The Strictest Common Relaxation of a Family of Risk Measures

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September 19, 2010

Abstract

Operations which form new risk measures from a collection of given (often simpler) risk measures have been used extensively in the literature. Examples include convex combination, convolution, and the worst-case operator. Here we study the risk measure that is constructed from a family of given risk measures by the best-case operator; that is, the newly constructed risk measure is defined as the one that is as restrictive as possible under the condition that it accepts all positions that are accepted under any of the risk measures from the family. In fact we define this operation for conditional risk measures, to allow a multiperiod setting. We show that the well known VaR risk measure can be constructed from a family of conditional expectations by a combination that involves both worst-case and best-case operations. We provide an explicit description of the acceptance set of the conditional risk measure that is obtained as the strictest common relaxation of two given conditional risk measures.

JEL Classification: D81.

Keywords: nonconvex risk measures, Value at Risk, best-case operator.

1 Introduction

Operations which construct a new risk measure out of a family of given (usually simpler) risk measures have been extensively studied in the literature. For example, a well known operation of this type is taking a convex combination of two or more risk measures, or

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more generally, forming an integral of a continuously parametrized family of risk measures. Such a mixture operation was applied by Gerber and Goovaerts [10] to Esscher principles. Kusuoka [12] showed that all coherent law-invariant risk measures, under a weak continuity condition and a technical condition on the probability space, can be represented as mixtures of expected shortfall risk measures. Another example of a combination is the convolution of risk measures, used for instance by Barrieu and El Karoui [3] and by Klöppel and Schweizer [11]. Several ways of combining coherent risk measures, including the worst-case operator, were studied by Delbaen [6].

In this paper we consider the *best-case operator*, which can be viewed as a natural counterpart of the worst-case operator but has received less attention in the literature, presumably because of the fact that it does not in general preserve convexity. We show however that the best-case operator is useful as a means of constructing risk measures; in particular, we show that Value at Risk can be described in terms of this operator. There are also direct applications of the concept. For instance, if an institution is insuring the five worst credit events among a given collection of names, but at the same time is buying insurance against the two worst credit events among the same names, then the remaining risk for the institution can be described as the "three best cases out of the five worst". For an example outside the domain of finance and insurance, it may be noted that at many universities the grade obtained by a student who takes part in several exams on the same subject is determined as the best result obtained in any of the trials; in other words, the results are combined on a best-case basis. In figure skating, the lowest among the grades awarded by the judges is dropped from the calculation that leads to the final result of contestants; to describe such an aggregation system, again the best-case operator can be used.

The best-case operator produces a *relaxation* of each of the risk measures on which it operates; that is, all positions are accepted that are accepted by at least one of the original risk measures. To make the result of the operation well-defined, we look for the most restrictive measure that has this property. In other words, the best-case operator is the operator that produces, starting from a given family of risk measures, the *strictest common relaxation* of the family.

Much of the recent literature on risk measures has focused on multiperiod models and consequently on *conditional* risk measures; see for instance [2, 9, 15, 17, 7]. In this paper too we work with conditional risk measures. It may be noted that the "partial information" case can also be viewed as a "partial aggregation" case, so that the idea of a conditional risk measure can not only be applied in multiperiod situations but also in any context in which evaluation takes place in several stages through successively higher levels of aggregation.

The most basic object related to a risk measure is its acceptance set, and in fact it has

been argued that the acceptance set is more fundamental than the risk measure itself [1]. Our main purpose in this paper is to characterize the acceptance set corresponding to the strictest common relaxation of two conditional risk measures. In the unconditional case this set is easily described as the union of the acceptance sets of the two given risk measures, but when we have only partial aggregation the set-theoretic union is in general not even the acceptance set of any conditional risk measure. For this reason we introduce a concept which we call the *conditional union*. The conditional union is a superset of the set-theoretic union, and we show that it gives the acceptance set of the strictest common relaxation.

The literature on risk measures is marked by differences in terminology and in conventions. Even the term "risk measure" as it has been used in the recent literature may be viewed as debatable, one of the reasons being that it refers to a focus on adverse outcomes which is in fact from a mathematical perspective largely immaterial. In this paper we will use the term "evaluation", following Peng [13]. The sign convention that we use is "positive/positive", meaning that positive outcomes of random variables are interpreted as gains rather than losses, and outcomes are evaluated in a way that preserves rather than inverts signs. Under these conventions, convex risk measures are replaced by concave evaluations, and the best-case operator is obtained by taking supremum rather than infimum.

We start with recalling some basic definitions and properties in the next section. All main results are in Section 3, and Section 4 concludes. There is an Appendix containing some technical material on the essential supremum which is needed in the proof of the main theorem in Section 3.

