise stable*** *if it is not blocked by any coalition formed by a single seller and a set of buyers, that is, if there does not exist any coalition $T = s_r \cup T_p$ with $s_r \in S$ and $T_p \subset P$, and any feasible matching $\widehat{\mu}$, such that*

$$\sum_{\substack{p_i \in T_p \\ \widehat{\mu}(p_i) = q_j}} \alpha_{ij} > w_r + \sum_{p_i \in T_p} u_i$$

[5]Note that we allow the blocking pair to possibly keep some of their respective previous partners. If we did not allow for this, we say that a buyer and a seller block a given outcome if $u_i + w_r < \max \alpha_{ij}$. This does not change any of the posterior results.

The following result states the relationship between the pairwise stable set and the setwise stable set in this market. Clearly, if an outcome is setwise stable for a given market then it is pairwise stable, that is, setwise stability implies pairwise stability. However, both definitions do not coincide in general. An illustrative example shows that setwise stability is not a necessary condition for pairwise stability.

Proposition 1 *Given a market M, every setwise stable outcome is in the pairwise stable set.*

Proof. From Definitions 5 and 7, it follows directly that a setwise stable outcome is always pairwise stable. Let $M \equiv (P, S, Q, f, \alpha)$, with $S = \{s_1\}$, $Q = Q_1 = \{q_1, q_2\}$, $P = \{p_1, p_2\}$, and $\alpha_{11} = \alpha_{22} = 10$, $\alpha_{12} = \alpha_{21} = 8$. Taking $(u, w, v; \mu) = ((2,2), 12, (6,6); \mu(q_1) = p_2, \mu(q_2) = p_1)$ is pairwise stable ($\alpha_{11} = \alpha_{22} = 10 < 8 + 6 = \alpha_{12} + v_1 = \alpha_{21} + v_2$, see Definition 5). But it is not group stable. Indeed, the grand coalition $T = \{s_1, p_1, p_2\}$ can be matched as follows: $\mu'(q_1) = p_1, \mu'(q_2) = p_2$, and by setting, for example, $v_1' = v_2' = 7$, all agents win more than in outcome $(u, w, v; \mu)$, since the new payoffs are: $(u', w') = ((3,3), 14) > ((2,2), 12) = (u, w)$.
■

We have proven that, given a market M, the set of setwise stable outcomes can be strictly contained in the set of pairwise stable outcomes, that is, setwise stability is a stronger condition than pairwise stability. This is due to the fact that we allow the objects of one seller to be different. In the special case where all the objects of the same seller are equal, that is, $\alpha_{ij} = \alpha_{ik}$ if $f(q_j) = f(q_k)$, for every $p_i \in P$ and $q_j, q_k \in Q$, we have coincidence between the pairwise and the setwise stable set: Suppose $\alpha_{ij} = \alpha_{ik}$ whenever $f(q_j) = f(q_k)$. In that case, if an outcome $(u, w, v; \mu)$ is not setwise stable, it means that there exists a coalition of a seller, say s_r, and a set of buyers that blocks it, where it is necessarily the case that at least one of the buyers, say p_i, is not buying from s_r under μ. This means that $u_i + v_j < \alpha_{ij}$ for some $q_j \in Q_r$ and $\mu(q_j) \in P$, or $u_i < \alpha_{ij}$ for some $q_j \in Q_r$ unsold, because otherwise there is no additional gain that the coalition can share. But this implies that the outcome $(u, w, v; \mu)$ is not pairwise stable.

Since in Camiña (2006) she proves that the setwise stable set is equivalent to the core of the game, an important Corollary of the previous result is the following:

Corollary 1 *Pairwise stability does not imply efficiency as setwise stability does.*

4 Connection with the Assignment Game and structure of the (setwise) stable set

Camiña (2006) shows that the setwise stable set of a market M is endowed with a lattice structure under the partial ordering of the buyers. If there is only one seller, as in her model, the same occurs under the partial ordering of the seller. But when we allow the market to have more than one seller, this result is no longer true.

