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Abstract

In 1957, Berge has introduced the *Berge equilibrium* for a normal form game based on the notion of equilibrium of a coalition structure P with respect to a set of coalitions K. This equilibrium did not receive any attention from game theory researchers for two decades. In the 80s, Zhukovskii V.I. and his followers started to study a special case of this equilibrium, we call it *simple Berge equilibrium* to avoid confusion. The most important feature of this equilibrium is that it captures cooperation in noncooperative settings. Later, Vaisman, a student of Zhukovskii, discovered that simple Berge equilibrium does not satisfy the individual rationality condition. Therefore, this condition has been added to the simple Berge equilibrium equilibrium, we call *Berge-Vaisman equilibrium* the obtained equilibrium. Past research has showed that the problem of existence of Berge equilibrium is difficult (compared to that of Nash). This paper is a contribution to the problem of existence and computation of Berge-Vaisman equilibrium and Berge equilibrium of a normal form game. Indeed, using the g-maximum equality, we establish the existence of these two equilibria. In addition, we give sufficient conditions for the existence of a Berge-Vaisman equilibrium which is also a Nash equilibrium. This allows us to get equilibria enjoying the properties of both concepts of solution. Finally, using these results, we provide two methods of computation of Berge-Vaisman

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equilibria: the first one computes Berge-Vaisman equilibria; the second one computes Berge-Vaisman equilibria which are also Nash equilibria.

1 Introduction

Berge [1957] has introduced the *Berge equilibrium* (see Definition 2.1 below) for a normal form game based on the notion of equilibrium of a coalition structure P with respect to a set of coalitions K. This equilibrium did not receive any attention from game theory researchers for two decades. In the 80s, Zhukovskii and his group of researchers started to study a special case of this equilibrium, we call it *simple Berge equilibrium* (see Definition 2.2 below). This equilibrium can be used as an alternative solution when Nash equilibrium (Nash [1951, 1950]) does not exist. In addition to this, it captures cooperation in noncooperative settings. In this equilibrium, the payoff of each player is maximized by the rest of players.

Gaidov wrote a pair of short papers on simple Berge equilibrium (Gaidov [1987, 1986]) in stochastic differential games. The volume entitled “Multicriteria Dynamical Problems Under Uncertainty”, a Collection of Scientific Works, published in Orekhovo-Zuevo, contains three papers on different aspects of simple Berge equilibrium: existence theorems (Dinovsky [1991]), simple Berge equilibrium in difference differential games (Boribekova and Jarkynbayev [1991]) and simple Berge equilibrium in bi-matrix games (Gintchev [1991]). Radjef [1988] has also studied the problem of existence of this equilibrium in differential games. In all the mentioned works the set of players is assumed to be finite and no procedure for the computation of simple Berge equilibrium is proposed.

In Zhukovskii *et al.* [1994], Vaisman constructed an example where simple Berge equilibrium does not satisfy the individual rationality condition, therefore, it has been added to the definition of simple Berge equilibrium (see Definition 2.3 below). To avoid confusion, we call *Berge-Vaisman equilibrium* the obtained equilibrium. Further, the existence of this equilibrium has been investigated in three person differential games with quadratic payoff functions Zhukovskii *et al.* [1994]. Zhukovskii [1999] has investigated the problem of existence of Berge-Vaisman equilibrium in the case of two and three person games involving uncertainty with strictly concave payoff functions; in the case of quadratic payoff functions, an explicit formula of Berge-Vaisman equilibrium is given. Thus, there are no general existence results of Berge-Vaisman equilibrium.

Abalo and Kostreva [2005, 2004, 1996a, 1996b] studied the Berge equilibrium as defined in Berge [1957]. They also provide theorems of existence of this equilibrium in the case of infinite set of players as Theorems 2, 3 in 2005, Theorems 3.1, 3.2 in 2004, Theorems 2, 3 in 1996a and Theorems 3.2, 3.4 in 1996b. It is to be noted that these theorems are based on an earlier paper of Radjef [1988] providing an existence theorem of simple Berge equilibrium. In Nessah *et al.* [2007], Larbani and Nessah [2008], we have showed that the above mentioned Abalo and

Kostreva's theorems are flawed, and proposed their corrections as well. The same remark can be made for the Radjef's Theorem.

In this paper we provide general sufficient conditions for the existence of Berge-Vaisman equilibrium and Berge equilibrium when the set of players may be infinite countable. Next, we provide a procedure for computation of Berge-Vaisman equilibrium. We also establish sufficient conditions for the existence of Berge-Vaisman equilibrium that is also Nash equilibrium (Berge-Vaisman-Nash equilibrium) and a method for its computation. Our approach is totally different from the existing ones, we use the g-maximum equality theorem (Nessah and Larbani [2005]).

This paper is organized as follows. In Section 2, we recall the definitions of Berge equilibrium, simple Berge equilibrium, Berge-Vaisman equilibrium and some of their properties. In Section 3, we provide sufficient conditions for the existence of Berge-Vaisman equilibrium (Subsection 3.1) and Berge-Vaisman-Nash equilibrium (Subsection 3.2). Then, from these two results, we derive two procedures for the determination of these equilibria. An existence theorem of Berge equilibrium is provided in Subsection 3.3 followed by a discussion. We end the paper with a conclusion in Section 4.

2 Berge Equilibrium

Consider the following non cooperative game in normal form

$$G = (X_i,\ u_i)_{i\in I}. \tag{2.1}$$

where I is the set of players, which we assume to be finite or infinite countable; $X = \prod\limits_{i\in I} X_i$ is the set of strategy profiles of the game, where X_i is the set of strategies of player i; $X_i \subset E_i$, E_i is a vector space; $u_i : X \longrightarrow \mathbb{R}$ is the payoff function of player i.

Let $\Im$ denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $R \in \Im$, we denote by $-R$; the set $-R = \{i \in I \text{ such that } i \notin R\}$: the complementary coalition of R, if R is reduced to a singleton $\{i\}$, then we denote by $-i$ the set $-R$. We also denote by $X_R = \prod\limits_{i\in R} X_i$ the set of strategy profiles of players in coalition R. If $\{K_i\}_{i\in\{1,..,s\}\subset\mathbb{N}}$ is a partition of the set of players I, then any strategy profile $x = (x_1, ..., x_n, ...) \in X$ can be written as $x = (x_{K_1}, x_{K_2}, .., x_{K_s})$ where $x_{K_i} \in X_{K_i} = \prod\limits_{j\in K_i} X_j$.

We denote by $\overline{A}$ the closure of a set A and by ∂A its boundary. Let Y_0 be a nonempty convex subset of a convex subset Y of a vector space and $y \in Y_0$, we denote by $H_{Y_0}(y)$, $T_{Y_0}(y)$ and $Z_{Y_0}(y)$, respectively, the following sets: $H_{Y_0}(y) = \bigcup\limits_{h>0} [Y_0 - y]/h$, $T_{Y_0}(y) = \overline{H_{Y_0}(y)}$ and $Z_{Y_0}(y) = [T_{Y_0}(y) + y] \cap Y$. Note that $T_{Y_0}(y)$ is called tangent cone to Y_0 at the point y.

Let us now give the existing different definitions of Berge equilibria. We start by the general definition of Berge equilibrium as introduced in Berge [1957].