2 Basic definitions and properties

In this section we list some basic definitions and properties and fix notation. The material in this section is well known (cf. [7, 5, 8]).

2.1 Standing assumptions and notation

Throughout the paper we use a probability space $(\Omega, \mathcal{F}, P)$. The terms "measurable" and "almost surely" without further specification mean $\mathcal{F}$-measurable and P-almost surely, respectively. The complement of an event $F \in \mathcal{F}$ is denoted by F^c. We write $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$. Elements of L^∞ will be referred to as random variables but also as "payoffs" or "positions". We work with bounded random variables for simplicity; cf. [4] for methods of generalization to the case of unbounded variables. The notation $Q \ll P$, when Q and P are measures, means that Q is absolutely continuous with respect to P.

Throughout the paper we work with a fixed sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, which we refer to as the *conditioning sub-σ-algebra*. The σ-algebra $\mathcal{G}$ is thought of as representing an intermediate

level of aggregation between the trivial σ-algebra $\{\emptyset, \Omega\}$ which represents full aggregation and the σ-algebra $\mathcal{F}$ which corresponds to complete disaggregation. All statements and properties that involve conditioning are taken with respect to $\mathcal{G}$, unless otherwise indicated.

We write $L^\infty_{\mathcal{G}}$ to indicate the subset of L^∞ consisting of $\mathcal{G}$-measurable random variables. Given a random variable $X \in L^\infty$, the random variable $\|X\|_{\mathcal{G}} \in L^\infty_{\mathcal{G}}$ defined by $\|X\|_{\mathcal{G}} = \operatorname{ess\,inf}\{m \in L^\infty_{\mathcal{G}} \mid m \geq |X|\} \in L^\infty_{\mathcal{G}}$ is referred to as the *conditional norm* of X. The notation $\|X\|$ (without subscript) refers to the usual L^∞-norm of X, which is the same as the $\mathcal{G}$-conditional norm when $\mathcal{G}$ is the trivial sub-σ-algebra $\{\emptyset, \Omega\}$. We have $\|X\|_{\mathcal{G}} \leq \|X\|$ for all $X \in L^\infty$.

All equalities and inequalities applied to random variables are understood to hold almost surely; also, convergence is taken in the almost sure sense unless indicated otherwise. We use $\inf X$ and $\sup X$ to refer to the essential infimum and the essential supremum, respectively, of an element X of L^∞. Given a nonempty set $\mathcal{S} \subset L^\infty$, $\operatorname{ess\,sup} \mathcal{S}$ is defined as the least element in the a.s.-equivalence classes of measurable functions from Ω to $\mathbb{R} \cup \{\infty\}$ that dominate all elements of $\mathcal{S}$ in the almost sure sense (see for instance [8]); $\operatorname{ess\,inf} \mathcal{S}$ is defined similarly.

2.2 Conditional evaluations

The definition below follows [5] (cf. also [7]). We follow [14] in using the term "translation equivariance" rather than the more often used phrase "translation invariance".

Definition 2.1 A *conditional evaluation* is a mapping ϕ from L^∞ to $L^\infty_{\mathcal{G}}$ that is monotonic (i.e. for $X, Y \in L^\infty$, if $X \geq Y$ then $\phi(X) \geq \phi(Y)$), normalized (i.e. $\phi(0) = 0$), and translation equivariant (i.e. for $X \in L^\infty$, if $C \in L^\infty_{\mathcal{G}}$ then $\phi(X + C) = \phi(X) + C$).

Conditional evaluations, just like conditional expectations, map $\mathcal{F}$-measurable functions to $\mathcal{G}$-measurable functions. Unlike conditional expectations, conditional evaluations can be nonlinear. The following concavity property is often considered in the literature but plays a lesser role in the current paper.

Definition 2.2 A conditional evaluation ϕ is said to be *concave* if

$$\phi(\Lambda X + (1 - \Lambda)Y) \geq \Lambda\phi(X) + (1 - \Lambda)\phi(Y) \tag{2.1}$$

for all $X, Y \in L^\infty$ and for all $\Lambda \in L^\infty_{\mathcal{G}}$ with $0 \leq \Lambda \leq 1$.

An important fact is the following.

