Centre d'Etude et de Recherche en Economie, Gestion, Modélisation et Informatique Appliquée
www.cereg

Document de travail 2010-05

Septembre 2010

"Pointvalue characterizations in multi-parameter algebras"

Maximilian F. HASLER & Jean-André MARTI

Université des Antilles et de la Guyane. Faculté de Droit et d'Economie de la Martinique. Campus de Schoelcher - Martinique FWI
B.P. 7209 - 97275 Schoelcher Cedex - Tél. : 0596. 72.74.00 - Fax. : 0596. 72.74.03

Université des Antilles et de la Guyane
CEREGMIA working paper 2010-05

Pointvalue characterizations in multi-parameter algebras

Maximilian F. Hasler* and Jean-André Marti
Laboratoire CEREGMIA, Univ. Antilles-Guyane, Schoelcher, Martinique (France)
(*revised version as of June 2010*)

We extend classical results from the Colombeau algebra, concerning point-value characterizations of generalized functions, to the more general case of multi-parameter (C,E,P)–algebras. Our investigations include considerations about different definitions of subspaces related to tempered generalized functions.

AMS Subject Classification: 46F30, 46T30, 46A19, 46H10.
Keywords: Algebra of generalized functions; pointvalue characterization; tempered generalized functions

1. Introduction

In this paper, we extend classical results from the Colombeau algebra, concerning point-value characterizations of generalized functions, to the more general case of multi-parameter $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras. Our investigations include an analysis of different definitions of tempered generalized functions and similar related subspaces.

The usefulness of pointvalue characterisations, in particular for proving existence and uniqueness of solutions to various differential problems, is well-established in the existing literature. As in the classical case, the well-definedness of (generalized) point-values is also relevant for considerations about the possibiliy of composition of generalized functions.

The extension with respect to known results is thus twofold: On one hand we consider multi-indices as regularisation parameters. This proves very useful in concrete differential problems with singular coefficients and data, which can be irregular concerning its behaviour as well as the geometry of its support. On the other hand we consider scales other than the polynomial scales, in particular those ("over")generated by a given set of nets, indexed by the beforementioned parameters. This setting allows a fine analysis which distinguishes the dependency of the singular spectrum of the solutions to a given problem, on the different sources of singularities [8, 13].

The results extend, mutatis mutandis, known results from the usual Colombeau algebra [2, 9, 15], which are of course reproduced in the corresponding case. Nevertheless, the consideration of several parameters and non-polynomial scales is not always completely straightforward. Asymptotic bounds usually given explicitely in terms of "ε going to zero", as for example in the notion of slow scale nets, do not make manifest in how far they correspond to the regularisation parameter going to zero, and to what extent the concrete expression is related to the choice of the polynomial scale. Since in our approach the parameters themselves cannot be used as a numerical value, the relation with the asymptotic scale is necessarily made manifest in an explicit manner.

* Corresponding author. e-mail: `mhasler@univ-ag.fr`

2. $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras

We consider the setting of $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebras [12], which is a special case of the asymptotic extension of topological algebras as described in [10].

Definition 2.1 *Let Λ be a set of indices on which is given a filter base $\mathcal{B}_\Lambda$, allowing to consider asymptotics on nets indexed by $\lambda\in\Lambda$: For $x,y\in\mathbb{K}^\Lambda$ with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, the notation $x=O(y)$ (resp. $x=o(y)$) means that there is (resp. for all) $c>0$ and some $\Lambda'\in\mathcal{B}_\Lambda$, $|x_\lambda|\le c\,|y_\lambda|$ for all $\lambda\in\Lambda'$. Then for any **solid** subring $S\subset\mathbb{K}^\Lambda$, i.e., a subring such that*

$$\forall(x,s)\in\mathbb{K}^\Lambda\times S : x=O(s)\Rightarrow x\in S\ , \tag{1}$$

and any semi-normed $\mathbb{K}$–vector space $(\mathcal{E},\mathcal{P})$, we define

$$\mathcal{H}_{(S,\mathcal{E},\mathcal{P})}=\left\{f\in\mathcal{E}^\Lambda\mid\forall p\in\mathcal{P}:\ p(f)\in S\right\}\ , \tag{2}$$

where $p(f)=(p(f_\lambda))_{\lambda\in\Lambda}\in\mathbb{R}_+^\Lambda\subset\mathbb{K}^\Lambda$. We will also consider $\mathcal{H}_{(S,K,\mathcal{P})}$ for any subset $K\subset\mathcal{E}$, which does not need to be a vector (sub)space.

Example 2.2 A left filtering partial order $\prec$ on Λ induces the base of filter $\mathcal{B}_\Lambda=\{\Lambda_\lambda;\lambda\in\Lambda\}$, where $\Lambda_\lambda=\{\lambda'\in\Lambda\mid\lambda'\prec\lambda\}$. Classical examples for $(\Lambda,\prec)$ are $(\mathbb{N},\ge)$ and $((0,1],\le)$. However, it can be very useful in practical applications to have several independent parameters, $\lambda=(\varepsilon,\eta,...)$, which may correspond to different processes of regularization, requiring different respective scales [8, 14]. It may also be of interest to consider more complex types of parameters, e.g. $\lambda=(\varepsilon,\varphi)\in(0,1]\times\mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$, the space of compactly supported smooth functions, would be equipped with an appropriate filter.

Example 2.3 The set of complex nets of at most polynomial growth indexed by $(0,1]$ can be written as $A=\left\{x\in\mathbb{C}^{(0,1]}\mid\limsup|x_\varepsilon|^{1/|\log\varepsilon|}<\infty\right\}$ [3]. For $\mathcal{E}=\mathcal{C}^\infty(\mathbb{R}^n)$ with the usual family of seminorms $\mathcal{P}=\{\,p_{K,\alpha}:f\mapsto\|\partial^\alpha f\|_{L^\infty(K)};\ K\Subset\mathbb{R}^n,\ \alpha\in\mathbb{N}^n\,\}$, this yields $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}=\mathcal{E}_M$, Colombeau's moderate nets.