DEFINITION 2.1 (Berge [1957]) Consider the game (2.1). Let $R = \{R_i\}_{i \in M} \subset \Im$ be a partition (coalition structure) of I and $S = \{S_i\}_{i \in M}$ be a set of subsets of I. A feasible strategy $\overline{x} \in X$ is an equilibrium point for the set R relative to the set S or a *Berge equilibrium* (BE) for (2.1) if

$$u_{r_m}(\overline{x}) \geq u_{r_m}(x_{S_m}, \overline{x}_{-S_m}),$$

for each given $m \in M$, any $r_m \in R_m$ and $x_{S_m} \in X_{S_m}$.

It is easy to see that when $M = I$, $R_i = \{i\}$ and $S = \{i\}$, for all $i \in I$, then BE is a Nash equilibrium. A strategy profile $\overline{x}$ is a BE if no player in any coalition R_m in R, can be better off when the players of corresponding coalition S_m in S deviate from their BE strategy profile $\overline{x}_{S_m}$. This means that at BE, the players in coalition S_m play a strategy profile that maximizes the payoff of the players in coalition R_m, but they neglect or ignore their own payoffs (when $S_m \bigcap R_m = \emptyset$)! This statement makes BE look unrealistic and irrational. In fact, the payoffs of the players in S_m are taken care of by some other players. Indeed, let $j \in S_m$, since the family of coalitions R is a partition of the set of players I, then there exists some $p \in M$ such that $j \in R_p$. According to the definition of BE, the players of the corresponding coalition S_p maximize the payoff functions of the players in R_p, since $j \in R_p$, the payoff of player j is also maximized by the players of S_p. It appears that at BE, globally, each player maximizes the payoff of at least one other players, in return his payoff is maximized by at least one other player. It is important to note that for some coalition structures R and sets of coalitions S, BE may not be individually rational as Vaisman pointed out in Zhukovskii *et al.* [1994] for the simple Berge equilibrium (see Definition 2.2 below). Therefore, for such BE, it is necessary to incorporate the individual rationality in their definition or select only BE that are individually rational in the process of game resolution. In general, the problem of individual rationality may occur when $S_m \bigcap R_m = \emptyset$, because in this case the players in R_m do not maximize their own payoff function.

As mentioned in the introduction, Abalo and Kostreva [2005, 2004, 1996a, 1996b] provide many theorems of existence of BE in the case of infinite set of players. After a deep investigation, we have found that the above mentioned Abalo and Kostreva's theorems are flawed Nessah *et al.* [2007], Larbani and Nessah [2008], then we proposed their corrections.

Next we present the simple Berge equilibrium, which is a special case of BE.

DEFINITION 2.2 (Zhukovskii [1985]) A strategy profile $\overline{x} \in X$ is a *simple Berge equilibrium* (SBE) of the game (2.1) if

$$u_i(\overline{x}) \geq u_i(x_{-i}, \overline{x}_i), \tag{2.2}$$

for each given $i \in I$ and $x_{-i} \in X_{-i}$.

We can see that this definition means that when a player $i \in I$ plays his strategy $\overline{x}_i$ from the SBE $\overline{x}$, he cannot obtain a maximum payoff unless the remaining players $-i$ willingly (or obliged) play the strategy $\overline{x}_{-i}$ from the SBE $\overline{x}$. In other words, if at least one of the players of coalition $-i$ deviates from his equilibrium strategy, the payoff of the player i in the resulting strategy profile would be at most equal to his payoff $u_i(\overline{x})$ in the resulting profile. To see that SBE is a special case of BE, we just need to assume $M = I$, $R_i = \{i\}, i \in I$ and $S_i = -i, i \in I$.

Many authors have investigated the SBE (Gaidov [1987, 1986], Dinovsky [1991], Boribekova and Jarkynbayev [1991], Gintchev [1991] and Radjef [1988]). In all the mentioned works the set of players is assumed to be finite and no procedure for its computation is proposed.
In Zhukovskii *et al.* [1994] an example where SBE does not satisfy the individual rationality condition is constructed, that is, at SBE, some of the players may get a payoff that is less than their security or maxmin level. In general the problem of individual rationality in BE may arise when $r_m \notin S_m$, for some $r_m \in R_m, m \in M$, which means that player r_m does not take care of his own payoff. This serious drawback makes it difficult to accept SBE as a solution concept for a normal form game. Therefore, the individual rationality condition has been added to the definition of SBE as follows.

DEFINITION 2.3 (Zhukovskii *et al.* [1994]) We say that a strategy profile $\overline{x} \in X$ is a *Berge-Vaisman equilibrium* (BVE) of the game (2.1) if

1. $\forall i \in I,\ \forall y_{-i} \in X_{-i},\ u_i(\overline{x}_i, y_{-i}) \leq u_i(\overline{x})$
2. $\forall i \in I,\ \alpha_i = \sup\limits_{x_i \in X_i} \inf\limits_{y_{-i} \in X_{-i}} u_i(x_i, y_{-i}) \leq u_i(\overline{x})$.

The first condition of Definition 2.3 means that BVE is an SBE (see Definition 2.2). The second condition of Definition 2.3 means that the strategy profile $\overline{x}$ is individually rational. In other words, for each player $i \in I$, BVE $\overline{x}$ yields a payoff that is greater or equal than his security level, denoted by α_i. We then say that BVE is individually rational.

Zhukovskii *et al.* [1994] have investigated the existence of BVE in three person differential games with quadratic payoff functions. Zhukovskii [1999] has investigated the problem of existence of BVE in the case of two and three person games involving uncertainty with strictly concave payoff functions; in the case of quadratic payoff functions, an explicit formula of BVE is given. Thus, there are no general existence results of BVE. Note that individual rationality of BE has not been discussed in Abalo and Kostreva [2004, 2005, 1996a, 1996b], this is also a major difference between our work and theirs.

The BVE is rarely mentioned (not to say used) by game theorists. One of the most important reasons for this is that Zhukovskii and his group of researchers published their results in Russian and within former USSR with local publishers only, so their results are not known world wide. The first paper published on SBE outside former USSR is Radjef [1988]. The first papers published on BE in well established international journals are (Abalo and Kostreva [2005, 2004, 1996a, 1996b]).

There are two main reasons that motivated the introduction of BVE as an alternative solution to Nash equilibrium (Zhukovskii and Chikrii [1994]). The first one is the absence of a concept of solution (in pure strategies) for games where there is no Nash equilibrium; the second one is the difficulty to choose a Nash equilibrium in games where there are more than one Nash equilibrium. The BVE can be used to study numerous non-cooperative models, more particularly coalition games. Furthermore, on the contrary to the Nash equilibrium, this concept allows to reach cooperative strategy profiles. Indeed, with this equilibrium it is no necessary to introduce behavioral assumptions to obtain cooperative strategy profiles, consequently, it becomes possible to reach cooperation in a non-cooperative framework. This property is very important for games like prisoner's dilemma. Let us give an example of a conflict situation where BVE equilibrium is the solution to which players will converge.

EXAMPLE 2.1 Consider the game illustrated by the following table.