Lemma 2.3 ([7, Prop. 1,2], [5, Prop. 3.3]) *Let ϕ be a conditional evaluation. Then ϕ satisfies the local property, that is to say, for all $G \in \mathcal{G}$ and $X, Y \in L^\infty$ we have*

$$\phi(1_G X + 1_{G^c} Y) = 1_G \phi(X) + 1_{G^c} \phi(Y). \tag{2.2}$$

When normalization (i.e. $\phi(0) = 0$) is taken as a part of the definition of a conditional evaluation as we do in this paper, the local property for conditional evaluations is equivalent, as shown in [7, Prop. 1], to the *regularity* property [13, 9, 5]

$$\phi(1_G X) = 1_G \phi(X) \qquad \text{for all } G \in \mathcal{G} \text{ and } X \in L^\infty. \tag{2.3}$$

2.3 Acceptance sets

The *acceptance set* of a conditional evaluation $\phi : L^\infty \to L^\infty_{\mathcal{G}}$ is defined by

$$\mathcal{A}(\phi) = \{X \in L^\infty \mid \phi(X) \geq 0\}.$$

Conversely, given an arbitrary set $\mathcal{S} \subset L^\infty$, one may define a mapping from L^∞ to $L^\infty_{\mathcal{G}}$ by

$$\phi_{\mathcal{S}}(X) = \operatorname{ess\,sup}\{Y \in L^\infty_{\mathcal{G}} \mid X - Y \in \mathcal{S}\}. \tag{2.4}$$

The mapping was introduced in [7] and is called the *conditional capital requirement* induced by $\mathcal{S}$. The following proposition states conditions under which the conditional capital requirement is a conditional evaluation.

Proposition 2.4 ([7], [5], [16, Prop. 2.3]) *If $\mathcal{S} \subset L^\infty$ is such that*

$$0 \in \mathcal{S} \tag{2.5}$$

$$X \in \mathcal{S},\ Y \in L^\infty,\ Y \geq X \ \Rightarrow\ Y \in \mathcal{S} \tag{2.6}$$

$$X \in L^\infty_{\mathcal{G}} \cap \mathcal{S} \ \Rightarrow\ X \geq 0 \tag{2.7}$$

then the conditional capital requirement $\phi_{\mathcal{S}}$ defined by (2.4) is a conditional evaluation.

The relation $\phi = \phi_{\mathcal{A}(\phi)}$ holds ([7, §2.3], [5, Prop. 3.9]), as well as the inclusion $\mathcal{A}(\phi_{\mathcal{S}}) \supset \mathcal{S}$. Necessary and sufficient conditions under which this inclusion is in fact an equality can be stated as follows.

Proposition 2.5 *For $\mathcal{S} \subset L^\infty$, the relation $\mathcal{A}(\phi_{\mathcal{S}}) = \mathcal{S}$ holds if and only if $\mathcal{S}$ satisfies the three properties (2.5–2.7) as well as the two additional properties*

$$1_G X + 1_{G^c} Y \in \mathcal{S} \quad \text{for all } X, Y \in \mathcal{S},\ G \in \mathcal{G} \tag{2.8}$$

$$X_n \in \mathcal{S}\ (n = 1, 2, \dots),\ \|X_n - X\|_{\mathcal{G}} \to 0 \ \Rightarrow\ X \in \mathcal{S} \quad (X \in L^\infty). \tag{2.9}$$

Condition (2.8) may be called the *local property* of subsets of L^∞, and (2.9) may be referred to as *conditional closedness*. The proposition above is an immediate consequence of the following result ([5, Prop. 3.10]).

PROPOSITION 2.6 *Let $\mathcal{S} \subset L^\infty$ satisfy (2.5–2.7), so that $\phi_{\mathcal{S}}$ is a conditional evaluation. Then $\mathcal{A}(\phi_{\mathcal{S}})$ is the smallest subset of L^∞ that contains $\mathcal{S}$, has the local property, and is conditionally closed.*

A property that is related to the local property is *closedness under isolation*:

$$1_G X \in \mathcal{S} \quad \text{for all } X \in \mathcal{S},\ G \in \mathcal{G}. \tag{2.10}$$

When $0 \in \mathcal{S}$, closedness under isolation is implied by the local property.

3 The strictest common relaxation

3.1 Definition

Let us say that a conditional evaluation ϕ is *at least as strict* as another conditional evaluation ϕ' if

$$\phi(X) \leq \phi'(X) \quad \text{for all } X \in L^\infty. \tag{3.1}$$

In this case we also say that ϕ' is a (possibly non-strict) *relaxation* of ϕ. We write $\phi \leq \phi'$ or equivalently $\phi' \geq \phi$. When Φ is a family of conditional evaluations, we write $\phi \geq \Phi$ in case $\phi \geq \phi'$ for all $\phi' \in \Phi$.

DEFINITION 3.1 Let Φ be a family of conditional evaluations. We say that a conditional evaluation ϕ is the *strictest common relaxation* of the conditional evaluations in the family Φ if $\phi \geq \Phi$, and $\phi \leq \phi'$ for any conditional evaluation ϕ' that satisfies $\phi' \geq \Phi$.