Proposition 2.4 *Consider Λ and $(\mathcal{E},\mathcal{P})$ as in the above Definition 2.1.*

1. *If A is a solid subring of $\mathbb{K}^\Lambda$, then $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ is an A–module for componentwise multiplication, and an A–algebra if $\mathcal{E}$ is a topological algebra.*
2. *If I is a solid ideal of A, then $\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ is an A–linear subspace of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, and an ideal of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ if $\mathcal{E}$ is a topological algebra.*
3. *As a consequence, the factor space $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\,\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ is again an A–module, but also an A/I–module (and an algebra, if $\mathcal{E}$ is a topological algebra).*
4. *For $(\mathcal{E},\mathcal{P})=(\mathbb{K},\{|\cdot|\})$, we get $\mathcal{H}_{(A,\mathbb{K},|\cdot|)}/\,\mathcal{H}_{(I,\mathbb{K},|\cdot|)}=A/I$.*

Remark 2.5 Requiring $\mathcal{E}$ to be a topological algebra means that multiplication in $\mathcal{E}$ is continuous for the topology defined by the family of seminorms. But we also consider the more primitive case of a vector space, relevant for the notion of generalized points.

Definition 2.6 *Consider $(\mathcal{E},\mathcal{P})$ and A, I as in the above Proposition 2.4, (1)–(2).*

1. *The factor ring $\mathcal{C}=A/I$ is called the ring of generalized numbers associated to A and I, and the $\mathcal{C}$–algebra $\mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P}):=\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\,\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ is called the $(\mathcal{C},\mathcal{E},\mathcal{P})$–algebra of generalized functions.*
2. *If $(\mathcal{E},\mathcal{P})$ is a sheaf of $\mathbb{K}$–algebras over a topological space X, then we define*

$$\mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P})\ :=\ \Omega\mapsto\mathcal{A}_\mathcal{C}(\mathcal{E}(\Omega),\mathcal{P}(\Omega))\ . \tag{3}$$

for all open $\Omega \subset X$.

Example 2.7 Assume that for all $a \in A$ there is $\bar{a} \in A^*$ with $a = O(\bar{a})$, where A^* is the set of invertible elements of the ring A. Then we have the "canonically" associated ideal $I_A := \{x \in A \mid \forall a \in A^* : x = o(a)\}$, which is solid if A is.
For A from the preceding Example 2.3, this yields the set of sequences decreasing to zero faster than any power, $I_A = \big\{x \in \mathbb{C}^{]0,1]} \mid \lim |x_\varepsilon|^{1/|\log \varepsilon|} = 0\big\}$. With $\mathcal{E}, \mathcal{P}$ as before, we get Colombeau's simplified (or "special" [9]) algebra $\mathcal{G}_s(\mathbb{R}^n)$ over the ring of generalized numbers $\overline{\mathbb{C}}$.

Example 2.8 ("Overgenerated" $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebras.) For any nonempty subset $B_0 \subset (\mathbb{R}_+^*)^\Lambda$, let $B = \langle B_0 \rangle$ be the closure of B_0 under addition and division (consisting of rational fractions of "linear combinations" with *positive* integer (or rational) coefficients of products of elements of B_0.) Then $A = A_{B_0} = \big\{x \in \mathbb{K}^\Lambda \mid \exists b \in B : x = O(b)\big\}$ is a solid ring, and $\mathcal{C} = \mathcal{C}_{B_0} = A/I_A$ is said to be generated by the set B_0, and $\mathcal{A}_{\mathcal{C}_{B_0}}(\mathcal{E}, \mathcal{P})$ the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebra generated by B_0. (In earlier publications, the term "overgenerated" had been used to describe this construction.) In practical applications, this construction is useful to construct the adequate algebra for a given differential problem [5–7]. For $B_0 = \big\{(\varepsilon)_{\varepsilon \in (0,1]}\big\}$ we get back Colombeau's polynomial scale. Sometimes we use the fact that B is countable whenever B_0 is countable or finite. (Actually every $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebra whose ideal is I_A as given in Example 2.7, is generated by the set $B_0 = A^* \cap \mathbb{R}_+^\Lambda$, but this set is uncountable except for pathological cases.)

Remark 2.9 The assignment $f \mapsto (f)_{\lambda \in \Lambda} + \mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ defines a map $\mathbf{i} : \mathcal{E} \to \mathcal{A}_\mathcal{C}(\mathcal{E})$ iff $\mathbb{1} = (1)_{\lambda \in \Lambda} \in A$, or equivalently, if A contains at least one (and thus any) nonzero constant sequence. Then this map is injective iff $(\mathcal{E}, \mathcal{P})$ is Hausdorff and $\mathbb{1} \notin I$ ($\Longleftrightarrow$ $I \neq A$). **We shall assume these three conditions to hold throughout the sequel of this paper.** (The condition $(x_\lambda)_{\lambda \in \Lambda} \in I \Rightarrow \lim (x_\lambda)_{\lambda \in \Lambda} = 0$ is sufficient but not necessary to have $\mathbb{1} \notin I$; and for $A = A_{B_0}$ and I_A as in Example 2.8, all these conditions on A an I are satisfied for arbitrary sets B_0.)