	A	B
A	(-1.40, 0.94)	(-0.99, 0.93)
B	(-1.01, 0.98)	(-1, 1)

There are two players I and II, and each has available two strategies. We list I's strategies as rows in the table, and II's strategies as columns. This game has no pure-strategy Nash equilibrium. On the other hand, the strategy profile (B, B) is a BVE. Let us explain this. The strategy A is attractive for player I because he may get his best payoff in the game, *i.e* -0.99, but in the case where player II chooses the strategy A, he gets his worst payoff in the game, *i.e* -1.40. In addition, strategy B is his maxmin strategy. Indeed, the minimum he gets by choosing A is -1.40, and by choosing B he gets -1.01. Thus, player I will tend to choose the strategy B. He can reach the SBE (B, B) in announcing that he has chosen the strategy B. Indeed, in this case player II will automatically choose the strategy B for which he will get his best payoff in the game, *i.e* 1. One can easily verify that (B, B) is also individually rational, that is, a BVE. The described resolution process involves an implicit reciprocal cooperation. Indeed, by playing strategy B, Player I maximizes the payoff of player II, and by playing B, player II maximizes the payoff of player I.

It is important to note that BE is totally different from strong Berge equilibrium that was also introduced in Berge [1957] as follows.

DEFINITION 2.4 (Berge [1957]) A strategy profile $\overline{x} \in X$ is said to be *strong Berge equilibrium* (STBE) of the game (2.1), if

$$\forall i \in I,\ \forall j \in -i,\ u_j(\overline{x}_i, y_{-i}) \leq u_j(\overline{x}),\ \forall y_{-i} \in X_{-i}. \tag{2.3}$$

Indeed, referring to Definition 2.1, let us try to see if STBE can be a special case of BE. In Definition 2.4, by construction, we have $M = I$, $R_i = \{-i\}$, for any $i \in I$ and and $S_i = -i$. It is obvious that the family $R = \{R_i\}_{i \in I}$ is not a partition of the set of players I, in games with more than two players. Therefore, in this case STBE cannot be a BE.

If a player i chooses his strategy $\overline{x}_i$ in a STBE $\overline{x}$, then the remaining players $-i$ cannot improve their earnings by deviating from $\overline{x}_{-i}$, *i.e.*, this equilibrium is stable against deviations of any coalition of type $-i, i \in I$.

Analyzing the game aspect in BE and STBE, we find that they are totally different. STBE is a refinement[1] of the Nash equilibrium Nash [1951] (see Larbani and Nessah [2001]), but in general, BE is not a Nash equilibrium. Let us compare STBE with BVE, which is a special case of BE. If only one player i adopts his strategy in a STBE, he obliges all the other players in the coalition $-i$ to choose their strategy in this equilibrium: the adoption of other strategies by any players in the coalition $-i$, would provide each of them a payoff at most equal to that they get in this equilibrium. In other words, if any player selects his strategy in a STBE, the other players have no other choice than to follow him by choosing their strategies from the same STBE. By contrast, if a player chooses his strategy in a BVE, he cannot oblige the other players to follow him; he gets a maximum payoff if the other players are willing or obliged by some circumstances to choose their strategies in the same BVE.

The reader can find a detailed study and interesting results about STBE in Larbani and Nessah [2001].

The next definition merges the properties of BVE and Nash equilibrium.

DEFINITION 2.5 (Abalo and Kostreva [2004]) A Berge equilibrium which is also Nash equilibrium is called *Berge-Nash equilibrium* or (B-Nash) equilibrium.

Similarly, we can define the simple Berge-Nash (SB-Nash) equilibrium, and Berge-Vaisman-Nash (BV-Nash) equilbrium.

It would be interesting to find sufficient conditions for the existence of BV-Nash equilibrium for such equilibrium enjoys the properties of both concepts of solution at the same time. We address this problem in Subsection 3.2.

3 Existence and Computation of Berge Equilibria

In this section we establish the existence of BVE (Definition 2.3), BV-Nash equilibrium (Definition 2.5) and BE (Definition 2.1). From these results we derive procedures for the computation of BVE.

[1] For more details, see the book of Van damme [1987]

3.1 Berge-Vaisman Equilibrium

In order to establish the existence of BVE for the game (2.1), we will use the following generalization of the Ky Fan minmax inequality (Ky Fan [1972]), which was established by Nessah and Larbani [2005] and called the g-Maximum Equality Theorem. Let us recall this theorem.

THEOREM 3.1 *(g-Maximum Equality Theorem (Nessah and Larbani [2005])) Let X be a nonempty subset of a metric space E, Y be a nonempty, compact and convex subset of a locally convex Hausdorff space F. Let Ω be a real valued function defined on $X \times Y$. Let X_0 be a nonempty compact subset of X and g be a continuous function defined from X_0 into Y such that:*

1. *$g(X_0)$ is a convex subset of Y,*
2. *the function $(x, y) \mapsto \Omega(x, y)$ is continuous on $X_0 \times Y$,*
3. *for all $x \in X_0$, the function $y \mapsto \Omega(x, y)$ is quasi-concave on Y,*
4. *for all $g(x) \in \partial g(X_0)$ and for all $y \in Y$, there exists $z \in Z_{g(X_0)}(g(x))$ such that $\Omega(x, y) \leq \Omega(x, z)$.*

Then, there exists $\overline{x} \in X_0$ such that

$$\sup_{y \in Y} \Omega(\overline{x}, y) = \Omega(\overline{x}, g(\overline{x})). \tag{3.1}$$

The following Lemmas will be used thereafter.

LEMMA 3.1 *(Choquet [1984]) A product of convex sets is a convex set.*

LEMMA 3.2 *(Schwartz [1970]) A finite or countable product of metric spaces is a metric space.*

LEMMA 3.3 *(Schwartz [1970]) A product of locally convex spaces is a locally convex space.*

LEMMA 3.4 *(Kolmogorov and Fomine [1977]) A Hausdorff topological vector space, locally convex and locally bounded is a normable space.*

Let us consider the following set

$$A = \{\overline{x} \in X \text{ such that } \alpha_i = \max_{x_i \in X_i} \min_{y_{-i} \in X_{-i}} u_i(x_i, y_{-i}) \leq u_i(\overline{x}), \forall i \in I\}. \tag{3.2}$$

The set A represents the set of individually rational strategy profiles of the game (2.1). We have the following Lemma.

LEMMA 3.5 *Suppose that the following conditions are satisfied:*

1) for all $i \in I$, the set X_i is non empty, convex and compact in the Hausdorff locally convex space E_i,

2) for all $i \in I$, the function u_i is continuous and quasiconcave on X.

Then, the set A defined in (3.2) *is nonempty, convex and compact.*

PROOF. A is a nonempty set. The conditions 1) and 2) of Lemma 3.5 imply that $\forall i \in I$, $\alpha_i = \sup_{x_i \in X_i} \inf_{y_{-i} \in X_{-i}} u_i(x_i, y_{-i})$ exists. Since the functions u_i, $i \in I$ are continuous over the compact X, then $\forall i \in I$, $\exists \widetilde{x}_i \in X_i$ such that

$$\alpha_i = \sup_{x_i \in X_i} \inf_{y_{-i} \in X_{-i}} u_i(x_i, y_{-i}) = \inf_{y_{-i} \in X_{-i}} u_i(\widetilde{x}_i, y_{-i}).$$

Let be $\widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n, ...) \in X$, we have then:

$$\forall i \in I,\ u_i(\widetilde{x}) = u_i(\widetilde{x}_i, \widetilde{x}_{-i}) \geq \inf_{y_{-i} \in X_{-i}} u_i(\widetilde{x}_i, y_{-i}) = \alpha_i.$$

Thus $A \neq \emptyset$.