The definition does not immediately ensure that the strictly common relaxation of any given family does indeed exist, but this fact is easily established. Given a family Φ of conditional evaluations, we can define a mapping $\bigvee\Phi$ from L^∞ to $L^\infty_{\mathcal{G}}$ by

$$(\bigvee\Phi)(X) = \operatorname{ess\,sup}\{\phi(X) \mid \phi \in \Phi\} \qquad (X \in L^\infty). \tag{3.2}$$

It follows from elementary properties of the essential supremum that $\bigvee\Phi$ is a conditional evaluation. This leads to the following conclusion.

PROPOSITION 3.2 *Let Φ be a family of conditional evaluations. The strictest common relaxation of Φ exists and is given by the essential supremum $\bigvee\Phi$.*

If $\mathcal{G}$ is trivial and the collection Φ is finite, then $(\bigvee\Phi)(X)$ is simply the maximum of all evaluations $\phi(X)$ with $\phi \in \Phi$. In other words, the strictest common relaxation is a best-case operator. It is a natural counterpart of the worst-case operator given by $(\bigwedge\Phi)(X) = \operatorname{ess\,inf}\{\phi(X) \mid \phi \in \Phi\}$. Another related operator is the *convolution* [6, 3, 11] defined (for a finite collection $\Phi = \{\phi_1, \ldots, \phi_N\}$ of conditional evaluations) by

$$(\Box\Phi)(X) = \operatorname{ess\,sup}\Big\{\sum_{i=1}^{N} \phi_i(X_i) \mid X_i \in L^\infty \ (i = 1, \ldots, N),\ \sum_{i=1}^{N} X_i = X\Big\}.$$

We have $\Box\Phi \geq \bigvee\Phi$, but equality does not hold in general. Indeed, it may happen that the convolution is infinite, whereas the strictest common relaxation of two conditional evaluations is always finite. On the other hand, if all conditional evaluations in Φ are concave, then their convolution (if finite) is also concave [11, Thm. 4.1], whereas the strictest common relaxation in general does not preserve concavity.

3.2 Application to VaR

Value at Risk (VaR) can be defined as follows [8, Ex. 4.11]:

$$\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R} \mid P(X + m < 0) \leq 1 - \alpha\}.$$

This is an unconditional risk measure, that is to say, the conditioning sub-σ-algebra $\mathcal{G}$ is trivial. Of course it would be possible to consider conditional versions, but our purpose here is to show how VaR can be defined in terms of simpler operations (namely conditional expectations) by means of sup and inf operations.

Proposition 3.3 *The following relation holds, with $F \in \mathcal{F}$ being understood:*

$$\mathrm{VaR}_\alpha(X) = \bigvee_{P(F)\geq\alpha} \bigwedge_{Q\ll P} E^Q[X \mid F]. \tag{3.3}$$

Proof We already know that the operation on the right gives a conditional evaluation, so it is sufficient to show that the acceptance sets of the mappings on the left and on the right are the same. First, assume that $\mathrm{VaR}_\alpha(X) \geq 0$. By definition, this means that $P(X \geq 0) \geq \alpha$, so that the set $F' := \{X \geq 0\}$ is among the sets that define the supremum at the right hand side of (3.3). Clearly we have $E^Q[X \mid F'] \geq 0$ for all $Q \ll P$, so that $\bigwedge_{Q\ll P} E^Q[X \mid F'] \geq 0$ and consequently

$$\bigvee_{P(F)\geq\alpha} \bigwedge_{Q\ll P} E^Q[X \mid F] \geq 0. \tag{3.4}$$

Conversely, suppose that (3.4) holds. To show that $\mathrm{VaR}_\alpha(X) \geq 0$, assume that the opposite is true, so that $P(X < 0) > 1 - \alpha$. Then in fact there must be an $n \in \mathbb{N}$ such that $P(X < -\frac{1}{n}) > 1 - \alpha$. For every $F \in \mathcal{F}$ such that $P(F) \geq \alpha$, we then must have $P(F \cap \{X < -\frac{1}{n}\}) > 0$. Consequently there exists a measure $Q' \ll P$ such that $Q'(F \cap \{X < -\frac{1}{n}\}) = 1$,

which implies $E^{Q'}[X \mid F] < -\frac{1}{n}$. It follows that $\bigwedge_{Q \ll P} E^Q[X \mid F] < -\frac{1}{n}$ for all F with $P(F) \geq \alpha$, so that $\bigvee_{P(F) \geq \alpha} \bigwedge_{Q \ll P} E^Q[X \mid F] < -\frac{1}{n}$. We have a contradiction. □

3.3 Corresponding operation on acceptance sets

The order relation between conditional evaluations is related in a natural way to the inclusion relation between acceptance sets.