Proposition 2.10 *If* $(\mathcal{E}, \mathcal{P})$ *is a presheaf of semi-normed* $\mathbb{K}$*–algebras over a topological space* X*, i.e.,*

(1) for any open $\Omega \subset X$*, the algebra* $\mathcal{E}(\Omega)$ *is endowed with the set* $\mathcal{P}(\Omega)$ *of seminorms such that, if* $\Omega_1 \subset \Omega_2 \subset \Omega$ *and* ρ_1^2 *is the restriction from* Ω_2 *to* Ω_1*, then for each* $p \in \mathcal{P}(\Omega_1)$*, we have* $p \circ \rho_1^2 \in \mathcal{P}(\Omega_2)$*.*
(2) for any open covering $(U_i)_i$ *of an open set* $\Omega \subset X$ *and each* $p \in \mathcal{P}(\Omega)$*, there is a finite subfamily* $(U_{i_1}, ..., U_{i_n})$ *of* $(U_i)_i$ *and* $p_1 \in \mathcal{P}(U_{i_1})$*, ...,* $p_n \in \mathcal{P}(U_{i_n})$ *such that for all* $u \in \mathcal{E}(\Omega)$*,* $p(u) \leq p_1(u|_{U_{i_1}}) + ... + p_n(u|_{U_{i_n}})$ *,*

then $\mathcal{A}_\mathcal{C}(\mathcal{E}, \mathcal{P})$ *defined in* (3) *is again a presheaf.*
Moreover, if $\mathcal{E}$ *is a fine sheaf, then* $\mathcal{A}_\mathcal{C}(\mathcal{E}, \mathcal{P})$ *also is a fine sheaf.*

The proof is given in [12], and, for the last statement, in [10].

3. Multiparameter algebras of tempered generalized functions

We first study the relations between two closely related definitions of spaces of tempered generalized functions, which generalize the "simplified" version $\mathcal{G}_\tau(\Omega)$ of the corresponding space introduced by Colombeau in [1]. An important property of functions in $\mathcal{G}_\tau(\Omega)$ is that their point-values in (not necessarily compactly supported) generalized points are well-defined. This is also relevant when considering the possibility of composition of generalized functions. In the previously introduced framework it is most natural to consider

$$\mathcal{A}_{\mathcal{C}}(\mathcal{O}_M)(\Omega) := \mathcal{A}_{\mathcal{C}}(\mathcal{O}_M(\Omega), \mathcal{P}_\tau(\Omega)) ,$$

the $\mathcal{C}$–extension of Schwartz' space $\mathcal{O}_M(\Omega)$ of "multipliers" or slowly increasing functions, with topology given by the family of semi-norms

$$\mathcal{P}_\tau(\Omega) = \{ p_{\varphi,\alpha} : f \mapsto \|\varphi \cdot \partial^\alpha f\|_{L^\infty(\Omega)} \ ; \ \varphi \in \mathcal{S}(\Omega), \ \alpha \in \mathbb{N}^n \} .$$

The elements of $\mathcal{O}_M(\Omega)$ are the smooth functions for which all of the above seminorms are finite,

$$\mathcal{O}_M(\Omega) = \{ f \in \mathcal{C}^\infty(\Omega) \mid \forall \alpha \in \mathbb{N}^n \ \ \forall \varphi \in \mathcal{S}(\Omega) : p_{\varphi,\alpha}(f) < \infty \} .$$

For the sequel, it is also important to note that $\mathcal{O}_M(\Omega)$ is a topological algebra, which is trivial if Ω is bounded, but else (and in particular for $\Omega = \mathbb{R}^n$) requires a rather lengthy proof of Lemma 4 given in [2][1].

It is well known [11] that for $\Omega = \mathbb{R}^n$, we have $\mathcal{O}_M(\mathbb{R}^n) = \mathcal{O}_M^{\mathrm{g}}(\mathbb{R}^n)$, where

$$\mathcal{O}_M^{\mathrm{g}}(\Omega) = \{ f \in \mathcal{C}^\infty(\Omega) \mid \forall \alpha \in \mathbb{N}^n \ \ \exists r \in \mathbb{N} : q_{-r,\alpha}(f) < \infty \} ,$$

with $q_{r,\alpha} : f \mapsto \sup\{ |(1 + \|x\|)^r \, \partial^\alpha f(x)| \, ; \ x \in \Omega \}$.

Obviously, the $q_{r,\alpha}$ are not seminorms on the whole of $\mathcal{O}_M^{\mathrm{g}}(\Omega)$, which could be written as projective limit of the inductive limit of the spaces $E_{r,\alpha}$ on which these seminorms are finite. For the same reason, the corresponding factor algebra

$$\mathcal{G}_{\tau,\mathcal{C}}(\Omega) = \mathcal{M}_{\tau,A}(\Omega) / \mathcal{M}_{\tau,I}(\Omega) \tag{4}$$

where for any $S \subset \mathbb{K}^\Lambda$,

$$\mathcal{M}_{\tau,S}(\Omega) = \left\{ f \in (\mathcal{O}_M^{\mathrm{g}}(\Omega))^\Lambda \mid \forall \alpha \in \mathbb{N}^n \ \ \exists r \in \mathbb{N} : (q_{-r,\alpha}(f_\lambda))_\lambda \in S \right\} , \tag{5}$$

does not fit in the framework of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$–algebras as defined in Def. 2.6. (It is included, however, in the more general concept reviewed in [3].) Since we will not apply the construction of Def. 2.6 with this space, we do not need to know whether $\mathcal{O}_M^{\mathrm{g}}(\Omega)$ is a topological algebra. The obvious estimates using the $q_{r,\alpha}$ are sufficient to establish $\mathcal{M}_{\tau,A}(\Omega)$ as an algebra and $\mathcal{M}_{\tau,I}(\Omega)$ as an ideal thereof.

Remark 3.1 In the above definition, the integer $r \in \mathbb{N}$ must not depend on $\lambda \in \Lambda$, *i.e.*, for any representative $u \in \mathbf{u}$, the whole net $u = (u_\lambda)_\lambda$ must lie in a subspace of $\mathcal{C}^\infty(\Omega)$ on which some $q_{-r,\alpha}$ is finite, for given $\alpha \in \mathbb{N}$.

Remark 3.2 Even though we have $\mathcal{O}_M(\mathbb{R}^n) = \mathcal{O}_M^{\mathrm{g}}(\mathbb{R}^n)$, we do not claim that the topologies induced on this space by $\mathcal{P}_\tau(\mathbb{R}^n)$ resp. $\mathcal{Q}_\tau = \{q_{r,\alpha}\}$ are the same.