A is convex in X.

Let $\overline{x}$ and $\overline{\overline{x}}$ be two elements in A and let $\lambda \in [0,1]$. Let us show that $\lambda\overline{x} + (1-\lambda)\overline{\overline{x}} \in A$. $\overline{x}, \overline{\overline{x}}$ are two elements in A, then $\alpha_i \leq u_i(\overline{x})$ and $\alpha_i \leq u_i(\overline{\overline{x}})$, $\forall i \in I$, hence

$$\alpha_i \leq \min\{u_i(\overline{x}), u_i(\overline{\overline{x}})\},\ \forall i \in I.$$

Since the functions u_i, $i \in I$ are quasiconcave over X, then

$$\alpha_i \leq u_i(\lambda\overline{x} + (1-\lambda)\overline{\overline{x}}),\ \forall i \in I,\ \forall \lambda \in [0,1].$$

Therefore, $\lambda\overline{x} + (1-\lambda)\overline{\overline{x}} \in A$.

A is compact in X.

Since X is compact, then it sufficient to prove that A is closed. Let $\{x^p\}_{p \geq 1}$ a sequence of elements in A converging to $\overline{x}$. Let us show that $\overline{x} \in A$. We have $\forall p \geq 1$, $x^p \in A$, then

$$\forall p \geq 1,\ \forall i \in I,\ \alpha_i \leq u_i(x^p).$$

Taking into account the condition 1) and the continuity of u_i of Lemma 3.5, when $p \to \infty$, we obtain: $\forall i \in I$, $\alpha_i \leq u_i(\overline{x})$, *i.e.* $\overline{x} \in A$. ■

Let us introduce the following functions

$$g : A \to \widehat{X}$$

defined by $x \mapsto g(x) = (x_{-1}, ..., x_{-n}, ...)$.

$$\Gamma : A \times \widehat{X} \to \mathbb{R}$$

defined by $(x, \widehat{y}) \mapsto \Gamma(x, \widehat{y}) = \sum_{i \in I}\{u_i(x_i, y_{-i}) - u_i(x)\}$ where $\widehat{y} = (y_{-1}, ..., y_{-n}, ...) \in \widehat{X} = \prod_{i \in I} X_{-i}$, where $X_{-i} = \prod_{j \in -i} X_j$, $\forall i \in I$.

REMARK 3.1 For all $x \in A$, we have

$$\sup_{\widehat{y}\in\widehat{X}}\Gamma(x,\widehat{y}) \geq 0.$$

We have the following Lemma.

LEMMA 3.6 *If for all $i \in I$, the set X_i is nonempty, convex and compact in the Hausdorff locally convex space E_i, then the following assertions are true.*

1) The function g is continuous on A.

2) If A is convex and compact, then $g(A)$ is also convex and compact.

PROOF. The fact that the function g is continuous is a consequence of its definition and the construction of the set $\widehat{X}$. The compactness of the set $g(A)$ is a consequence of Weierstrass Theorem. The convexity of $g(A)$ is a consequence of the linearity of g, which can be easily verified. ∎

The following Lemma establishes the relation between BVE of the game (2.1) and the functions Γ and g.

LEMMA 3.7 *The following two assertions are equivalent.*

1) $\overline{x}$ is a BVE of the game (2.1).

2) $\overline{x} \in A$ and $\sup\limits_{\widehat{y}\in\widehat{X}}\Gamma(\overline{x},\widehat{y}) = 0$.

PROOF. *Sufficiency*. Let $\overline{x} \in A$ such that $\max\limits_{\widehat{y}\in\widehat{X}}\Gamma(\overline{x},\widehat{y}) = 0$, this equality implies $\forall\widehat{y} \in \widehat{X}$, $\Gamma(\overline{x},\widehat{y}) = \sum\limits_{i\in I}\{u_i(\overline{x}_i,\widehat{y}_{-i}) - u_i(\overline{x})\} \leq 0$. For an arbitrarily fixed $i \in I$, we have $\forall\widehat{y} \in \widehat{X}$, $\Gamma(\overline{x},\widehat{y}) = \{u_i(\overline{x}_i,\widehat{y}_{-i}) - u_i(\overline{x})\} + \sum\limits_{\substack{j\neq i\\ j\in I}}\{u_j(\overline{x}_j,\widehat{y}_{-j}) - u_j(\overline{x})\} \leq 0$.

For $\widehat{y} \in \widehat{X}$ such that $\widehat{y}_{-i}$ is arbitrarily chosen in X_{-i} and $\widehat{y}_{-j} = \overline{x}_{-j}, \forall j \neq i$, we have $\sum\limits_{\substack{j\neq i\\ j\in I}}\{u_j(\overline{x}_j,\widehat{y}_{-j}) - u_j(\overline{x})\} = 0$. Then from the last inequality we deduce that $\forall\widehat{y}_{-i} \in X_{-i}$, $u_i(\overline{x}_i,\widehat{y}_{-i}) \leq u_i(\overline{x})$. Since i is arbitrarily chosen in I, we have $\forall i \in I$, $\forall y_{-i} \in X_{-i}$, $u_i(\overline{x}_i,y_{-i}) \leq u_i(\overline{x})$, hence, taking into account the fact that $\overline{x} \in A$, we deduce that $\overline{x}$ is a BVE of the game (2.1).

Necessity. Let $\overline{x} \in X$ be a BVE of the game (2.1). The second condition of Definition 2.3 implies that $\overline{x} \in A$. The first condition of Definition 2.3 implies $u_i(\overline{x}_i,t_{-i}) \leq u_i(\overline{x})$, $\forall t_{-i} \in X_{-i}$, $\forall i \in I$, hence $\Gamma(\overline{x},\widehat{y}) = \sum\limits_{i\in I}\{u_i(\overline{x}_i,\widehat{y}_{-i}) - u_i(\overline{x})\} \leq 0$, $\forall\widehat{y} \in \widehat{X}$, *i.e.* $\max\limits_{\widehat{y}\in\widehat{X}}\Gamma(\overline{x},\widehat{y}) \leq 0$. Taking into account Remark 3.1, we obtain $\max\limits_{\widehat{y}\in\widehat{X}}\Gamma(\overline{x},\widehat{y}) = 0$. ∎

REMARK 3.2 Lemma 3.7 transforms the problem of finding BVE of the game (2.1) into a problem of finding a strategy profile $\overline{x} \in A$ satisfying $\sup\limits_{\widehat{y}\in\widehat{X}} \Gamma(\overline{x}, \widehat{y}) = 0$.

We will now establish the existence of BVE by g-Maximum Equality Theorem (Theorem 3.1).