PROPOSITION 3.4 *Let ϕ and ϕ' be conditional evaluations. We have*

$$\phi \leq \phi' \quad \Longleftrightarrow \quad \mathcal{A}(\phi) \subset \mathcal{A}(\phi'). \tag{3.5}$$

PROOF The inequality relation between the conditional evaluations obviously implies the inclusion relation between their acceptance sets. Conversely, assume that the inclusion relation holds, and take $X \in L^\infty$. Write $Y = X - \phi(X)$; then $\phi(Y) = 0$ so that $Y \in \mathcal{A}(\phi)$ which by assumption implies that $Y \in \mathcal{A}(\phi')$ or in other words $\phi'(Y) \geq 0$. This in turn implies $\phi'(X) \geq \phi(X)$ due to the conditional translation equivariance of ϕ' and the fact that $\phi(X)$ is $\mathcal{G}$-measurable. □

It is easy to verify (as noted in [6]) that $\mathcal{A}(\phi_1 \wedge \phi_2) = \mathcal{A}(\phi_1) \cap \mathcal{A}(\phi_2)$, where $\wedge$ denotes the operation of taking the essential infimum. However, when the conditioning sub-σ-algebra $\mathcal{G}$ is nontrivial, the supremum operation (3.2) on conditional evaluations does *not* in general correspond to the set-theoretic union of acceptance sets. We therefore propose the following operation.

DEFINITION 3.5 Given two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of L^∞, the set

$$\begin{aligned}\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2 := \{X \in L^\infty \mid \text{there exist } G_1, G_2 \in \mathcal{G}, \text{ with } G_1 \cap G_2 = \emptyset \text{ and } G_1 \cup G_2 = \Omega, \\ \text{and } X_1 \in \mathcal{S}_1,\ X_2 \in \mathcal{S}_2 \text{ s.t. } X = 1_{G_1} X_1 + 1_{G_2} X_2\}\end{aligned}$$

is called the *conditional union* of $\mathcal{S}_1$ and $\mathcal{S}_2$.

It is straightforward to establish that, when we have three subsets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$,

$$\begin{aligned}(\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2) \cup_{\mathcal{G}} \mathcal{S}_3 = \{X \in L^\infty \mid \text{there exist } G_i \in \mathcal{G},\ X_i \in \mathcal{S}_i\ (i = 1, 2, 3) \text{ s.t.} \\ G_i \cap G_j = \emptyset\ (i \neq j), G_1 \cup G_2 \cup G_3 = \Omega,\ X = 1_{G_1} X_1 + 1_{G_2} X_2 + 1_{G_3} X_3\}\end{aligned}$$

so that we can unequivocally speak of the conditional union of three subsets, and more generally the conditional union of any finite number of subsets can be defined. The conditional union contains the set-theoretic union (let all G_i's be empty except one) but is in general larger, unless $\mathcal{G} = \{\emptyset, \Omega\}$ (the case of full aggregation).

Clearly, the notion of conditional union is closely related to the local property. Below it is shown that, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets having the local property, then their conditional union is the smallest set that contains both $\mathcal{S}_1$ and $\mathcal{S}_2$ and that itself has the local property. As another illustration, consider the following proposition.

PROPOSITION 3.6 *A subset $\mathcal{S} \subset L^\infty$ has the local property if and only if*

$$\mathcal{S} = \mathcal{S} \cup_{\mathcal{G}} \mathcal{S}.$$

PROOF Suppose first that the condition holds. Take $G \in \mathcal{G}$ and $X, Y \in \mathcal{S}$. Then $1_G X + 1_{G^c} Y \in \mathcal{S} \cup_{\mathcal{G}} \mathcal{S} = \mathcal{S}$. Conversely, assume that $\mathcal{S}$ has the local property. Take $X = 1_{G_1} X_1 + 1_{G_2} X_2 \in \mathcal{S} \cup_{\mathcal{G}} \mathcal{S}$ ($G_1, G_2 \in \mathcal{G}$, $G_1 \cap G_2 = \emptyset$, $G_1 \cup G_2 = \Omega$, $X_1, X_2 \in \mathcal{S}$). Then $G_2 = G_1^c$ and it follows that $X \in \mathcal{S}$. □

The conditional union $\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$ preserves several properties of interest that the set $\mathcal{S}_1$ and $\mathcal{S}_2$ may have, as shown in the following proposition.

PROPOSITION 3.7 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subsets of L^∞. If $\mathcal{S}_1$ and $\mathcal{S}_2$ both have one of the following properties:*

(i) the local property (2.8)

(ii) closedness under isolation (2.10)

(iii) conditional nonnegativity (2.7)

then the conditional union $\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$ has the same property. If the two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are closed under isolation and solid (i.e. (2.6) is satisfied), then $\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$ is solid as well.