Definition 8 *A* ***payoff vector*** (u, w) *is* ***setwise stable*** *for a market M if there exists a vector of prices $v \in \Re^n$ and a feasible matching μ such that $(u, w, v; \mu)$ is a setwise stable outcome. We say that v is compatible with (u, w).*

The partial orders $\geq_P$ and $\geq_S$ are defined as follows: For any two setwise stable payoffs (u, w) and (u', w'), $(u, w) \geq_P (u', w')$ if $u_i \geq u_i'$ for all p_i in P, and $(u, w) \geq_S (u', w')$ if $w_r \geq w_r'$ for all s_r in S.

Definition 9 *Take (u, w) and (u', w') setwise stable payoffs and denote by v and v' the compatible price vectors for (u, w) and (u', w'), respectively, for the optimal matching μ.*[6] *We define $\overline{w}_S$ and $\underline{u}_S$ as follows:*

(i) for every $s_r \in S$, $\overline{w}_{Sr} = \max \{w_r, w_r'\}$.

(ii) for every $p_i \in P$, $\underline{u}_{\mathrm{Si}}(\mu) = \begin{cases} u_i, & \text{if } \overline{w}_{Sf(\mu(p_i))} = w_r \\ u_i', & \text{if } \overline{w}_{Sf(\mu(p_i))} = w_r'. \end{cases}$

Similarly, we define $\underline{w}_S$ and $\overline{u}_S(\mu)$.

Proposition 2 *Let (u, w) and (u', w') be two setwise stable payoffs. Then, the payoffs $(\underline{u}_S(\mu), \overline{w}_S)$ and $(\overline{u}_S(\mu), \underline{w}_S)$ defined for an optimal matching μ may not even be pairwise stable.*

Proof. Take the following market. Let $M \equiv (P, S, Q, f, \alpha)$ be $S = \{s_1, s_2\}$, $Q = Q_1 = \{q_1, q_2, q_3\}$, with $f(q_1) = f(q_2) = s_1$ and $f(q_3) = s_2$, $P = \{p_1, p_2, p_3\}$, and $\alpha_{11} = 4$, $\alpha_{22} = 6$, $\alpha_{12} = 6$, $\alpha_{21} = 2$, $\alpha_{31} = 4.5$, $\alpha_{32} = 6$, $\alpha_{13} = 4$, $\alpha_{23} = 1$ and $\alpha_{33} = 6$. The payoffs $(u, w) = ((3.5, 3, 4), (3.5, 2))$ and $(u', w') = ((4, 1, 5), (5, 1))$ are setwose stable. To check it, take the following matching μ, with $\mu(p_1) = q_1$, $\mu(p_2) = q_2$ and $\mu(p_3) = q_3$, and prices $(v_1, v_2, v_3) = (0.5, 3, 2)$, $(v_1', v_2', v_3') = (0, 5, 1)$. The vector of prices $\overline{v}$ that makes the outcome $(\underline{u}_S(\mu), \overline{w}_S, \overline{v}; \mu)$ feasible is the following:

[6] A feasible matching is optimal for a market if it maximizes the gain of the whole set of players.

for every $v_j \in V$, $\overline{v}_j = \begin{cases} v_j, & \text{if } \overline{w}_{Sf(q_j)} = w_r \\ v'_j, & \text{if } \overline{w}_{Sf(q_j)} = w'_r. \end{cases}$

In this market, $(\underline{u}_S(\mu), \overline{w}_S, \overline{v}; \mu) = ((4,1,4),(5,2),(0,5,2);\mu)$, is not even pairwise stable, because the pair of agents (s_1, p_3) blocks the outcome since $\underline{u}_{S3} + \overline{v}_1 = 4 + 0 < \alpha_{31} = 4.5$.

A similar thing happens with $(\overline{u}_S(\mu), \underline{w}_S)$. ■

Given a market $M \equiv (P, S, Q, f, \alpha)$, we can define the "one-to-one" market (an Assignment Game), $M' \equiv (P, S', Q, f', \alpha)$, as follows:

$Q = \{q_1, q_2, ..., q_n\}$, set of objects
$S' = \{s'_1, s'_2, ..., s'_n\}$, set of sellers with $f'(q_j) = s'_j$, for all $j = 1, ..., n$.
$P = \{p_1, p_2, ..., p_m\}$, set of buyers.

Given a feasible outcome $(u, w, v; \mu)$ in the Generalized Assignment Game, the *transformed* outcome in the Assignment Game is given by $(u, w', v; \mu)$ with:

$$w'_j = \begin{cases} v_j, & \text{if } \mu(q_j) \in P \\ 0, & \text{if } \mu(q_j) = q_j, \end{cases}$$

for all $j = 1, ..., n$.