THEOREM 3.2 *Assume that (1) the sets X_i, $i \in I$ are non empty compact and convex subsets of locally convex Hausdorff spaces, (2) $\forall i \in I$, the function u_i is continuous and concave on X, and (3) for all $g(x) \in \partial g(A)$, for all $\widehat{y} \in \widehat{X}$, there exists $\widehat{z} \in Z_{g(A)}(g(x))$ such that $\Gamma(x, \widehat{y}) \leq \Gamma(x, \widehat{z})$. And in addition if I is infinite countable, assume that the function Γ is continuous on $A \times \widehat{X}$. Then, the game* (2.1) *has at least one BVE (Definition 2.3).*

PROOF. The assumptions of Theorem 3.2 imply that those of Lemma 3.5 are satisfied. Then the set A is nonempty, convex and compact, and the function $\widehat{y} \mapsto \Gamma(x, \widehat{y})$ is concave on $\widehat{X}$. Then, from Lemmas 3.1-3.4 and the non emptiness, convexity and compactness of A, we conclude that all the conditions of the Theorem 3.1 are satisfied. Consequently,

$$\exists \overline{x} \in A \text{ such that } \sup_{\widehat{y}\in\widehat{X}} \Gamma(\overline{x}, \widehat{y}) = \Gamma(\overline{x}, g(\overline{x})) = 0. \tag{3.3}$$

Then by Lemma 3.7, $\overline{x}$ is a BVE of the game (2.1). ■

Taking into account Remark 3.1 and Lemma 3.7, we deduce the following proposition for games with a finite number of players.

Let

$$\mu = \inf_{x\in A} \left[\sup_{\widehat{y}\in\widehat{X}} \Gamma(x, \widehat{y}) \right]. \tag{3.4}$$

PROPOSITION 3.1 *Assume that the set of players is finite in the game (2.1). Suppose that Γ is continuous on $A \times \widehat{X}$ and the sets X_j, $j \in I$ are compact. Then, the game* (2.1) *has at least one BVE if and only if $\mu = 0$.*

Proposition 3.1 actually provides a method for the determination of BVE of game (2.1) under certain conditions (see Algorithm 1).

Let us now illustrate this Algorithm by examples.

EXAMPLE 3.1 Let us consider the following game: $I = \{1, 2, 3\}$, $X_1 = [0, 1]$, $X_2 = [1, 2]$, $X_3 = [-1, 1]$ and $x = (x_1, x_2, x_3)$.

$$u_1(x) = -x_2^2 - x_3^2,$$
$$u_2(x) = -x_3^2 + x_2,$$
$$u_3(x) = -x_3^3 x_1 - 3x_1^2 - x_3^2 x_2^2.$$

The conditions (1)-(2) of Theorem 3.2 are satisfied. Let us verify the condition (3).

Algorithm 1 Procedure for the Computation of a BVE.

Require: Suppose that all conditions of the Proposition 3.1 are satisfied.

Require: Determine the security levels $\alpha_i, \forall i \in I$.

Require: Calculate the value μ in (3.4)

if $\mu > 0$, **then**

the game (2.1) has no BVE.

else

the strategy profiles $\overline{x} \in A$ satisfying $\max_{\widehat{y} \in \widehat{X}} \Gamma(\overline{x}, \widehat{y}) = 0$ are BVE of the game (2.1).

end if

a) $\forall x \in X$, with $x_1 \in X_1$, $x_2 \in X_2$ and $-1 \leq x_3 < 0$, $\exists \overline{y} = (\frac{-x_3^3}{6}, 1, 0) \in X$ such that $u_i(x_i, t_{-i}) \leq u_i(x_i, \overline{y}_{-i}), \forall t_{-i} \in X_{-i}$, $\forall i \in I$.

b) $\forall x \in X$, with $x_1 \in X_1$, $x_2 \in X_2$ and $0 \leq x_3 \leq 1$, $\exists \overline{y} \in X$, $\overline{y} = (0, 1, 0)$ such that $u_i(x_i, t_{-i}) \leq u_i(x_i, \overline{y}_{-i}), \forall t_{-i} \in X_{-i}$, $\forall i \in I$.

Hence, a) and b) imply

$$\forall x \in X, \exists y \in X, \quad u_i(x_i, t_{-i}) \leq u_i(x_i, y_{-i}), \forall t_{-i} \in X_{-i}, \forall i \in I. \tag{3.5}$$

Now let us prove that both $\overline{y} = (0, 1, 0)$ and $\overline{y} = (\frac{-x_3^3}{6}, 1, 0)$ with $-1 \leq x_3 < 0$, are in the set A. Indeed, we have

$$u_1(0, 1, 0) = -1,\ u_1(\tfrac{-x_3^3}{6}, 1, 0) = -1 \text{ and } \alpha_1 = -5,$$
$$u_2(0, 1, 0) = 1,\ u_2(\tfrac{-x_3^3}{6}, 1, 0) = 1 \text{ and } \alpha_2 = 1,$$
$$u_3(0, 1, 0) = -3,\ u_3(\tfrac{-x_3^3}{6}, 1, 0) = -3\tfrac{x_3^6}{36} \text{ and } \alpha_3 = -3.$$

Hence both $\overline{y} = (0, 1, 0)$ and $\overline{y} = (\frac{-x_3^3}{6}, 1, 0)$ are in the set A. Taking into account (3.5), we deduce that the condition (3) of Theorem 3.2 is satisfied. Thus, according to Theorem 3.2, this game has at least one BVE.

From the preceding result we have

$\max_{y_{-1}} u_1(0, y_{-1}) = u_1(0, 1, 0)$, $\max_{y_{-2}} u_2(1, y_{-2}) = u_2(0, 1, 0)$, $\max_{y_{-3}} u_3(0, y_{-3}) = u_3(0, 1, 0)$.

Hence, $\sum_{i=1}^{3} \max_{y_{-i}} \ u_i(\overline{x}_i, y_{-i}) = \sum_{i=1}^{3} u_i(\overline{x})$ with $\overline{x} = (0, 1, 0)$ which is equivalent to $\max_{\widehat{y} \in \widehat{X}} \sum_{i=1}^{3} u_i(\overline{x}_i, \widehat{y}_{-i}) = \sum_{i=1}^{3} u_i(\overline{x})$, *i.e.* $\max_{\widehat{y} \in \widehat{X}} \Gamma(\overline{x}, \widehat{y}) = 0$. Since we have proved above that $\overline{x} = (0, 1, 0) \in A$, then according Algorithm 1, $\overline{x} = (0, 1, 0)$ is a BVE of this game.

3.2 Berge-Vaisman-Nash Equilibrium

In this section, we establish the existence of BV-Nash (Berge-Vaisman-Nash) equilibrium of the game (2.1) by using Theorem 3.1. Let us consider the following functions:

$$\widetilde{g} : X \to \widehat{X} \times X$$

defined by $x \mapsto \widetilde{g}(x) = ((x_{-1}, ..., x_{-n}, ...), x)$ and

$$\widetilde{\Gamma} : X \times (\widehat{X} \times X) \to \mathbb{R}$$

defined by $(x, (\widehat{y}, z)) \mapsto \widetilde{\Gamma}(x, (\widehat{y}, z)) = \sum\limits_{i \in I} [u_i(x_i, y_{-i}) + u_i(x_{-i}, z_i)]$.