PROOF Write $\mathcal{S} = \mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$, and suppose that both $\mathcal{S}_1$ and $\mathcal{S}_2$ have the local property. Take $X, Y \in \mathcal{S}$ and $G \in \mathcal{G}$; we want to prove that $1_G X + 1_{G^c} Y \in \mathcal{S}$. By definition of the conditional union, we can write

$$X = 1_H X_1 + 1_{H^c} X_2, \qquad Y = 1_J Y_1 + 1_{J^c} Y_2$$

for some $H, J \in \mathcal{G}$, $X_1, Y_1 \in \mathcal{S}_1$, and $X_2, Y_2 \in \mathcal{S}_2$. We have

$$\begin{aligned} 1_G X + 1_{G^c} Y &= 1_G(1_H X_1 + 1_{H^c} X_2) + 1_{G^c}(1_J Y_1 + 1_{J^c} Y_2) = \\ &= (1_{H \cap G} + 1_{J \cap G^c})(1_G X_1 + 1_{G^c} Y_1) + (1_{H^c \cap G} + 1_{J^c \cap G^c})(1_G X_2 + 1_{G^c} Y_2). \end{aligned}$$

Note that $1_G X_i + 1_{G^c} Y_i \in \mathcal{S}_i$ ($i = 1, 2$) by the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the local property. The fact that

$$1_{H \cap G} + 1_{J \cap G^c} + 1_{H^c \cap G} + 1_{J^c \cap G^c} = 1$$

now implies that $1_G X + 1_{G^c} Y \in \mathcal{S}$.

Preservation of closedness under isolation is easily shown. Indeed, take $X \in \mathcal{S}$ and write $X = 1_H X_1 + 1_{H^c} X_2$ with $H \in \mathcal{G}$, $X_1 \in \mathcal{S}_1$, $X_2 \in \mathcal{S}_2$. Then, for any $G \in \mathcal{G}$,

$$1_G X = 1_H 1_G X_1 + 1_{H^c} 1_G X_2 \in \mathcal{S}$$

since $1_G X_1 \in \mathcal{S}_1$ and $1_G X_2 \in \mathcal{S}_2$.

Now assume that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are conditionally nonnegative. Take $X = 1_G X_1 + 1_{G^c} X_2 \in \mathcal{S}$ ($G \in \mathcal{G}$, $X_1 \in \mathcal{S}_1$, $X_2 \in \mathcal{S}_2$), and suppose that $X \in L^\infty_{\mathcal{G}}$. Then $1_G X = 1_G X_1 \in L^\infty_{\mathcal{G}} \cap \mathcal{S}_1$ so that $1_G X \geq 0$, by the conditional nonnegativity of $\mathcal{S}_1$. Likewise it follows that $1_{G^c} X \geq 0$, so that $X = 1_G X + 1_{G^c} X \geq 0$.

Finally, assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ both are closed under isolation and solid. To prove the solidness of $\mathcal{S}$, take $X = 1_G X_1 + 1_{G^c} X_2 \in \mathcal{S}$ ($G \in \mathcal{G}$, $X_1 \in \mathcal{S}_1$, $X_2 \in \mathcal{S}_2$), and suppose $Y \in L^\infty$ is such that $Y \geq X$. Then we also have $1_G Y \geq 1_G X = 1_G X_1$, which by the solidness and the closedness under isolation of $\mathcal{S}_1$ implies that $1_G Y \in \mathcal{S}_1$. Similarly it follows that $1_{G^c} Y \in \mathcal{S}_2$, so that $Y = 1_G(1_G Y) + 1_{G^c}(1_{G^c} Y) \in \mathcal{S}$. □

Proposition 3.8 *Suppose that $\mathcal{S}_1, \mathcal{S}_2 \subset L^\infty$ both have the local property. Then the conditional union $\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$ is the smallest set that has the local property and that contains both $\mathcal{S}_1$ and $\mathcal{S}_2$.*

Proof It has already been shown in Prop. 3.7 that $\mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$ has the local property. Suppose now that $\tilde{\mathcal{S}} \subset L^\infty$ has the local property and is such that $\tilde{\mathcal{S}} \supset \mathcal{S}_1 \cup \mathcal{S}_2$. Then in particular we have $1_G X_1 + 1_{G^c} X_2 \in \tilde{\mathcal{S}}$ for all $G \in \mathcal{G}$, $X_1 \in \mathcal{S}_1$, and $X_2 \in \mathcal{S}_2$, which means that $\tilde{\mathcal{S}} \supset \mathcal{S}_1 \cup_{\mathcal{G}} \mathcal{S}_2$. □

The following theorem establishes that the conditional union is the operation on acceptance sets that corresponds to taking the strictest common relaxation of two conditional evaluations.