THEOREM 3.3 *(i) Consider $\mathcal{C} = A/I$ as in Def. 2.6. Then, for $S = A$ and $S = I$, we have $\mathcal{M}_{\tau,S}(\mathbb{R}^n) \subset \mathcal{H}_{(S,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n)$.*
(ii) Assume the additional hypothesis that the base of filter $\mathcal{B}_\Lambda$ is countable, and that A and I are given as $A = \left\{ x \in \mathbb{K}^\Lambda \mid \exists \ell \in \mathbb{Z} : x = O(b^{(\ell)}) \right\}$, $I = \left\{ x \in \mathbb{K}^\Lambda \mid \forall \ell \in \mathbb{Z} : x = o(b^{(\ell)}) \right\}$ in terms of a countable set $\left\{ b^{(k)} \ ; \ k \in \mathbb{Z} \right\} \subset \mathbb{R}_+^\Lambda$ such that $\forall k, \ell \in \mathbb{Z} : k < \ell \Rightarrow b^{(k)} = o(b^{(\ell)})$. Then we have: $\mathcal{M}_{\tau,A}(\mathbb{R}^n) =$

[1] Personal communication by the author of [2].

$\mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n)$ and therefore $\mathcal{A}_{\mathcal{C}}(\mathcal{O}_M)(\mathbb{R}^n)$ can be seen as $\mathcal{G}_{\tau,\mathcal{C}}(\mathbb{R}^n)$ modulo the canonical image, in $\mathcal{G}_{\tau,\mathcal{C}}(\mathbb{R}^n)$, of the larger ideal $\mathcal{H}_{(I,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n)$.

Remark 3.4 Such a countable set $\{b^{(\ell)}\}$ exists in particular for asymptotic algebras [4] and thus in the Colombeau case. In practical applications, when A/I is to be generated by a finite number of nets, we can usually choose a subsequence of the set B mentioned in example 2.8, which has the required property. On the other hand, the rather restrictive hypothesis on $\{b^{(\ell)}\}$ could be significantly relaxed. However, for the scope of this short paper, we feel obliged to confine ourselves to this somehow limited framework, leaving a more general treatment as future work.

To prove the Theorem, we will use the following Lemma:

LEMMA 3.5 *Consider $f \in \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}(\Omega)$, with A as in Theorem 3.3. We have $f \in \mathcal{M}_{\tau,A}(\Omega)$ iff*

$$\forall\alpha \in \mathbb{N}^n \;\; \exists \ell, r \in \mathbb{N} \;\; \exists K \Subset \Omega \;\; \exists \Lambda' \in \mathcal{B}_\Lambda \;\; \forall \lambda \in \Lambda' \;\; \forall x \notin K : \\ (1+\|x\|)^{-\ell}\left(\partial^\alpha f_\lambda(x)\right)_\lambda \le b_\lambda^{(r)} \; . \tag{6}$$

Proof From the definition (5) of $\mathcal{M}_{\tau,A}(\Omega)$, it is clear that (6) is satisfied for any $f \in \mathcal{M}_{\tau,A}(\Omega)$, with *any* $K \Subset \Omega$, and $\ell = p$, $r = \ell'$, where $b^{(\ell')}$ is dominating $q_{-p,\alpha}(f)$ in (5). Conversely, assume that (6) holds for some $f \in \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}$. We have to show that for each $\alpha \in \mathbb{N}^n$, there is r'' such that the analogous relation is verified also inside K. For this, it is sufficient to consider the definition of $\mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}$ with the seminorm $p_{\varphi,\alpha}$ for $\varphi \in \mathcal{D}(\Omega) \subset \mathcal{S}(\Omega)$ equal to 1 on K: This implies that $p_{\varphi,\alpha}(f)$ is an element of A, which by hypothesis is dominated by some $b^{(r')}$.
Multiplying by $(1+\|x\|)^{-\ell}$ and restricting x to K makes the left hand side only smaller. Thus, for $\ell'' = \max\{\ell, \ell'\}$ Choosing r'' such that $b^r + b^{r'} = O(b^{r''})$ we have the inequality in (6) for all $x \in \Omega$, *i.e.*, $f \in \mathcal{M}_{\tau,A}(\Omega)$. ■

Proof of the Theorem. (i) We show that $\mathcal{M}_{\tau,X}(\Omega) \subset \mathcal{H}_{(X,\mathcal{O}_M,\mathcal{P}_\tau)}(\Omega)$ for $X = A$ and $X = I$. From the definitions (of $\mathcal{S}$ in particular), this inclusion is obvious in both cases: For any α, if such p exists in (5), then, since any $\varphi \in \mathcal{S}$ decreases faster than $(1+\|x\|)^{-p}$, one has $p_{\varphi,\alpha} \le C\, q_{-p,\alpha}$ (with $C = \sup|(1+\|x\|)^p \varphi(x)|$), and since X is solid, $q_{-p,\alpha}(f) \in X \Rightarrow p_{\varphi,\alpha}(f) \in X$.
(ii) For the converse inclusion with $X = A$ and $\Omega = \mathbb{R}^n$, we assume that $\mathcal{B}_\Lambda$ has an equivalent countable base $\Lambda_1 \supset \Lambda_2 \supset ....$ Then, in view of the Lemma, if $f \notin \mathcal{M}_{\tau,A}(\Omega)$ then

$$\exists \alpha \in \mathbb{N}^n \;\; \forall \ell, r \in \mathbb{N} \;\; \forall K \Subset \Omega \;\; \forall \Lambda' \in \mathcal{B}_\Lambda \;\; \exists \lambda \in \Lambda' \;\; \exists x \notin K \;\; : \\ (1+\|x\|)^{-\ell}\,|\partial^\alpha f_\lambda(x)| > b_\lambda^{(r)} \; .$$