REMARK 3.3 By definition, for all $x \in X$, we have

$$\sup_{(\widehat{y}, z) \in \widehat{X} \times X} \widetilde{\Gamma}(x, (\widehat{y}, z)) \geq \widetilde{\Gamma}(x, \widetilde{g}(x)).$$

LEMMA 3.8 *If for all $i \in I$, the set X_i is nonempty, convex and compact in the Hausdorff locally convex space E_i, then the following propositions are true.*

1. *The function $\widetilde{g}$ is continuous on X.*
2. *The set $\widetilde{g}(X)$ is convex and compact.*

PROOF. The proof of this lemma is similar to that of Lemma 3.6 ■

The following Lemma establishes the relation between BV-Nash equilibria of the game (2.1) and the functions $\widetilde{\Gamma}$ and $\widetilde{g}$.

LEMMA 3.9 *The following two assertions are equivalent:*

1. $\sup\limits_{(\widehat{y}, z) \in \widehat{X} \times X} \widetilde{\Gamma}(\overline{x}, (\widehat{y}, z)) = \widetilde{\Gamma}(\overline{x}, \widetilde{g}(\overline{x})).$
2. *$\overline{x}$ is a Berge-Nash equilibrium of the game* (2.1).

PROOF. *Sufficiency*. Suppose that $\sup\limits_{(\widehat{y}, z) \in \widehat{X} \times X} \widetilde{\Gamma}(\overline{x}, (\widehat{y}, z)) = \widetilde{\Gamma}(\overline{x}, \widetilde{g}(\overline{x}))$, *i.e.*

$$\sum_{i \in I} [u_i(\overline{x}_i, y_{-i}) + u_i(\overline{x}_{-i}, z_i)] \leq \sum_{i \in I} [u_i(\overline{x}) + u_i(\overline{x})], \ \forall (\widehat{y}, z) \in \widehat{X} \times X \tag{3.6}$$

If we take $y_{-i} = \overline{x}_{-i}, \forall i \in I$ in (3.6), we conclude that $\sum\limits_{i \in I} u_i(\overline{x}_{-i}, z_i) \leq \sum\limits_{i \in I} u_i(\overline{x}), \ \forall z \in X$, which implies that $\overline{x}$ is Nash equilibrium of the game (2.1).

If we take $z = \overline{x}$ in (3.6), we conclude that $\overline{x}$ verifies the property 1) of Definition 2.5 and since $\overline{x}$ is a Nash equilibrium, it is also individually rational. We conclude that $\overline{x}$ is a BVE of the game (2.1).

Necessity. Suppose that $\overline{x}$ is a BV-Nash equilibrium of the game (2.1). The fact that $\overline{x}$ is a Nash equilibrium of the game (2.1) implies

$$\max_{z\in X}\sum_{i\in I}u_i(\overline{x}_{-i},z_i)=\sum_{i\in I}u_i(\overline{x}). \tag{3.7}$$

The fact that $\overline{x}$ is a BVE of the game (2.1) implies

$$\max_{\widehat{y}\in\widehat{X}}\sum_{i\in I}u_i(\overline{x}_i,y_{-i})=\sum_{i\in I}u_i(\overline{x}). \tag{3.8}$$

The two equalities (3.7) and (3.8) imply

$$\max_{(\widehat{y},z)\in\widehat{X}\times X}\widetilde{\Gamma}(\overline{x},(\widehat{y},z))=\widetilde{\Gamma}(\overline{x},\widetilde{g}(\overline{x}))$$

■

It is to be noted that in Lemma 3.9 we have deliberately omitted the condition $\overline{x}\in A$ of individual rationality for it is well known that a Nash equilibrium is always individually rational. We have the following Theorem.

THEOREM 3.3 *Suppose that (1) the sets X_i, $i\in I$ are nonempty, compact and convex subsets of Hausdorff locally convex vector spaces, (2) the function u_i is continuous on X and the functions $y_{-i}\mapsto u_i(x_i,y_{-i})$ and $z_i\mapsto u_i(z_i,x_{-i})$ are concave on X_{-i} and on X_i, respectively, $\forall x\in X$ and $\forall i\in I$, and (3) $\forall\widetilde{g}(x)\in\partial\widetilde{g}(X)$, $\forall(\widehat{y},z)\in\widehat{X}\times X$, $\exists(\widehat{p},q)\in Z_{\widetilde{g}(X)}(\widetilde{g}(x))$ such that $\widetilde{\Gamma}(x,(\widehat{y},z))\leq\widetilde{\Gamma}(x,(\widehat{p},q))$. In addition if I is infinite countable, assume that the function $\widetilde{\Gamma}$ is continuous on $X\times(\widehat{X}\times X)$. Then the game* (2.1) *has at least one BV-Nash equilibrium (Definition 2.5).*

PROOF. The conditions of Theorem 3.3 imply that the function $\widetilde{\Gamma}$ satisfies all conditions of Theorem 3.1, consequently, $\exists\overline{x}\in X$ such that

$$\sup_{(\widehat{y},z)\in\widehat{X}\times X}\widetilde{\Gamma}(\overline{x},(\widehat{y},z))=\widetilde{\Gamma}(\overline{x},\widetilde{g}(\overline{x})).$$

By Lemma 3.9, the strategy profile $\overline{x}$ is a BV-Nash equilibrium of the game (2.1). ■

From Remark 3.3 and Lemma 3.9, we deduce the following proposition.

Let

$$\beta=\inf_{x\in X}\left[\sup_{(\widehat{y},z)\in\widehat{X}\times X}\widetilde{\Gamma}(x,(\widehat{y},z))-\widetilde{\Gamma}(x,\widetilde{g}(x))\right]. \tag{3.9}$$

PROPOSITION 3.2 *Suppose that the function $\widetilde{\Gamma}$ is continuous on $X\times(\widehat{X}\times X)$ and the sets X_j are compact. Then, the game* (2.1) *has at least one BV-Nash equilibrium if and only if $\beta=0$.*

Since the function $\widetilde{\Gamma}$ is a series of functions, the calculation of the value β may be difficult, but in the case where the set of players is finite, Proposition 3.2 can be used to verify if a BV-Nash equilibrium exists or not. From this Proposition we deduce the method presented in Algorithm 2 for the computation of a BV-Nash equilibrium of the game (2.1).

Algorithm 2 Procedure for the determination of a BV-Nash equilibrium.

Require: Suppose that the conditions of Proposition 3.2 are satisfied.

Require: Calculate the value β in (3.9).

if $\beta > 0$, **then**

the game (2.1) has no BV-Nash equilibrium.

else

the strategy profiles $\overline{x} \in X$ satisfying $\sup\limits_{(\widehat{y},z)\in\widehat{X}\times X} \widetilde{\Gamma}(\overline{x},(\widehat{y},z)) = \widetilde{\Gamma}(\overline{x},\widetilde{g}(\overline{x}))$ are BV-Nash equilibria of the game (2.1).

end if

3.3 Berge Equilibrium

In this section, to establish the existence of a Berge equilibrium (Definition 2.1) of the game (2.1), we will use the Theorem 3.1.

Let $R = \{R_i\}_{i\in M} \subset \Im$ be a partition of I and $S = \{S_i\}_{i\in M}$ be a set of subsets of I.