Theorem 3.9 *Let ϕ_1 and ϕ_2 be conditional evaluations. Then*

$$\mathcal{A}(\phi_1 \vee \phi_2) = \mathcal{A}(\phi_1) \cup_{\mathcal{G}} \mathcal{A}(\phi_2).$$

Proof The set $\mathcal{A}(\phi_1 \vee \phi_2)$ has the local property, since it is the acceptance set of a conditional evaluation; moreover it contains both $\mathcal{A}(\phi_1)$ and $\mathcal{A}(\phi_2)$. It therefore follows from Prop. 3.8 that

$$\mathcal{A}(\phi_1 \vee \phi_2) \supset \mathcal{A}(\phi_1) \cup_{\mathcal{G}} \mathcal{A}(\phi_2).$$

To prove the reverse inclusion, it is enough, by Lemma 5.3 in the Appendix, to show that the assumption

$$1_{\{\phi_1(X)<0\}} \phi_2(X) \geq 0 \tag{3.6}$$

for $X \in L^\infty$ implies that $X \in \mathcal{A}(\phi_1) \cup_{\mathcal{G}} \mathcal{A}(\phi_2)$. Write $G = \{\phi_1(X) < 0\}$, and note that $G \in \mathcal{G}$. We have $\phi_1(1_{G^c}X) = 1_{G^c}\phi_1(X) \geq 0$ by definition of G, and $\phi_2(1_G X) = 1_G \phi_2(X) \geq 0$ by assumption (3.6). It follows that $X = 1_{G^c}(1_{G^c}X) + 1_G(1_G X) \in \mathcal{A}(\phi_1) \cup_{\mathcal{G}} \mathcal{A}(\phi_2)$. □

4 Conclusions

One way to combine two evaluations is to take the best of the two. From a conservative point of view such an operation may be looked upon as dangerous, and indeed concavity is not preserved in general. Still under some circumstances this way of combining evaluations can be reasonable, and the best-case operator can also be of use as an instrument of description as we have seen in the case of Value at Risk. The acceptance set of the maximum of two unconditional (fully aggregated) evaluations is simply the set-theoretic union of the acceptance sets of the separate evaluations, but this statement is no longer true in general when we consider conditional (partially aggregated) evaluations. In this paper we have identified the operation on acceptance sets that corresponds to the combination of two conditional evaluations by the best-case operator.

A limitation in the theory that was developed here is that we have only considered combinations of two evaluations. The extension to finite collections of evaluations is straightforward, but we have not answered the question how to describe the acceptance set of the strictest common relaxation of an infinite family of conditional evaluations.

5 Appendix

In this appendix we prove some general results concerning the essential supremum of a family of random variables. We work in the same context as in the body of the paper.

Lemma 5.1 *Let $\mathcal{Z} \subset L^\infty$ be nonempty and bounded. For any $F \in \mathcal{F}$, we have*

$$\operatorname{ess\,sup} 1_F \mathcal{Z} = 1_F \operatorname{ess\,sup} 1_F \mathcal{Z} \tag{5.1}$$

where $1_F\mathcal{Z}$ denotes $\{1_F Z \mid Z \in \mathcal{Z}\}$.

Proof Write $Y = \operatorname{ess\,sup} 1_F \mathcal{Z}$; we have to show that $Y = 1_F Y$. The inequality $Y \geq 1_F Z$ which holds for all $Z \in \mathcal{Z}$ implies $1_F Y \geq 1_F Z$, so that $1_F Y$ is an upper bound for the family $1_F\mathcal{Z}$. Since Y is the least upper bound for this family, we have $1_F Y \geq Y$. To prove the converse inequality, take any $Z \in \mathcal{Z}$. The inequality $Y \geq 1_F Z$ implies that $1_{F^c} Y \geq 1_{F^c} 1_F Z = 0$. □

Lemma 5.2 *Let $\mathcal{Z} \subset L^\infty$ be nonempty and bounded. For any $F \in \mathcal{F}$, we have*

$$1_F \operatorname{ess\,sup} \mathcal{Z} = 1_F \operatorname{ess\,sup} 1_F \mathcal{Z}. \tag{5.2}$$