For $\Omega = \mathbb{R}^n$, this allows to construct, for some $\alpha \in \mathbb{N}^n$, sequence $(x_\ell)_{\ell\in\mathbb{N}}$ and $(\lambda_\ell)_{\ell\in\mathbb{N}}$ such that $\|x_{\ell+1}\| \ge \|x_\ell\| + 2$, $\lambda_\ell \in \Lambda_\ell$ and $(1+\|x_\ell\|^2)^{-\ell}\,|\partial^\alpha f_{\lambda_\ell}(x_\ell)| \ge b_{\lambda_\ell}^{(r)}$ for all $\ell \in \mathbb{N}$. Let us consider the element $\varphi \in \mathcal{S}$ which consists of "bumps" of height 1 centered in these x_ℓ,

$$\varphi(x) = \sum_{\ell\in\mathbb{N}} (1+\|x_\ell\|^2)^{-\ell} \rho(x - x_\ell) \; , \quad \rho \in \mathcal{D}(\mathbb{R}^n), \;\; \operatorname{supp}\rho \subset B_1(o), \;\; 0 \le \rho \le 1 = \rho(o) \; .$$

Obviously it is such that $p_{\varphi,\alpha}(f_{\lambda_\ell}) \ge b_{\lambda_\ell}^{(\ell)}$ for every ℓ, therefore $\left(p_{\varphi,\alpha}(f_\lambda)\right)_\lambda$ is not dominated by any $a \in A$ and thus $f \notin \mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)}$. ■

We have the following characterization of the ideal $\mathcal{H}_{(I,\mathcal{O}_M,\mathcal{P}_\tau)}$:

Lemma 3.6 *Under the same hypotheses as in part (ii) of Theorem 3.3,*

$$\mathcal{H}_{(I,\mathcal{O}_M,\mathcal{P}_\tau)}(\mathbb{R}^n) = \left\{u \in \mathcal{O}_M(\mathbb{R}^n)^\Lambda \mid \forall\alpha \in \mathbb{N}^n\ \forall\ell \in \mathbb{Z}\ \exists p \in \mathbb{N} : q_{-p,\alpha}(u_\lambda) = o(b_\lambda^{(\ell)})\right\}.$$

Proof With the quantifiers and asymptotics exchanged, the proof of the nontrivial inclusion is here the same as for $\mathcal{H}_{(A,\mathcal{O}_M,\mathcal{P}_\tau)} \subset \mathcal{M}_{\tau,A}$ in the preceding Theorem. ■

4. Generalized points and point values of generalized functions

Here we generalize classical results concerning point values in the Colombeau algebra, as given, *e.g.*, in [9], to the multiparametric algebras introduced above.

Definition 4.1 *For a given ring of generalized numbers $\mathcal{C} = A/I$, the **generalized points** in $\Omega \subset \mathbb{R}^n$, $\widetilde{\Omega} = \Omega_A/\sim$, are equivalence classes of A–moderate sequences $x \in \Omega_A = \mathcal{H}_{(A,\Omega,\|\cdot\|)} = \left\{x \in \Omega^\Lambda \mid (\|x_\lambda\|)_\lambda \in A\right\}$ modulo the equivalence relation*

$$x \sim y \iff (\|x_\lambda - y_\lambda\|)_\lambda \in I \iff x - y \in \mathcal{H}_{(I,\mathbb{R}^n,\|\cdot\|)}\ .$$

*The **compactly supported** points in $\widetilde{\Omega}$ are those having a representative in a compact set, $\widetilde{\Omega}_c = \widetilde{\Omega} \cap \left\{\,\widetilde{x}\ ;\ x \in K^\Lambda,\ K \Subset \Omega\,\right\}$, or, equivalently, having a compact **support*** $\operatorname{supp}\widetilde{x} = \{y \in \mathbb{R}^n \mid \forall V \in \mathcal{V}(y)\ \ \forall\Lambda' \in \mathcal{B}_\Lambda\ \ \exists\lambda \in \Lambda' :\ x_\lambda \in V\}$.

(The support of a generalized point is thus nothing else than the set of cluster points of any of its representatives.)

Remark 4.2 Since an open set $\Omega \subsetneq \mathbb{R}^n$ is not a vector space, we cannot write $\widetilde{\Omega}$ as quotient vector space, but have to use the set-theoretic formulation modulo an equivalence relation. However, for applications (where we are only interested in the behaviour for "λ small enough"), it amounts to the same to consider points of $\widetilde{\mathbb{R}^n} = \mathcal{A}_\mathcal{C}(\mathbb{R}^n, \|\cdot\|)$ which have a representative in Ω^Λ. Since elements of I have zero limit, this implies that, for open Ω, *all* representatives of such points lie in Ω for λ small enough. (However, for some values of λ, we may have $x_\lambda \notin \Omega$. Then, an expression $f(x_\lambda)$ is not defined for these λ, if the domain of f is Ω.)

We now prove the following generalization of Proposition 1.2.45 in [9]:

Theorem 4.3 *Let $\mathcal{C} = A/I$ be a ring of generalized numbers, $\mathcal{E}$ the space of C^∞ functions on a connected open $\Omega \subset \mathbb{R}^n$, with topology given by the supremum norms of all derivatives on compact sets, $\mathcal{P} = \left\{p_{K,\alpha} : f \mapsto \|\partial^\alpha f\|_{L^\infty(K)}\right\}$. Then, for any $\mathbf{u} \in \mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P})$ and $\tilde{x} \in \widetilde{\Omega}_c$, $\mathbf{u}(\widetilde{x})$ is a well defined element of $\mathcal{C} = \widetilde{\mathbb{K}}$.*

This means that the sequence $(u_\lambda(x_\lambda))_\lambda$ is an element of A, for any representatives $(u_\lambda)_\lambda$ resp. $(x_\lambda)_\lambda$ of $\mathbf{u}$ resp. $\tilde{x}$, and that its class modulo I is independent of the choice of these representatives.