We can define feasibility and stability for these *transformed* markets in the same way as before, since they are a subset of our generalized markets. Note that the pairwise stable set and the setwise stable set coincide in the one-to-one market, but the concept of pairwise stability used is different to that used for the many-to-one market, since now condition (i) in Definition 5 is necessary for all pairs $(p_i, q_j) \in P \times Q$. Note also that the optimal matchings for a given market M coincide with the optimal matchings for the transformed one-to-one market M'.

The following proposition states the relationship between stable outcomes of a given market M and stable outcomes in the corresponding Assignment Game M'.

Proposition 3 *Take a market M, and its corresponding one-to-one market M'. If the outcome $(u, w', v; \mu)$ is (pairwise) stable for M', then $(u, w, v; \mu)$ is a setwise stable outcome for M, where $w_r = \sum_{\substack{q_j \in Q_r \\ \mu(q_j) \in P}} v_j$ for every $s_r \in S$.*

Proof. By contradiction, suppose that the outcome $(u, w, v; \mu)$ is not setwise stable for market M. We prove that the outcome $(u, w', v; \mu)$ is also not (pairwise) stable for M'.

Since $(u, w, v; \mu)$ is not setwise stable, there exists a coalition T formed by, say, seller s_r and a subset of buyers T_p and a feasible matching μ', such that, $w'_r > w_r$, i.e.,

$$\sum_{\substack{q_j \in Q_r \\ \mu'(q_j) \in P}} v'_j > \sum_{\substack{q_j \in Q_r \\ \mu(q_j) \in P}} v_j$$

and

$$\alpha_{i\mu'(p_i)} - v'_{\mu'(p_i)} > \alpha_{i\mu(p_i)} - v_{\mu(p_i)}, \text{ for every } p_i \in T_p,$$

where v' is the new vector of prices.

This means that there exists $q_j \in Q_r$ with $\mu'(q_j) \in T_p$ such that, either $v'_j > 0$ and $\mu(q_j) = q_j$, or $v'_j > v_j$, $\mu(q_j) \in P$, and $\mu'(q_j) \neq \mu(q_j)$. In both cases, we must have $u'_{\mu'(q_j)} > u_{\mu'(q_j)}$. Therefore, in M', the pair $(\mu'(q_j), q_j)$ blocks the outcome $(u, w', v; \mu)$. ■

The previous Proposition proves that (pairwise) stability in the Assignment Game is a sufficient condition for setwise stability.

From this result, we can restore the dual lattice structure in the Assignment Game for a proper subset of the set of setwise stable payoffs. This is formally stated in the following Proposition.

Proposition 4 *Take a market M and its corresponding market M'. The set of setwise stable payoffs in M that are pairwise stable in M' forms a complete dual lattice under the partial orderings $\geq_P$ and $\geq_S$.*

Proof. Note that Proposition 3 implies that, given a market M and its corresponding market M', the set of pairwise stable outcomes in M' is strictly contained in the setwise stable set of M. We know, from Shapley and Shubik (1972), that this set forms a complete dual lattice. Also note that, given two stable payoffs in M', if all sellers in M' prefer one to the other, this implies that all sellers in M prefer the same payoff over the other. Also, the duality of the two orderings is satisfied. ■

References

[1] Blair, C., 1988, The lattice structure of the set of stable matchings with multiple partners, Mathematical of Operations Research 13, 619-628.

[2] Camiña, E., 2006, A Generalized Assignment Game, Mathematical Social Sciences 52, 152-161.

[3] Martínez, R., Massó, J., Neme, A., Oviedo, J., 2001, On the lattice structure of the set of stable matchings for a many-to-one model, Optimization 50, 439-457.

[4] Roth, A., 1985, Conflict and coincidence of interest in job matching: some new results and open questions, Mathematics of Operations Research 10, 379-389.

[5] Sotomayor, M., 1992, The multiple partners game, in: M. Majumdar, Ed., Equilibrium and Dynamics: Essays in Honor to David Gale Macmillian, 322-336.

[6] Shapley, L. and M. Shubik, 1972, The assignment game I: the core, International Journal of Game Theory 1, 111-130.