Proof Write $Y = \operatorname{ess\,sup} 1_F \mathcal{Z}$. From the fact that $\operatorname{ess\,sup} \mathcal{Z}$ is an upper bound for the family $\mathcal{Z}$ it follows that $1_F \operatorname{ess\,sup} \mathcal{Z}$ is an upper bound for $1_F \mathcal{Z}$; consequently we have $1_F \operatorname{ess\,sup} \mathcal{Z} \geq Y$ which implies $1_F \operatorname{ess\,sup} \mathcal{Z} \geq 1_F Y$. To prove the converse inequality, note that $Z = 1_F Z + 1_{F^c} Z \leq Y + 1_{F^c} Z$ for all $Z \in \mathcal{Z}$. It follows that

$$\operatorname{ess\,sup} \mathcal{Z} \leq Y + \operatorname{ess\,sup} 1_{F^c} \mathcal{Z} = Y + 1_{F^c} \operatorname{ess\,sup} 1_{F^c} \mathcal{Z}$$

where the translation invariance of the essential supremum is used in the first step and the previous lemma in the second. From the above, we have $1_F \operatorname{ess\,sup} \mathcal{Z} \leq 1_F Y$ as required. □

As an immediate corollary of the two lemmas, we have what might be called the regularity property of the essential supremum:

$$\operatorname{ess\,sup} 1_F \mathcal{Z} = 1_F \operatorname{ess\,sup} \mathcal{Z}. \tag{5.3}$$

We use this to prove a more specialized result.

Lemma 5.3 *The following equivalence relation holds for $Z_1, Z_2 \in L^\infty$:*

$$\operatorname{ess\,sup}\{Z_1, Z_2\} \geq 0 \;\Leftrightarrow\; 1_{\{Z_1<0\}} Z_2 \geq 0. \tag{5.4}$$

Proof Write $F = \{Z_1 < 0\} \in \mathcal{F}$. Assume first that $1_F Z_2 \geq 0$. To prove that $\operatorname{ess\,sup}\{Z_1, Z_2\} \geq 0$, it suffices to show that for every $\varepsilon > 0$ we have $P(\operatorname{ess\,sup}\{Z_1, Z_2\} \leq -\varepsilon) = 0$. So, take $\varepsilon > 0$ and define $G = \{\operatorname{ess\,sup}\{Z_1, Z_2\} \leq -\varepsilon\}$. By the regularity of the essential supremum, this implies that $1_G Z_1 \leq -\varepsilon 1_G$ and $1_G Z_2 \leq -\varepsilon 1_G$. Moreover we have $1_{F^c} Z_1 \geq 0$ and $1_F Z_2 \geq 0$ by definition and by assumption respectively, so that we can write

$$0 \leq 1_{F^c \cap G} Z_1 \leq -\varepsilon 1_{F^c \cap G} \leq 0$$

as well as

$$0 \leq 1_{F \cap G} Z_2 \leq -\varepsilon 1_{F \cap G} \leq 0.$$

It follows that all inequalities in the above are actually equalities, so that in particular $1_{F^c \cap G} = 0$ and $1_{F \cap G} = 0$. Consequently we have $1_G = 0$, or in other words $P(G) = 0$.

For the converse part of the proof, assume now that $\operatorname{ess\,sup}\{Z_1, Z_2\} \geq 0$. Take $\varepsilon > 0$, and define $G = \{1_F Z_2 \leq -\varepsilon\}$. We then have

$$1_G 1_F Z_2 \leq -\varepsilon 1_G. \tag{5.5}$$

Take $k \in \mathbb{N}$ and define $F_k = \{Z_1 \leq -\frac{1}{k}\}$. Since $F_k \subset F$, we have from (5.5)

$$1_{F_k} 1_G Z_2 \leq -\varepsilon 1_{F_k} 1_G.$$

By definition of F_k, we also have

$$1_{F_k} 1_G Z_1 \leq -\tfrac{1}{k} 1_{F_k} 1_G.$$

Therefore we can write

$$0 \leq 1_{G \cap F_k} \operatorname{ess\,sup}\{Z_1, Z_2\} = \operatorname{ess\,sup}\{1_{G \cap F_k} Z_1, 1_{G \cap F_k} Z_2\} \leq \max(-\varepsilon, -\tfrac{1}{k})\, 1_{G \cap F_k} \leq 0.$$

From this it follows that $1_{G \cap F_k} = 0$, or in other words, $P(G \cap F_k) = 0$. Because $G \cap F = \cup_{k=1}^{\infty} (G \cap F_k)$, we obtain $P(G \cap F) = 0$. To show that the equality $P(G \cap F^c) = 0$ holds as well, multiply both sides of (5.5) by the indicator function of F^c to obtain

$$0 = 1_{F^c} 1_G 1_F Z_2 \leq -\varepsilon 1_{F^c} 1_G \leq 0$$

which indeed implies that $1_{F^c} 1_G = 0$. We conclude that $P(G) = 0$. Since $\varepsilon > 0$ was arbitrary, it follows that $1_F Z_2 \geq 0$, which is what we had to prove. $\square$

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