Proof Consider representatives $(u_\lambda)_\lambda$, $(v_\lambda)_\lambda$ of $\mathbf{u}$ and $(x_\lambda)_\lambda$, $(y_\lambda)_\lambda$ of $\widetilde{x}$. Let us first show that $(u_\lambda(x_\lambda))_\lambda \in A$. Indeed, we can assume that for all "sufficiently small" λ, x_λ lies in some compact K. Then, since for all compact sets K and $\alpha \in \mathbb{N}^n$, $p_{K,\alpha}(u_\lambda) \in A$, we have that $(u_\lambda(x_\lambda))_\lambda \in A$. In the same way we have for any $j \in \mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$, $(j_\lambda(x_\lambda))_\lambda \in I$. We use this in

$$u_\lambda(x_\lambda) - v_\lambda(y_\lambda) = u_\lambda(x_\lambda) - u_\lambda(y_\lambda) + \underbrace{u_\lambda(y_\lambda) - v_\lambda(y_\lambda)}$$

to see that the second part is an element of I. As to the first part, we use

$$u_\lambda(x_\lambda)-u_\lambda(y_\lambda)=\int_{x_\lambda}^{y_\lambda}\operatorname{grad}u_\lambda(\xi)\cdot\mathrm{d}\xi=\int_0^1\operatorname{grad}u_\lambda(x_\lambda+s(y_\lambda-x_\lambda))\cdot(y_\lambda-x_\lambda)\ \mathrm{d}s\ .$$

(Since we have $x_\lambda - y_\lambda \to 0$ following $\mathcal{B}_\Lambda$, all segments connecting x_λ and y_λ eventually lie in u_λ's domain Ω.) Thus

$$|u_\lambda(x_\lambda)-u_\lambda(y_\lambda)|\le\|y_\lambda-x_\lambda\|\int_0^1\|\operatorname{grad}u_\lambda(x_\lambda+s(y_\lambda-x_\lambda))\|\,\mathrm{d}s\ ,$$

and using that $(\|y_\lambda-x_\lambda\|)_\lambda\in I$ and $(p_{K,\alpha}(u_\lambda))_\lambda\in A$ (for $|\alpha|=1$ and some compact K containing the segments $[x_\lambda,y_\lambda]$, which exists since both x and y are in $\widetilde{\Omega}_c$), we finally get $u_\lambda(x_\lambda)-v_\lambda(y_\lambda)\in I$, *i.e.*, the required independence of respective representatives. ■

The following Lemma, which generalizes Theorem 1.2.3 in [9], will be used to prove Theorem 4.6:

LEMMA 4.4 (Characterization of the ideal by 0-order estimate) *Assume that $I=I_A$ (cf. Example 2.7) and for every $x\in A$ and $a\in A^*$, there is $b\in A^*$ such that $b\,x=o(a)$. Then we have $\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}=\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}\cap\mathcal{H}_{(I,\mathcal{E},\mathcal{P}_0)}$ where $\mathcal{P}_0=\{p_{K,0};K\Subset\Omega\}$, $p_{K,0}=\|\cdot\|_{L^\infty(K)}$. In other words, for $u\in\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ we have $u\in\mathcal{H}_{(I,\mathcal{E},\mathcal{P})}$ iff for every $K\Subset\Omega$, $\left(\|u_\lambda\|_{L^\infty(K)}\right)_\lambda\in I$.*

Remark 4.5 The second assumption is satisfied whenever every $x\in A$ are dominated by some $y\in A^*$, thus in particular in algebras generated (as in Example 2.8) by a set B_0 having an element going to 0 or to infinity.

Proof We only have to show the inclusion $\supset$. Consider $u=(u_\lambda)\in\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ such that $p_{K,0}(u)\in I$ for all $K\Subset\Omega$. It is enough to show that for any partial derivative ∂_i, we still have $p_{K,0}(\partial_i u)\in I$ for all $K\Subset\Omega$. Then, since $\partial_i u$ is still in $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, the result holds for any derivative by immediate recurrence. Let $K\Subset\Omega$ and $a\in A^*$ be given. We will show that $p_{K,0}(\partial_i u)=o(a)$. As usual, we let $L=K+\overline{B}_{\delta/2}(0)$, where $\delta=\min(\operatorname{dist}(K,\partial\Omega),1)$. We know that $\partial_i^2 u\in\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$, thus, by assumption, there exists $h\in A^*:p_{L,0}(h\,\partial_i^2 u)=o(a)$, and we can assume that $|h_\lambda|<\delta/2$ for all $\lambda\in\Lambda$. By Taylor's theorem, $\partial_i u(x)=h^{-1}\left(u(x+h\,e_i)-u(x)\right)-\frac12 h\,\partial_i^2 u(x+\tilde h\,e_i)$, with $\tilde h_\lambda\in[0,h_\lambda]$. From this we get, as required,

$$p_{K,0}(\partial_i u)\ \le\ \underbrace{|h^{-1}|}_{\in A}\ 2\ \underbrace{p_{L,0}(\hat u)}_{\in I}+\tfrac12\ \underbrace{|h|\,p_{L,0}(\partial_i^2\hat u)}_{=o(a)}\ =\ o(a)\ .$$

■

THEOREM 4.6 *Under the assumptions of Lemma 4.4, if $\mathbf{u}\in\mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})$, then*

$$\mathbf{u}=0\in\mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})\iff\forall\tilde x\in\Omega_c:\mathbf{u}(\tilde x)=0\in\mathcal{C}\,.$$

Proof The implication "$\Rightarrow$" is a consequence of Theorem 4.3. Let us show "$\Leftarrow$" by contraposition: Assume $\mathbf{u}\ne0$. This means that for some $K\Subset\Omega$ and some representative $(u_\lambda)\in\mathbf{u}$, $(p_{K,0}(u_\lambda))_\lambda\notin I$ (using the preceding Lemma 4.4). Now, if we let $x_\lambda\in K$ such that $u_\lambda(x_\lambda)=\|u_\lambda\|_{L^\infty(K)}$, then $\tilde x\in\Omega_c$ and $\mathbf{u}(\tilde x)\ne0$. ■

The requirement of compactly supported points can be dropped if we confine ourselves to tempered generalized functions defined in (4), in analogy to Proposition 1.2.45 in [9].