Let us consider the following functions:

$$h : X \to \widetilde{X}$$

defined by $x \mapsto h(x) = ((\overbrace{x_{S_m}, ..., x_{S_m}}^{r_m - times}),\ m \in M)$ and

$$F : X \times \widetilde{X} \to \mathbb{R}$$

defined by $(x, \widetilde{y}) \mapsto F(x, \widetilde{y}) = \sum\limits_{m\in M}\sum\limits_{j\in R_m} \{u_j(x_{-S_m}, y_{S_m}) - u_j(x)\}$, where $\widetilde{X} = \prod\limits_{m\in M}\prod\limits_{j\in R_m} X^j_{S_m}$ and $X^j_{S_m} = X_{S_m}, \forall j \in R_m$.

LEMMA 3.10 *If for all $i \in I$, the set X_i is nonempty, convex and compact in the Hausdorff locally convex space E_i, then the following assertions are true.*

1. *The function h is continuous on X.*

2. *The set $h(X)$ is convex and compact.*

PROOF. The proof of this lemma is similar to that of Lemma 3.6 ■

The following Lemma establishes the relation between BE of the game (2.1) and the functions F and h.

LEMMA 3.11 *The following two propositions are equivalent:*

1. $\sup\limits_{\widetilde{y}\in\widetilde{X}} F(\overline{x}, \widetilde{y}) = 0.$

2. *$\overline{x}$ is a BE of the game* (2.1).

PROOF. The proof of this lemma is similar to that of Lemma 3.7. ■

Finally, we have the following existence theorem.

THEOREM 3.4 *Suppose that (1) the sets X_i, $i \in I$ are nonempty, compact and convex subsets of Hausdorff locally convex vector spaces, (2) the function u_i is continuous over X, and the functions $y_{S_m} \mapsto \sum_{i \in R_m} u_i(x_{-S_m}, y_{S_m})$ are concave on X_{S_m}, $\forall x_{-S_m} \in X_{-S_m}$ and $\forall m \in M$, and (3) $\forall h(x) \in \partial h(X)$, $\forall \widetilde{y} \in \widetilde{X}$, $\exists \widetilde{p} \in Z_{h(X)}(h(x))$ such that $F(x, \widetilde{y}) \leq F(x, \widetilde{p})$. In the case if I is infinite countable, in addition to assumption (1)-(3), assume that the function F is continuous on $X \times \widetilde{X}$. Then, the game* (2.1) *has at least one BE (Definition 2.1).*

PROOF. The proof of this theorem is similar to that of Theorem 3.2. ■

By Lemma 3.11, we deduce the following proposition. Let

$$\gamma = \inf_{x \in X} \left[\sup_{\widetilde{y} \in \widetilde{X}} F(x, \widetilde{y}) \right]. \tag{3.10}$$

PROPOSITION 3.3 *Suppose that the function F is continuous on $X \times \widetilde{X}$ and the sets X_j are compact. Then, the game* (2.1) *has at least one berge equilibrium (Definition 2.1) if and only if $\gamma = 0$.*

From this proposition we deduce the method presented in Algorithm 3 for the computation of a Berge equilibrium of the game (2.1).

Algorithm 3 Procedure for the determination of a Berge equilibrium.

Require: Suppose that the conditions of Proposition 3.3 are satisfied.

Require: Calculate the value γ in (3.10).

if $\gamma > 0$, **then**

the game (2.1) has no Berge equilibrium.

else

The strategy profiles $\overline{x} \in X$ satisfying $\sup_{\widetilde{y} \in \widetilde{X}} F(\overline{x}, \widetilde{y}) = 0$ are Berge equilibria of the game (2.1).

end if

3.4 Discussion

From Theorems 3.2, 3.3, 3.4, one can see that the existence of SBE (simple Berge equilibrium), BVE (Berge-Vaisman equilibrium), BV-Nash (Berge-Vaisman-Nash equilibrium) and BE (Berge equilibrium) requires much stronger conditions than the existence of Nash equilibrium which can be established by common conditions on the primitives of the normal form game as continuity, compactness, convexity and (quasi-) concavity. To illustrate this fact, the following counter example shows that a game may not possess a SBE (Definition 2.2) even if it is continuous, compact, convex and (quasi-) concave.

EXAMPLE 3.2 Consider a three person game such that $I = \{1, 2, 3\}$, $X_1 = X_2 = X_3 = [0, 1]$ with

$$\begin{aligned} u_1(x) &= x_1 + x_2 + x_3, \\ u_2(x) &= -x_1 + x_2 - x_3, \\ u_3(x) &= x_1 - x_2 + x_3. \end{aligned}$$

It can be easily seen that this game is compact, convex, continuous and concave. However, it has no SBE. We will demonstrate this fact by two ways. The first way is direct by using the Definition 2.2 of SBE itself. The second is by Algorithm 3.

Let us start with the first way. It is easy to calculate the following maximums

$$\begin{aligned} &\max_{y_{-1}} u_1(x_1, y_{-1}) = u_1(x_1, 1, 1) = x_1 + 2, \\ &\max_{y_{-2}} u_2(x_2, y_{-2}) = u_2(0, x_2, 0) = x_2, \\ &\max_{y_{-3}} u_3(x_3, y_{-3}) = u_3(1, 0, x_3) = x_3 + 1, \end{aligned} \tag{3.11}$$

for all $x \in X$.

Assume that $x \in X$ is an SBE of the considered game, then by Definition 2.2 and (3.11), we have

$$\begin{aligned} &\max_{y_{-1}} u_1(x_1, y_{-1}) = u_1(x_1, 1, 1) = u_1(x_1, x_2, x_3), \\ &\max_{y_{-2}} u_2(x_2, y_{-2}) = u_2(0, x_2, 0) = u_2(x_1, x_2, x_3), \\ &\max_{y_{-3}} u_3(x_3, y_{-3}) = u_3(1, 0, x_3) = u_3(x_1, x_2, x_3). \end{aligned} \tag{3.12}$$

Then based on the uniqueness of the maximums in (3.11) and the equalities (3.12), we deduce that

$$(x_2, x_3) = (1, 1), \quad (x_1, x_3) = (0, 0), \quad (x_1, x_2) = (1, 0). \tag{3.13}$$

The first equation of (3.13) implies that $x_2 = 1$, however, the third equation implies that $x_2 = 0$. This contradiction shows that x cannot be an SBE of the considered game.

Let us now prove that the considered game has no SBE by Algorithm 3. An SBE is a Berge equilibrium in the case where $M = I$, $R_i = \{i\}, i \in I$ and $S_i = -i, i \in I$. Then, $\widetilde{X}$ becomes $\widetilde{X} = X_{-1} \times X_{-2} \times X_{-3}$ and

$$F(x, \widetilde{y}) = [u_1(x_1, y_{-1}) - u_1(x)] + [u_2(x_2, y_{-2}) - u_2(x)] + [u_3(x_3, y_{-3}) - u_3(x)], \tag{3.14}$$

where $x \in X$ and $\widetilde{y} = (y_{-1}, y_{-2}, y_{-3}) \in \widetilde{X}$.