Theorem 4.7 *For* $\mathbf{u} \in \mathcal{G}_{\tau,\mathcal{C}}(\Omega)$ *and* $\tilde{x} \in \widetilde{\Omega}$, $\mathbf{u}(\tilde{x})$ *is a well-defined element of* $\mathcal{C}$.

Proof Let u resp. x be representatives of $\mathbf{u}$ resp. $\widetilde{x}$. We have that $r \in \mathbb{N}$ such that $a_\lambda = \sup_{\xi\in\Omega}(1+|\xi|)^{-r}|u_\lambda(\xi)|$ defines an element $a = (a_\lambda)_\lambda$ of A, and $b = (\|x_\lambda\|)_\lambda$ is also in A. Replacing ξ by x_λ, we get $|u_\lambda(x_\lambda)| \leq (1+b_\lambda)^r a_\lambda$, and since A is a solid ring, we also have $(u_\lambda(x_\lambda))_\lambda \in A$. As in the previous proof, $|u_\lambda(x_\lambda) - u_\lambda(y_\lambda)| \in I$ if y is another representative of $\widetilde{x}$ and thus $x - y \in I$, and in the same way $|u_\lambda(x_\lambda) - v_\lambda(x_\lambda)| \in I$ for any other representative v of $\mathbf{u}$, achieving the proof. ■

The following Lemma generalizes Theorem 1.2.25 in [9, p.27]:

Lemma 4.8 (Characterization of $\mathcal{M}_{\tau,I}$ by 0-order estimates.)
Under the assumptions of Lemma 4.4, and the additional hypothesis that Ω *is an* n*–dimensional box, we have* $\mathcal{M}_{\tau,I} = \mathcal{M}_{\tau,A} \cap \mathcal{M}_{\tau^*,I}$, *where*

$$\mathcal{M}_{\tau^*,I} = \Big\{ f \in (\mathcal{C}^\infty(\Omega))^\Lambda \mid \exists p \in \mathbb{N} : \big(\sup_{\Omega}|(1+\|x\|)^{-p} f_\lambda|\big)_{\lambda\in\Lambda} \in I \Big\} .$$

Proof For $u \in \mathcal{M}_{\tau,A} \cap \mathcal{M}_{\tau^*,I}$, we will show that $q_{-p,0}(\partial_i u) = o(a)$ for some $p \in \mathbb{N}$ and all $a \in A^*$. Let $p \in \mathbb{N}$ such that $q_{-p,0}(u) \in I$ and $q_{-p,0}(\partial_i^2 u) \in A$, and let $a \in A^*$ be given. Using the assumption, there is $h \in A^*$ such that $h\, q_{-p,0}(\partial_i^2 u) = o(a)$ (and we can assume that $h_\lambda \to 0$). Again, by Taylor's theorem, $\partial_i u(x) = h^{-1}\,(u(x + h\,e_i) - u(x)) - \frac{1}{2} h\,\partial_i^2 u(x + \tilde{h}\,e_i)$, with $\tilde{h}_\lambda \in [0, h_\lambda]$. (Since Ω is a box, for each e_i the sign of h_λ can be chosen such that the segments $[x, x + h\,e_i]$ lie in Ω.) In the expression of $q_{-p,0}$ we use $\|x\| \geq \|x + h_\lambda\,e_i\| - \|h_\lambda\|$ and $(1 + \|x + h\,e_i\| - \|h\|)^{-p} = (1 + \|x + h\,e_i\|)^{-p}(1 + O(h))$ to get

$$q_{-p,0}(\partial_i u) \;\leq\; \underbrace{|h^{-1}|}_{\in A}\; \underbrace{q_{-p,0}(u)\,(2 + O(h))}_{\in I} + \tfrac{1}{2}\, \underbrace{|h|\, q_{-p,0}(\partial_i^2 \hat{u})(1 + O(h))}_{=o(a)} \;=\; o(a) \;.$$

■

Theorem 4.9 *Under the hypotheses of Lemma 4.8, and assuming that* $\mathcal{B}_\Lambda$ *is countable (or cofinal to a countable filter base), we have that* $\mathbf{u} \in \mathcal{G}_{\tau,\mathcal{C}}(\Omega)$ *is zero if, and only if,* $\mathbf{u}(\widetilde{x}) = 0 \in \mathcal{C}$ *for all* $\widetilde{x} \in \widetilde{\Omega}$.

Remark 4.10 The condition on the shape of Ω can be significantly relaxed; as in [9, Thm. 1.2.50], the result holds also if Ω is a moderate open set.

Proof The sense ($\Rightarrow$) is a consequence of Theorem 4.7, e.g., by taking as representative of $\mathbf{u}$ the sequence identically equal to zero. Now consider ($\Leftarrow$), by contraposition. Assume that $\mathbf{u} \in \mathcal{G}_{\tau,\mathcal{C}} \setminus \{o\}$, *i.e.*, $(u_\lambda)_\lambda \in \mathcal{M}_{\tau,A} \setminus \mathcal{M}_{\tau^*,I}$ (using the Lemma 4.8). By definition and assumptions made on A, I_A, this means that

$$\forall \alpha \in \mathbb{N}^d,\ \exists p \in \mathbb{N},\ \exists a \in A,\ \forall \Lambda' \in \mathcal{B}_\Lambda,\ \exists \lambda \in \Lambda' : \sup_{x\in\Omega} |(1 + \|x\|)^{-p}\, \partial^\alpha u_\lambda(x)| \leq a_\lambda \tag{$*$}$$