Using (3.11), we get $\sum_{i=1}^{3}\max_{y_{-i}} u_i(x_i, y_{-i}) = x_1 + x_2 + x_3 + 3$. On the other hand, using (3.14), we get

$$\gamma = \inf_{x\in X}\left[\sup_{\widetilde{y}\in\widetilde{X}} F(x,\widetilde{y})\right] = \inf_{x\in X}\left[\sum_{i=1}^{3}\max_{y_{-i}} u_i(x_i, y_{-i}) - \sum_{i=1}^{3} u_i(x)\right].$$

Then we obtain

$$\gamma = \inf_{x\in X}\ [(x_1 + x_2 + x_3 + 3) - (x_1 + x_2 + x_3)] = 3.$$

Since $\gamma = 3 > 0$, then according to Algorithm 3, the considered game has no SBE. Consequently, it does not have a BVE as well.

REMARK 3.4 The counter Example 3.2 shows also that the assumptions of Theorem 9 [2] and Theorem 10 [3] in Abalo and Kostreva [2006] are not sufficient for the existence of BE because these theorems state that a compact, convex, continuous and (quasi-) concave game has a BE.

In fact, the mentioned Theorems 9-10 in Abalo and Kostreva [2006] are based on the following theorem which is flawed as well.

THEOREM 3.5 *(Abalo and Kostreva [2006]) Let I be an indexed set, finite or infinite, and $S = \{S_i\}_{i\in I}$ be a set of non-empty pairwise distinct sets such that $I = \underset{i\in I}{\cup} S_i$. Let $\{L_i\}_{i\in I}$ be a family of separated locally convex topological vector spaces. For each $i \in I$, let X_i be a non-empty compact convex set in L_i. Let $\{E_i\}_{i\in I}$ be a family of subsets of X. If for each $i \in I$,*

(i) the section $E_i(x_{S_i}) = \{x_{-S_i} \in X_{-S_i} : \ (x_{S_i}, x_{-S_i}) \in E_i\}$ is open in X_{-S_i}, for each $x_{S_i} \in X_{S_i}$,

(ii) the section $E_i(x_{-S_i}) = \{x_{S_i} \in X_{S_i} : \ (x_{S_i}, x_{-S_i}) \in E_i\}$ is nonempty and convex for each $x_{-S_i} \in X_{-S_i}$.

Then, $\bigcap_{i\in I} E_i \neq \emptyset$.

[2]Theorem 9. Let I be an indexed set, finite or infinite. Let $S = \{S_i\}_{i\in I}$ be a set of non-empty pairwise distinct sets such that $I = \underset{i\in I}{\cup} S_i$. Let $\{L_i\}_{i\in I}$ be a family of separated locally convex topological vector spaces. For each $i \in I$, let X_i be a non-empty compact convex set in L_i. Let $\{u_i\}_{i\in I}$ be a family of real-valued continuous functions defined on the set X such that for each index $i \in I$ and for each $x_{-S_i} \in X_{-S_i}$, $u_i(., x_{-S_i})$ is a quasi-concave function on X_{S_i}. Then, there is a point $\overline{x} \in X$ such that for $i \in I$, $\max_{x_{S_i}\in X_{S_i}} u_i(x_{S_i}, \overline{x}_{-S_i}) = u_i(\overline{x})$.

[3]Theorem 10. Let I be an indexed set, finite or infinite. Consider a game (2.1) with an (S, R, M)-system. For each $i \in I$, let X_i be a non-empty compact convex set in a real separated locally convex topological vector space, u_i be a real-valued continuous functions defined on the set X. If for each m, each $r_m \in R_m$ and each $x_{-S_m} \in X_{-S_m}$, $J_{r_m}(x_{-S_m}, .)$ is a quasi-concave function on X_{S_m}. Then, there is a point $\overline{x} \in X$ such that for all m, each $r_m \in R_m$, $\max_{x_{S_m}\in X_{S_m}} u_{r_m}(x_{S_m}, \overline{x}_{-S_m}) = u_{r_m}(\overline{x}_S)$.

The following counterexample shows also that the Theorem 3.5 is not correct.

EXAMPLE 3.3 Consider a three person game with $I = \{1, 2, 3\}$, $X_1 = X_2 = X_3 = [0, 1]$, $x = (x_1, x_2, x_3)$ and

$$E_1(x) = \{x \in X : \ x_2 + x_3 > 1.99\},$$
$$E_2(x) = \{x \in X : \ x_1 + x_3 < 0.1\},$$
$$E_3(x) = \{x \in X : \ x_1 - x_2 > 0.99\}.$$

For $i = 1, 2, 3$, let $S_i = -i, i \in I$.

It can be easily seen that the the section $E_i(x_{S_i})$ is open in X_{-S_i}, for each $x_{S_i} \in X_{S_i}$, and the section $E_i(x_{-S_i})$ is nonempty [4] and convex for each $x_{-S_i} \in X_{-S_i}$. Thus by Theorem 3.5 , we have $\bigcap_{i \in I} E_i \neq \emptyset$.

Let $\overline{x} \in \bigcap_{i \in I} E_i$, then $\overline{x} \in E_i$ for each $i \in I$. Since $\overline{x} \in E_1$, then $\overline{x}_2 + \overline{x}_3 > 1.99$, *i.e.* $\overline{x}_2 > 1.99 - \overline{x}_3 \geq 0.99$. we have also $\overline{x} \in E_3$, then $\overline{x}_1 - \overline{x}_2 > 0.99$, *i.e.* $\overline{x}_1 > 0.99 + \overline{x}_2 \geq 1.98$. Therefore, $\overline{x}_1 > 1.98$ and $\overline{x}_1 \in X_1 = [0, 1]$ which is impossible.

4 Conclusion

In this paper we dealt with the problem of existence and computation of Berge-Vaisman, Berge-Vaisman-Nash and Berge equilibria. For the general case of games with an infinite countable number of players, we have used the g-Maximum Equality Theorem to derive general sufficient conditions for their existence in Theorems 3.2-3.4 respectively. From these theorems we have derived Algorithms 1-3 respectively, for their computation.

In Subsection 3.4, we have shown that the problem of existence of Berge equilibria is a challenging problem. In general the existence of such equilibria cannot be established based on the conditions that guarantee the existence of Nash equilibrium. Therefore, the weakening of existence conditions of Berge equilibria could be one future direction of research. The study of Berge equilibria in differential games may be a worthy direction of research as well.

The Definition 2.1 of Berge equilibrium is very general, so only Nash equilibrium and Berge-Vaisman equilibrium have been investigated as special cases of it. It would be very interesting to explore more special cases of this equilibrium both from theoretical an application points of view. We expect that some Berge equilibria may be very useful in social sciences (political, regional and global issues and conflicts).

[4] Let $x = (x_1, x_2, x_3) \in X$.

1) If $i = 1$, let $y_2 = y_3 = 1$, then $(x_1, y_2, y_3) \in E_1$. Thus $E_1(x_1) \neq \emptyset$

2) If $i = 2$, let $y_1 = y_3 = 0$, then $(y_1, x_2, y_3) \in E_2$. Thus $E_2(x_2) \neq \emptyset$

3) If $i = 3$, let $y_1 = 1, y_2 = 0$, then $(y_1, y_2, x_3) \in E_3$. Thus $E_3(x_3) \neq \emptyset$

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