(where $a \in A$ can be taken invertible, $a \in A^*$, without loss of generality), and

$$\forall q \in \mathbb{N},\ \exists j \in A \setminus I,\ \forall \Lambda' \in \mathcal{B}_\Lambda,\ \exists \lambda \in \Lambda' : \sup_{x\in\Omega} |(1 + \|x\|)^{-q}\, u_\lambda(x)| \geq j_\lambda \;, \tag{$**$}$$

where j can be taken in A^*, according to the assumption.
Now take $\alpha = 0$ and $p \in \mathbb{N}$, $a \in A$ as in ($*$), and $j \in A^*$ such that ($**$) holds with $q = p + 1$. Then we have $(1 + \|x\|)^{-p-1}\,|u_\lambda(x)| \leq (1 + \|x\|)^{-1}\, a_\lambda < j_\lambda$ whenever $\lambda \in \Lambda_0$ and $\|x\| \geq a_\lambda\, j_\lambda^{-1}$. This means, in view of ($**$), that

$$\forall \Lambda' \subset \Lambda_0,\ \exists \lambda \in \Lambda' : \sup_{\|x\| \leq a_\lambda j_\lambda^{-1}} |u_\lambda(x)| \geq \sup_{\|x\| \leq a_\lambda j_\lambda^{-1}} (1 + \|x\|)^{-p-1}\, |u_\lambda(x)| \geq j_\lambda \;.$$

Thus there exists a sequence $(\Lambda_k)_k$ which can be taken to be decreasing and cofinal to $\mathcal{B}_\Lambda$, $(\lambda_k)_k$ (with $\lambda_k \in \Lambda_k$), and $(x_k)_k \in \Omega^{\mathbb{N}}$ such that $\|x_k\| \leq a_{\lambda_k} j_{\lambda_k}^{-1}$ and $|u_{\lambda_k}(x_k)| \geq \frac{1}{2} j_{\lambda_k}$. If we let $x_\lambda = x_k$ for $\lambda \in \Lambda_k \setminus \Lambda_{k+1}$, then $(\|x_\lambda\|)_\lambda \in A$, thus $\widetilde{x} \in \widetilde{\Omega}$, and $(u_\lambda(x_\lambda))_\lambda \notin I$, *i.e.*, $\mathbf{u}(\widetilde{x}) \neq 0 \in \mathcal{C}$, which achieves the proof of the "if" part of the Theorem. ∎

We can establish an analogon of the pointvalue characterizations in $\mathcal{A}_\mathcal{C}(\mathcal{O}_M, \mathcal{P}_\tau)$ known for the simplified Colombeau case[1], using

DEFINITION 4.11 *A generalized point $\widetilde{x} \in \widetilde{\Omega}$ is said to be of slow scale (c.f. [15]) if*

$$\exists a \in A^* \quad \forall n \in \mathbb{N} \quad |x_\lambda|^n = O(a_\lambda)\ . \tag{7}$$

It is easily seen that (7) is independent of the chosen representative $x \in \widetilde{x}$. The detailed theorems and proofs will be given in a forthcoming paper.

Acknowledgements. The authors wish to thank the anonymous referee for a very careful analysis of the submitted manuscript, and for many detailed and useful suggestions, which helped to significantly improve the quality of the paper.

References

[1] J.-F. Colombeau, *Elementary Introduction to New Generalized Functions.* North-Holland, Amsterdam, 1985. MR 86c:46042

[2] A. Delcroix, *A new approach to temperate generalized functions.* Publ. Inst. Math. (Beograd) **84**, 98 (2008) 109–121.

[3] A. Delcroix, M. F. Hasler, S. Pilipović, V. Valmorin, *Sequence spaces with exponent weights—Realizations of Colombeau type algebras.* Diss. Math. 447 (2007) pp. 1–56.

[4] A. Delcroix, D. Scarpalézos, *Asymptotic scales, asymptotic algebras.* Integral transforms and special functions 6 (1997) pp. 157–166.

[5] A. Delcroix, V. Devoué, J.-A. Marti, Generalized solutions of singular differential problems. Relationship with classical solutions. J. Math. Anal. Appl. **353**, 386–402 (2009).

[6] V. Dévoué, Generalized solutions to a non-Lipschitz Goursat problem. Diff. Equa & Appl. **1**/2, 153–178 (2009)

[7] V. Dévoué, Generalized solutions to a non-Lipschitz Cauchy problem. J. Appl. Anal. **15**/1, 1–32 (2009)

[8] V. Dévoué, M. Hasler, J.-A. Marti, *Multidimensional asymptotic spectral analysis and applications.* to appear in Applicable Analysis, 2010.

[9] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer, *Geometric Theory of Generalized Functions with Applications to General Relativity.* Kluwer Academic Publ., 2001.

[10] M. F. Hasler, *Asymptotic extension of topological modules and algebras.* Integral Transforms and Special Functions 20 (2009) pp. 291–299.

[11] V.-K. Khoan, *Distributions, analyse de Fourier, opérateurs aux dérivées partielles, tome II.* Vuibert, Paris, 1972.

[12] J.-A. Marti, *$(\mathcal{C}, \mathcal{E}, \mathcal{P})$-sheaf structures and applications.* Nonlinear theory of generalized functions. (Grosser & al., eds.), Research notes in mathematics. Chapman & Hall/CRC, 1999, 175–186.

[13] J.-A. Marti, *Regularity, Local and Microlocal Analysis in Theories of Generalized Functions.* Act. Appl. Math. 105 (2009) pp. 267–302.

[14] J.-A. Marti, *Nonlinear algebraic analysis of delta shock wave solutions to Burgers' equation.* Pacific Journal of Mathematics 210 (2003) pp. 165–187.

[15] H. Vernaeve, *Pointwise characterizations in generalized function algebras.* Monatshefte f. Mathematik 158 (2009) pp. 195–213.

[1]H. Vernaeve, personal communication.